A Constructive Study of Markov Equilibria in Stochastic Games with Strategic Complementarities*

Lukasz Balbus† Kevin Reffett‡ Lukasz Wożny§

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Abstract

We study a class of infinite horizon, discounted stochastic games with strategic complementarities. In our class of games, we prove the existence of a Stationary Markov Nash equilibrium, as well as provide methods for constructing this least and greatest equilibrium via a successive approximation scheme. We also provide equilibrium monotone comparative statics results relative to ordered perturbations of the space of stochastic games. Under slightly stronger assumptions, we prove the stationary Markov Nash equilibrium values form a complete lattice, with least and greatest equilibrium value functions being the uniform limit of these successive approximations starting from pointwise lower and upper bounds.

keywords: Markov equilibria, stochastic games, constructive methods
JEL codes: C62, C73

1 Introduction and related literature

Since the class of infinite horizon discounted stochastic games was introduced by Shapley (1953), the question of existence and characterization of equilibrium has been the object of extensive study in game theory\(^{1}\) (see Duggan, 2012; Levy, 2012, e.g.). Moreover, recently stochastic games have become a fundamental tool for studying strategic interactions in dynamic economic models, where agents possess some form of limited commitment over time. Examples of such situations in the literature include work in such diverse fields as: (i) equilibrium models of stochastic growth without commitment (e.g., Balbus, Reffett, and Wożny (2012) along with

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†Faculty of Mathematics, Computer Sciences and Econometrics, University of Zielona Góra, Poland.
‡Department of Economics, Arizona State University, Phoenix, USA.
§Department of Theoretical and Applied Economics, Warsaw School of Economics, Warsaw, Poland. Address: al. Niepodległości 162, 02-554 Warszawa, Poland. E-mail: lukasz.wozny@sgh.waw.pl.

1For example, see Raghavan, Ferguson, Parthasarathy, and Vrieze (1991) or Neyman and Sorin (2003) for an extensive survey of results, along with references.
references), (ii) international lending and sovereign debt (Atkeson, 1991), (iii) optimal Ramsey taxation (Phelan and Stacchetti, 2001), (iv) models of savings and asset prices with hyperbolic discounting (Harris and Laibson, 2001), (v) dynamic political economy (e.g., see Lagunoff (2009), and references contained within), (vi) dynamic negotiations with status quo (see Duggan and Kalandrakis (2012)) or (vii) dynamic oligopoly models (see Cabral and Riordan (1994)), among others.

A focal point of a great deal of applied work has been on minimal state space Markov stationary Nash equilibrium (MSNE). Apart from the existence and characterization of such equilibrium, more recently though, new questions concerning its computation have become a central focus of applied researchers, who seek to either calibrate or estimate particular parameters of stochastic games. When constructing approximate solutions for an equilibrium in any dynamic/stochastic game/economy, a prerequisite for the rigorous implementation of numerical methods is to have access to a sharp set of theoretical tools that both characterize the structure of elements of the set of MSNE in the economy under study, as well as identify how the equilibrium set vary in deep parameters of the game. In such a situations, one must be concerned first with constructive fixed point methods that can be tied directly to numerical approximation schemes. For finite games, the question of existence and computation of MSNE has been essentially resolved. Unfortunately, for infinite games, although the equilibrium existence question has received a great deal of attention, results that provide characterization of the MSNE set (and how it varies in deep parameters) are needed, not only to address the question of accuracy of approximation methods, but also to develop notions of qualitative and quantitative stability.

The aim of this paper is to address all of these issues (i.e., existence, computation, and equilibrium comparative statics) within the context of a single unified methodological approach. Our methods are constructive and monotone, where our notion of monotonicity is defined using pointwise partial order on function spaces of values or pure strategies. To obtain sufficient conditions for such constructive monotone methods relative to the set of MSNE, we study an important subclass of stochastic games, namely these with strategic complementarities and positive externalities. This class of games is important among others, as equilibrium is known to exists relative to general assumptions, but also as many other classes of games are simply uncomputable.

For these infinite horizon stochastic supermodular games, we prove a number of new results relative to Curtat (1996), Amir (2005) or Nowak (2007). First, we prove the existence of MSNE in broader spaces of (bounded, measurable) pure strategies. Second, and perhaps most importantly per applications, we develop reasonable sufficient conditions for MSNE to exist over very general state spaces. Third, we give sufficient conditions under which the set of MSNE values forms a complete lattice of Lipschitz continuous functions. Fourth, we contribute to the literature that studies specific forms of transition kernels, and we are able to show the full power of the mixing

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2In this paper, we study both the construction of Markov stationary Nash equilibrium strategies (MSNE) and their associated values (MSNE values). The pair (MSNE, MSNE value) is what we refer to as a SMNE.

3By “finite game” we mean a dynamic/stochastic game with a (a) finite horizon and (b) finite state and strategy space game. By an “infinite game”, we mean a game where either (c) the horizon is countable (but not finite), or (d) the action/state spaces are uncountable. We shall focus on stochastic games where both (c) and (d) are present.

4For example, relative to existence, see Federgruen (1978); for computation of equilibrium, see Herings and Peeters (2004); and, finally, for estimation of deep parameters, see Aguirregabiria and Mira (2007), Pesendorfer and Schmidt-Dengler (2008) or Pakes, Ostrovsky, and Berry (2007).

5There are exceptions to this remark. For example, in Balbus and Nowak (2004), a truncation argument for constructing a SMNE in symmetric games of capital accumulation is proposed. In general, though, in this literature, a unified approach to approximation and existence has not been addressed.
assumption studied extensively by Nowak and coauthors in a class of stochastic games. Finally, we prove our results using minimal assumptions (i.e. we are also able to present counterexamples that violate both our assumptions and results).

Along this line, our results contribute to the recent literature on the (non-)existence of MSNE in a class of discounted, stochastic games with absolute continuity conditions (ACC) (see Levy, 2012) or with additional noise (see Duggan, 2012). Specifically, as shall be clear in the sequel, we provide results on the existence MSNE in a stochastic game over uncountable state space, where transition between states is not absolutely continuous. This result complements the recent result of Levy (2012) concerning the importance of ACC; namely, that this condition is neither sufficient or necessary for the existence of MSNE. Moreover, we obtain our existence results per MSNE in pure strategies without introducing additional correlation or noise as is done in the work of Nowak and Raghavan (1992) or Duggan (2012).

Finally, unlike the existing work in the literature, our methods provide a unified approach to both finite and infinite horizon games. That is, we give conditions under which infinite horizon MSNE are simply the limits of equilibria in truncated finite horizon stochastic games. This fact is particularly important for numerical implementations. These can be seen, hence, as a direct generalization of Amir (1996b) results on optimal policies for discounted dynamic lattice programming models to equilibria in a general class of stochastic supermodular games. As in Amir is his decision-theoretic setting, we are able to provide monotone comparative statics results on the set of MSNE equilibria for the infinite horizon stochastic game, as well as describe how equilibrium comparative statics can be computed. This is particularly important, when one seeks to construct a stable selection of the set of MSNE that are numerically (and theoretically) tractable as functions of the deep parameters of the economy/game. See also Amir (1996a) for one of the first attempts to use a theory of supermodularity for a study of stochastic games.

The rest of the paper is organized as follows. Section 2 presents a motivating example that highlights the applicability of our results. Then section 3 states the formal definition of an infinite horizon, stochastic game. Under general conditions, in section 3.2, we propose a method for MSNE existence and computation. In section 3.4, we present related equilibrium comparative statics and equilibrium dynamics results. In section 4, we present applications of our results.

2 Motivating example

We start with a simple motivating example of computing extremal MSNE in a two player stochastic game with uncountable and two-dimensional state space the highlights the essential issues raised by this paper. Each period a pair of states \((s_1, s_2) \in [0,1] \times [0,1] =: S\) is drawn and after observing it players choose one of two actions 1 or 0. The payoffs from a stage game are given in the following table:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(s_1, s_2)</td>
<td>(s_1 - c, b)</td>
</tr>
<tr>
<td>0</td>
<td>(b, s_2 - c)</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

This game, for example, can be thought of as a stylized partnership game in which players choose to keep putting effort into their partnership (cooperate) or to quit. That payoffs/states \(s_1, s_2\) represent the expected returns to each player from putting effort into the partnership. Parameter \(c\) represents the losses associated with staying in the partnership, when the other player walks out; parameter \(b\) represents a potential benefit from cheating on a cooperating
partner. Assume $1 > c > b > 0$. This stage game is clearly a supermodular game with positive externalities. For such a one-shot game, we have the following pure strategy Nash equilibria depending on parameters $s_1, s_2$:

$$\frac{s_1}{s_2} < b \quad \in (b, c) \quad > c$$

<table>
<thead>
<tr>
<th>$s_1 / s_2$</th>
<th>$&lt; b$</th>
<th>$\in (b, c)$</th>
<th>$&gt; c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt; b$</td>
<td>$(0, 0)$</td>
<td>$(0, 0)$</td>
<td>$(0, 1)$</td>
</tr>
<tr>
<td>$\in (b, c)$</td>
<td>$(0, 0)$</td>
<td>Two NE $(1, 1)$ and $(0, 0)$</td>
<td>$(1, 1)$</td>
</tr>
<tr>
<td>$&gt; c$</td>
<td>$(1, 0)$</td>
<td>$(1, 1)$</td>
<td>$(1, 1)$</td>
</tr>
</tbody>
</table>

Let state $s = (s_1, s_2)$ be drawn from distribution $Q(\cdot|s, a)$ parameterized by current action $a = (a_1, a_2)$. Now, say the stochastic transition of state $s$ is given by a transition kernel $Q(\cdot|s, a) = g(s, a)\lambda(\cdot|s) + (1 - g(s, a))\delta_0(\cdot)$, where $\lambda(\cdot|s)$ is a measure on $S$ and $\delta_0$ is a Delta Dirac concentrated at $(0, 0) \in S$. In this case, the main results of this paper show there exists the greatest and the least MSNE (see theorem 3.1), which also can be used from a numerical viewpoint to bound the set of all MSNE (see lemma 3.1). Moreover, our theorems show we can develop a simple successive approximation scheme to compute extremal MSNE (see 3.1).

To see how easy it is to apply our computational techniques, let us specify some parameters for the game, and compute extremal MSNE. For example, let $c = .8, b = .2$, discount factor $\beta = .9$ and assume that the function in the above specification of $Q$ is given by $g(s, a) = \frac{(s_1 + s_2)(a_1 + a_2)}{2}$, with $\lambda$ uniformly distributed on $[0, 1] \times [0, 1]$. In figure 1, we present the results of the computations of the greatest and the least expected values (as well as the iterations to these values): $\int_S v(s)\lambda(ds)$. Having computed the expected values, we also construct both the greatest and the least MSNE (figure 2). Apart from equilibrium existence and computation results, later in the paper, we prove an equilibrium monotone comparative statics theorem 3.4, which in principle could easily be computed for this game. The discussion of these results is in subsection 3.3.

This game is quite simple to compute using our techniques. Observe however, that celebrated APS approach, first suggested in the seminal work of Abreu, Pearce, and Stacchetti (1990), and later finding use in many papers on stochastic games (see Chakrabarti (1999) or Mertens and Parthasarathy (2003)) cannot be used to analyze this game. This is true for at least two reason. First, APS type methods fail in the case of stochastic games with uncountable two

Figure 1: Convergence of iterations (expected values) from above and below to extremal MSNE expected values. Iterations in expected value space and MSNE value bounds (left panel); speed of convergence (right panel).
dimensional state spaces (as conditions that guarantee the existence of an non-empty, compact set of equilibrium values for multidimensional state space are not known). Second, APS-type methods focus on the sequential or nonstationary Markov equilibria, and have little to say about the structure of MSNE. Finally, it bears mentioning that other topological techniques using continuity conditions also fail here as the transition probability $Q$ need not be absolutely continuous (see discussion in Levy, 2012).

### 3 Main results

#### 3.1 Definitions and assumptions

Consider an $n$-player discounted infinite horizon stochastic game in discrete time. The primitives of the class of games are given by a tuple $\{S, (A_i, \tilde{A}_i, \beta_i, u_i)^n_{i=1}, Q, s_0\}$, where $S = [0, \bar{S}] \subset \mathbb{R}^k$ is the state space, $A_i \subset \mathbb{R}^{k_i}$ player $i$ action space with $A = \times_i A_i$, $\beta_i$ is the discount factor for player $i$, $u_i : S \times A \to \mathbb{R}$ is the one-period payoff function, and $s_0 \in S$ the initial state of the game. For each $s \in S$, the set of feasible actions for player $i$ is given by $\tilde{A}_i(s)$, which is assumed to be compact Euclidean interval in $\mathbb{R}^{k_i}$ for each $s \in S$. By $Q$, we denote a transition function that specifies for any current state $s \in S$ and current action $a \in A$, a probability distribution over the realizations of next period states $s' \in S$.

Using this notation, we can provide a formal definition of a (Markov, stationary) strategy, payoff, and a Nash equilibrium. A strategy for a player $i$ is denoted by $\Gamma_i = (\gamma^1_i, \gamma^2_i, \ldots)$, where $\gamma^t_i$ specifies an action to be taken at stage $t$ as a function of history of all states $s^t$, as well as actions $a^t$ taken till stage $t$ of the game. If a strategy depends on a partition of histories limited to the current state $s_t$, then the resulting strategy is referred to as Markov. If for all stages $t$, we have a Markov strategy given as $\gamma^t_i = \gamma_i$ , then strategy $\Gamma_i$ for player $i$ is called a Markov-stationary strategy, and denoted simply by $\gamma_i$.

For a strategy profile $\Gamma = (\Gamma_1, \Gamma_2, \ldots, \Gamma_n)$, and initial state $s_0 \in S$, the expected payoff for player $i$ can be denoted by:

$$U_i(\Gamma, s_0) = (1 - \beta_i) \sum_{t=0}^{\infty} \beta^t_i \int u_i(s_t, a_t) dm^t_i(\Gamma, s_0),$$
where $m^t_i$ is the stage $t$ marginal on $A_i$ of the unique probability distribution induced on the space of all histories for $\Gamma$, given by Ionescu–Tulcea’s theorem. A strategy profile $\Gamma^* = (\Gamma^*_i, \Gamma^*_{-i})$ is then a Markov Stationary Nash equilibrium (MSNE) if and only if $\Gamma^*$ is feasible, and for any $i$, and all feasible $\Gamma_i$, we have

$$U_i(\Gamma^*_i, \Gamma^*_{-i}, s_0) \geq U_i(\Gamma_i, \Gamma^*_{-i}, s_0).$$

The aim of this section is two-fold. We first prove the existence of a MSNE. We then provide a successive approximation scheme for computing particular (extremal) elements of the set of MSNE.

Before that, we state some initial conditions on the primitives of the game that are required for the methods we discuss in this section.

Assumption 1 (Preferences) For $i = 1, \ldots, n$ let:

- $u_i$ be continuous on $A$ and measurable on $S$, with $0 \leq u_i(s, a) \leq \bar{u}$,
- $(\forall a \in A) u_i(0, a) = 0$,
- $u_i$ be increasing in $a_{-i}$,
- $u_i$ be supermodular in $a_i$ for each $(a_{-i}, s)$, and has increasing differences in $(a_i; a_{-i})$,
- for all $s \in S$ the sets $\tilde{A}_i(s)$ be nonempty, compact intervals and $s \to \tilde{A}_i(s)$ be a measurable correspondence.

Assumption 2 (Transition) Let $Q$ be given by:

- $Q(\cdot|s, a) = g_0(s, a)\delta_0(\cdot) + \sum_{j=1}^L g_j(s, a)\lambda_j(\cdot|s)$, where
- for $j = 1, \ldots, L$ the function $g_j : S \times A \to [0, 1]$ is continuous on $A$ and measurable on $S$, increasing and supermodular in $a$ for fixed $s$, and $g_j(0, a) = 0$ (clearly $\sum_{j=1}^L g_j(\cdot) + g_0(\cdot) \equiv 1$),
- $(\forall s \in S, j = 1, \ldots, L) \lambda_j(\cdot|s)$ is a Borel transition probability on $S$,
- $\delta_0$ is a probability measure concentrated at point 0.

3.2 Existence and computation of SMNE

We first study the existence and computation of MSNE in a space of bounded measurable functions. Let $\text{Bor}(S, \mathbb{R}^n)$ be the set of Borel measurable functions from $S$ into $\mathbb{R}^n$, and consider its following subset:

$$\mathcal{B}^n(S) := \{ v \in \text{Bor}(S, \mathbb{R}^n) : \forall i v_i(0) = 0, ||v_i|| \leq \bar{u} \}.$$

Equip the space $\text{Bor}(S, \mathbb{R}^n)$ with a pointwise partial order, and the subset $\mathcal{B}^n(S)$ with its relative partial order. We are now prepared to state the first main theorem of this section, which concerns the existence and approximation of MSNE.

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6This means among other that function $s \to \int_S v(s')\lambda_j(ds'|s)$ is measurable for any integrable $v$. 

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Theorem 3.1 (Existence and approximation of SMNE) Let assumptions 1 and 2 be satisfied. Then

- there exists the greatest $\psi^*$ and least $\phi^*$ SMNE in $\mathcal{B}^n(S)$ and their associated values $w^*$ and least $v^*$,

- both greatest and least MSNE and their associated values can be pointwise approximated as limits of monotone sequences,

- let $\gamma^*$ be an arbitrary MSNE. Then $(\forall s \in S)$ we have the equilibrium bounds $\phi^*(s) \leq \gamma^*(s) \leq \psi^*(s)$. Further, if $\mu^*$ is equilibrium payoff associated with any stationary Markov Nash equilibrium $\gamma^*$, then $(\forall s \in S)$ we have the bounds: $v^*(s) \leq \mu^*(s) \leq w^*(s)$.

To provide some insight into our construction of MSNE, first define for a vector of continuation values $v = (v_1, v_2, \ldots, v_n) \in \mathcal{B}^n(S)$, we can consider the auxiliary one-period, $n$-player game $G_n^*$ with action sets $A_i(s)$ and payoffs given as follows:

$$
\Pi_i(v_i, s, a_i, a_{-i}) := (1 - \beta_i)u_i(s, a_i, a_{-i}) + \beta_i \int_s v_i(s')Q(ds'|s, a_i, a_{-i}).
$$

Then note that under assumptions 1 and 2, the auxiliary game $G_n^*$ is a supermodular game for any $(v, s)$, hence it possesses a greatest $\pi(s, v)$ and least $a(s, v)$ (measurable\(^7\)) pure strategy Nash equilibrium (see, Topkis (1979) and Vives (1990)) (as well as corresponding greatest $\Pi^*(v, s)$ and least $\Pi^*(v, s)$ equilibrium values), where the equilibrium payoffs are given as $\Pi^*(v, s) = (\Pi^1_1(v, s), \Pi^2_1(v, s), \ldots, \Pi^*_n(v, s))$. From these equilibrium payoffs, we can define a pair of extremal value operators $T(v)(s) = \Pi^*(v, s)$ and $T(v)(s) = \Pi^*(v, s)$ (as well as $T^j(v)$, which denote the $j$-iteration/orbit of the operator $T(v)$ from $v$). We can then generate recursively a sequence of lower (resp., upper) bounds for equilibrium values \{\{v^j\}_{j=0}^\infty\} (resp., \{w^j\}_{j=0}^\infty\}) where $v^j+1 = T(v^n)$ for $j \geq 1$ from the initial guess $v^0(s) = (0, 0, \ldots, 0)$ (resp., $w^j+1 = T(w^n)$ from initial guess $w^0(s) = (\bar{u}, \bar{u}, \ldots, \bar{u})$) for $s > 0$ and $w^0(0) = 0)$. For both lower (resp., upper) value iterations, we can then associate sequences of pure strategy Nash equilibrium strategies \{\phi^j\}_{j=0}^\infty (resp., \{\psi^j\}_{j=0}^\infty), which are defined recursively by $\phi^j = a(s, v^j)$ (resp., $\psi^j = a(s, w^j)$). Existence and computation of MSNE reduces then to studying the limiting properties of this collection of (monotone) iterative processes. With this, our main results in this section lead to a useful lemma.

Lemma 3.1 (The successive approximation of SMNE) Under assumptions 1 and 2 we have

1. (for fixed $s \in S$) $\phi^j(s)$ and $\psi^j(s)$ are increasing sequences and $\psi^j(s)$ and $w^j(s)$ are decreasing sequences,

2. for all $t$ we have $\phi^j \leq \psi^j$ and $v^j \leq w^j$ (pointwise),

3. the following limits exist: $(\forall s \in S) \lim_{t \rightarrow \infty} \phi^t(s) = \phi^*(s)$ and $(\forall s \in S) \lim_{n \rightarrow \infty} \psi^j(s) = \psi^*(s),

4. the following limits exist $(\forall s \in S) \lim_{n \rightarrow \infty} v^j(s) = v^*(s)$ and $(\forall s \in S) \lim_{n \rightarrow \infty} w^j(s) = w^*(s),

\(^7\)Lemma 5.3 shows that both extremal equilibria and their corresponding values are measurable.
5. $\phi^*$ and $\psi^*$ are stationary Markov Nash equilibria in the infinite horizon stochastic game. Moreover, $v^*$ and $w^*$ are equilibria payoffs associated with $\phi^*$ and $\psi^*$ respectively.

As our existence result in Theorem 3.1 is obtained under different assumptions than those found in the existing literature for stochastic supermodular games (e.g., Curtat (1996), Amir (2005) or Nowak (2007)), we feel it is useful to provide a detailed discussion of the central differences between our results and those found in the existing literature.

First, relative to Curtat (see also Amir, 2002), we do not require the payoffs or the transition probabilities to be Lipschitz continuous (which is an assumption used in both of those papers). Such conditions appear to be very strong relative to many economic applications. We also do not impose any conditions on payoffs and stochastic transitions that imply "double increasing differences" in the sense of Granot and Veinott (1985) in payoff structures, or any strong concavity conditions such as strict diagonal dominance to obtain our existence result. Moreover, and equally critical, we do not assume any increasing differences between actions and states. This last difference is also critical when comparing our results versus Amir (2005). That is, we do not require monotone Markov equilibrium to obtain existence (rather, we just need enough complementarity to construct monotone operators). So our conditions are able to distinguish between the role of monotonicity conditions needed for the existence and computation of MSNE (e.g., to obtain monotone operators in sufficiently chain complete partially ordered sets) from those conditions needed for the existence of increasing MSNE. That is not done in the existing literature. Also, as compared to Amir (2005) we do not require the class of games to have a single dimensional state space. Finally, there are critical differences between our conditions and theirs per the specification of the stochastic transition $Q$. For example, in Amir (2005), to obtain existence in the infinite horizon game, he requires strong stochastic equicontinuity conditions for the distribution function $Q$ relative to the actions $a$, which is critical for his existence argument. We do not need this latter assumption.

Second, to obtain our results, though, we must impose a very important conditions on the stochastic transitions $Q$, which are stronger than needed for existence in the work of Curtat and Amir. In particular, we assume the transition structure induced by $Q$ can be represented as a convex combination of $L + 1$ probability measures, of which one measure is a delta Dirac concentrated at 0. As a result, with probability $g_0$, we set the next period state to zero; with probability $g_j$, the distribution is drawn from the non-degenerate distribution $\lambda_j$ (where, in this latter case, this distribution does not depend on the vector of actions $a$, but is allowed to depend on the current state $s$). Also, although we assume each $\lambda_j$ is stochastically ordered relative to

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8For example, such conditions rule out payoffs that are consistent with Inada type assumptions (e.g., Cobb-Douglas utility).

9Both of these sets of assumptions (e.g., double increasing differences and strong diagonal dominance) are required by both of these authors for existence. That is, they each need to obtain unique Lipschitz Nash equilibrium in the stage game that is continuous with continuation $v$ for their eventual application of a topological fixed point theorem per existence. Similar NE uniqueness conditions are required by Nowak (2007). The difference with our setup and theirs is of utmost importance. Specifically without equilibrium uniqueness in the stage game with continuation $v$, authors cannot construct an upper-hemicontinuous correspondence $T$, a condition necessary to apply Fan-Glicksberg fixed point theorem. Similar issues arise when trying to apply Schauder’s theorem (e.g. as in Curtat (1996)).

10Also versus Amir (2005), our feasible action correspondences $\tilde{A}_i$, and payoff/transition structures $u_i$ and $g_j$ are only required to be measurable with $s$, as opposed to upper semicontinuous as in Amir (2005).

11Also, as we do not require any form of continuity of $\lambda_j$ with respect to the state $s$, we do not satisfy Amir’s assumption T1. Further, although we both require that the stochastic transition structure $Q$ is stochastically supermodular with $a$, we do not require increasing differences with $(a,s)$ as Amir does, nor do we require the stochastic monotonicity conditions for $Q$ in $s$. 8
the Dirac delta $\delta_0$, we do not impose stochastic orders among the various measures $\lambda_j$. This "mixing" assumption for transition probabilities has been discussed extensively in the literature. Surprisingly, the main strength of this assumption has not been fully used (see Nowak, 2007) until the work of Balbus, Reffett, and Woźny (2013) in the context of paternalistic altruism economies, as well as in this present paper. Clearly, the restrictive part of our assumption is that we require existence of an absorbing state 0 that gives the minimal value for any $v \in \mathcal{B}^n(S)$. This is required by our techniques as we need to show that operators $\mathcal{T}, \overline{\mathcal{T}}$ are well defined transformations of $\mathcal{B}^n(S)$ (and hence, require $\mathcal{T}(v)(0) = \overline{\mathcal{T}}(v)(0) = 0$). This assumption can be potentially very restrictive in some applications, still allows for some generalizations. For example, our assumptions can be easily generalized to allow any absorbing state to be potentially very restrictive in some applications, still allows for some generalizations. For this reason, our assumptions can be easily generalized to allow any absorbing state $\underline{g} \in S$, such that $v(\underline{g}) = z$, where $z = \min_{s \in S} v(s)$ for any $v \in \mathcal{B}^n(S)$, and the unique Nash equilibrium value in the auxiliary game $G^*_T$ has for any integrable continuation $v$ with $v(\underline{g}) = z$. Moreover, $\underline{g}$ need not be minimal in $S$ unless $v$ is monotone. Also, we can allow for other absorbing states (and hence, the probability of reaching 0 can be reduced to zero (see theorem 3.5)).

3.3 Uniform error bounds for Lipschitz continuous SMNE

We now turn to error bounds for approximate solutions. To the best of our knowledge, this is a question that has not been addressed in the current literature. We initially give two motivations for the importance of our results in this section. First, notice that the limits of iterations computing in theorem 3.1 and lemma 3.1 are only relative to pointwise convergence. With slightly stronger assumptions, we can obtain for those limits uniform convergence to least and greatest MSNE. Second, to obtain uniform error bounds, we need make some stronger assumptions on the primitives which will allow us to address the question of Lipschitz continuity of equilibrium strategies. The assumptions we will use are common in applications (e.g., compare the assumptions we impose to those in Curtat (1996)).

In this section, we assume that the state space $S$ is endowed with a taxi-norm $|| \cdot ||_1$. The spaces $A_i$ and $A$ are endowed with a natural sup-norm. Each function $f: S \to A$ is said to be $M$-Lipschitz continuous if and only if, for all $i = 1, \ldots, n$ $||f_i(x) - f_i(y)|| \leq M||x - y||_1$. Note, if $f_i$ is differentiable, then $M$-Lipschitz continuity is equivalent to that each partial derivative being bounded above by $M$. To obtain corresponding uniform convergence and uniform approximation results, we need some additional structure, that is discussed in the following assumption.

**Assumption 3** For all $i, j$:

- $u_i, g_j$ are twice continuously differentiable on an open set containing $S \times A$.

- $u_i$ is increasing in $(s, a_{-i})$ and satisfies cardinal complementarity\(^{14}\) in $a_i$ and $(a_{-i}, s)$,

- $u_i$ satisfy a strict dominant diagonal condition in $a_i, a_{-i}$ for fixed $s \in S, s > 0$, i.e. if we denote $a_i \in \mathbb{R}^{k_i}$ as $a_i := (a_i^1, \ldots, a_i^{k_i})$, then

$$\forall i = 1, \ldots, n \forall j = 1, \ldots, k_i \sum_{\alpha=1}^{k_i} \sum_{\beta=1}^{k_j} \frac{\partial^2 u_i}{\partial a_i^\alpha \partial a_j^\beta} < 0,$$

\(^{12}\)Taxi-norm of vector $x = (x_1, \ldots, x_k)$ is defined as $||x||_1 = \sum_{i=1}^{k} |x_i|$.

\(^{13}\)Note that this implies that $u_i$ and $g_j$ are bounded Lipschitz continuous functions on compact $S \times A$.

\(^{14}\)That is, the payoffs are supermodular in $a_i$ and have increasing differences in $(a_i, s)$ and in $(a_i, a_{-i})$. 

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• $g_j$ is increasing in $(s,a)$ and has cardinal complementarity in $a_i$ and $(s,a_{-i})$,
• $g_j$ satisfy a strict dominant diagonal condition in $a_i,a_{-i}$ for fixed $s \in S,s > 0$,
• $\lambda_j$ has a Feller property,
• for each increasing, Lipschitz and bounded by $\pi$ function $f$, the function $\eta'_j(s) := \int f(s')\lambda_j(ds'|s)$ is increasing and Lipschitz-continuous with a constant $\bar{\eta}$,\footnote{This condition is satisfied if each of measures $\lambda_j(ds'|s)$ has a density $\rho_j(s'|s)$ and the function $s \to \rho_j(s'|s)$ is Lipschitz continuous uniformly in $s'$.}
• $\tilde{A}_i(s) := [0,\tilde{a}_i(s)]$ and each function $s \to \tilde{a}_i(s)$ is Lipschitz continuous and isotope function\footnote{Each coordinate $\tilde{a}_i$ is Lipschitz continuous. Notice, this implies that the feasible actions are Veinott strong set order isotone.}.

Define the set $CM^N$ of $N$-tuples of increasing Lipschitz continuous functions with some constant $M$ on $S$. Equip $CM^N$ with a partial order, i.e.: for $w,v \in CM^N$, where $w = (w_1,\ldots,w_N)$ and similarly for $v$, let
\[ w \geq v \text{ iff } (\forall i = 1,\ldots,N)(\forall s \in S) w_i(s) \geq v_i(s). \]
Clearly, $CM^N$ is a complete lattice.

It is also nonempty, convex and compact in the sup norm. It is also important to note that $CM^N$ is closely related to the space where equilibrium is constructed in Curtat (1996). There are two key differences, though, between our game and those studied in these two papers. First, we allow the choice set $A$ to depend on $s$ (where, these authors assume $A_i$ independent of $s$). Second, under our assumptions, the auxiliary game $G^0_v$ has a continuum of Nash equilibria, and hence we need to close the Nash equilibrium correspondence in state 0. These technical differences are addressed in lemma 5.4 and proof of the next theorem.

Before we proceed, we provide a few additional definitions. For each twice continuously differentiable function $f : A \to \mathbb{R}$, we define the following mappings:
\[
L_{i,j}(f) := -\sum_{\alpha=1}^n \sum_{j=1}^k \frac{\partial^2 f}{\partial a_i^\alpha \partial a_j^\beta}, \quad U_{i,j,l}^2 := \sup_{s \in S, a \in A(s)} \frac{\partial^2 u_i}{\partial a_i^\alpha \partial s_l}(s,a),
\]
\[
G_{i,j}^1 := \sup_{s \in S, a \in A(s)} \sum_{\alpha=1}^L \frac{\partial g_a}{\partial a_i^\alpha}(s,a), \quad G_{i,j,l}^2 := \sup_{s \in S, a \in A(s)} \sum_{\alpha=1}^L \frac{\partial^2 g_a}{\partial a_i^\alpha \partial s_l}(s,a),
\]
\[
M_0 := \max \left\{ \frac{(1 - \beta_i)U_{i,j,l}^2 + \beta_i \tilde{\eta}_G_{i,j} + \beta_i \tilde{\eta}_G_{i,j,l}}{- (1 - \beta_i)L_{i,j}(u_i)} : i = 1,\ldots,n, j = 1,\ldots,k_i, l = 1,\ldots,k \right\}.
\]
With these definitions in mind, we can now prove our main result on Lipschitz continuous MSNE:

**Theorem 3.2 (Lipschitz continuity)** Let assumptions 1, 2, 3 be satisfied. Assume additionally that each $\tilde{a}_i(\cdot)$ is Lipschitz continuous with a constant less then $M_0$. Then, stationary Markov Nash equilibria $\phi^s, \psi^s$ and corresponding values $v^s, w^s$ are all Lipschitz continuous.
We can now study the uniform approximation of MSNE. Our results appeal to a version of Amann’s theorem (e.g., Amann (1976), Theorem 6.1). For this argument, denote by \( \mathbf{0} \) (by \( \bar{u} \) respectively) the \( n \)-tuple of function identically equal to 0 (\( \bar{u} \) respectively) for all \( s > 0 \). Observe, as under assumptions 3, the auxiliary game has a unique NE value, we have

\[
\mathcal{T}(v) = \mathcal{T}(v) := T(v).
\]

The result is as follows:

**Corollary 3.1 (Uniform approximation of extremal SMNE)** Let assumptions 1, 2 and 3 be satisfied. Then \( \lim_{j \to \infty} \| T_j^i \mathbf{0} - v^* \| = 0 \) and \( \lim_{j \to \infty} \| T_j^i \bar{u} - w^* \| = 0 \), with \( \lim_{j \to \infty} \| \phi_j^i - \phi^* \| = 0 \) and \( \lim_{j \to \infty} \| \psi_j^i - \psi^* \| = 0 \).

Notice, the above corollary assures that the convergence in theorem 3.1 is uniform. We also obtain a stronger characterization of the set of MSNE in this case, namely, the set of MSNE equilibrium value functions form a complete lattice.

**Theorem 3.3 (Complete lattice structure of SMNE value set)** Under assumptions 1, 2 and 3, the set of Markov stationary Nash equilibrium values \( v^* \) in \( CM^n \) is a nonempty complete lattice.

The above result provides a further characterization of a MSNE strategies, as well as their corresponding set of equilibrium value functions. From a computational point of view, not only are the extremal values and strategies Lipschitzian (as known from previous work), they can also can be uniformly approximated by a simple algorithm. Also observe, is it not clear if the set of MSNE in \( CM_{\sum_i k_i} \) is necessarily a complete lattice.

### 3.4 Monotone comparative dynamics

We now study the question of sufficient conditions under which our games exhibit equilibrium monotone comparative statics relative to both extremal fixed point values \( v^*, w^* \), as well as the corresponding pure strategy extremal equilibria \( \phi^*, \psi^* \). We also consider the question of ordered equilibrium stochastic dynamics. With the question of equilibrium comparative statics in mind, we first parameterize our stochastic game by a set of parameters \( \theta \in \Theta \), where \( \Theta \) is some partially ordered set. One way to interpret \( \theta \) is a vector whose elements include parameters representation ordered perturbations to any of the following primitive data of the game: (i) period payoffs \( u_i \), (ii) the stochastic transitions \( g_j \) and \( \lambda_j \), and (iii) feasibility correspondence \( A_i \). Alternatively, we can think of elements \( \theta \) as being policy parameters of the environment governing the setting of taxes or subsidies (as, for example, in a dynamic policy game with strategic complementarities).

Along those lines, consider parameterized versions of Assumptions 1 and 2 as follows:

**Assumption 4 (Parameterized preferences)** For \( i = 1, \ldots, n \) let:

- \( u_i : S \times A \times \Theta \to \mathbb{R} \) be a function and \( u_i(\cdot, s, \theta) \) continuous on \( A \) for any \( s \in S, \theta \in \Theta \) with \( u_i(\cdot) \leq \bar{u}, \) and \( u_i(\cdot, \cdot, \theta) \) is measurable for all \( \theta \),
- \( (\forall a \in A, \theta \in \Theta) u_i(0, a, \theta) = 0, \)
- \( u_i \) be increasing in \( (s, a_{-i}, \theta), \)
• $u_i$ be supermodular in $a_i$ for fixed $(a_{-i}, s, \theta)$, and has increasing differences in $(a_i; a_{-i}, s, \theta)$,

• for all $s \in S, \theta \in \Theta$, the sets $\tilde{A}_i(s, \theta)$ are nonempty, measurable (for given $\theta$), compact intervals and $\tilde{A}_i$ measurable multifunction that is both ascending in the Veinott’s strong set order\(^{17}\), and expanding under set inclusion\(^{18}\) with $\tilde{A}_i(0, \theta) = 0$.

**Assumption 5 (Parameterized transition)** Let $Q$ be given by:

- $Q(\cdot|s, a, \theta) = g_0(s, a, \theta)\delta_0(\cdot) + \sum_{j=1}^{L} g_j(s, a, \theta)\lambda_j(\cdot|s, \theta)$, where

- for $j = 1, \ldots, L$ function $g_j : S \times A \times \Theta \to [0, 1]$ is continuous with $a$ for a given $s, \theta$, measurable for given $\theta$, increasing in $(s, a, \theta)$, supermodular in $a$ for fixed $(s, \theta)$, and has increasing differences in $(a; s, \theta)$ and $g_j(0, a, \theta) = 0$ (clearly $\sum_{j=1}^{L} g_j(\cdot) + g_0(\cdot) \equiv 1$),

- $(\forall s \in S, \theta \in \Theta, j = 1, \ldots, L)\lambda_j(\cdot|s, \theta)$ is a Borel transition probability on $S$, with each $\lambda_j(\cdot|s, \theta)$ stochastically increasing with $\theta$ and $s$,

- $\delta_0$ is a probability measure concentrated at point 0.

Notice, in both of these assumptions, we have added increasing difference assumptions between actions and states (as, for example, in Curtat (1996)).

We first introduce some notation. For a stochastic game evaluated at parameter $\theta \in \Theta$, denote the least and greatest equilibrium values, respectively, as $v^*_\theta$ and $w^*_\theta$. Further, for each of these extremal values, denote the associated least and greatest SMNE pure strategies, respectively, as $\phi^*_\theta$ and $\psi^*_\theta$. Our first monotone equilibrium comparative statics theorem is then given in the next theorem:

**Theorem 3.4 (Monotone equilibrium comparative statics)** Let assumptions 4 and 5 be satisfied. Then, the extremal equilibrium values $v^*_\theta(s), w^*_\theta(s)$ are increasing on $S \times \Theta$. In addition, the associated extremal pure strategy stationary Markov Nash equilibrium $\phi^*_\theta(s)$ and $\psi^*_\theta(s)$ are increasing on $S \times \Theta$.

In the literature on infinite horizon stochastic games with strategic complementarities, we are not aware of any analog result to the above concerning monotone equilibrium comparative statics as in Theorem 3.4. In particular, because of the non-constructive approach to the equilibrium existence problem (that is typically taken in the literature), it is difficult to obtain such a monotone comparative statics without fixed point uniqueness. Therefore, one key innovation of our approach of the previous section is that for the special case of our games where SMNE are monotone Markov processes, we are able to construct a sequence of parameterized monotone operators whose fixed points are extremal equilibrium selections. As the method is constructive, this also allows us to compute directly the relevant monotone selections from the set of MSNE.

Finally, we state results on dynamics and invariant distribution started from $s_0$ and governed by a MSNE and transition $Q$. Before this, let us mention that by our assumptions delta Dirac concentrated at 0 is an absorbing state (and hence we can have a trivial invariant distribution).
As a result, we do not aim to prove a general result on the existence of an invariant distribution; rather, we will characterize a set of all invariant distributions, and discuss conditions when this limiting distribution is not a singleton. Along these lines, let \( \theta \) be given, and let \( s^t \) denote a stochastic process induced by \( Q \) and equilibrium strategy \( f \) (i.e., \( s_0 = s^f \) is an initial value and for \( t > 0 \), with \( s_{t+1} \) has a conditional distribution \( Q(\cdot|s_t, f(s_t)) \). By \( \geq \) we denote the first order stochastic dominance order on the space of probability measures.

We then have the following theorem:

**Theorem 3.5 (Invariant distribution)** Let assumptions 1, 2 and 3 be satisfied.

- Then the sets of invariant distributions for processes \( s^\phi^* \) and \( s^\psi^* \) are chain complete (with both greatest and least elements) with respect to (first-) stochastic order.

- Let \( \Pi(\phi^*) \) be the greatest invariant distribution with respect to \( \phi^* \) and \( \Pi(\psi^*) \) the greatest invariant distribution with respect to \( \psi^* \). If the initial state of \( s^\phi^* \) or \( s^\psi^* \) is a Dirac delta in \( \bar{S} \), then \( s^\phi^* \) converges weakly\(^{19} \) to \( \Pi(\phi^*) \), and \( s^\psi^* \) converges weakly to \( \Pi(\psi^*) \), respectively.

We make a few remarks. First, the above result is stronger than that obtained in a related theorem in Curtat (1996) (e.g., Theorem 5.2). That is, not only we do characterize the set of invariant distributions associated with extremal strategies (which he does not), but we also prove a weak convergence result per the greatest invariant selection. Second, it is worth mentioning if for almost all \( s \in S \), we have \( \sum_j g_j(s, \cdot) < 1 \), we obtain a positive probability of reaching zero (an absorbing state) each period, and hence the only invariant distribution is delta Dirac at zero. Hence, to obtain a nontrivial invariant distribution, one has to assume \( \sum_j g_j(s, \cdot) = 1 \) for all \( s \) in some subset of a state space \( S \) with positive measure, e.g. interval \([S', \bar{S}] \subset S \) (see Kamihigashi and Stachurski (2013)).

Second, Theorems 3.4 and 3.5 also imply results on monotone comparative dynamics (e.g., as defined by Huggett (2003)) with respect to the parameter vector \( \theta \) induced by extremal MSNE: \( \phi^*, \psi^* \). To see this, we define the greatest invariant distribution \( \Pi_\theta(\phi^*_\theta) \) induced by \( Q(\cdot|s, \phi^*_\theta, \theta) \), and greatest invariant distribution \( \Pi_\theta(\psi^*_\theta) \) induced by \( Q(\cdot|s, \psi^*_\theta, \theta) \), and consider the following corollary:

**Corollary 3.2** Assume 4, 5. Additionally let assumptions of theorem 3.5 be satisfied for all \( \theta \in \Theta \). Then \( \Pi_{\theta_2}(\phi^*_{\theta_2}) \geq \Pi_{\theta_1}(\phi^*_{\theta_1}) \) as well as \( \Pi_{\theta_2}(\psi^*_{\theta_2}) \geq \Pi_{\theta_1}(\psi^*_{\theta_1}) \) for any \( \theta_2 \geq \theta_1 \).

The above results points to the importance of having constructive iterative methods for both strategies/values, as well as limiting distributions associated with extremal SMNE. Without such monotone iterations, we could not close our monotone comparative statics results. Further, in conclusion, we stress the fact that by weak continuity of operators used to establish invariant distributions, we can also obtain results that lead us to develop methods to estimate parameters \( \theta \) using simulated moments methods (e.g., see Aguirregabiria and Mira (2007)), for discussion of how this is done, and why it is important.

\(^{19}\)That is their distributions converge weakly.

\(^{20}\)It is also worth mentioning that much of the existing literature does not consider the question of characterizing the existence of invariant distributions.
4 Applications

There are many applications of our new results. For example, they can be used to study many examples such as dynamic (price or quantity) oligopolistic competition, stochastic growth models without commitment (and related problems of dynamic consistency), models with weak social interaction among, dynamic policy games, and interdependent security systems. In this section, we discuss four such applications of our results. We first use our results to prove existence of Markov equilibrium in a dynamic oligopoly model. Next, we show how the methods can be used for analyzing the question of credible government public policies as in Stokey (1991). We conclude with a discussion of how our results can be used to generalize existing results per symmetric MSNE of symmetric stochastic games.

4.1 Price competition with durable goods

We begin with a price competition problem with durable goods. Consider an economy with \( n \) firms competing on customers buying durable goods, that are heterogeneous but substitutable to each other. Apart from price of a given good, and vector of competitors goods’ prices, demand for any commodity depends on demand parameter \( s \). Each period firms choose their prices, competing a la Bertrand with other’s prices. Our aim is to analyze the Markov Stationary Nash Equilibrium of such economy.

Payoff of firm \( i \), choosing price \( a_i \in [0, \bar{a}] \) is

\[
u_i(s, a_i, a_{-i}, \theta) = a_i D_i(a_i, a_{-i}, s) - C_i(D_i(a_i, a_{-i}, s), \theta),\]

where \( s \) is a (common) demand parameter, while \( \theta \) is a cost function parameter. As within period game is Bertrand with heterogeneous, but substitutable products, naturally the preference assumption 1 is satisfied if (a) demand \( D_i \) is increasing with \( a_{-i} \), has increasing differences in \( (a_i, a_{-i}) \), and (b) the cost function \( C_i \) is increasing and convex. As \([0, \bar{a}]\) is single dimensional, \( u_i \) is a supermodular in \( a_i \) trivially.

Concerning the interpretation of the assumptions placed on \( Q \) in the context of this model: letting \( s = 0 \) be an absorbing state means that there is a probability that demand will vanish and companies will be driven out of the market. The other assumptions on transition probabilities are also satisfied if \( Q(\cdot | s, a) = g_0(s, a) \delta_0(\cdot) + \sum_j g_j(s, a) \lambda_j(\cdot | s) \) and \( g_j, \lambda_j \) satisfy assumptions 2. Interpreting: high prices \( a \) today result in high probability for positive demand in the future, as the customer trades-off between exchanging the old product with the new one, and keeping the old product and waiting for lower prices tomorrow. Supermodularity in prices implies that the impact of a price increase on positive demand parameter tomorrow is higher when the others set higher prices. Indeed, when the company increases its price today, it may lead to a positive demand in the future (if the others have also high prices). But, if the others firms set low prices today, then such impact is definitely lower, as some clients may want to purchase the competitors good today instead. Such assumptions guarantee that the stochastic (extensive form) game has the supermodular structure for extremal strategies, the feature that is uncommon for general extensive form games (see Echenique (2004)). More specifically, if a strategy of a player is increased in the some period \( t + \tau \), it leads to a higher value of all players and by our mixing transition assumption increase period \( t \) extremal strategies.

The results of the paper (theorem 3.1) prove existence of the greatest and the least Markov stationary Bertrand Equilibrium and allow to compute the equilibria, by a simple iterative procedure. The results extend, therefore, the results obtained in Curtat (1996) paper per this
example to the non-monotone strategies, characterizing the monopolistic competition economy with substitutable durable goods and varying consumer preferences. Finally, our approximation procedure allow applied researcher to compute and estimate the stochastic properties of these models using the extremal invariant distributions (see theorem 3.5). Finally, if one adds assumptions of theorem 3.4 one obtain monotone comparative statics of the extremal equilibria and invariant distributions (see corollary 3.2), the results absent in the related work.

Note, to analyze such economy using the methods of Curtat (1996), one needs to assume increasing differences between \((a, s)\) and monotonicity in \(s\). Such method allow hence to study the monotone equilibria only. The interpretation of such assumption means that high demand today imply high demand in the future. To justify this assumption, Curtat (1996) argues: that "high level of demand today is likely to result in a high level of demand tomorrow because one can assume that not all customers will be served today in the case of high demand." Hence, in our paper, the SMNE existence is obtained under weaker assumption than those of Curtat (1996), i.e. where monotonicity assumption is questionable, as customer rationing is not a part of this game description. Hence, methods developed in section 3.2 are plausible.

4.2 Time-consistent public policy

We now consider a time-consistent policy game as defined by Stokey (1991) and analyzed more recently by Lagunoff (2008). Consider a (stochastic) game between a large number of identical households and the government. We will study equilibria that treat each household identically. For any state \(k \in S\) (capital level), households choose consumption \(c\) and investment \(i\) treating level of a government spending \(G\) as given. There are no security markets that household can share the risk for tomorrow capital level. The only way to consume tomorrow is to invest in the stochastic technology \(Q\). The within period preferences for the households are given by \(u(c)\) (i.e. household does not obtain utility from public spending \(G\)). The government raises revenue by levying flat tax \(\tau \in [0, 1]\) on capital income, to finance its public spending \(G \geq 0\). Each period the government budget is balanced and its within period preferences are given by: \(u(c) + J(G)\). The consumption good production technology is given by constant return to scale function \(f(k)\) with \(f(0) = 0\). The transition technology between states is given by a probability distribution \(Q(\cdot|i, k)\), where \(i\) denotes household investment. The timing of the game in each period is that the government and household choose their actions simultaneously. Observe, in this example a natural absorbing state is \(k = 0\).

To specify each players optimization problem, first we assume households and the government take price \(R\) as given, with profit maximization implying \(R = f'(k)\). Assume that \(u, J, f\) are increasing, concave and twice continuously differentiable and \(Q\) is given by assumption 2 (with \(L = 1\) to simplify notation). Each of the households then choose investment \(i\) to solve:

\[
\max_{i \in [0,(1-\tau)Rk]} u((1-\tau)Rk - i) + g(i)\beta \int_S v_H(s)\lambda(s|k).
\]

By a standard arguments, we see that the objective for the households is supermodular in \(i\) and has increasing differences in \((i, t)\), where \(t = 1 - \tau\) (noting \(-u''(\cdot) \geq 0\)). Moreover, the objective is increasing in \(t = 1 - \tau\) by monotonicity assumptions on \(u\).

The government is choosing \(t\) to solve:

\[
\max_{t \in [0,1]} u(tRk - i) + J(Rk(1 - t)) + g(i)\beta \int_S (v_H(s) + v_G(s))\lambda(s|k).
\]
That is, the government maximizes the household utility as well as the additional utility that it obtains from public spending $J$, and its continuation $v_G$. Again, objective is supermodular in $1 - \tau$ and has increasing differences in $(t = 1 - \tau, i)$ as $-u''(\cdot) \geq 0$. Moreover observe, although the objective is not increasing in $i$, along any Nash equilibrium of the auxiliary game, the government’s objective is increasing in $v_H$ by the envelope theorem. To see that, by $(i^*, t^*)(v_H)$ denote an extremal NE of the auxiliary game and observe that:

$$
\frac{\partial}{\partial i}
\left[
\begin{array}{l}
u(t^*(v_H)Rk - i) + J(Rk(1 - t^*(v_H))) + g(i)\beta \int_S (v_H(s) + v_G(s))\lambda(s|k)
\end{array}
\right]_{i = i^*(v_H)} =

\left[
\begin{array}{l}
-u'(t^*(v_H)Rk - i^*(v_H)) + g'(i^*(v_H))\beta \int_S v_H(s)\lambda(s|k)
\end{array}
\right] + \left[
\begin{array}{l}
g'(i^*(v_H))\beta \int_S v_G(s)\lambda(s|k)
\end{array}
\right] =

g'(i^*(v_H))\beta \int_S v_G(s)\lambda(s|k) \geq 0.
$$

So, interestingly, although this model’s general assumptions do not appear to satisfy the underlying sufficient conditions given in our paper, the same method developed in the paper can be extended easily to this case. That is, we are able to use our results to prove existence of MSNE, as well as compute the least and the greatest MSNE. Specifically, we can construct an operator on the space of values, that would be monotone (as the within period game is supermodular and the Nash equilibrium of such game is monotone in $v_H, v_C$).

Some additional interesting points of departure from this above basic specification can also be worked out, including: (i) elastic labor supply choice, or more importantly (ii) adding security markets, investment/insurance firms possessing $Q$ and proving existence of prices decentralizing optimal investment decision $i^*$. Still observe, however, that here we are able to offer weak assumptions for existence of a stationary credible policy, as well as offer a variety of tools allowing for its constrictive study and computation.

4.3 Symmetric equilibria in symmetric stochastic games

Finally, we consider a special case of our stochastic game, namely, one where all players have identical preferences $u := u_i$ and action sets $\tilde{A} := \tilde{A}_i \subset \mathbb{R}$. With slight abuse of notation, we denote payoff of a player choosing $a_i$, when others choose $a_{-i}$ in state $s$ by $u(s, a_i, a_{-i})$. Now observe that for such a special case we can obtain results of theorem 3.1, corollary 3.1 and others from section 3.2 for symmetric equilibria dispensing assumption 1 of increasing differences of $u$ in $(a_i, a_{-i})$ and supermodularity of $g$ in $a$ in assumption 2. Instead, to guarantee existence of the NE of the auxiliary game we need to add concavity of $g$ in $a$, and concavity of $u$ in $a_i$. Indeed, under such additional assumptions the auxiliary game $G^*_a$ has the greatest and the least symmetric Nash equilibrium, both monotone in $v$ by Corollary 2 of Milgrom and Roberts (1994). Hence, we can still construct two monotone operators $T_1, T_2$ and reconstruct the proofs of theorem 3.1 and corollary 3.1. Such modification is important, as it allows one to dispense with the restrictive assumption of (within period) strategic complementarities between players while allowing to obtain (between period) strategic complementarities (at least for selected extremal NE values), a necessary feature for our constructive arguments.

An immediate example of the importance of this generalization can be seen, when studying of symmetric SMNE in a stochastic version of a private provision of public good game. Let $u(c_i, Y)$ be a payoff from consumption of a private $c_i$ and public good $Y$. Assume marginal utilities are decreasing, and both goods are complements. Endow consumer with income $w$ to be distributed between $c_i$ and private provision $y_i$. Let a public good be produced using
technology \( Y = F(\sum_i y_i, s) \), where \( F \) is increasing and concave in the first argument. Observe that the function \( (y_i, y_{-i}) \rightarrow u(w-y_i, F(\sum_j y_j, s)) \) does not have increasing differences, but has positive externalities due to free rider problem. Let \( s \) parameterize public good stock (i.e., a draw representing a stock from the previous period) or its productivity, while \( Q \) represents a process allowing to reduce a future probability of a zero output / productivity, by higher provisions \((y_1, \ldots, y_n)\) today. By theorem 3.1 and corollary 3.1 we can prove existence and approximate the greatest and least symmetric MSNE of such a game.

Finally, using this generalization, we can reconsider symmetric MSNE of a Bertrand competition with durable good example (see subsection 4.1), and relax increasing differences assumption of demand \( D_i \) with \((p_i, p_{-i})\) and supermodularity of \( g \).

5 Proofs

We first state three lemmata that prove useful in verifying the existence of SMNE in our game, and in addition, are also useful in characterizing monotone iterative procedures for constructing least and greatest SMNE (relative to pointwise partial orders on \( B^n(S) \)). More specifically, these lemmas concern the structure of Nash equilibria (and their associated corresponding equilibrium payoffs) in our auxiliary game \( G^*_v \).

Lemma 5.1 (Monotone Nash equilibria in \( G^*_v \)) Under assumptions 1 and 2, for every \( s \in S \) and value \( v \in B^n(S) \), the game \( G^*_v \) has the maximal Nash equilibrium \( \overline{a}(v, s) \), and minimal Nash equilibrium \( \underline{a}(v, s) \). Moreover, both equilibria are increasing in \( v \).

Proof of lemma 5.1: Without loss of generality fix \( s > 0 \). Define auxiliary one shot game, say \( \Delta(\tau) \), with an action space \( A \), and payoff function for player \( i \) given as

\[
H_i(a, \tau) := (1 - \beta_i)u_i(s, a_i, a_{-i}) + \beta_i \sum_{j=1}^{L} \beta_{i,j}g_j(s, a_i, a_{-i}),
\]

where \( \tau := [\tau_{i,j}]_{i=1,\ldots,n; j=1,\ldots,L} \in \mathcal{T} := \mathbb{R}^{n \times L} \) is endowed with the natural pointwise order. As supermodularity of a function on a sublattice of a directed product of lattices implies increasing differences (see Topkis (1998) theorem 2.6.1) clearly, for each \( \tau \in \mathcal{T} \), the game \( \Delta(\tau) \) is supermodular, and satisfies all assumptions of Theorem 5 in Milgrom and Roberts (1990). Hence, there exists a complete lattice of Nash equilibria, with the greatest Nash equilibrium given by \( \overline{NE}\Delta(\tau) \), and the least Nash equilibrium given by \( \underline{NE}\Delta(\tau) \). Moreover, for arbitrary \( i \), the payoff function \( H_i(a, \tau) \) has increasing differences in \( a_i \) and \( \tau \); hence, \( \Delta(\tau) \) also satisfies conditions of Theorem 6 in Milgrom and Roberts (1990). As a result, both \( \overline{NE}\Delta(\tau) \) and \( \underline{NE}\Delta(\tau) \) are increasing in \( \tau \).

Step 2: For each \( s \in S \), the game \( G^*_v \) is a special case of \( \Delta(\tau) \) where \( \tau_{i,j} = \int_S v_i(s') \lambda_j(ds'|s) \). Therefore, by the previous step, least and greatest Nash equilibrium \( \underline{a}(v, s) \) and \( \overline{a}(v, s) \) are increasing in \( v \), for each \( s \in S \).

In our next lemma, we show that for each extremal Nash equilibrium (for state \( s \) and continuation \( v \)), we can associate an equilibrium payoff that preserves monotonicity in \( v \). To do this, we first compute the values of greatest (resp., least) best responses given a continuation values \( v \) and state \( s \) as follows:

\[
\Pi^*_i(v, s) := \Pi_i(v_i, s, \overline{a}_i(v, s), \overline{a}_{-i}(v, s))
\]

\[
\Pi^*_i(v, s) := \Pi_i(v_i, s, \overline{a}_i(v, s), \overline{a}_{-i}(v, s))
\]
and similarly
\[ \Pi^v_i(v, s) := \Pi_i(v_i, s, a_i(v, s), a_{-i}(v, s)). \]

We now have the following lemma:

**Lemma 5.2 (Monotone values in \(G_v^r\))** Under assumptions 1 and 2 we have: \(\Pi^v_i(v, s)\) and \(\Pi^v_i (v, s)\) are monotone in \(v\).

**Proof of lemma 5.2:** Function \(\Pi_i\) is increasing with \(a_{-i}\) and \(v_i\). For \(v_2 \geq v_1\) by Lemma 5.1, we have \(a(v_2, s) \geq a(v_1, s)\). Hence,
\[
\Pi^v_i(v^2, s) = \max_{a_i \in A_i(s)} \Pi_i(v^2_i, s, a_i, a_{-i}(v^2, s)) \geq \max_{a_i \in A_i(s)} \Pi_i(v^1_i, s, a_i, a_{-i}(v^2, s)) \geq \max_{a_i \in A_i(s)} \Pi_i(v^1_i, s, a_i, a_{-i}(v^1, s)) = \Pi^v_i(v^1, s).
\]

A similar argument proves the monotonicity of \(\Pi^v_i(v, s)\).

To show that \(\overline{T}() = \Pi()\) and \(\overline{T}() = \Pi()\) are well-defined transformations of \(\mathcal{L}(S)\) we use standard measurable selection arguments.

**Lemma 5.3 (Measurable equilibria and values of \(G_v^r\))** Under assumptions 1 and 2 we have:

- \(\overline{T} : \mathcal{L}^n(S) \to \mathcal{L}^n(S)\) and \(\overline{T} : \mathcal{L}^n(S) \to \mathcal{L}^n(S)\),
- functions \(s \to \pi(v, s)\) and \(s \to a(v, s)\) are measurable for any \(v \in \mathcal{L}^n(S)\).

**Proof of lemma 5.3:** For \(v \in \mathcal{L}^n(S)\) and \(s \in S\), define the function \(F_v : A \times S \to \mathbb{R}\) as follows:
\[
F_v(a, s) = \sum_{i=1}^{n} \Pi_i(v_i, s, a) - \sum_{i=1}^{n} \max_{z_i \in A_i(s)} \Pi_i(v_i, s, z_i, a_{-i}).
\]

Observe \(F_v(a, s) \leq 0\). Consider the problem:
\[
\max_{a \in \times^n A_i(s)} F_v(a, s).
\]

By assumption 1 and 2, the objective \(F_v\) is a Carathéodory function, and the (joint) feasible correspondence \(\tilde{A}(s) = \times_i A_i(s)\) is weakly-measurable. By a standard measurable maximum theorem (e.g. theorem 18.19 in Aliprantis and Border (2003)), the correspondence \(N_v : S \to \times_i A_i(s)\) defined as:
\[
N_v(s) := \arg \max_{a \in \tilde{A}(s)} F_v(a, s),
\]
is measurable with nonempty compact values. Further, observe that \(N_v(s)\), by definition, is a set of all Nash equilibria for the game \(G_v^r\). Therefore, to finish the proof of our first assertion, for some player \(i\), consider a problem \(\max_{a \in N_v(s)} \Pi_i(v_i, s, a)\). Again, by the measurable maximum theorem, the value function \(\Pi^v_i(v, s)\) is measurable. A similar argument shows each \(\Pi^v_i(v, s)\) is measurable. Therefore by theorem 4.1 in Himmelberg (1975), the product operators are also measurable, giving: \(\overline{T} : \mathcal{L}^n(S) \to \mathcal{L}^n(S)\) and \(\overline{T} : \mathcal{L}^n(S) \to \mathcal{L}^n(S)\).

To show the second assertion of the theorem, for some player \(i\), again consider a problem of \(\max_{a \in N_v(s)} a_i^j\) for some \(j \in \{1, 2, \ldots, k\}\). Again, appealing to the measurable maximum theorem and theorem 4.1 in Himmelberg (1975), the product of (maximizing) selections \(\sigma(v, s)\) (respectively, \(\sigma(v, s)\) is measurable with \(s\).
Proof of lemma 3.1: Proof of 1. Clearly \( \phi^1 \leq \phi^2 \) and \( v^1 \leq v^2 \). Suppose \( \phi^t \leq \phi^{t+1} \) and \( v^t \leq v^{t+1} \). By the definition of the sequence \( \{v^t\} \) and lemma 5.2, we have \( v^{t+1} \leq v^{t+2} \). Then, by Lemma 5.1, definition of \( \{\phi^t\} \), and the induction hypotheses, we obtain \( \phi^{t+1}(s) = g(v^{t+1}, s) \leq g(v^{t+2}, s) = \phi^{t+2}(s) \). Similarly, we obtain monotonicity of \( \psi^t \) and \( w^t \).

Proof of 2: Clearly, the thesis is satisfied for \( t = 1 \). By induction, suppose that the thesis is satisfied for some \( t \). Since \( v^t \leq w^t \), by Lemma 5.2, we obtain

\[
v^{t+1}(s) = \Pi^*(v^t, s) \leq \Pi^*(w^t, s) \leq \Pi^*(w^{t+1}, s) = w^{t+1}(s).
\]

Then, by Lemma 5.1, we obtain

\[
\phi^{t+1}(s) = g(v^{t+1}, s) \leq g(w^{t+1}, s) \text{ and hence} \leq \pi(w^{t+1}, s) = \pi^{t+2}(s).
\]

Proof of 3-4: It is clear since for each \( s \in S \), the sequences of values \( v^t \), \( w^t \) and associated pure strategies \( \phi^t \) and \( \psi^t \) are bounded. Further, by previous step, they are monotone.

Proof of 5: By definition of \( v^t \) and \( \phi^t \), we obtain

\[
v^{t+1}_i(s) = (1 - \beta_i)u_i(s, \phi^t(s)) + \beta_i \sum_{j=1}^{L} g_j(s, \phi^t(s)) \int_S v^t_i(s') \lambda_j(ds'|s)
\]

\[\geq (1 - \beta_i)u_i(s, a_i, \phi^t_{-i}(s)) + \beta_i \sum_{j=1}^{L} g_j(s, a_i, \phi^t_{-i}(s)) \int_S v^t_i(s') \lambda_j(ds'|s),\]

for arbitrary \( a_i \in \tilde{A}_i(s) \). By the continuity of \( u_i \) and \( g \) and the Lebesgue Dominance Theorem, if we take a limit \( t \to \infty \), we obtain

\[
v^*_i(s) = (1 - \beta_i)u_i(s, \phi^*(s)) + \beta_i \sum_{j=1}^{L} g_j(s, \phi^*(s)) \int_S v^*_i(s') \lambda_j(ds'|s)
\]

\[\geq (1 - \beta_i)u_i(s, a_i, \phi^*_{-i}(s)) + \beta_i \sum_{j=1}^{L} g_j(s, a_i, \phi^*_{-i}(s)) \int_S v^*_i(s') \lambda_j(ds'|s),\]

which, by lemma 5.3, implies that \( \phi^* \) is a pure stationary (measurable) Nash equilibrium, and \( v^* \) is its associated (measurable) equilibrium payoff. Analogously, we have \( \psi^* \) a pure strategy (measurable) Nash equilibrium, and \( w^* \) its associated (measurable) equilibrium payoff.

Proof of theorem 3.1: First and second points result from lemma 3.1. To prove the third one we proceed in steps. Step 1. We prove the desired inequality for equilibria payoffs. Since \( 0 \leq \mu^* \leq \bar{\mu} \), by Lemma 5.2 and definition of \( v^t \) and \( w^t \), we obtain

\[
v_1 \leq \mu^* \leq w_1.
\]
Lemma 5.4

Let assumptions 1, 2, 3 be satisfied and constraint functions be Lipschitz continuous with a constant less than \( M \) and \( \gamma^* \leq \gamma(s) \leq \gamma^*(s) \leq \gamma(w^*, s) = \psi^* \).

Step 2: By previous step and Lemma 5.1, we obtain:

\[
\phi^*(s) = \frac{a(v^*, s) \leq a(\mu^*, s)}{\mu^*(s) \leq \widetilde{\Pi}(\mu^*, s) \leq \widetilde{\Pi}(w^*, s) = \psi^*}.
\]

For fixed continuation value \( v \) let:

\[
M_{i,j,l} := \frac{\partial^2 \Pi_i}{\partial a_i \partial s_l} + \sum_{i=1}^{n} \sum_{j=1}^{k_i} \frac{\partial^2 \Pi_i}{\partial a_i \partial a_i},
\]

\[
M := \max \{ M_{i,j,l} : i = 1, \ldots, n, \text{ and } j = 1, \ldots, k_i, \text{ and } l = 1, \ldots, k \}.
\]

By assumption 3, the constant \( M \) is a strictly positive real number.

Lemma 5.4 Let assumptions 1, 2, 3 be satisfied and constraint functions be Lipschitz continuous with a constant less that \( M \). Fix \( v \in B(S) \), and assume it is Lipschitz continuous. Consider an auxiliary game \( G^v \). Then, there is a unique Nash equilibrium in this game \( a^*(v, s) \) and belongs to \( CM^k \).

Proof of lemma 5.4: Let \( s > 0 \) and let \( v \in B(S) \) be Lipschitz continuous function. To simplify we drop \( v \) from our notation. Let \( x^1(s) = \tilde{a}(s) \) and \( x^t+1(s) := \arg \max_{a_i \in \tilde{A}_i(s)} \Pi_i(s, a_i, x^t(s)) \) for \( n \geq 1 \). This is well defined by strict concavity of \( \Pi_i \) in \( a_i \). Clearly, \( x^1 \) is nondecreasing and Lipschitz continuous with a constant less that \( M \). By induction, assume that this thesis holds for \( t \in \mathbb{N} \). Note that \( (s, a_i) \rightarrow \Pi_i(s, a_i, x^t(s)) \) has increasing differences. Indeed if we take \( s_1 \leq s_2 \) and \( y_1 \leq y_2 \) then \( x^t-1(s_1) \leq x^t-2(s_2) \) and

\[
\Pi_i(s_1, y_2, x^t-1(s_1)) - \Pi_i(s_1, y_1, x^t-1(s_1)),
\]

\[
\leq \Pi_i(s_1, y_2, x^t-2(s_2)) - \Pi_i(s_1, y_1, x^t-2(s_2)),
\]

\[
\leq \Pi_i(s_2, y_2, x^t-2(s_2)) - \Pi_i(s_2, y_1, x^t-2(s_2)).
\]

Therefore, since \( \tilde{A}_i(\cdot) \) is ascending in the Veinott strong set order, by Theorem 6.1 in Topkis (1978) we obtain that \( x^t+1(\cdot) \) is isotonone. We show that \( x^t+1(\cdot) \) is Lipschitz continuous with a constant \( M \). To do this we check hypotheses of Theorem 2.4(ii) in Curtat (1996). Define \( \varphi(s) = s_1 + \ldots + s_k \). Define \( \mathbf{1}_i := (1, 1, \ldots, 1) \in \mathbb{R}^k \). We show that the function \( (s, y) \rightarrow \Pi^*(s, y) := \Pi_i(s, \varphi(s) \mathbf{1}_i - y, x^t_i(s)) \) has increasing differences. Note that \( M\varphi(s) - \tilde{a}_i(s) \leq y \leq M\varphi(s) \). We show that the collection of the sets \( Y(s) := [M\varphi(s) - \tilde{a}_i(s), M\varphi(s)] \) is ascending in the Veinott strong set order. Let \( s_1 \geq s_2 \) in product order. Then,

\[
M\varphi(s_1) - \tilde{a}_i(s_1) - (M\varphi(s_2) - \tilde{a}_i(s_2)) = M||s_1 - s_2||_1 - |\tilde{a}_i(s_1) - \tilde{a}_i(s_2)| \geq 0.
\]
This, therefore, implies that lower bound of $Y(s)$ is increasing with $s$. Clearly upper bound of $Y(s)$ is increasing as well. Hence $Y(s)$ is ascending in the Veinott’s strong set order.

Note that since for all $s_t \rightarrow x^t(s)$ is monotone and continuous, hence must be differentiable almost everywhere (Royden (1968)). By $M$ Lipschitz property of $x^t$ we conclude that each partial derivative is bounded by $M$. Hence we have for all $l = 1, \ldots, k, i = 1, \ldots, n$ and $j = 1, \ldots, k_i$:

$$\frac{\partial \Pi^*_i}{\partial y^i_l} = -\frac{\partial \Pi_i}{\partial a^j_l}$$

Next we have (for fixed $s_{-k}$):

$$\frac{\partial^2 \Pi^*_i}{\partial y^i_l \partial s_k} = -\frac{\partial^2 \Pi_i}{\partial a^j_l \partial s_k} - M \sum_{a=1}^{k_i} \frac{\partial^2 \Pi_i}{\partial a^j_l \partial a^c} \frac{\partial \varphi}{\partial s_k} \sum_{\alpha=1}^{k_i} \sum_{j=1}^{k_i} \frac{\partial^2 \Pi_i}{\partial a^j_l \partial a^j_i} \frac{\partial x^*_j}{\partial s_k}.$$

$$\geq -\frac{\partial^2 \Pi_i}{\partial a^j_l \partial s_k} - M \sum_{a=1}^{k_i} \frac{\partial^2 \Pi_i}{\partial a^j_l \partial a^c} \sum_{\alpha=1}^{k_i} \sum_{j=1}^{k_i} \frac{\partial^2 \Pi_i}{\partial a^j_l \partial a^j_i} M$$

$$= -\frac{\partial^2 \Pi_i}{\partial a^j_l \partial s_k} - \sum_{i=1}^{k_i} \sum_{j=1}^{k_i} \frac{\partial^2 \Pi_i}{\partial a^j_l \partial a^j_i} M$$

$$= \left( -\sum_{i=1}^{k_i} \sum_{j=1}^{k_i} \frac{\partial^2 \Pi_i}{\partial a^j_l \partial a^j_i} \right) \left( M - \frac{\partial^2 \Pi_i}{\partial a^j_l \partial s_k} - \sum_{i=1}^{k_i} \sum_{j=1}^{k_i} \frac{\partial^2 \Pi_i}{\partial a^j_l \partial a^j_i} \right) \geq 0$$

almost everywhere. Since $\frac{\partial \Pi^*_i}{\partial a^j_l}$ is continuous, by Theorem 6.2. in Topkis (1978) the solution of the optimization problem $y \rightarrow \Pi^*(v, s, y)$ (say $y^*(s, v)$) is isotone. From definition of $y^*(s, v)$ and $x^{t+1}$ if $s_1 \leq s_2$ we have

$$0 \leq x^t_{i,j}(s_1) - x^t_{i,j}(s_2) \leq M(\varphi(s_1) - \varphi(s_2)) = M||s_1 - s_2||_1.$$  

Analogously we prove appropriate inequality whenever $s_1 \geq s_2$. If $s_1$ and $s_2$ are incomparable then

$$x^t_{i,j}(s_1) - x^t_{i,j}(s_2) \leq x^t_{i,j}(s_1 \lor s_2) - x^t_{i,j}(s_1 \land s_2) \leq M||s_1 - s_2||_1$$

since $||s_1 - s_2||_1 = ||s_1 \lor s_2 - s_1 \land s_2||_1$. Similarly we prove that:

$$x^t_{i,j}(s_1) - x^t_{i,j}(s_2) \geq -M||s_1 - s_2||_1.$$  

But this implies that $x^{t+1}$ is also $M$-Lipschitz continuous, which implies that each $x^t$ is $M$-Lipschitz continuous. Since $\Pi_i$ has increasing differences in $(a_i, a_{-i})$ hence by Theorem 6.2 in Topkis (1978) we know that the operator $x \rightarrow \arg \max_{y_i \in A_i(s)} \Pi_i(s_i, y_i, x_{-i})$ is increasing. Therefore $x^t(s)$ must be decreasing in $t$. This implies that there exists $a^* = \lim_{n \to \infty} x^t$ which is isotone and $M$-Lipschitz continuous. Uniqueness of Nash Equilibria follows from assumption 3 and Gabay and Moulin (1980), hence $a^* = a^*(s, v)$ for $s > 0$.

Finally $\Pi(0, a) = 0$ for all $a \in A$ hence we can define $a^*(0) := \lim_{s \to 0^+} a^*(s)$ and obtain a unique Nash equilibrium $a^*(s, v)$ that is isotone and $M$-Lipschitz continuous in $s$. □
Proof of Theorem 3.2: To simplify notation, let \( L = 1 \) and hence \( g(s,a) := g_1(s,a) \) and \( \eta(f)(s) := \eta^f_1(s) \). Let \( v \in B^n(S) \) be Lipschitz continuous. Under assumptions 3, \( a^*(s,v) \) is a well defined (for \( s > 0 \)) as auxiliary game satisfies conditions of Gabay and Moulin (1980). Let \( \pi_i(v_i,s,a) = (1 - \beta_i)u_i(s,a) + \beta_i\eta_i(v_i)g(s,a) \), and observe that \( \pi_i(v_i,s,a) \) has also strict diagonal property, and obviously has cardinal complementarities. Here, note that

\[
L_{i,j}(\pi_i(v_i,s,a)) = (1 - \beta_i)L_{i,j}(u_i(s,a)) + \beta_i\eta_i(v_i)\eta_i(s,a)L_{i,j}(g(s,a)) < 0.
\]

Note further that applying Royden (1968) and continuity of the left side of expression below we have

\[
- \sum_{i} \sum_{j} (1 - \beta_i)\frac{\partial^2 \pi_i(v,s,c)}{\partial \alpha_i \partial \alpha_i} \leq \frac{(1 - \beta_i)u_i(s,a)}{(1 - \beta_i)^2u_i(s,a)} + \beta_i\frac{\partial \eta_i(v)(s)}{\partial \alpha_i} + \beta_i\eta_i(v)(s)\frac{\partial^2 g(s,a)}{\partial \alpha_i \partial \alpha_i} \leq M_0.
\]

By Lemma 5.4, we know that \( a^*(\cdot,v) \in CM_0^{l \sum_i k_i} \). The following argument shows that \( T\eta(s) \) is Lipschitz continuous.

\[
|T\eta(s_1) - T\eta(s_2)| \leq (1 - \beta_i)|u_i(s_1,a^*(s_1,v)) - u_i(s_2,a^*(s_2,v))|
\]

\[
+ \beta_i|\eta_i(v_i(s_1) - \eta_i(v_i(s_2))|g(s_1,a^*(s_1,v))|
\]

\[
+ \beta_i\eta_i(v_i(s_2)|g(s_1,a^*(s_1,v)) - g(s_2,a^*(s_2,v))|
\]

\[
\leq (U_1 + M_0U_2 + \bar{\eta} + (G_1 + G_2M_0)\bar{u})||s_1 - s_2||_1
\]

where

\[
U_1 := \sum_{i=1}^h \sup_{s \in S, a \in A(s)} \frac{\partial u_i}{\partial s_i}(s,a), \quad U_2 := \sum_{i=1}^m \sum_{j=1}^{k_i} \sup_{s \in S, a \in A(s)} \frac{\partial u_i}{\partial a_i}(s,a),
\]

\[
G_1 := \sum_{i=1}^h \sup_{s \in S, a \in A(s)} \frac{\partial g}{\partial s_i}(s,a), \quad G_2 := \sum_{i=1}^m \sum_{j=1}^{k_i} \sup_{s \in S, a \in A(s)} \frac{\partial g}{\partial a_i}(s,a),
\]

and \( M_1 := U_1 + M_0U_2 + \bar{\eta} + (G_1 + G_2M_0)\bar{u} \). Hence image of operator \( T \) is a subset of \( CM_1^n \). Therefore the thesis is proven.

Proof of Theorem 3.3: On \( CM^n \) define \( T(v)(s) = \Pi^*(v,s) \). By a standard argument (e.g. Curtat (1996)) \( T : CM^n \rightarrow CM^n \) is continuous. By our lemma 5.2 it is also increasing on \( CM^n \). By Tarski (1955) theorem, it therefore has a nonempty complete lattice of fixed points, say \( FP(T) \). Further, for each fixed point \( v^*(\cdot) \in FP(T) \), there is a corresponding unique stationary Markov Nash equilibrium \( a^*(v^*,\cdot) \).

Lemma 5.5 Let \( X \) be a lattice, \( Y \) be a poset. Assume \( (i) \) \( F : X \times Y \rightarrow \mathbb{R} \) and \( G : X \times Y \rightarrow \mathbb{R} \) have increasing differences, \( (ii) \) that \( \forall y \in Y, G(\cdot,y) \) and \( \gamma : Y \rightarrow \mathbb{R}_+ \) are increasing functions. Then, function \( H \) defined by \( H(x,y) = F(x,y) + \gamma(y)G(x,y) \) has increasing differences.
Proof of lemma 5.5: Under the hypotheses of the lemma, it suffices to show that $\gamma(y)G(x, y)$ has increasing differences (as increasing differences is a cardinal property and closed under addition). Let $y_1 > y_2$, $x_1 > x_2$ and $(x_i, y_i) \in X \times Y$. By the hypothesis of increasing differences of $G$, and monotonicity of $\gamma$ and $G(\cdot, y)$, we have the following inequality
$$\gamma(y_1)(G(x_1, y_1) - G(x_2, y_1)) \geq \gamma(y_2)(G(x_1, y_2) - G(x_2, y_2)).$$
Therefore,
$$\gamma(y_1)G(x_1, y_1) + \gamma(y_2)G(x_2, y_2) \geq \gamma(y_1)G(x_2, y_1) + \gamma(y_2)G(x_1, y_2).$$

Proof of theorem 3.4: Step 1. Let $v_\theta$ be a function $(s, \theta) \to v_\theta(s)$ that is increasing. By assumption 5 and lemma 5.5, the payoff function $\Pi_i(v_\theta, s, a, \theta)$ has increasing differences in $(a_i, \theta)$. Further, $\Pi_i$ clearly also has increasing differences in $(a_i, a_{-i})$. As $\tilde{A}_i()$ is ascending in Veinott’s strong set order, by Theorem 6 in Milgrom and Roberts (1990), the greatest and the least Nash equilibrium in the supermodular game $G^*_s(\cdot, \theta)$ are increasing selections. Further, by the same argument as in Lemma 5.2, as $\tilde{A}_i()$ is also ascending under set inclusion by assumption, we obtain monotonicity of corresponding equilibria payoff.

Step 2: Note, for each $\theta$, the parameterized stochastic game satisfies conditions of Theorem 3.1. Further, noting the initial values of the sequence of $w^t_\theta(s)$ and $v^t_\theta(s)$ (constructed in Theorem 3.1) do not depend on $\theta$ and are isotone in $s$, by the previous step, each iteration of both sequences of values is increasing with respect to $(s, \theta)$. Also, each of the iterations of $\phi^t_\theta(s)$ and $\psi^t_\theta(s)$ are also increasing in $(s, \theta)$. Therefore, as the pointwise partial order is closed, the limits of these sequences preserve this partial ordering, and the limits are increasing with respect to $(s, \theta)$.

For each equilibrium strategy $f$, define the operator
$$T^*_f(\eta)(A) = \int_S Q(A|s, f(s))\eta(ds).$$
(1)
where $\eta^*$ is said to be invariant with respect to $f$ if and only if it is a fixed point of $T^*_f$.

Proof of theorem 3.5: By theorem 3.4 both $\phi^*$ and $\psi^*$ are increasing function. Let $v$ be increasing function. By Assumption 2
$$\int_S v(s)T^*_\phi(\eta)(ds) = \sum_{j=1}^L g_j(s, \phi^*(s)) \int_S v(s')\lambda_j(ds'|s)\eta(ds).$$
Since by assumption, for each $s \in S$, the function under integral is increasing, the right-side increase pointwise whenever $\eta$ is stochastically increase. Moreover, as the family of probability measures on a compact state space $S$ ordered by $\succeq$ (first order stochastic dominance) is chain complete (as it is a compact ordered topological space, e.g., Amann (1977), lemma 3.1 or corollary 3.2). Hence, $T^*_\phi$ satisfies conditions of Markowsky (1976) theorem (Theorem 9), and we conclude that the set of invariant distributions is a chain complete with greatest and least
invariant distributions (see also Amann (1977), Theorem 3.3). By a similar construction, the
same is true for the operator $T_{\phi^*}^\alpha$.

To show the second assertion, we first prove that $T_{\phi^*}^\alpha(\cdot)(A)$ is weakly continuous (i.e. if $\eta_t \to \eta$ weakly then $T_{\phi^*}^\alpha(\eta_t) \to T_{\phi^*}^\alpha(\eta)$ weakly). Let $\eta_t \to \eta$ weakly, and $v$ be continuous. By Feller property of $\lambda_j(\cdot|s)$, we have $s \to \int_s v(s') \lambda_j(s'|s)$ continuous. Therefore,

$$\int_S \int_S v(s') \lambda_j(ds'|s) \eta_t(ds) \to \int_S \int_S v(s') \lambda_j(ds'|s) \eta(ds).$$

This, in turn, implies $T_{\phi^*}^\alpha(\eta_t) \to T_{\phi^*}^\alpha(\eta)$ weakly. Let $\eta_t^{\phi^*}$ be a distribution of $s_t^{\phi^*}$ and $\eta_1^{\phi^*} = \delta_S$. By the previous step, $\eta_t$ is stochastically decreasing. It is, therefore, weakly convergent to some $\eta^*$. By continuity of $T^\alpha$, we have $\eta^* = T_{\phi^*}^\alpha(\eta^*)$. By definition of $\bar{\eta}(\phi^*)$, we immediately obtain $\bar{\eta}(\phi^*) \leq \eta^*$. By the stochastic monotonicity of $T_{\phi^*}^\alpha(\cdot)$, we can recursively obtain that $\delta_S \succeq \eta_t^{\phi^*} \succeq \bar{\eta}(\phi^*)$, and hence $\eta^* \succeq \bar{\eta}(\phi^*)$.

As a result, we conclude $\eta^* = \bar{\eta}(\phi^*)$. Similarly, we show convergence of the sequence of distributions $s_t^{\psi^*}$.

**Proof of corollary 3.2:** By theorem 3.5, there exists greatest fixed points for $T_{\phi^*}^{\alpha,\beta_2}$ and $T_{\phi^*}^{\alpha,\beta_1}$. Also, $T_{\phi^*}^{\alpha,\beta}$ is weakly continuous. Further, $\theta \to T_{\phi^*}^{\alpha,\beta}$ is an increasing map under first stochastic dominance on a chain complete poset of probability measures on the compact state set.

Consider a sequence of iterations from a $\delta_S$ generated on $T_{\phi^*}^{\alpha,\beta}$ (the operator defined in (1) but associated with $Q(\cdot|s,a,\theta)$). Observe, by the Tarski-Kantorovich theorem (Dugundji and Granas (1982), theorem 4.2), we have

$$\sup_t T_{\phi^*}^{t,\alpha,\beta_2} = \bar{\eta}_{\theta_2} \text{ and } \sup_t T_{\phi^*}^{t,\alpha,\beta_1} = \bar{\eta}_{\theta_1}.$$

As for any $t$, we also have $T_{\phi^*}^{t,\alpha,\beta_2} \succeq T_{\phi^*}^{t,\alpha,\beta_1}$. Therefore, by weak continuity (and the fact that $\succeq$ is a closed order), we obtain:

$$\bar{\eta}_{\theta_2} = \sup_t T_{\phi^*}^{t,\alpha,\beta_2} \succeq \sup_t T_{\phi^*}^{t,\alpha,\beta_1} = \bar{\eta}_{\theta_1}.$$

Similarly we proceed for $\psi^*$.

**Remark 1** Since norms $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent on $\mathbb{R}^h$ and for all $s \in S$ $||s||_1 \leq h||s||$, hence we have that $v^*$, $w^*$, $\phi^*$ and $\psi^*$ each Lipschitz continuous when we endow $S$ with the maximum norm.

**References**


