Pricing Assets in an Economy with Two Types of People

By Roger E.A. Farmer*

This paper constructs a general equilibrium model with two types of people where asset price fluctuations are caused by random shocks to the price level that reallocate consumption across generations. In this model, asset prices are volatile, and price-earnings ratios are persistent, even though there is no fundamental uncertainty and financial markets are sequentially complete. I show that the model can explain a substantial risk premium while generating smooth time series for consumption and financial assets across types. In my model, asset price fluctuations are Pareto inefficient and there is a role for treasury or central bank intervention to stabilize asset prices.

There is a large literature in finance that seeks to reconcile features of financial data with features of aggregate time series (Gabaix, 2012). Typically, these explanations combine some version of the rare disasters hypothesis of Rietz (1988) and Barro (2006) with the variable long-run risk model of Bansal and Yaron (2004). To successfully explain financial data in an equilibrium model, the theorist must explain why the price of risk is highly volatile, while consumption data are smooth.

Most equilibrium explanations of financial data are based on the assumption that all shocks to the economy are fundamental. For example, in the long-run risk model of Bansal and Yaron (2004), consumption growth is exogenous and has a small highly persistent component. In the rare disaster model of Rietz (1988) and Barro (2006) there is, occasionally, a large negative shock to the aggregate endowment.

This paper explores an alternative approach. I build on the work of David Cass and Karl Shell (1983), by constructing a model where asset price fluctuations are caused by non-fundamental shocks to people's beliefs. Cass and Shell presented this idea in a two-period real model. I show that a calibrated version of their

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model can explain real world data.

I construct a general equilibrium model with two types of people where asset price fluctuations are caused by random shocks to the price level that reallocate consumption across generations. In this model, asset prices are volatile and price-earnings ratios are persistent even though there is no fundamental uncertainty and financial markets are sequentially complete. I refer to the random variable that drives asset prices in equilibrium as a *belief shock*. Because I am interested in the ability of non-fundamental shocks to explain asset prices, my model has no fundamental uncertainty of any kind.

My work differs in three ways from standard asset pricing models. First, I allow for birth and death by exploiting Blanchard's (1985) concept of perpetual youth. Second, there are two types of people that differ in the rate at which they discount the future. Third, my model contains an asset, government debt, denominated in dollars. All three assumptions are necessary to generate my main results.

I model a government with two branches; a central bank and a treasury. The central bank operates an interest rate peg and the treasury adjusts the tax rate periodically to ensure the government remains solvent. Because debt is denominated in dollars, this policy leads to indeterminacy of the initial equilibrium price level. For every initial price in a certain set, there is a different allocation of the present value of taxes between current and future generations. For each allocation of taxes, there is a different non-stationary perfect foresight equilibrium price sequence. Each of these sequences converges to the same steady-state equilibrium.

I exploit the indeterminacy of the set of perfect foresight equilibria to construct a rational expectations equilibrium in which non-fundamental shocks cause asset price fluctuations. The people in my model believe the future price level is a random variable, driven by a belief shock, and they write financial contracts contingent on its realization. In equilibrium, their beliefs turn out to be correct. Because the unborn cannot buy or sell contracts traded before they are born, belief shocks have real effects that reallocate resources between people of different generations.

I am pursuing an alternative explanation for asset price fluctuations because I am ultimately interested in a normative question. Should the treasury and or the central bank intervene in asset markets to reduce volatility? If all asset market fluctuations are caused by the responses of a representative agent to unavoidable endowment shocks, the government should not seek to intervene. If instead, a large component of asset price fluctuations is due to Pareto inefficient re-allocations of consumption goods between people of different generations, there is a potential role for an active financial policy, of the kind discussed in Farmer and Zabczyk (2016), to stabilize those fluctuations.

I. Antecedents

An active body of scholars seek to explain asset price data using the representative agent model. Some of the modifications to this model that have been tried include richer utility specifications (Abel, 1990; Constantinides, 1990; Campbell and Cochrane, 1999; Bansal and Yaron, 2004) adding technology shocks with exogenous time-varying volatility (Bansal and Yaron, 2004), and assuming that technology is occasionally hit by rare disasters (Rietz, 1988; Barro, 2005, 2006; Wachter, 2013; Gourio, 2012; Gabaix, 2012). Here, I take an alternative approach.

I build on the idea that non-fundamental shocks can have real effects when there is incomplete participation in asset markets. David Cass and Karl Shell (1983) refer to non-fundamental shocks as 'sunspots' and Costas Azariadis (1981) and Roger Farmer and Michael Woodford (1997) call them 'self-fulfilling prophecies'. Although the term 'sunspot' is widely understood by economic theorists, I have found that it represents a source of confusion when explaining the idea to a lay audience and I use the term 'belief shock' in this paper to mean non-fundamental uncertainty that may have real effects.

My work is closely related to four working papers, Farmer (2002b,c, 2014) and Farmer et al. (2012). Farmer (2002c) develops a version of Blanchard's (1985) perpetual youth model with capital and aggregate uncertainty, Farmer (2002b) adds nominal government debt to explain asset price volatility and Farmer et al. (2012) construct a model with multiple types.[1] The current paper relies on all three of these pieces; perpetual youth, nominal debt and multiple types.[2]

The idea of constructing stationary stochastic rational expectations equilibria by randomizing over multiple steady states is due to Azariadis (1981). The first paper to exploit randomizations across indeterminate perfect foresight paths in a monetary model is by Farmer and Woodford (1997). Farmer et al. (2015) show how to solve models with indeterminacy using standard solution methods by redefining a belief shocks to be a new fundamental and I draw on a non-linear extension of their technique in this paper.

This is not the only paper to explore heterogeneous-agent models to understand asset pricing data. Challe (2004) generates return predictability in an overlapping generations model and Guvenen (2009) constructs a production economy that he solves computationally. Constantinides and Duffie (1996) exploit cross-section heterogeneity of the income process to show that uninsurable income risk across consumers can potentially explain *any* observed process for asset prices and Kubler and Schmedders (2011) construct a heterogeneous-agent overlapping generations model with sequentially complete markets. By dropping the rational expectations assumption, they are able to generate substantial asset price volatility. In a related paper, Feng and Hoelle (2014) generate large welfare distortions from sunspot fluctuations.

[1] Farmer et al. (2012) claim to generate equilibria, driven by non-fundamental shocks. That claim is incorrect as their model fails to equate the marginal rates of substitution of each type of agent in every state and consequently, the paper does not fulfill its claim to generate sunspot equilibria. I am grateful to Markus Brunnermeir and Valentin Haddad for discussions on this point.

[2] An earlier version of the current paper appeared with the title "Global Sunspots and Asset Prices in a Monetary Economy" (Farmer, 2015). The current version is different, in a number of important ways and for that reason I have changed the title.

Gârleanu et al. (2012) build a two-agent life-cycle model where the agents have recursive preferences but a common discount factor and they show that this model generates inter-generational shifts in consumption patterns that they call 'displacement risk'. In a related paper Gârleanu and Panageas (2014) study asset pricing in a continuous time stochastic overlapping generations model. These papers focus on fundamental equilibria and they adopt the common assumption of Epstein-Zin preferences (Epstein and Zin, 1989, 1991). Gârleanu and Panageas (2016) introduce non-fundamental shocks in a real model where there are multiple equilibria as a result of an investment technology in which newborn people are born with new production techniques. Their work is the closest to mine and has a lot in common with the ideas I discuss here.

II. Data I Would Like to Explain

In Figure 1 I have plotted annual data on five aggregate time series for the period from 1948 through 2008. The top left panel is the average twelve month percentage increase in the CPI and the bottom left panel depicts the ratio of consumption to GDP. The raw data are from the Bureau of Economic Analysis. The top right panel plots the one-year real return on a short bond and the one-year real holding return to the S&P 500. The bottom right panel is Robert Shiller's Cyclically Adjusted Price Earnings Ratio, (CAPE). These data are from Shiller's website (Shiller, 2014).

A number of features of these data are striking. The consumption to GDP ratio is almost constant. In contrast, the cyclically adjusted price-earnings ratio varied from a low of 7, in 1982 to a high of 44, in 1999. In a market economy, the value of the stock market reflects the present discounted value of future dividends and, in an endowment economy, dividends are equal to consumption. A representative agent endowment economy has a very difficult time explaining these data with any simple, plausible, model of preferences.

The volatility of stock prices, relative to any measure of the income flow generated by stock ownership, was pointed out by Shiller (1981) and Leroy and Porter (1981) who refer to the problem of explaining the volatility of asset prices as the excess volatility puzzle. Any explanation of excess volatility must explain why the state dependent price that people are willing to pay for a risky claim, the so called *stochastic discount factor*, can be so volatile.

In the model I construct in this paper, fluctuations in the stochastic discount factor are one and the same thing as fluctuations in the inflation factor and the volatility of inflation is of the same order of magnitude as that depicted in Figure 1.

A second feature that stands out in these data is the smoothness of the real return to holding treasury securities, this is the dashed line in the top right panel, compared to the volatility of the stock market return; this is the solid line. These series do not just differ in volatility, they also differ in their means.

In Table 1, I record the average risky and safe interest rates for the period from

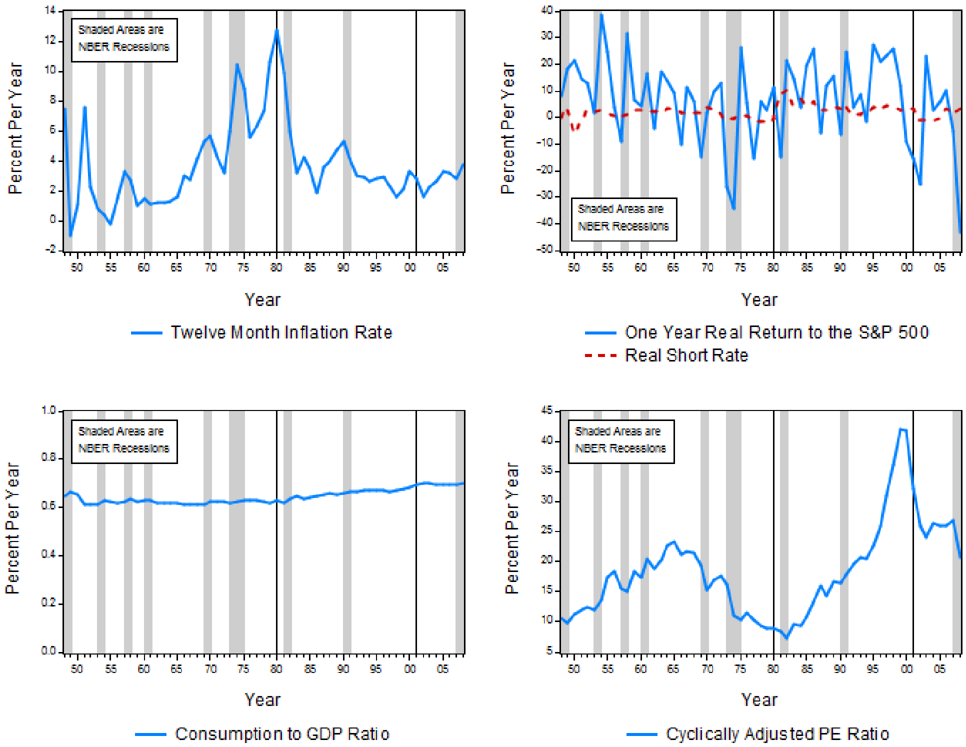

FIGURE 1. SIXTY YEARS OF US DATA

Variable	Symbol	Value
Stock Market Return	r^r	6.5%
Return to a One-Year Bond	r^s	1.9%
Sharpe Ratio	S	0.29

TABLE 1—SAFE AND RISKY RETURNS IN THE DATA

1948 through 2008. The average return to holding the stock market was 6.5%, the average real return to holding a one-year treasury was 1.9% and the standard deviation of the stock market return was 16. These data imply that the premium for holding a risky asset was equal to 4.6% and the risk adjusted excess return, the so called Sharpe ratio, was equal to 0.29. As pointed out by Mehra and Prescott (1985), a risk premium of this magnitude is a puzzle for representative agent macro models with standard preference assumptions.

One approach to solving the excess volatility and the equity premium puzzles is to combine exotic preference specifications (Backus et al., 2005) with assumptions about the long-run properties of fundamentals. In the rest of the paper, I will explore a different approach. I build a model that explains these features of data as belief shocks that re-allocate consumption between generations.

III. Structure and Assumptions

I describe an endowment economy with no fundamental uncertainty in which there are complete financial markets and where money is used as a unit of account. I show that this model has a continuum of non-stationary perfect foresight equilibria in which asset prices are described by a persistent difference equation that converges slowly to a steady state. I construct a set of stationary rational expectations equilibria by randomizing across these equilibria.

A. Structure

Time proceeds in a sequence of periods indexed by $t = 1, 2, \ldots \infty$. There are two types of people and a measure 1 of each type. People survive into the subsequent period with age-independent probability, π, that is the same for each type. A person of type $i \in \{1, 2\}$ receives an endowment of μ_i units of a unique consumption good in every period in which he is alive. I refer to the consumption good as an apple and I assume that

$$\mu_1 + \mu_2 = 1.$$

This assumption implies the aggregate endowment of apples of the people born at date t, summed over types, is equal to 1.

Every period a measure $1 - \pi$ of type i people from each generation dies and is replaced by a measure $1 - \pi$ of new people of the same type. This assumption implies there is a measure 1 of each type alive in every period and the social endowment of apples is constant and equal to 1.

The claim to a person's endowment can be traded in the financial markets. It is a random sequence of payments that I refer to as a tree. The earnings of a single tree is a stochastic process because its owner may die. The earnings of a unit measure of trees is a deterministic variable, equal to the fraction, π, of people.

People maximize the present discounted sum of expected future utility, where utility is represented by a homogeneous Von-Neumann Morgenstern utility function, u_i, discounted by a factor, β_i. I assume that type 1 people are more patient than type 2 people; that is,

$$0 < \beta_2 < \beta_1 < 1.$$

The assumptions that survival probabilities are age independent and utility functions are homogeneous allows me to summarize the consumption of each type of person by a linear function of their aggregate wealth. The assumption that people have different discount factors is important and is key to understanding the mechanism for persistence of asset prices.

B. *Financial Intermediaries, Annuities, and Life Insurance*

In a model with birth and death one must make an assumption about the dispersion of a person's assets when he dies. I follow Blanchard (1985), by assuming the existence of a perfect annuities market.

Although the endowment is constant, the present value of tax liabilities is a random variable. People alive at date t trade a complete set of financial securities, mediated by a set of financial intermediaries. The assets of financial intermediaries consist of a portfolio of state-contingent one-period consumption loans to type 2 people. Their liabilities consist of state-contingent one-period consumption loans from type 1 people.

In a rational expectations equilibrium, a newly born type 1 person becomes a lender and a newly born type 2 person becomes a borrower. A young type 1 person starts out life by consuming less than his endowment. If he has a long life, a type 1 person eventually consumes more than his endowment as he earns additional income from his accumulated assets. A young type 2 person starts out life by consuming more than his endowment. If he has a long life, a type 2 person eventually consumes less than his endowment as he repays his accumulated debts.

In addition to acting as intermediaries between buyers and sellers of financial assets, financial intermediaries provide life insurance to cohorts. The size of the endowment owned by any given cohort shrinks geometrically by a factor $\pi \in (0, 1)$ as the members of the cohort die. The present value of the endowment owned by any individual in the cohort is a random variable. Financial intermediaries provide perfect insurance against idiosyncratic shocks to the length of life.

To implement the efficient insurance contract, financial intermediaries offer annuities contracts to savers and life-insurance contracts to borrowers. A saver receives a discount in state ε when he buys a state-contingent claim to one apple in state ε'. That claim costs him $\pi Q(\varepsilon, \varepsilon')$, where $Q(\varepsilon, \varepsilon')$ is the price of the claim and π is the price of an annuity that pays 1 apple to the financial institution in the event that the person dies and 0 apples if he lives.

On the other side of this market there is a borrower. A borrower receives a loan of $\pi Q(\varepsilon, \varepsilon')$ in state ε in return for a promise to repay one apple in state ε'. The amount he can borrow against a promise to repay 1 apple is reduced by a fraction, π. The fraction π represents the price of a life-insurance contract that pays the debt of the borrower in the event of his death. Because there are complete markets for aggregate shocks, and because I assume free entry to the financial services industry, financial intermediaries make zero profits in equilibrium.

C. *Financial Markets and Tax Liabilities*

In this subsection I explain the source of uncertainty and I discuss the market structure that allows people to insure against shocks. All shocks in the model arise from non-fundamental shifts in beliefs. In a rational expectations equilibrium, belief shocks are reflected in different realizations of the dollar price of apples that are anticipated by participants in the financial markets. Importantly, newborn people of both types are unable to participate in the financial markets that open before they are born.

Each period, the price level is a function of a publicly observable random variable, ε, drawn from a time-invariant distribution χ with support Ω. The variable ε might represent the opinions of the editors of an influential newspaper or the views of a financial journalist. The mapping from ε to p is common knowledge.

At date t, type 1 and type 2 people are aware that prices will fluctuate in period $t+1$ and they trade a complete set of Arrow securities where security ε' is a promise to pay one apple in period $t+1$ if and only if state ε' occurs. I denote the price, at date t in state ε for delivery of an apple at date $t+1$ in state ε' by $Q(\varepsilon, \varepsilon')$. I refer to the function $\tilde{Q}(\varepsilon, \varepsilon') \equiv \frac{Q(\varepsilon, \varepsilon')}{\chi(\varepsilon')}$ as the *pricing kernel*.

A price level shock has real effects because it changes the net present value of tax liabilities. A person born into a state with a low price p, relative to nominal debt, B^N, will be born with a high tax liability. Once born, he is perfectly insured against all future fluctuations in p. But he cannot insure against the state of the world he is born into.

Price fluctuations result in wealth transfers between generations. Because different types of people have different propensities to consume out of wealth, random wealth transfers trigger persistent fluctuations in the stochastic discount factor.

D. Government

There is a government with two branches; a central bank and a treasury. The central bank operates an interest rate peg by standing ready to buy or sell one-period dollar-denominated pure-discount bonds for price $\bar{Q}^N$.[3] This policy is equivalent to setting the money interest rate, i, at $i = \frac{1}{\bar{Q}^N} - 1$.

It is well known that dynamic monetary models with an interest-rate peg lead to indeterminacy of equilibrium. One might hope that this problem is just a theoretical curiosity. Alas, that is not the case. Jordi Galí and Mark Gertler (2000) have shown that, in the period before 1979, the US Federal Reserve operated a *passive monetary policy* that is predicted, in New Keynesian models, to lead to indeterminacy of the price level.[4]

The US Federal Reserve Board and the Bank of England have pegged the money interest rate at, or close to, zero, for the past eight years. Japan has had a zero interest rate for several decades. And during the Great Depression, the short-term interest rate in the United States was at or close to zero for more than a decade. I conclude that it is not unrealistic to study a model in which the central bank pegs

[3]Although I model monetary policy, my model is one where no one actually holds money as a means of exchange. Michael Woodford has popularized the idea of modeling what he calls the "cashless limit", Woodford (2003). I adopt that strategy here. Although bonds are denominated in dollars, the dollar serves only as a unit of account.

[4]A passive monetary policy is an interest rate rule in which the central bank responds to inflation by raising or lowering the interest rate by less than one-for-one (Leeper, 1991).

the interest rate at a fixed number. An interest rate peg at zero is an extreme example of a passive monetary policy, and it is a policy that has characterized the operation of real-world monetary policy for much of recent world history.

The treasury begins the period with debt of $B(\varepsilon) = \frac{B^N}{p(\varepsilon)}$, where B^N is debt denominated in dollars and $p(\varepsilon)$ is the dollar price of an apple. It operates the tax-transfer policy,

$$T(\varepsilon) = p(\varepsilon)\tau + (1-\delta)B^N, \tag{1}$$

where τ is a real valued tax or transfer levied on, or distributed equally to, all persons alive at that date. A positive value of τ denotes a tax and a negative value denotes a transfer. The dollar value of this tax-transfer is equal to $p(\varepsilon)\tau$, where $p(\varepsilon)$ is the dollar price of an apple.

The treasury levies an additional tax of $(1-\delta)B^N$, where B^N is the outstanding value of government liabilities and $\delta \in [0,1]$. This feature of tax policy represents adjustments to the tax rate made to prevent government debt from becoming too large as a fraction of GDP.[5]

The real value of government debt follows the equation

$$\mathbb{E}\left[\tilde{Q}'(\varepsilon,\varepsilon')B(\varepsilon')\right] = B(\varepsilon) - \tau - (1-\delta)B(\varepsilon).$$

Period $t+1$ debt is a random variable, conditional on date t information, because the price level in period $t+1$ is a function of ε'.

Building on Woodford (1994), an extensive literature has developed which argues that, under some circumstances, government debt will remain bounded even in the case when the government makes no active attempt to remain solvent. This argument, called the fiscal theory of the price level (FTPL), is used to select a unique equilibrium in monetary models in which the price level is otherwise indeterminate.

I find the arguments that have been presented for the FTPL to be unpersuasive. If the FTPL held in practice, I do not believe that legislatures would be as concerned with budget balance as they have proven to be in the real world. In Section (IV.D) I will present an alternative way of dealing with the problem of how to determine what actually happens in models where there is a continuum of perfect foresight competitive equilibria.

In summary, the model I construct has a continuum of perfect foresight equilibria because the interest rate rule of the central bank is passive and because the treasury actively adjusts the tax rate to prevent the real value of debt from exploding.

[5]An example of a policy of this kind in the real world is the Omnibus Budget Reconciliation Act of 1993, which raised top US marginal tax rates and helped to stabilize the debt to GDP ratio. Before the passage of the Act, the top individual tax rate of 31% applied to all income over $51,900. The Act created a new bracket of 36% for income above $115,000, and 39.6% for income above $250,000.

E. Individual Choice Problems

A person of type i solves the problem,

PROBLEM 1:

$$v_i[a_i^s(\varepsilon)] = \max_{\{a_i^s(\varepsilon')\}_{\varepsilon'\in\Omega}} \left\{u_i[c_i^s(\varepsilon)] + \beta_i \pi \mathbb{E}\left[v_i(a_i^s(\varepsilon'))\right]\right\},$$

$$\pi\mathbb{E}\left[\tilde{Q}(\varepsilon,\varepsilon')a_i^s(\varepsilon')\right] + c_i^s(\varepsilon) \leq \mu_i[1-\mathcal{T}(\varepsilon)] + a_i^s(\varepsilon),$$

$$a_i^s(\varepsilon_s) = 0.$$

Here, the symbol v_i is the value function, $\mathcal{T}(\varepsilon) \equiv \tau + (1-\delta)B(\varepsilon)$ is a tax on the endowment and u_i, given by the expression,

$$u_i(x) = \begin{cases} \frac{x^{1-\rho_i}}{1-\rho_i} & \textit{if} \quad \rho_i > 0 \ , \ \rho_i \neq 1, \\ \log(x) & \textit{if} \quad \rho_i = 1, \end{cases}$$

is a Von-Neumann Morgenstern utility function.

The symbol $a_i^s(\varepsilon)$ is the value of financial assets, held as Arrow securities, by a person of type i born at date s at the beginning of period t. In addition to his financial assets, this person receives an after-tax income of $\mu_i[1-\mathcal{T}(\varepsilon)]$ and he may borrow against future after-tax income which has a net-present value of $\mu_i H(\varepsilon)$.

During the period, each person may allocate his wealth to consumption, $c_i^s(\varepsilon)$, or to the accumulation of a portfolio of Arrow securities, $a_i^s(\varepsilon')$, $\varepsilon' \in \Omega$, where security ε' costs $\pi Q(\varepsilon,\varepsilon')$.

I show in Appendix (A) that, when people solve this problem, their consumptions in consecutive states satisfy the Euler equation,

$$\tilde{Q}(\varepsilon,\varepsilon') = \beta_i \left(\frac{c_i^s(\varepsilon)}{c_i^s(\varepsilon')}\right)^{\rho_i}, \tag{2}$$

and optimal consumption is a linear function of wealth,

$$X_i(\varepsilon)c_i^s(\varepsilon) = a_i^s(\varepsilon) + \mu_i H(\varepsilon), \tag{3}$$

where, $H(\varepsilon)$ is the after-tax price of a tree, $a_i^s(\varepsilon)$ is financial wealth and $X_i(\varepsilon)$ is the inverse propensity to save. $X_i(\varepsilon)$ is defined by the recursion,

$$X_i(\varepsilon) = 1 + \pi\beta_i^{\frac{1}{\rho_i}}\mathbb{E}\left[\tilde{Q}(\varepsilon,\varepsilon')^{\frac{\rho_i-1}{\rho_i}}X_i(\varepsilon')\right]. \tag{4}$$

The absence of arbitrage opportunities implies the after-tax price of a tree is related to the pricing kernel and the future price of a tree by the valuation equation,

$$H(\varepsilon) = 1 - [\tau + (1-\delta)B(\varepsilon)] + \pi\mathbb{E}\left[\tilde{Q}(\varepsilon,\varepsilon')H(\varepsilon')\right],$$

where $\mathcal{T}(\varepsilon) \equiv [\tau + (1-\delta)B(\varepsilon)]$, the real value of taxes, is found by dividing Equation (1) by $p(\varepsilon)$.

IV. Rational Expectations Equilibrium

In this section I define a rational expectations equilibrium and I show that it can be characterized by seven stochastic difference equations in seven aggregate variables.

A. Definition of Equilibrium

A government policy is a triple $\{Q^N, \tau, \delta\}$. A *rational expectations equilibrium* is a government policy, a stochastic process for $\{p, Q\}$, and a consumption allocation for each person at each date such that; 1) people of all generations and at all dates choose consumption and assets to solve Problem (1): 2) The sum of financial assets over all people alive at every date t equals government debt: 3) The consumption of all people alive at every date t is equal to the social endowment: 4) The real value of government debt follows a stationary stochastic process: and 5) The value of debt is less than the value of after-tax human wealth at every date.

An equilibrium is a complicated object that involves a wealth distribution across types and cohorts that evolves through time. Because preferences are homogeneous and because life-expectancy is independent of age, the evolution of aggregate variables is independent of this distribution. In the remainder of the paper, I exploit this independence to describe the properties of aggregate variables in equilibrium. I generate artificial data, driven by belief shocks, and I show these data mimic the properties of the US data from 1948 through 2008 that I displayed in Figure 1.

B. Definition of Aggregate Variables

I will characterize the behavior of the variables $C_i(\varepsilon)$, $A_i(\varepsilon)$, $X_i(\varepsilon)$, $B(\varepsilon)$, $H(\varepsilon)$ and $Q(\varepsilon,\varepsilon')$. $C_i(\varepsilon)$ is the consumption of all type i people at date t in state ε. $A_i(\varepsilon)$ is the sum of the value of the ε Arrow securities owned by all type i people at the beginning of period t. $X_i(\varepsilon)$ is the inverse propensity to save out of wealth of type i people. $B(\varepsilon)$ is the real value of government debt. $H(\varepsilon)$ is the after-tax price of a tree and $Q(\varepsilon,\varepsilon')$ is the price of an Arrow security at date t in state ε for delivery of an apple at date $t+1$ in state ε'.

Consumptions of types 1 and 2 are linked by the goods market clearing equation

$$C_1(\varepsilon) + C_2(\varepsilon) = 1,$$

and the Arrow security holdings of types 1 and 2 are related to the outstanding value of government debt by the asset market clearing equation,

$$A_1(\varepsilon) + A_2(\varepsilon) = B(\varepsilon).$$

In my description of equilibrium I define

$$C(\varepsilon) \equiv C_1(\varepsilon), \quad \text{and} \quad A(\varepsilon) \equiv A_1(\varepsilon),$$

and I do not explicitly model $C_2(\varepsilon)$ or $A_2(\varepsilon)$. These variables are implicitly defined by goods and asset market clearing.

C. *Seven Equations that Characterize Equilibrium*

Using the facts that consumption is linear in wealth and that the newborn cohort does not own financial assets, one may derive the following expression which connects the aggregate consumptions and financial assets of each type in consecutive states,[6]

$$\tilde{Q}(\varepsilon, \varepsilon') = \left(\frac{C_i(\varepsilon) \pi \beta_i^{\frac{1}{\rho}i}}{\pi C_i(\varepsilon') + (1-\pi) X_i(\varepsilon)^{-1} A_i(\varepsilon')} \right)^{\rho_i}.$$

I call this the *modified Euler equation.*

Using the goods and asset market clearing equations, and the definitions, of $C(\varepsilon)$ and $A(\varepsilon)$, the modified Euler equations for each type are given by equations

$$\tilde{Q}(\varepsilon, \varepsilon') = \left(\frac{C(\varepsilon) \pi \beta_1^{\frac{1}{\rho}1}}{\pi C(\varepsilon') + (1-\pi) X_1(\varepsilon)^{-1} A(\varepsilon')} \right)^{\rho_1}, \tag{5}$$

and

$$\tilde{Q}(\varepsilon, \varepsilon') = \left(\frac{[1 - C(\varepsilon)] \pi \beta_2^{\frac{1}{\rho}2}}{\pi [1 - C(\varepsilon')] + (1-\pi) X_2(\varepsilon)^{-1} [B(\varepsilon') - A(\varepsilon')]} \right)^{\rho_2}. \tag{6}$$

In addition to equations (5) and (6), we may add up the policy function, Equation (3), over all type 1 people to give,

$$X_1(\varepsilon) C(\varepsilon) = A(\varepsilon) + \mu_1 H(\varepsilon). \tag{7}$$

We also know, from the no-arbitrage pricing of assets, that the price of a tree

[6]For the derivation of this result, see Appendix (B), which draws on the results of Farmer et al. (2011).

is related to its own future price, and to the stochastic discount factor, $\tilde{Q}(\varepsilon, \varepsilon')$, by the equation,

$$H(\varepsilon) = 1 - [\tau + (1-\delta)B(\varepsilon)] + \pi\mathbb{E}\left[\tilde{Q}(\varepsilon, \varepsilon')H(\varepsilon')\right], \tag{8}$$

and government debt is related to its own future values by the budget equation,

$$B(\varepsilon) = [\tau + (1-\delta)B(\varepsilon)] + \mathbb{E}\left[\tilde{Q}(\varepsilon, \varepsilon')B(\varepsilon')\right]. \tag{9}$$

To complete the dynamic equations of the model, the inverse savings propensities are described by the equations,

$$X_1(\varepsilon) = 1 + \pi\beta_1^{\frac{1}{\rho_1}}\mathbb{E}\left[\tilde{Q}(\varepsilon, \varepsilon')^{\frac{\rho_1-1}{\rho_1}}X_1(\varepsilon')\right], \tag{10}$$

and

$$X_2(\varepsilon) = 1 + \pi\beta_2^{\frac{1}{\rho_2}}\mathbb{E}\left[\tilde{Q}(\varepsilon, \varepsilon')^{\frac{\rho_2-1}{\rho_2}}X_2(\varepsilon')\right]. \tag{11}$$

Equations (5), (6), (7), (8), (9), (10) and (11) constitute a system of seven equations in the seven unknown variables C, A, B, H, X_1, X_2 and Q'. These equations are also associated with three boundary conditions that I turn to next.

D. Boundary Conditions, Indeterminacy and Belief Shocks as Fundamentals

The model has three boundary conditions. One of these is a terminal condition; the other two are initial conditions.

The terminal condition asserts that the variables H and B must remain bounded. This rules out paths that remain consistent with equations (5) – (11) for a finite number of periods but that eventually become unbounded. An unbounded path for H is inconsistent with market clearing because it leads to an infinite demand for commodities and an unbounded path for B would mean that either the government or the private sector has become insolvent. Neither variable can become unbounded in equilibrium.

The first of the two initial conditions is given by the expression,

$$A_1 = \bar{a}, \tag{12}$$

where $\bar{a}$ is the net financial assets owned by type 1 people in the first period of the model. This initial value may be positive or negative.

The second initial condition is given by the equation,

$$B_1 = \frac{B_1^N}{p_1}. \tag{13}$$

This equation asserts that the initial value of government debt, measured in units of apples, depends on the period 1 price level. In models where the central bank sets an interest rate peg, there is an open set of possible values for p_1, all of which are consistent with the existence of a non-stationary perfect foresight equilibrium and all of which converge to the same steady state (Sargent and Wallace, 1975).

How should we deal with the existence of multiple equilibria? I believe it is a mistake to think of the price level as an initial condition. A better way to think of price level determination is that the price in period $t+1$, call this p', is different from the date t belief of what p' will be; call this $p^{E'}$. The date t price, p, is connected to the belief about the date $t+1$ price, $p^{E'}$, by the Fisher equation,

$$p = \frac{p^{E'} \bar{Q}^N}{Q'}.$$

This way of formulating the 'problem' of multiple equilibria identifies clearly why the economist is unable to make a clear prediction: He has failed to explain how beliefs are determined. In a model with multiple equilibria, it is not enough to assert that people have perfect foresight. The economist must write down an equation to determine beliefs. The perfect foresight assumption is a consistency requirement which asserts that beliefs should be correct in the steady state of a perfect foresight model. It is not a substitute for a formal description of how beliefs are formed.

This argument can be extended to a world where there are shocks, either to preferences and endowments, or simply to beliefs. In a model where there are multiple perfect foresight equilibria, there is no reason that people should hold point expectations. In this paper I assume instead that $p^{E'}$ is drawn from a known probability distribution, χ. When people believe that the period $t+1$ price is a random variable, the period t price is determined by the equation,

$$p(\varepsilon) = \frac{\bar{Q}^N}{\mathbb{E}\left[\frac{\tilde{Q}(\varepsilon, \varepsilon')}{p^E(\varepsilon')}\right]}.$$

I propose to resolve the indeterminacy of equilibrium by introducing a new random variable, $p^E(\varepsilon')$, which has a probability distribution χ and a support, Ω. $p^E(\varepsilon')$ is the price that people think will occur in period $t+1$ and state ε'. The variable $p^E(\varepsilon')$ is a new fundamental that has the same methodological status as preferences or endowments.

I have not yet said anything about whether the belief that $p^E(\varepsilon')$ will be drawn from χ will be validated in equilibrium. That requires an additional assumption and I propose to assume that the belief about the distribution of future prices has the same distribution as the realized distribution of future prices. The people in my model have rational expectations.

Robert Lucas (1972) claimed that we do not need to ask *how* people come to have rational expectations. We simply need to assume that they are clever enough not to be consistently fooled. In my view, in a stationary environment, if the steady state equilibrium is locally determinate, this argument makes a lot of sense and I will make that assumption here. By making beliefs fundamental, I have turned a model with a continuum of indeterminate equilibria, into a model with a unique determinate equilibrium. In my model, beliefs are both fundamental and rational at the same time.[7]

V. Simplifying the Model

In this section I simplify the model by finding two variables that summarize the behavior of the other five. I call the variables that summarize the dynamics, the *state* variables and I call the remaining variables, the *auxiliary* variables. Because of the functional dependencies among the seven variables, there is no unique way of choosing state variables.

The behavior of people at date t in state ε depends on two factors. The first, is their belief about future prices. This is coded into their willingness to pay more or less for a claim to the future endowment and is summarized by the value of the variable H. The second is the value of the Arrow securities held by type 1 people which depends on the consumption of type 1 and type 2 people in period $t-1$. The easiest way to code that behavior into the model is to introduce the period $t-1$ value of consumption, I call this C_L, as an additional state variable. It is defined by the equation,

$$C(\varepsilon) = C_L'(\varepsilon).$$

A. *Three Auxiliary Functions*

To characterize equilibria I will reduce the dimension of the problem still further by finding three functions, f_A, f_Q and f_B, that explain A, Q, and B as functions of C_L, H, X_1 and X_2. This reduces the model to system of equations with a 2×1 vector of state variables, x, where $x \equiv \{H, C_L\}$ and a 3×1 vector of auxiliary variables, y, where $y \equiv \{X_1, X_2, C\}$.

Using Equation (7), the function f_A is defined as

$$A = f_A(C, H, X_1) \equiv X_1 C - \mu_1 H. \tag{14}$$

Replacing A by $f_A(C, H, X_1)$ in Equation (5), lagged one period, gives the func-

[7]This argument says nothing about how a particular equilibrium is enforced. In other work (Farmer, 2002a) I have shown how to implement a given rational expectations equilibrium with a *belief function*.

tion, $f_Q:\ [0,1]^2 \times \mathbb{R}_+^2 \to \mathbb{R}_+$,

$$\tilde{Q} = f_Q(C_L, C, H, X_1) \equiv \left(\frac{C_L \pi \beta_1^{\frac{1}{\rho_1}}}{\pi C + (1-\pi) {X_1}^{-1} f_A(C,H)} \right)^{\rho_1}. \tag{15}$$

Finally, the function $f_B : [0,1] \times [0,1] \times \mathbb{R}_+^3 \to \mathbb{R}$, is implicitly defined by the expression,

$$\left(\frac{\pi \beta_1^{\frac{1}{\rho_1}} C_L}{\pi C + (1-\pi) {X_1}^{-1} f_A(C,H,X_1)} \right)^{\rho_1} = \left(\frac{\pi \beta_2^{\frac{1}{\rho_2}} (1 - C_L)}{\pi(1-C) + (1-\pi) {X_2}^{-1} [f_B(C_L, C, H, X_1, X_2) - f_A(C, H, X_1)]} \right)^{\rho_2}. \tag{16}$$

For given values of C_L, C, H, X_1 and X_2, $B \equiv f_B(C_L, C, H, X_1, X_2)$, is the real value of government debt that equates the marginal rates of substitution of types 1 and 2.

It is important to note that Equation (16) holds for all pairs of states $\{\varepsilon, \varepsilon'\}$. The fact that B' is a function of ε' is recognized in advance by type 1 and type 2 people in period t who equate their marginal rates of substitution by trading a complete set of Arrow securities.

B. *Solving the Model*

Let $\mathcal{S} \equiv \mathbb{R}_+ \times [0,1]$ be the *state space* and let $\mathcal{Y} \equiv \mathbb{R}_+^2 \times [0,1]$ be the *control space*. Using the functions f_A, f_Q, and f_B, together with equations (8) – (11), I define a function $\mathcal{S} \times \mathcal{Y} \times \mathcal{S} \times \mathcal{Y} \to \mathbb{R}^5$,

$$\tilde{F}(C_L, H, C, X_1, X_2, C_L', H', C', X_1', X_2') \equiv \begin{bmatrix} H - 1 + \tau + (1-\delta) f_B(C_L, C, H, X_1, X_2) - \pi f_Q(C, C', H', X_1') H' \\ f_Q(C, C', H', X_1') f_B(C, C', H', X_1', X_2') - f_B(C_L, C, H, X_1, X_2) \\ \qquad + \tau + (1-\delta) f_B(C_L, C, H, X_1, X_2) \\ X_1 - 1 - \pi \beta_1^{\frac{1}{\rho_1}} f_Q(C, C', H', X_1')^{\frac{\rho_1 - 1}{\rho_1}} X_1' \\ X_2 - 1 - \pi \beta_2^{\frac{1}{\rho_2}} f_Q(C, C', H', X_1')^{\frac{\rho_2 - 1}{\rho_2}} X_2' \\ C - C_L' \end{bmatrix}$$

More compactly,

$$F(x, y, x', y') \equiv \tilde{F}(C_L, H, C, X_1, X_2, C'_L, H', C', X'_1, X'_2).$$

A rational expectations equilibrium is characterized by functions $f : \mathcal{S} \times \Omega \to \mathcal{S}$ and $g : \mathcal{S} \to \mathcal{Y}$ that satisfy the functional equation,

$$\mathbb{E}\{F(x, g(x), f(x, \varepsilon'), g[f(x, \varepsilon')])\} = 0. \tag{17}$$

I now turn to the problem of solving this functional equation numerically using a local approximation.

C. *Approximating the Rational Expectations Equilibrium with Perturbation Methods*

Define a *steady state* to be a state vector $\bar{x}$ and a control vector $\bar{y}$ that satisfies the equation

$$F(\bar{x}, \bar{y}, \bar{x}, \bar{y}),$$

and let $\bar{J}_1$ and $\bar{J}_2$ be the Jacobian matrices of F with respect to $[x, y]$ and $[x', y']$ evaluated at $\{\bar{x}, \bar{y}\}$.

Local uniqueness of a solution to Equation (17) requires that two of the generalized eigenvalues of the matrix pencil $\{\bar{J}_1, \bar{J}_2\}$ are inside the unit circle.[8] That condition guarantees that there is a two-dimensional *stable manifold* $g(x)$ with the property that for any $x_1 \in \mathcal{N}(\bar{x})$ all initial conditions $y_1 = g(x_1)$ that begin on the stable manifold, converge asymptotically to the steady state $\{\bar{x}, \bar{y}\}$.

To a first approximation, g coincides with the linear subspace spanned by the two eigenvectors associated with the two stable eigenvalues and f coincides with the linearized dynamics of x on the stable manifold. When F is differentiable, higher order approximations may be obtained by extending the solution along the stable manifold using second, third and higher order derivatives.

To study the properties of rational expectations equilibria, in the next section I provide numerical values for the model parameters and approximate the solution using perturbation methods around a steady-state equilibrium.

VI. The Properties of a Calibrated Model

A. *Two Different Calibrations*

To study the properties of the model, I calibrated it in two different ways. In both calibrations I chose common values for π, β_1, β_2, μ_1, τ and δ. These values

[8] The matrix pencil of the matrices $\{\bar{J}_1, \bar{J}_2\}$ is the matrix valued function $\mathbb{L}(\lambda) = \bar{J}_1 - \lambda \bar{J}_2$, for $\lambda \in \mathbb{C}$. The generalized eigenvalues of $\{\bar{J}_1, \bar{J}_2\}$ are solutions to the polynomial equation $|\bar{J}_1 - \lambda \bar{J}_2| = 0$, and the generalized eigenvectors, v, are solutions to the equation $\bar{J}_1 v = \lambda \bar{J}_2 v$, where v is a vector and λ is a scalar (Gantmacher, 2000, Chapter XII). Both v and λ may take complex values. The generalized eigenvalue problem is used to solve linearized systems where $\bar{J}_1$ and or $\bar{J}_2$, or both, are singular. It is widely applied to the solution of linear rational expectations models. See Sims (2001) for a description of the method.

are reported in Table 2. The values are not chosen systematically to match first and second moments of US data. They are designed to show that a model, with the features I have built into it, has promise as a theoretical vehicle to replicate data if one were to conduct a more serious econometric exercise.

In the first calibration I set the risk aversion parameter of both types to 1. This corresponds to logarithmic preferences. In the second calibration I set the risk aversion parameter of both types to 8. I experimented by making the risk aversion of type 1 people different from the risk aversion of type 2 people but I found that differences in risk aversion across types made little qualitative difference to the results.

Concept	Symbol	Calibrated Values
Life expectancy	$\frac{1}{1-\pi}$	50 years
Survival probability	π	0.98
Discount factor of type 1 people	β_1	0.98
Discount factor of type 2 people	β_2	0.94
Fraction of type 1 people	μ_1	0.5
Tax ($-$ve value denotes Transfer)	τ	-0.01
Debt Repayment Parameter	δ	0.94
Nominal Discount Factor	$\bar{Q}^N$	0.93

Table 2—Common Parameter Values for Two Calibrations of the Model

I chose π to be 0.98. That value implies that an average person has an age-independent life expectancy of 50 years.

I chose the discount factor of type 1 people to equal 0.98 and the discount factor of type 2 people to be 0.94. I chose those numbers to bracket a steady state riskless discount factor of 0.97 which is consistent with a mean real rate of 3%. I found, in my simulations, that if the difference between β_1 and β_2 is too large, the steady state may not exist. I chose a relatively large range that still ensures the existence of a steady state.

I experimented with different values for μ_1 and found that the qualitative features of the model are similar for calibrations in the range 0.25 to 0.75. The results I report are for $\mu_1 = 0.5$.

I chose $Q^N = 0.93$ for the monetary policy parameter which implies the central bank targets a nominal interest rate of 7%. The only steady-state variable influenced by Q^N, is the level of the inflation factor which is scaled up or down by changes in Q^N.

For fiscal policy, I chose a value of $\tau = -0.01$, which means that the treasury sets a state-independent real transfer of 1% of GDP and I set $\delta = 0.94$, which implies that the treasury adjusts the tax rate each period to repay 6% of outstanding debt.[9]

B. The Steady States of the Two Calibrations

In Table 3 I report some statistics associated with my two calibrations. In the first row I report the stable generalized eigenvalues of $\{\bar{J}_1, \bar{J}_2\}$. For both calibrations there are two eigenvalues inside the unit circle and in each calibration, both roots are close to 1. This fact accounts for the ability of the model to explain persistent movements in PE ratios. The table shows that an increase in the risk aversion parameter, from 1 to 8, increases the persistence of the state variables. The largest root increases from 0.97, for log preferences, to 0.99, for CRRA preferences with $\rho = 8$.

Variable	Symbol	$\rho = 1$	$\rho = 8$
Stable Generalized Eigenvalues	λ_1, λ_2	0.96,0.97	0.98,0.99
Real Interest Rate	$100(\frac{1}{Q} - 1)$	2.9%	4.5%
Inflation Rate	$100(\Pi - 1)$	4.0%	2.4%

Table 3—Steady Values of the Interest Rate for Two Different Calibrations of Risk Aversion

In rows 2 and 3 of the table I report steady state values of the real interest rate $100(\frac{1}{Q} - 1)$ and the inflation rate $100(\Pi - 1)$. For the logarithmic calibration, the steady state real interest rate is 2.9% and the steady state inflation rate is 4.0%. Raising risk aversion to 8 increases the real interest rate to 4.5% with an associated steady state inflation rate of 2.4%. In both cases, the inflation rate is found by subtracting the real rate from the targeted nominal interest rate which I set at 7%.

C. Two Stochastic Simulations Using the Same Draw for the Shocks

I simulated 90 years of data for the inflation rate, the stochastic discount factor, the real safe rate of return and the real risky rate of return and I discarded the first 30 observations to remove the influence of initial conditions. The results of this simulation are reported in Figure 2. On all four panels, the solid curves represent a simulation for logarithmic preferences and the dashed curves represent a simulation for CRRA preferences with $\rho = 8$. For both simulations I used the same sequence of stochastic shocks.

I simulated data using the function

$$x' = \hat{f}(x) + \eta\epsilon',$$

[9]For existence of equilibrium, it is not necessary that the tax rate be adjusted every period. It simply needs to be adjusted occasionally to prevent debt from exploding.

where $\hat{f}$ is a second order approximation to the policy function computed using the code from Levintal (2016) and η is a 2×1 vector that allocates shocks to state variables.[10] To ensure that H does not become negative, I defined the state variable to be $\log(H)$ rather than H.

I chose the vector $\eta = [\sigma, 0]^T$ and I set $\sigma = 0.075$. A zero appears in the second element of η to reflect the fact that C_L is predetermined. The parameter σ is the standard deviation of a belief shock that causes people to revise their view of the current price of a tree. The implied belief about the price shock, $p(\varepsilon)$, can be recovered from the definition

$$p(\varepsilon) = \frac{B^N}{B}.$$

The top left panel of Figure 2 reports the inflation rate, measured in percent per year, and the top right panel is the stochastic discount factor which is a pure number. These series are identical up to a scale normalization, because, when the central bank pegs the nominal interest rate, fluctuations in the realized stochastic discount factor are caused by fluctuations in the realized inflation rate. The stochastic discount factor has a lower mean and a higher standard deviation under CRRA preferences than under logarithmic preferences. In the simulation reported in this figure, the realized stochastic discount factor fluctuated between 0.96 and 0.98 when preferences are logarithmic and between 0.935 and 0.998 for the CRRA calibration.

The bottom left panel of Figure 2 is the safe interest rate, defined by the expression

$$r^s = 100 \left(\frac{1}{\mathbb{E}\left[\tilde{Q}(\varepsilon, \varepsilon')\right]} - 1 \right).$$

I computed this statistic by adding an addition auxiliary variable, $R^s \equiv 1 + r^s$, and an additional auxiliary equation

$$\frac{1}{R^s} - \mathbb{E}\left[f_Q(C, C', H', X_1')\right] = 0,$$

to the model.

The bottom right panel is the realized after-tax one-period holding return for buying a measure 1 of trees in period t and selling it in period $t+1$. It is defined by the expression

$$r^r = 100 \left(\frac{\pi H(\varepsilon')}{H(\varepsilon) - [1 - \mathcal{T}(\varepsilon)]} - 1 \right).$$

The safe return, r^s, and the risky return, r^r, are identical in a non-stochastic steady state. They are very different in the stochastic steady state.

[10] Levintal's code is based on Schmitt-Grohé and Uribe (2004). It allows for approximations up to fifth order and it is faster and more efficient than the Schmitt-Grohé-Uribe code. I found no noticeable difference in simulations using second and higher order approximations and, because the second-order approximation is faster, I used it in my simulations.

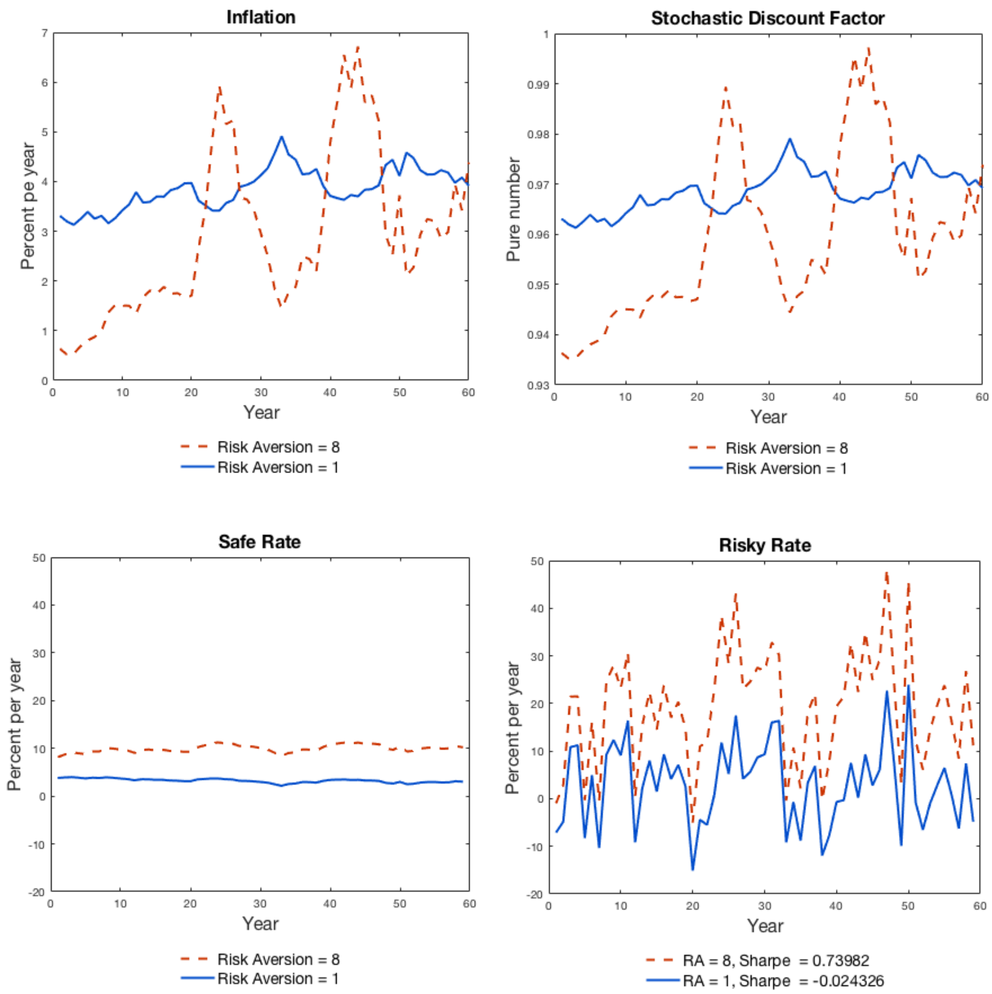

FIGURE 2. INFLATION AND INTEREST RATES FOR TWO DIFFERENT CALIBRATIONS

In US data reported in Table 1, the average yield from holding the stock market is 4.6% higher than the average return from holding government bonds. Because it is possible to increase the risky return by assuming more risk, it is common to report the Sharpe ratio which is a measure of excess return normalized by risk. The Sharpe ratio, S, is defined as,

$$S \equiv \frac{\mathrm{mean}(r^r - r^s)}{\mathrm{std}(r^r)}.$$

In data reported in Table1, the Sharpe ratio is 0.29. This value is sensitive to the definition of the risky asset and the time period over which it is computed and alternative definitions and periods give numbers between 0.25 and 0.5 (Cochrane, 2001). In Table 4 I report the Sharpe ratio and the means of the safe and risky returns for the simulations from Figure 2.

Variable	Symbol	$\rho = 1$	$\rho = 8$
Mean of Safe Rate in One Simulation	$\bar{r}^s$	3.24%	9.92%
Mean of Risky Rate in One Simulation	$\bar{r}^r$	3.03%	18.92%
Sharpe Ratio in One Simulation	S	-0.024	0.74

Table 4—Mean Safe and Risky Rates for a Single Simulation

The numbers reported in this table refer to a single draw of shocks. When people have logarithmic preferences, this draw led to an average safe interest rate in 60 years of data of 3.24% and an average risky rate of 3.03%. The Sharpe ratio for this draw, for the case of logarithmic preferences, is -0.024.

When people have CRRA preferences, with risk aversion parameter $\rho = 8$, this same sequence of stochastic shocks led to a mean safe return of 9.92% and a mean risky return of 18.92%. The equity premium for this calibration is 9% and the Sharpe ratio is 0.74. The mean returns are different from the steady-state value of 4.5%, reported in Table 2, because the model has an eigenvalue close to 1 and mean reversion is extremely slow. The average risky return in 60 years of data is a random variable with a high variance that becomes very large as the largest root of the system approaches 1.

To check whether a high Sharpe ratio was a freak draw from a particular realization of the shocks, I simulated 100,000 time series, of length 60 years, and for each series I computed the Sharpe ratio and the means of the safe and risky interest rates. The results are reported in Figure 3 for two different calibrations of the model. For both calibrations, I used the same sequence of random shocks for every simulation.

The top panel of Figure 3 shows that, when $\rho = 1$, the distribution of risky rates has a higher variance than the distribution of safe rates, but both are centered around 3% and both distributions are approximately symmetric. The bottom panel shows that, for the logarithmic calibration, the empirical distribution of Sharpe ratios is centered, to a first approximation, on 0. There is, nevertheless, a fair amount of probability mass in the interval $[0.2, 0.3]$. Even if preferences are logarithmic, we still might have observed a Share ratio of 0.25 in 60 years of data, purely by chance. The figure also illustrates that, if preferences are logarithmic, the probability of observing a Sharpe ratio of 0.5 is close to zero.

Contrast that with the case of $\rho = 8$. The top panel of Figure 3 shows that drawing a mean safe rate of 9% and a mean risky rate of 18% is indeed, an outlier. Most of the probability mass for both safe and risky rates lies between 5% and

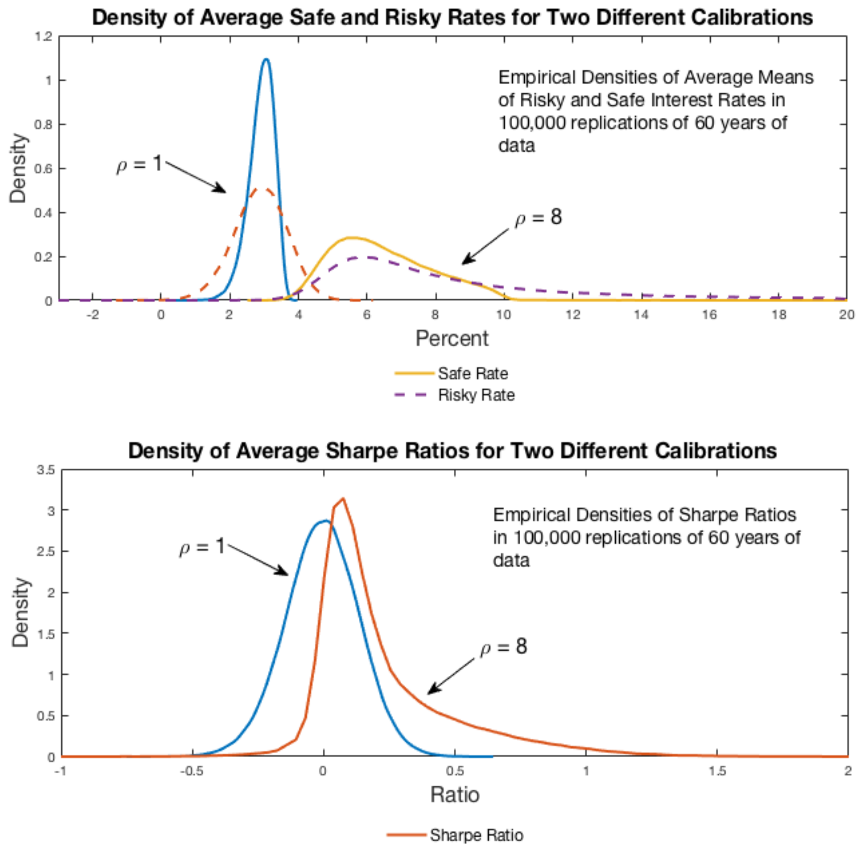

FIGURE 3. EMPIRICAL FREQUENCY OF INTEREST RATES AND SHARPE RATIOS IN 100,000 SIMULATIONS

8%. The lower panel shows that, nevertheless, a Sharpe ratio between 0.25 and 3 is not unusual and, when $\rho = 8$, there is fair amount of probability mass between 0.5 and 1. For this calibration, the probability of drawing a Sharpe ratio of 0.5 is high.

I infer, from these simulations, that this model is a good candidate for helping understand the equity premium puzzle (Mehra and Prescott, 1985).

D. Consumption, Asset Holdings and Government Debt

In Table 5 I report the type 1 and type 2 savings propensities, $\bar{X}_1^{-1}$ and $\bar{X}_2^{-1}$, the steady-state values of type 1 consumption, $\bar{C}$, type 1 asset holdings, $\bar{A}$, and the steady-state PE ratio, $\bar{H}$. I also report the steady-state value of government debt and two different concepts of the steady-state government budget deficit.

Consider first, the savings propensities from rows 1 and 2 of Table 5. For

Variable	Symbol	$\rho = 1$	$\rho = 8$
Savings Propensity of Type 1 People	$\bar{X}_1^{-1}$	3.96%	6.0%
Savings Propensity of Type 2 People	$\bar{X}_2^{-1}$	7.88%	6.46%
Consumption of Type 1 People	$\bar{C}$	68%	55%
Financial Assets of Type 1 People	$\bar{A}$	6.8	1.36
Price-earnings Ratio	$\bar{H}$	20.98	15.57
Government Debt	$\bar{B}$	31%	60%
Primary Budget Surplus	$(1-\delta)\bar{B}+\tau$	0.9%	2.6%
Secondary Budget Surplus	$(Q^N-\delta)\bar{B}+\tau$	−1.2%	−1.3%

Table 5—Steady Values of Consumption and Assets Positions

the case of logarithmic preferences, type 1 people consume 3.96% of steady state wealth in every period while type 2 people consume 7.88%. In the CRRA case, the mean savings propensities are much closer together, 6.0% for type 1 people and 6.46% for type 2. The increased savings propensity for type 2 people is reflected in a difference in steady state consumption across the two calibrations.

Row 3 of Table 5 shows that, when both types have logarithmic preferences, type 1 people consume 68% of the social endowment in the steady state. When they have CRRA preferences, type 1 people consume 55% of the social endowment. The difference occurs because, when $\rho = 8$, both types are not only highly averse to fluctuations across states, they are also highly averse to fluctuations of consumption across time. That feature reduces the willingness of type 2 people to have a tilted consumption profile and it reduces the gains from trade that occur from having different discount rates.

The reduced gains from trade are reflected in A, the financial asset position of type 1 people. Row 4 of Table 5 shows that, for logarithmic preferences, type 1 people hold assets equal to 6.8 times GDP in the steady state. This falls to 1.36 when both types are averse to risk and to intertemporal substitution.

Row 5 shows the steady state after-tax price-earnings ratio. This is equal to 20.98 for logarithmic preferences and 15.57 in the CRRA case. This difference reflects the difference in steady state real interest rates reported in Table 3.

Rows 6, 7 and 8 report the steady state value of government debt and two different measures of the steady-state budget deficit. For the logarithmic calibration, the government holds debt equal to 31% of GDP in the steady state. For the CRRA calibration that figure increases to 60%. In both cases the government runs a small primary surplus. When interest payments on the steady-state debt are deducted from the primary surplus, government, in each calibration, runs a small secondary deficit.

The top left panel of Figure 4 reports type 1 consumption, measured as a fraction of GDP, and the top right panel is the PE ratio for the same sequence of random shocks used to construct Figure 2. Since the endowment is constant and

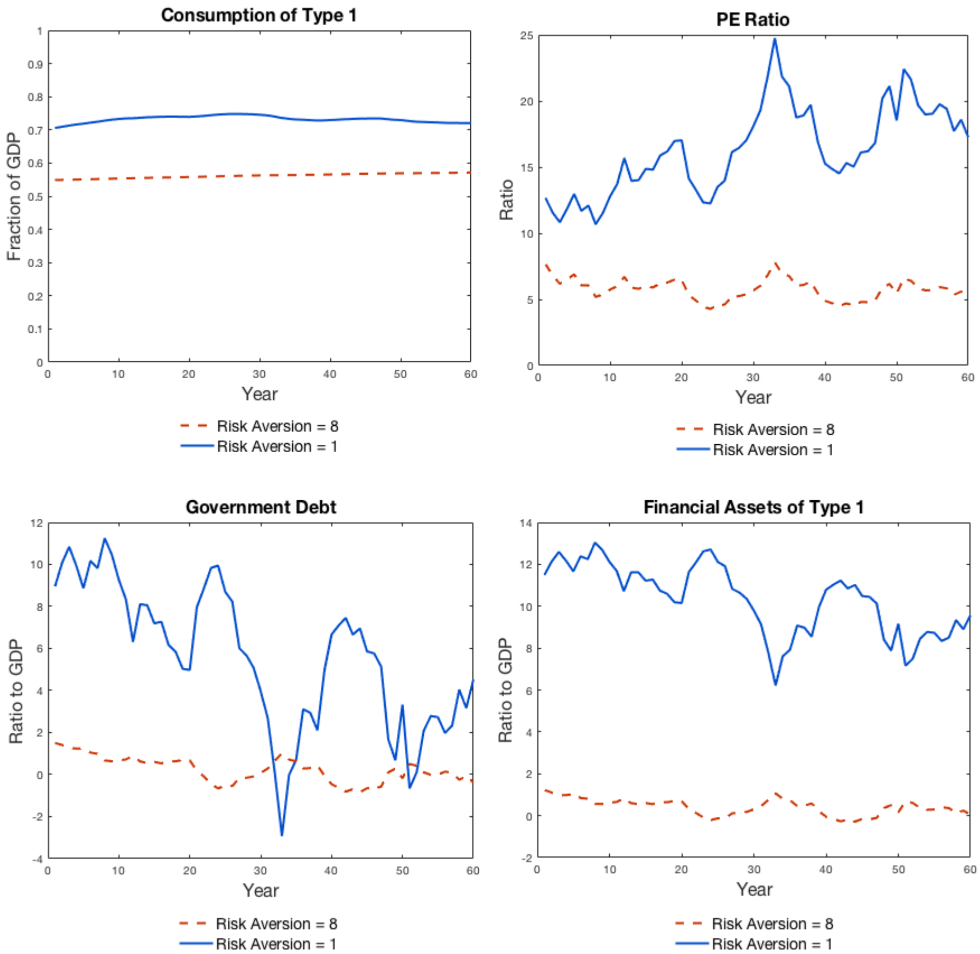

Figure 4. Consumption and Asset Demands for Two Different Calibrations

equal to 1, the PE ratio is equal to the price of a tree. The important takeaway from these two graphs is that the consumption of type 1 people is almost constant, in both calibrations, but there is substantial variation in asset values. The PE ratio varies between 10 and 25 when preferences are logarithmic and between 4 and 8 for the CRRA case. In the real world, the price earnings ratio displays considerable persistence and has been as low as 5, in 1920, and close to 50 in the 1990s.

The bottom two panels of Figure 4 show that differences in risk aversion also have substantial implications for both the level and volatility of the private and government sector asset positions. When people have logarithmic preferences, government debt fluctuates between negative two and plus 12 as a ratio to GDP. For CRRA preferences debt never exceeds 2 times GDP and most of the time is between 0 and 1. The private sector asset position of type 1 people is both higher and more volatile when both types are less risk averse.

E. A Discussion of the Results

In the introduction, I claimed that the paper relies on three features. First, there must be birth and death. Second, there must at least two types of people, and third, there must be nominally denominated government debt. Here, I discuss why each of these features is important and I speculate on the ability of a model in this class to provide a vehicle for more formal empirical work.

The assumption that there are two types of Blanchard-Yaari consumers is important to generate the mechanism that causes belief shocks to be persistent. In a purely real version of the model, there is a unique perfect-foresight equilibrium. That model has a single initial condition, represented by the debt of type 1 people to type 2 people in the first period. A natural assumption to make is that that first period private-sector debt is equal to zero.

How do people behave in this purely real world? The initial type 1 people choose to lend part of their endowment to type 2 people. The initial interest rate is different from its steady state value because the initial age distribution of types is different from the steady state distribution. A population heavily weighted to young people is associated with a different interest rate from a population with a steady state distribution of cohorts because as people age, type 2 people acquire debts and type 1 people acquire assets. Asymptotically, the interest rate converges to a unique steady state.

Adding money to the model leads to steady state indeterminacy. The real value of government debt must equal the discounted net present value of tax receipts. For any initial price level in a certain set, there is a initial value of debt that is consistent with an equilibrium. Government debt is net wealth to the current generation because they must be persuaded to hold it at equilibrium prices. But although current generations own the government debt, part of it is repaid by future generations. Different initial price levels are associated with different allocations of the debt burden among generations.

In a rational expectations equilibrium, the one-period-ahead price level is a random variable. Type 1 and type 2 people alive at date t recognize that the future price is random and they insure each other against price level fluctuations. A person of either type who is born into a low price state is worse off for her entire life than a person born into a high price state. She is forever saddled with a high tax obligation to existing generations. Once born, she is insured against all future price level fluctuations. But she is not insured against the state of the world she is born into.

Price level shocks generate random transfers from the newborn people to or from existing generations. A type 1 person who receives a wealth transfer from the newly born, will choose to lend part of it to a type 2 person. The price shock, in a monetary model, acts like a shift in the initial condition of the purely real model and it sets off a persistent mean reverting movement in the discount factor.

I have shown, in this section, that this mechanism can explain many features of real world data. The fit is not perfect. For example, the logarithmic calibration misses features of the return data but displays large swings in PE ratios. The CRRA model, with $\rho = 8$, captures rate-of-return discrepancies at the cost of shifting and dampening fluctuations in PE ratios. But although the model is off in some dimensions, I did not try to fit the data in a systematic way and a more formal attempt at estimation would correct some, if not all, of the places where the calibrated model is inaccurate.[11]

I conclude that a model in which asset price fluctuations are caused by belief shocks has the potential to explain, quantitatively, many features of real world data.

VII. Conclusion

In this paper, I have presented a theory that explains asset pricing data in a new way. In contrast to much of the existing literature in both macroeconomics and finance, my work is based on the idea that most asset price fluctuations are caused by non-fundamental shocks to beliefs. My model produces data that display volatile asset prices and a sizable risk premium. If one accepts the argument that a simpler explanation is a better one, the fact that I am able to reproduce these empirical facts in a model with CRRA preferences and no fundamental shocks suggests that the model is on the right track.

My model is rich in its implications. It provides a simple theory of the pricing kernel that can be used to price other assets. The model is open to more rigorous econometric testing and its parameters can be estimated, rather than calibrated, using non-linear methods. It provides a theory of the term structure of interest rates that can be tested against observed bond yields and by adding a richer theory, in which output fluctuates as a consequence of labor supply or because of movements in the unemployment rate, the theory can be estimated using data from both bond prices and equity markets. I view all of these extensions as grist for the mill of future research. Conducting these extensions is important because my model is not just a positive theory of asset prices; it is ripe with normative implications.

In my baseline calibration, I chose parameters to match key features of the data and I generated simulated data series that closely mimic observed interest rates and asset prices in the real world. In these simulations, asset price fluctua-

[11]It would also be possible to solve a version of the model that separates risk-aversion from intertemporal elasticity of substitution by moving to Epstein-Zin preferences (Epstein and Zin, 1989) and that is likely to improve the fit. I have chosen not to take that step here because there are already a lot of new features in the model. It is, however, on my research agenda for a future empirical paper.

tions cause Pareto inefficient re-allocations of wealth between current and future generations and these re-allocations lead to substantial fluctuations in welfare. If my model is correct, and these fluctuations are the main reason why asset prices move in the real world, stabilizing asset prices through monetary and fiscal interventions will be unambiguously welfare improving.

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Appendix A

In an optimum, the budget constraint must hold with equality,

$$c_i^s(\varepsilon) = \mu_i[1 - \mathcal{T}(\varepsilon)] + a_i^s(\varepsilon) - \pi\mathbb{E}\left[\tilde{Q}(\varepsilon,\varepsilon')a_i^s(\varepsilon')\right]. \tag{A1}$$

From the envelope condition we have that

$$\frac{\partial v_i}{\partial a_i^s(\varepsilon)} = \frac{\partial u_i}{\partial c_i^s(\varepsilon)}\frac{\partial c_i^s(\varepsilon)}{\partial a_i^s(\varepsilon)} \equiv \frac{\partial u_i}{\partial c_i^s(\varepsilon)}, \tag{A2}$$

where I have used the fact that, from (A1), $c_i^s(\varepsilon)$ is a linear function of $a_i^s(\varepsilon)$.

From the Euler equation, using the definition of u_i^s,

$$\frac{\partial u_i}{\partial c_i^s(\varepsilon)}\frac{\partial c_i^s(\varepsilon)}{\partial a_i^s(\varepsilon')} = \pi\beta_i\chi(\varepsilon)\frac{\partial v_i}{\partial a_i^s(\varepsilon')}. \tag{A3}$$

Using (A1) and (A2) in (A3) and the functional form for utility,

$$\pi\tilde{Q}(\varepsilon,\varepsilon')c_i^s(\varepsilon)^{-\rho_i} = \pi\beta_i c_i^s(\varepsilon')^{-\rho_i}. \tag{A4}$$

Rearranging this equation gives

$$\tilde{Q}(\varepsilon,\varepsilon') = \beta_i\left(\frac{c_i^s(\varepsilon)}{c_i^s(\varepsilon')}\right)^{\rho_i}, \tag{A5}$$

which is Equation 2 from the solution to Problem 1.

Now conjecture that the policy function takes the form,

$$X_i(\varepsilon)c_i^s(\varepsilon) = \mu_i H(\varepsilon) + a_i^s(\varepsilon) \tag{A6}$$

where

$$X_i(\varepsilon) = 1 + \pi\beta_i^{\frac{1}{\rho_i}}\mathbb{E}\left[\tilde{Q}(\varepsilon,\varepsilon')^{\frac{\rho_i-1}{\rho_i}}X_i(\varepsilon')\right]. \tag{A7}$$

and

$$H(\varepsilon) = 1 - [\tau + (1-\delta)B(\varepsilon)] + \pi\mathbb{E}\left[\tilde{Q}(\varepsilon,\varepsilon')H(\varepsilon')\right], \tag{A8}$$

From Equation (A6),

$$X_i(\varepsilon')c_i^s(\varepsilon') = \mu_i H(\varepsilon') + a_i^s(\varepsilon'). \tag{A9}$$

Multiplying Equation (A9) by $\pi\tilde{Q}(\varepsilon,\varepsilon')$, taking expectations and using equations

(A1) and (A8),

$$\pi \mathbb{E}\left[\tilde{Q}(\varepsilon,\varepsilon')X_i(\varepsilon')c_i^s(\varepsilon')\right] = \pi \mathbb{E}\left[\tilde{Q}(\varepsilon,\varepsilon')\mu_i H(\varepsilon')\right] + \pi \mathbb{E}\left[\tilde{Q}(\varepsilon,\varepsilon')a_i^s(\varepsilon')\right]. \tag{A10}$$

Next, replace $c_i^s(\varepsilon')$ by $c_i^s(\varepsilon)\beta_i^{\frac{1}{\rho_i}}\tilde{Q}(\varepsilon,\varepsilon')^{\frac{-1}{\rho_i}}$ from (A5) and use (A1) and (A8) to give

$$\begin{aligned} \pi c_i^s(\varepsilon)\mathbb{E}\left[\beta_i^{\frac{1}{\rho_i}}\tilde{Q}(\varepsilon,\varepsilon')^{\frac{\rho_i-1}{\rho_i}}X_i(\varepsilon')c_i^s(\varepsilon')\right] \\ = \pi \mathbb{E}\left[\tilde{Q}(\varepsilon,\varepsilon')\mu_i H(\varepsilon')\right] + \pi \mathbb{E}\left[\tilde{Q}(\varepsilon,\varepsilon')a_i^s(\varepsilon')\right] \\ = \mu_i H(\varepsilon) + a_i^s(\varepsilon) - c_i^s(\varepsilon) = [X_i(\varepsilon)-1]c_i^s(\varepsilon), \end{aligned} \tag{A11}$$

where the final equality uses the policy function, Equation (A5).

Cancelling $c_i^s(\varepsilon)$ from both sides of (A11) and rearranging terms gives

$$X_i(\varepsilon) = 1 + \pi \mathbb{E}\left[\beta_i^{\frac{1}{\rho_i}}\tilde{Q}(\varepsilon,\varepsilon')^{\frac{\rho_i-1}{\rho_i}}X_i(\varepsilon')c_i^s(\varepsilon')\right], \tag{A12}$$

which establishes that the conjectured policy function, (A5) solves Problem 1 when $X_i(\varepsilon)$ is given by Equation (A12).

Appendix B

Let $c_i^o(\varepsilon)$ be the consumption at date t in state ε of a representative type i person who was alive at date $t-1$ and let $c_i^y(\varepsilon)$ be the consumption of a representative newborn person of type i. Adding up over all people of type i, gives

$$C_i(\varepsilon) = \pi c_i^o(\varepsilon) + (1-\pi)c_i^y(\varepsilon). \tag{B1}$$

A person who is alive in two consecutive periods obeys the Euler equation,

$$c_i(\varepsilon)\beta_i^{\frac{1}{\rho_i}} = c_i^o(\varepsilon')\tilde{Q}(\varepsilon,\varepsilon')^{\frac{1}{\rho_i}} \tag{B2}$$

Newborn people consume

$$c_i^y(\varepsilon') = \mu_i X_i(\varepsilon')^{-1}H(\varepsilon'), \tag{B3}$$

Aggregating Equation (B2), using (B1).

$$\tilde{Q}(\varepsilon,\varepsilon')^{\frac{1}{\rho_i}} = \frac{C_i(\varepsilon)\pi\beta_i^{\frac{1}{\rho_i}}}{C_i(\varepsilon') - (1-\pi)X_i^{-1}\mu_i H(\varepsilon')}, \tag{B4}$$

Using the fact that,

$$\mu_i X_i^{-1} H(\varepsilon') = C_i(\varepsilon') - A_i(\varepsilon'),$$

leads to,

$$\tilde{Q}(\varepsilon,\varepsilon') = \left(\frac{C_i(\varepsilon)\pi\beta_i^{\frac{1}{\rho_i}}}{\pi C_i(\varepsilon') + (1-\pi)X_i(\varepsilon')^{-1}A_i(\varepsilon')}\right)^{\rho_i}, \tag{B5}$$

which is the modified Euler equation.