An experimental verification of Bayes' Theorem

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Abstract
Thomas Bayes, an English Presbyterian minister, identified a method for computing conditional probabilities after new information has been added. This method is useful for determining the probabilities in "shell games", otherwise known as the Monty Hall problem.

Introduction

Bayes' theorem formalizes a procedure for computing probability after new information has been added to a problem. Rev. Bayes' paper was published posthumously in 1763 in the journal Philosophical Transactions.[3]

Modern applications of Bayes' Theorem include subjectivist approaches to epistemology, statistics, and inductive logic.[1] A more commonplace application can be found in shell games, where an object is hidden in some fashion, and one of its possible locations is revealed. Bayes' Theorem predicts the change in probability associated with that new information.

Theoretical Section

Bayes' theorem can be simply derived from basic statistical principles. Given two mutually exclusive and exhaustive probabilities

$$P(A)+P(A')=1 \quad (1)$$

and another non-zero probability

$$P(B) \quad (2)$$

the conditional probability can be calculated

$$P(A/B)=\frac{P(A\cap B)}{P(B)} \quad (3)$$

Bayes' theorem is a generalized extension of this concept for a set of mutually exclusive events A_1, A_2,…A_n, and another event B, such that the conditional probability $P(B/A_i)$ is known, but $P(A_i/B)$ is the desired quantity.

Equation (3) can be manipulated to give

$$P(A_i\cap B)=P(A_i/B)*P(B) \quad (4)$$

and

$$P(A_i\cap B)=P(B/A_i)*P(A_i) \quad (5)$$

These two equations can be set equal to one another

$$P(A_i/B)*P(B)=P(B/A_i)*P(A_i) \quad (6)$$

and then divided by $P(B)$ to gain the desired quantity

$$P(A_i/B)=\frac{P(B/A_i)*P(A_i)}{P(B)} \quad (8)$$

However, $P(B)$ is an unknown. It can be computed by summing

$$P(B)=\sum_{i=1}^{n}P(A_i\cap B) \quad (9)$$

substituting eq. (4) for the right hand side gives

$$P(B)=\sum_{i=1}^{n}\mathrm{P}(\mathrm{A}_{\mathrm{i}}/\mathrm{B})\ *P(A_i) \quad (10)$$

which is then substituted into the bottom of equation (8)

$$P(A_i/B)=\frac{P(B/A_i)*P(A_i)}{\sum_{i=1}^{n}\mathrm{P}(\mathrm{B}/\mathrm{A}_{\mathrm{i}})*P(A_i)} \quad (11)$$

Experimental Section

A simple experimental verification of Bayes' theorem was conducted using a human test subject, three paper cups, and a bag of peanut M&M's.

Fig. 1 Test subject

A peanut M&M was placed under a paper cup outside of the subject's view. The three cups were then revealed to the subject. The subject was asked to choose a cup. Of the two remaining cups, one was lifted to show the M&M was not beneath it. The subject was then asked whether they would keep their first choice, or switch their preference to the other cup. After the choice was made, the chosen cup was lifted and the subject was rewarded if the M&M was underneath the chosen cup.

Fig. 2 Experimental setup

Data for this experiment was recorded for fifty trials. The data consisted of whether the subject switched their preference after an empty cup was revealed, and whether the M&M was found on that trial.

The following probabilities can be assigned to the events in this experiment:

M_i: the probability the M&M is hidden under cup i.
R_i: the probability cup i is revealed.

For the sake of discussion, assume the subject chose cup 1. What then is the probability of finding the M&M given the subject retains the original choice, or chooses the other cup? The following compound probabilities result:

$$P(R_2 / M_2) = 0$$
$$P(R_2 / M_1) = 0.5$$
$$P(R_2 / M_3) = 1.0$$
$$P(R_3 / M_2) = 0.5$$
$$P(R_3 / M_1) = 0.5$$
$$P(R_3 / M_3) = 0.5$$

Using these probabilities and eqn. 11, the probability of success can be found in each case.

Case 1: Stick
Further assume that cup 2 is revealed. This implies:

$$P(M_1 / R_2) = \frac{P(R_2 / M_1) * P(M_1)}{\sum_{i=1}^{n} [\mathrm{P(R_2/M_i)} * P(M_i)]}$$

$$P(M_1 / R_2) = \frac{\frac{1}{2} * \frac{1}{3}}{\frac{1}{2} * \frac{1}{3} + 0 * \frac{1}{3} + 1 * \frac{1}{3}}$$

$$P(M_1 / R_2) = \frac{\frac{1}{2} * \frac{1}{3}}{\frac{1}{3}} = \frac{1}{3}$$

Case 2: Switch

$$P(M_3 / P_2) = \frac{P(R_2 / M_3) * P(M_3)}{P(R_2)}$$

$$P(M_3 / P_2) = \frac{1 + \frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

By symmetry we can see that the mathematics is exactly the same for any combination of initial choice and cup revealed.

Results and Discussion

The observed probability of finding the M&M sticking with the original choice was 0.317, which closely approximates the expected value of 1/3. Whereas the probability of winning with a switch was only 1/3. The test subject indicated a preference for sticking, so switching was only selected nine times. This provides an explanation for the deviation from the expected value of 2/3, due to an insufficient sample space. Further experimentation to increase the sample size was precluded due to chocolate overdose.

Conclusion

It is apparent that a larger sample space would produce data with higher reliability. Nevertheless, the data for sticking with the original choice conforms to expectation due to a larger number of trials, and can be considered reliable. Sticking with the original choice produces no better results than the 1/3 chance the M&M was hidden beneath that cup.

A simpler experiment design could facilitate future tests of Bayes' theorem by decreasing the time necessary. Reducing the chocolate intake of the test subject will also increase the sample space, and is highly recommended for future studies.

References

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