Note 5 Projectile Motion

**Projectile motion** is two-dimensional motion in which the only acceleration is due to gravity. This is an acceleration of “g” pointed toward the center of the Earth. The trajectory looks like this. It is parabolic.

![Projectile Motion Diagram](image1.png)

In the horizontal direction, the motion is shown on the left below. The spacing is even so the velocity is constant. This is **constant motion**.

In the vertical direction, the motion is shown on the right below. The spacing decreases as the object is going up implying a downward acceleration. The spacing increases as the object is going down also implying a downward acceleration. The acceleration is due to gravity only and this motion is called **free fall**.

![Vertical Motion Diagram](image2.png)

Thus, projectile motion is a combination of constant motion and free fall.

For projectile motion, the acceleration in the horizontal direction is zero. Therefore, the equations of motion reduce to simpler equations. The first says that the initial and final horizontal velocities are the same. The second says that the displacement is just the constant, initial velocity multiplied by the time.

In the vertical direction, the acceleration is “–g” using up as positive.

\[
\begin{align*}
\text{x-direction} & \\
v_{i,x} &= v_{f,x} \\
\Delta x &= v_{i,x} \Delta t \\

\text{y-direction} & \\
\Delta v_y &= -g \Delta t \\
\Delta y &= v_{i,y} \Delta t - \frac{1}{2} g \Delta t^2 \\
\Delta (v_y^2) &= -2g \Delta y
\end{align*}
\]
Initial Velocity

Here are the details about the initial and final velocities. The velocities can be written in terms of the magnitudes and angles or in terms of the components

\[ \vec{v}_i (v_i, \theta_i) \text{ and } \vec{v}_f (v_f, \theta_f) \]

\[ \vec{v}_i (v_{ix}, v_{iy}) \text{ and } \vec{v}_f (v_{fx}, v_{fy}) \]

They are related to each other in these ways.

\[ v_{ix} = v_i \cos \theta_i \text{ and } v_{iy} = v_i \sin \theta_i \]

\[ v_{fx} = v_f \cos \theta_f \text{ and } v_{fy} = v_f \sin \theta_f \]

In terms of the magnitudes of the initial and final velocities and the initial and final angle of travel, the equations of motion for projectile motion, in general, are these.

\[
\begin{align*}
\text{x-direction} & \quad \text{y-direction} \\
\Delta x & = v_i \cos \theta_i \Delta t & \Delta y & = v_i \sin \theta_i \Delta t - \frac{1}{2} g \Delta t^2 \\
v_f \cos \theta_f & = v_i \cos \theta_i & v_f \sin \theta_f - v_i \sin \theta_i & = -g \Delta t \\
\end{align*}
\]
Finally, here is the slightly simplified information table for projectile motion with down as negative direction.

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>−g</td>
</tr>
<tr>
<td>ti</td>
<td>tf</td>
</tr>
<tr>
<td>xi</td>
<td>xf</td>
</tr>
<tr>
<td>vi•cosθi</td>
<td>vi•cosθf</td>
</tr>
<tr>
<td>vi•sinθi</td>
<td>vi•sinθf</td>
</tr>
</tbody>
</table>

**Special Cases 1: Same Height**

If a projectile is **launched and comes back down to the same height**, then there are shortcut equations that tell us about some of the characteristics of the motion.

The equations of motion for this situation are the following. Note that the vertical displacement is zero as dictated by the above condition. Also, the final vertical velocity has the same magnitude as the initial velocity but it is in the opposite direction.

\[
\begin{align*}
\left( -v_i \sin \theta_i \right) - \left( v_i \sin \theta_i \right) &= -gt_f \\
x_f &= v_i \cos \theta_i t_f \\
\text{and} \quad 0 = v_i \sin \theta_i t_f - \frac{1}{2} g t_f^2 \\
\left( v_i^2 \sin^2 \theta_i \right) - \left( v_i^2 \sin^2 \theta_i \right) &= 0
\end{align*}
\]

**Flight Time**

This is the total time of flight. Both the first and the second equations in the y direction say this.

\[
\begin{align*}
\left( -v_i \sin \theta_i \right) - \left( v_i \sin \theta_i \right) &= -gt_f \\
-2v_i \sin \theta_i = -gt_f \quad &\Rightarrow \quad t_f = \frac{2v_i \sin \theta_i}{g}
\end{align*}
\]

Also,

\[
\begin{align*}
0 = v_i \sin \theta_i t_f - \frac{1}{2} g t_f^2 \quad &\Rightarrow \quad \left( v_i \sin \theta_i - \frac{1}{2} g t_f \right) t_f = 0 \quad &\Rightarrow \quad v_i \sin \theta_i - \frac{1}{2} g t_f = 0 \\
\frac{1}{2} g t_f &= v_i \sin \theta_i \quad &\Rightarrow \quad t_f = \frac{2v_i \sin \theta_i}{g}
\end{align*}
\]

**Range**

The range is the horizontal displacement at the same height. Use the displacement in the horizontal direction is described by the x equation. Using the flight time above,

\[
x_f = v_i \cos \theta_i t_f \quad \text{and} \quad t_f = \frac{2v_i \sin \theta_i}{g}
\]

\[
x_f = v_i \cos \theta_i \frac{2v_i \sin \theta_i}{g} = \frac{v_i^2}{g} 2 \sin \theta_i \cos \theta_i = \frac{v_i^2}{g} \sin(2\theta_i)
\]
**Maximum Range**

Given the range above, we can also find the angle from which the maximum range can be achieved.

$$\max(x_f) = \max \left( \frac{v_i^2}{g} \sin(2\theta_i) \right) = \frac{v_i^2}{g} \max \left( \sin(2\theta_i) \right)$$

The argument that returns the maximum value for the sine function is 90°. Thus,

$$2\theta_i = 90° \implies \theta_i = 45°$$

**Special Case 2: Maximum Height**

If the projectile reaches the maximum height, the final vertical velocity goes to zero.

$$-(v_i \sin \theta_i) = -gt_f$$

$$x_f = v_i \cos \theta_i t_f \quad \text{and} \quad y_f = v_i \sin \theta_i t_f - \frac{1}{2}gt_f^2$$

$$-(v_i^2 \sin^2 \theta_i) = -2gy_f$$

**Flight Time**

The flight time to maximum height is this.

$$-(v_i \sin \theta_i) = -gt_f \implies t_f = \frac{v_i \sin \theta_i}{g}$$

**Maximum Height**

The actual maximum height is this.

$$-(v_i^2 \sin^2 \theta_i) = -2gy_f \implies y_f = \frac{v_i^2 \sin^2 \theta_i}{2g}$$

**Special Case 3: Zero Launch Angle**

When a projectile is launched at zero angle, the initial velocity is all and only in the horizontal direction.

$$v_f = -gt_f$$

$$x_f = v_i t_f \quad \text{and} \quad y_f = -\frac{1}{2}gt_f^2$$

$$v_{fy} = -2gy_f$$
**Flight Time**

The flight time of the projectile is this if the projectile is launched at a height $h$ from the lower landing point.

\[
y_f = -\frac{1}{2}gt_f^2 \quad \Rightarrow \quad \frac{1}{2}gt_f^2 = -y_f \quad \Rightarrow \quad t_f = \sqrt{\frac{-2y_f}{g}}
\]

The time is positive and real since the values of final $y$ position is also negative.

**Horizontal Displacement**

The displacement in the horizontal direction is this. Using the time above,

\[
x_f = v_i t_f = v_i \sqrt{\frac{-2y_f}{g}}
\]