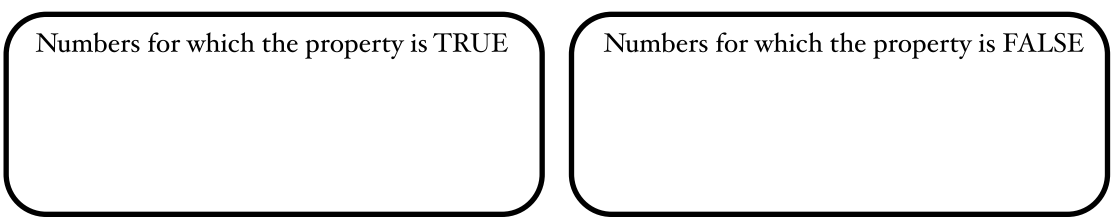
Induction Strand

1. Recall the formula for the sum of the first n natural numbers, you should almost know this by heart, but it is $1+2+...+n=\frac{n(n+1)}{2}$. One way to prove this formula is to use *proof by contradiction.*

   You may recall that the first step in an *indirect proof* is to assume that what you are trying to prove is false. So let's do that. Assume Not - We are going to assume that there is some number p for which the statement is not true... Hence: $1+2+...+p \neq \frac{p(p+1)}{2}$. If we can disprove this our work is done and we will have proved our original formula.

   Let's begin with what we know, one thing we know for sure is that the statement is true for $n = 1$, show this. Also show the statement is true for n = 2. Clearly we could keep doing this but it would not finish our problem —explain why.

2. (Continued) We have established that the statement we are trying to prove is, for any number, either true or false, a *mutually exclusive* situation that we can model with a Venn diagram.



   Your task is to put the following numbers into their proper boxes. Justify each placement. You will want to use the *Well Ordering Principle* which states that if you have a non-empty set of positive integers then that set will have a least element.
   a. the numbers 1 and 2
   b. the number p
   c. the number m, where m is the minimum number for which the statement is false.
   d. the number m -1

3. (Continued) This is good, we are (believe it or not) making progress. We know our statement is true for $n=m-1$. Write this out fully. Next, using this information reconsider the situation for $n = m$. Finally, arrive at a contradiction and finish the proof.

4. (Continued) A proof by *mathematical induction* shortens the rather long process we just completed above into 3 easy steps.

   Step 1: Prove the statement is true for $n = 1$ or some other base case
   Step 2: Assume that the statement is true for $n = k$.
   Step 3: Use 1 and 2 to Prove the statement is true for $n = k + 1$

   When you are doing a proof by induction, you *must* write out each of the three steps before you do them. They are an essential part of the proof. Use induction to prove $2 + 4 + 6 + ... + 2n = n^2 + n$ for all positive integers n.

5. Prove $\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1)$ for all positive integers by induction. Many find it easier to start by expanding the left hand side.

6. Prove $\sum_{k=1}^{n} 2k - 1 = ??$ by induction. You need to figure out what goes in for the question marks.

7. Prove: $\sum_{k=1}^{n} k^3 = \frac{n^2 (n+1)^2}{4}$ for all positive integers by induction. Many find it easier to start by expanding the left hand side.

8. Prove the formula for the sum of the first terms of a geometric sequence:
   $S_n = \frac{a_1 (1 - r^n)}{(1 - r)}$ recall that a_1 is the first term and $r, r \neq 1$ is the common ratio.

9. Find a formula for the series, and then prove it by induction:
   $\frac{1}{1 \bullet 2} + \frac{1}{2 \bullet 3} + \frac{1}{3 \bullet 4} + ... + \frac{1}{n \bullet (n+1)}$

10. Oh divisibility, prove that $9^n - 1$ is divisible by 8 for all n≥1 where n is an integer.

11. Prove that $3^{2n} - 1$ is a multiple of 8, for integers $n \geq 1$

12. In mathematical shorthand an upside-down A means "for all." Knowing this prove that 2^{3n-1} is divisible by 7 $\forall n \in \mathbb{Z}^+$.

13. Prove DeMoivre's Theorem for all positive integers, i.e.

$\left(\cos\theta + i\sin\theta\right)^n = \cos n\theta + i\sin n\theta$. Use induction.

14. Using mathematical induction prove that

$\frac{d^n}{dx^n}\left(\cos x\right) = \cos\left(x + \frac{n\pi}{2}\right)$ for all positive integers.

15. Prove that $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix}$ for all positive integers using mathematical induction.

16. Consider $D = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$, form a conjecture for D^n . Prove your conjecture with mathematical induction.

17. The function f is defined by $f\left(x\right) = e^{px}\left(x+1\right)$, where $p \in \mathbb{R}$. Show that

$f'\left(x\right) = e^{px}\left(p\left(x-1\right)+1\right)$. Next let $f^{(n)}\left(x\right)$ denote the result of differentiating $f\left(x\right)$ with respect to x, n times. Use mathematical induction to prove that

$f^{(n)}\left(x\right) = p^{n-1}e^{px}\left(p\left(x+1\right)+n\right), n \in \mathbb{Z}^+$.