GENERALIZED FOURIER TRANSFORMS AND THEIR APPLICATIONS

A Thesis

Submitted to the Faculty

in partial fulfillment of the requirements for the

degree of

Doctor of Philosophy

in

Mathematics

by

Sarah Wolff

DARTMOUTH COLLEGE

Hanover, New Hampshire

May 2015

Examining Committee:

Daniel Rockmore, Chair

Thomas Shemanske

Sergi Elizalde

Richard Foote

F. Jon Kull, Ph.D.
Dean of Graduate Studies

Abstract

This thesis centers around a generalization of the classical discrete Fourier transform. We first present a general diagrammatic approach to the construction of efficient algorithms for computing the Fourier transform of a function on a finite group or semisimple algebra. By extending work which connects Bratteli diagrams to the construction of Fast Fourier Transform algorithms [65], we make explicit use of the path algebra connection to the construction of Gel'fand-Tsetlin bases and work in the setting of general semisimple algebras and quivers. We relate this framework to the construction of a *configuration space* derived from a Bratteli diagram. In this setting the complexity of an algorithm for computing a Fourier transform reduces to the calculation of the dimension of the associated configuration space.

We give explicit counting results to find the dimension of these configuration spaces, and thus the complexity of the associated Fourier transform. Our methods give improved upper bounds for the general linear groups over finite fields, the classical Weyl groups, and homogeneous spaces of finite groups, while also recovering the best known algorithms for the symmetric group and compact Lie groups. We extend these results further to semisimple algebras, giving the first results for non-trivial upper bounds for computing Fourier transforms on the Brauer and Birman-Murakami-Wenzl (BMW) algebras.

The extension of our algorithm to Fourier transforms on semisimple algebras is motivated by emerging applications of such transforms. In particular, Fourier transforms on the Iwahori-Hecke algebras have been used to analyze Metropolis-based systematic scanning strategies for generating Coxeter group elements [25]. We consider the Metropolis algorithm in the context of the Brauer and BMW monoids and provide systematic scanning strategies for generating monoid elements. As the BMW monoid consists of tangle diagrams, these scanning strategies can be rephrased as random walks on links and tangles. We translate these walks into left multiplication operators in the corresponding BMW algebra. Taking this algebraic perspective enables the use of tools from representation theory to analyze the walks; in particular, we develop a norm arising from a trace function on the BMW algebra to analyze the time to stationarity of the walks.

Acknowledgements

First and foremost, I'd like to thank my advisor, Dan Rockmore, for believing in me long before I ever could. Thank you for your advice and guidance, for somehow making time no matter how busy your schedule, and for your ability to spark a month's worth of ideas in thirty minutes (or less!).

A huge thank you as well to the members of my thesis committee: Sergi Elizalde, Richard Foote, and Tom Shemanske, for your helpful comments, questions, and ideas for future research directions. Thank you to Tom for being the person to run to for any kind of math/life advice, despite never actually volunteering for this position. Thank you to Sergi for not only being someone to look up to professionally but also for showing me that it is possible to fit some soccer games into a successful mathematical career. A million thanks as well to Arun Ram for inspiration, encouragement, and helpful suggestions.

Thank you to my friends and teammates in Hanover for making the last five years so much fun, and to my friends outside of Hanover for supporting me in my 'quest to understand paranomials.' Thanks to Megan, Nathan, Lin, Emily, Jessie, Hannah, and Ali. And to my Texan, for always being there, even when neither of us knew.

Above all else, thank you to my family, especially my mom and dad, for their love and support. I love you and could not have done this without you.

Contents

Chapter 1

Introduction

This thesis unites discrete mathematics with representation theory and algebraic combinatorics through the analysis and application of the construction of efficient algorithms for the computation of a generalization of the classical discrete Fourier transform (DFT) in the setting of finite groups and semisimple algebras.

Since Cooley and Tukey's (re-)discovery of Gauss' efficient algorithm for computing Fourier transforms, there have been far-reaching applications of the classical DFT in many areas, including (but not limited to) digital signal processing, image processing, spectral analysis, data compression, and astrophysics [2, 3, 11, 19, 31, 89, 91]. A *generalized Fourier transform* extends the concept behind the DFT to algebraic structures wherein applications have been found in many domains, including VLSI design, the design of filters, machine learning, conditional probability models, group convolution algorithms, and crossover designs (see, e.g., [4, 6, 14, 80, 53, 55, 90, 97]).

In this thesis, we focus on generalized Fourier transforms; in particular, we develop efficient algorithms to compute Fourier transforms on finite groups and semisimple algebras, and also analyze applications to random walks on the Brauer and Birman-

Murakami-Wenzl (BMW) algebras.

An algebraic framing of the DFT allows for its generalization to groups and semisimple algebras. We view the DFT map that arises from expressing a function $f : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ as a sum of complex exponentials:

$$f(j) = \sum_{k=0}^{N-1} \hat{f}(k)\zeta_k(j).$$

Here, $\mathbb{Z}/N\mathbb{Z} = \{0, 1, \ldots, N-1\}$ is identified with the cyclic group of order N and $\zeta_k : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ is defined by $\zeta_k(j) = \frac{1}{\sqrt{N}} e^{2\pi ijk/N}$. The coefficient $\hat{f}(k)$ is called the *kth Fourier coefficient of f*.

The *group algebra* of $\mathbb{Z}/N\mathbb{Z}$, denoted $\mathbb{C}[\mathbb{Z}/N\mathbb{Z}]$, is the space of formal complex linear combinations of group elements. By identifying complex-valued functions on $\mathbb{Z}/N\mathbb{Z}$ with elements of the group algebra

$$f \longleftrightarrow \sum_{j \in \mathbb{Z}/N\mathbb{Z}} f(j)j,$$

the DFT can be seen as a change of basis of $\mathbb{C}[\mathbb{Z}/n\mathbb{Z}]$ from the natural basis of indicator functions $\{\delta_k \mid k \in \mathbb{Z}/N\mathbb{Z}\}$ to the orthonormal basis $\{\zeta_k \mid k \in \mathbb{Z}/N\mathbb{Z}\}$:

$$\sum_{j \in \mathbb{Z}/n\mathbb{Z}} f(j)j \xrightarrow{DFT} \sum_{j \in \mathbb{Z}/n\mathbb{Z}} \hat{f}(j)\zeta_j$$

(for $\zeta_j = \sum_{k \in \mathbb{Z}/n\mathbb{Z}} \zeta_j(k)k$). In signal processing, $f(j)$ is identified with a sampole of some continuous function $f : \mathbb{R} \to \mathbb{R}$ and the DFT effects a Fourier representation of the function f, thought of as a map from the *time domain* to the *frequency domain* [71].

There is a natural inner product on $\mathbb{C}[\mathbb{Z}/N\mathbb{Z}]$ given by $\langle f, g\rangle = \sum_{j=0}^{N-1} f(j)\overline{g(j)}$. The orthonormality of the complex exponentials $\{\zeta_k \mid k \in \mathbb{Z}/N\mathbb{Z}\}$ under this inner product allows for the determination of the kth Fourier coefficient as an inner product,

$$\hat{f}(k) = \langle f, \zeta_k\rangle = \sum_{j=0}^{N-1} f(j)\overline{\zeta_k(j)}.$$

Moreover, the functions $\{\zeta_k\}$ comprise a complete set of *inequivalent irreducible representations* of $\mathbb{Z}/N\mathbb{Z}$. It is this beautiful result that underlies the generalization of the DFT to finite groups and semisimple algebras: consider an arbitrary finite group G and let f be a complex-valued function on G. Given a matrix representation ρ of G, the *(generalized) Fourier transform of f at ρ* is given by the complex matrix sum:

$$\hat{f}(\rho) = \sum_{x\in G} f(x)\rho(x).$$

The computation of the set of Fourier transforms of f at each representation in a complete set R of inequivalent irreducible matrix representations of G is the *computation of a (generalized) Fourier transform of f at R.* The problem of efficient computation of generalized Fourier transforms is an area of active research (see e.g. [6, 13, 15, 26, 54]). Some of the first results in this area are due to Willsky [94], motivated by a search for efficient algorithms for filter design.

Let $T_G(R)$ denote the computational complexity of the Fourier transform on a group G at a set of inequivalent irreducible representations R. This is basically the least upper bound for the number of arithmetic operations needed to compute the Fourier transform of an arbitrary complex-valued function f on G at R. We let $C(G)$

denote the *complexity of the group* G, defined as

$$C(G) := \min_R\{T_G(R)\}.$$

For N a "highly composite" number, Cooley and Tukey's algorithm gives $C(\mathbb{Z}/N\mathbb{Z}) \leq O(N\log_2 N)$. More recently, Johnson and Frigo [51] and Lundy and Van Buskirk [59] have independently shown that $C(\mathbb{Z}/N\mathbb{Z}) \leq \left(\frac{8}{3}+o(1)\right)N\log_2 N$ for $N = 2^m$. More generally, for A an abelian group of size N, it is known that $C(A) \leq O(N\log_2 N)$ [23], versus the naive bound, $C(A) \leq N^2$ that comes from considering a direct computation.

Cooley and Tukey give a divide-and-conquer algorithm whose key step rewrites the DFT on $\mathbb{Z}/N\mathbb{Z}$ as a linear combination of DFTs on subgroups. This step applies in the nonabelian group setting as well: if H is a subgroup of any group G, the Fourier transform of a complex-valued function on G reduces to a linear combination of Fourier transforms on H. This basic generalization of Cooley and Tukey's algorithm has been used in [65] to give efficient algorithms for generalized Fourier transforms and has been extended to develop efficient algorithms for computing Fourier transforms on finite inverse semigroups [61].

A more sophisticated approach that builds on the subgroup chain is given in [63, 65], where a general algorithm is presented for computing Fourier transforms on semisimple algebras as a sequence of bilinear maps written in terms of the *path algebra* associated to a derived *Bratteli diagram* — a directed graph based on a subgroup chain which depicts how representations factor when restricted to subgroups. In the path algebra, basis vectors correspond to paths in the Bratteli diagram and matrix elements (indexed by pairs of vectors) correspond to very specific kinds of subgraphs. Matrix element multiplication also has a diagrammatic formulation. Factorizations of the associated Fourier transform correspond to decompositions of the Bratelli diagram in terms of recurrent subgraphs. The complexity counts for an associated algorithm then amount to counting the number of morphisms of subgraphs into the Bratteli diagram.

The work in this thesis builds on these earlier ideas by using the formalism of *quivers* [36]. We produce efficient algorithms for computing the Fourier transform of functions on the general linear groups over finite fields, classical Weyl groups, and their homogeneous spaces, improving on [63, 65]. We also provide new results for general semisimple algebras including the Brauer and BMW algebras.

These new results are motivated in part by emerging applications of Fourier transforms on semisimple algebras. For example, in [25] Diaconis and Ram consider the problem of systematically generating elements of a finite Coxeter group, W. This problem stated in terms of the group algebra $\mathbb{C}[W]$ is equivalent to generating elements of the basis W of $\mathbb{C}[W]$. They show that a Metropolis-type formulation gives rise to systematic *scanning strategies* — choices of an ordering of a succession of Markov chains. This corresponds to the successive multiplication of group generators — which translates into an applications of left multiplication operators in the Iwahori-Hecke algebra. The algebraic setting enables the use of Fourier analysis on the Iwahori-Hecke algebra to analyze the time to stationarity of the corresponding random walks. Their work gives a comparison between systematic scan algorithms and random scan algorithms. This is of interest in the study of Ising models [10] and Gibbs sampling for image processing [34].

We extend these Metropolis-type ideas to *diagram algebras* in the consideration of random walks on the monoid bases of the Brauer and BMW algebras. Elements of these monoid bases have a natural interpretation as diagrams in which multiplication becomes concatenation of diagrams. The Metropolis algorithm in the context of the Brauer and BMW monoids gives rise to systematic scanning strategies for generating basis elements via multiplication of generators. As the diagrams forming the BMW monoid basis of the BMW algebra are tangles (see [40] or Chapter 3 for a description of this basis), scanning strategies for generating BMW monoid elements have relations to physics: random generation of links and tangles has been of use in [28, 60, 96]. As in [25] the walks arising from the Metropolis algorithm can be reinterpreted as left multiplication operators in the BMW algebra and studied via a new trace norm. This allows us to use the tools of representation theory to analyze the time to stationarity.

The two parts of this thesis split naturally into two self-contained chapters. In Chapter 2, we develop the Separation of Variables (SOV) approach for the efficient computation of Fourier transforms on finite groups and semisimple algebras. We end the chapter by using the algorithm to provide efficiency counts for the Brauer and BMW algebras. In Chapter 3, we explore the application of Fourier transforms on these algebras to random walks on the Brauer and BMW monoids. We close in Chapter 4 with some thoughts about future directions of work.

Chapter 2

Fourier Transforms on Finite Groups and Semisimple Algebras

2.1 Introduction

The (classical) *Fast Fourier Transform* (FFT) remains among the most important family of algorithms in information processing [81]. In its simplest form, it effects the efficient computation of the *discrete Fourier transform* (DFT), which up to normalization and indexing is the computation, for each $k \in \mathbb{Z}/N\mathbb{Z}$, of the sums

$$\hat{f}(k) = \sum_{j=0}^{N-1} f(j) e^{2\pi ijk/N}, \tag{2.1}$$

as described in Chapter 1. While this calculation can be framed in a number of ways, we take a representation theoretic point of view and cast the DFT as a change of basis in $\mathbb{C}[\mathbb{Z}/N\mathbb{Z}]$, the complex group algebra of the cyclic group of order N (naturally identified with $\mathbb{Z}/N\mathbb{Z}$), from the natural basis of indicator functions to a basis of irreducible matrix elements for $\mathbb{Z}/N\mathbb{Z}$. This perspective suggests a generalization of the DFT to finite nonabelian groups G, that is, as the computation of a change of basis in $\mathbb{C}[G]$ from the basis of indicator functions to a basis of irreducible matrix elements, with concomitant questions of computational complexity (see e.g., [67]).

In the case of finite abelian groups, the various fast Fourier transform algorithms can be pooled together in such a way that it is possible to say that the complexity of the DFT on an abelian group of size N is bounded above by $O(N \log_2 N)$ [23]. Using different approaches, Lundy and Van Buskirk [59] and Johnson and Frigo [51] independently determined complexity bounds of $(\frac{8}{3} + o(1))N \log_2 N$ for the case $N = 2^m$. The deep and ongoing study of this problem has been motivated by a wide range of applications in digital signal processing and beyond (see e.g. [2, 3, 11, 19, 31, 89, 91]).

The Cooley-Tukey algorithm is undoubtedly the most famous of the FFTs [17]. It is a divide-and-conquer algorithm, in fact, first recorded by Gauss in unpublished work (see e.g. [47] for a brief history of the algorithm). The key step is to rewrite the DFT on a cyclic group $\mathbb{Z}/N\mathbb{Z}$ as a linear combination of DFTs on $m\mathbb{Z}/N\mathbb{Z} < \mathbb{Z}/N\mathbb{Z}$ (for $N = mn$). Iterating this step for a chain of subgroups of $\mathbb{Z}/N\mathbb{Z}$ yields efficient algorithms. In this chapter we continue a line of work that generalizes this approach to nonabelian groups [64, 80, 65, 66], finding subgroup factorizations compatible with bases of irreducible matrix elements that combine to provide the elementary ingredients of efficient DFT algorithms for arbitrary finite groups and more general algebraic structures.

In particular, we build on the "Separation of Variables" (SOV) approach [65] by integrating the approach detailed in [63] with the formalism of *quivers*. This gives a general algorithm for computing Fourier transforms on semisimple algebras as a sequence of bilinear maps written in terms of the *path algebra* associated to a derived *Bratteli diagram* — a directed graph based on a subgroup chain which depicts how representations factor when restricted to subgroups. In the path algebra, basis vectors correspond to paths in the Bratteli diagram, matrix elements correspond to very specific kinds of subgraphs. Factorizations of the associated Fourier transform correspond to decompositions of the Bratelli diagram in terms of recurrent subgraphs. The complexity counts for an associated algorithm amount to counting the number of morphisms of these subgraphs into the Bratteli diagram.

We use quivers to define spaces associated to morphisms of a graph with dimension equal to the number of morphisms of the graph into the Bratteli diagram. The Fourier transform is then realized as an accumulation of sequences of bilinear maps on these "configuration spaces". The associated complexity of such an algorithm is derived by identifying each subgraph in a "gluing process" and computing the dimension of its configuration space. As in [65], if our choice of factorization makes use of subgroup and centralizer algebra structure, the configuration spaces of the subgraphs identified in the gluing process have low dimension. With well chosen subgroup (subalgebra) chains, we improve on upper bounds for the complexity of computing DFTs on the general linear groups over finite fields, classical Weyl groups, and the homogeneous spaces of these groups. Moreover, our results hold for arbitrary chains of semisimple algebras rather than just chains of group algebras.

In Section 2.2 we outline the preliminaries needed for our results, defining the Fourier transform and complexity and discussing the main ideas behind the SOV approach. We then state our new complexity results for Fourier transforms on the general linear groups $Gl_n(\mathbb{F}_q)$, the Weyl groups B_n and D_n, and their homogeneous spaces. In Section 2.3 we present the SOV approach in detail, rewriting an iterated product in the path algebra as a sequence of bilinear maps on the newly defined configuration spaces. The complexity of the algorithm is determined by the dimensions of the configuration spaces and we generalize some results of Stanley on differential posets [87, 88] to give explicit methods for finding these dimensions.

In Section 2.4 we give specific factorizations and counts to prove the specific group complexity results and also recover previously known bounds for S_n [63] and compact Lie groups [62]. Finally, we extend these ideas in Section 2.5 to general semisimple algebras and work out complexity bounds for computing DFTs on the Brauer and BMW algebras. The last of these provides material and motivation for Chapter 3.

2.2 Background

The usual discrete Fourier transform of a finite data sequence may be viewed as a special case of Fourier transforms on finite groups, which in turn arise as a special case of Fourier transforms on semisimple algebras. Here we review the basic concepts and definitions. For more background on the representation theory of finite groups we refer the reader to [86], while for semisimple algebras see [76]. We'll work in the general semisimple setting.

Definition 2.1. *A* ***matrix representation*** *of a $\mathbb{C}$-algebra A is an algebra homomorphism*

$$\rho : A \to M_d(\mathbb{C}),$$

where $M_d(\mathbb{C})$ denotes the complex algebra of $d \times d$ matrices with entries in $\mathbb{C}$. We *call d the **dimension** of ρ.*

*An algebra A is **simple** if $A \cong M_n(\mathbb{C})$ for some $n \geq 1$ and **semisimple** if it decomposes as a direct sum of simple algebras:*

$$A \cong \bigoplus_{\lambda \in \Lambda} M_\lambda(\mathbb{C}),$$

for a finite index set Λ.

Definition 2.2. *Let A be a semisimple algebra, $\{a_i\}_{i \in I}$ a basis for A and $f = \sum_{i \in I} f(a_i)a_i \in A$.*

*(i) Let ρ be a matrix representation of A. Then the **Fourier transform of** f **at** ρ, denoted $\hat{f}(\rho)$, is the matrix sum*

$$\hat{f}(\rho) = \sum_{i \in I} f(a_i)\rho(a_i).$$

*(ii) Let R be a set of matrix representations of A. Then the **Fourier transform of** f **on** R is the direct sum of Fourier transforms of f at the representations in R:*

$$\mathcal{F}_R(f) = \bigoplus_{\rho \in R} \hat{f}(\rho) \in \bigoplus_{\rho \in R} M_{\dim \rho}(\mathbb{C}).$$

Example 2.1. Our main interest will be the case in which $A = \mathbb{C}[G]$, the complex group algebra for a finite group G. The group algebra $\mathbb{C}[G]$ is the space of all formal complex linear combinations of group elements under the product

$$\left(\sum_{s \in G} f(s)s\right)\left(\sum_{t \in G} h(t)t\right) = \sum_{s,t \in G} f(s)h(t)st.$$

Elements of $\mathbb{C}[G]$ are in one-to-one correspondence with complex-valued functions on G, and the group algebra product corresponds to convolution of functions. The Fourier transform of $\sum_{s\in G} f(s)s \longleftrightarrow f : G \to \mathbb{C}$ at a matrix representation ρ of G is

$$\hat{f}(\rho) = \sum_{s\in G} f(s)\rho(s).$$

This is equivalent to the d_ρ^2 individual Fourier transforms at the corresponding matrix elements

$$\hat{f}(\rho_{ij}) = \sum_{s\in G} f(s)\rho_{ij}(s).$$

When we compute the Fourier transform for a complete set of inequivalent irreducible representations R of G we refer to the calculation as the **computation of a Fourier transform on** G (with respect to R).

Definition 2.3. *Let G be a finite group, R a set of matrix representations of G.*

*(i) A **straight-line program** is a list of instructions for performing the operations $\times, \div, +, -$ on inputs and precomputed values [9].*

*(ii) The **arithmetic complexity** of a Fourier transform on R, denoted $T_G(R)$, is the minimum number of complex arithmetic operations needed to compute the Fourier transform of f on R via a straight-line program for an arbitrary complex-valued function f defined on G. Let $C(G)$ denote the **complexity of the group** G:*

$$C(G) := \min_R \{T_G(R)\},$$

where R varies over all complete sets of inequivalent irreducible representations of G.

*(iii) The **reduced complexity**, denoted $t_G(R)$, is defined by*

$$t_G(R) = \frac{1}{|G|} T_G(R).$$

Given a finite-dimensional semisimple algebra A, we have analogous definitions for the arithmetic complexity, $T_A(R)$, and reduced complexity, $t_A(R) = \frac{1}{\dim(A)} T_A(R)$. As in [63] we will always define the arithmetic complexity of an algorithm to be the maximum of the number of complex multiplications and the number of complex additions.

A complete set R of inequivalent irreducible matrix representations of a group G determines a basis for $\mathbb{C}[G]$ and a Fourier transform determines f through the Fourier inversion formula.

Theorem 2.1 (Fourier inversion (see e.g. [24]))**.** *Let G be a group, f a complex-valued function on G, and R a complete set of inequivalent irreducible matrix representations of G. Then*

$$f(s) = \frac{1}{|G|} \sum_{\rho \in R} \dim_\rho \operatorname{Trace}(\hat{f}(\rho)\rho(s^{-1})).$$

Thus, the Fourier transform is a change of basis algorithm to a basis of irreducible matrix elements. Reorganization using the obvious change of basis matrix gives a naive complexity bound. Let $\rho_1, \ldots, \rho_m$ be a complete set of inequivalent irreducible matrix representations of a group G of dimensions $d_1, \ldots, d_m$, respectively. A direct computation of a Fourier transform would require at most $|G| \sum_i d_i^2 = |G|^2$ arithmetic operations. Rewriting, for a direct computation we have

$$C(G) \leq T_G(R) \leq |G|^2.$$

Fast Fourier transforms (FFTs) are algorithms for computing Fourier transforms that improve on this naive upper bound. A priori, the number of operations needed to compute the Fourier transform may depend on the specific representations used.

Example 2.2. The classical DFT and FFT. For $G = \mathbb{Z}/N\mathbb{Z}$, identified with the cyclic group of order N, all irreducible representations are 1-dimensional, defined by $\zeta_k(j) = e^{2\pi ijk/N}$ for $k = 0, \dots, N-1$. The corresponding Fourier transform on $\mathbb{Z}/N\mathbb{Z}$ is the usual discrete Fourier transform. Cooley and Tukey's algorithm showed that for N 'highly composite,' i.e., N factors completely as a product of small prime numbers, $C(\mathbb{Z}/N\mathbb{Z}) \leq O(N \log_2 N)$ [17].

2.2.1 Gel'fand-Tsetlin Bases and Bratteli diagrams

The fundamental idea underlying the Cooley-Tukey FFT is a divide-and-conquer approach dependent on the use of a subgroup. In the best-known case of the Fourier transform on $\mathbb{Z}/N\mathbb{Z}$, when $N = nm$ the subgroup $m\mathbb{Z}/N\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z}$ enables a factorization of the associated DFT matrix and related computation. This can be framed in terms of a factorization of the path algebra of the *Bratteli diagram* of the associated subgroup chain $\mathbb{Z}/N\mathbb{Z} > m\mathbb{Z}/N\mathbb{Z} > \{0\}$, an idea which extends to the computation of the DFT for general group algebras and their associated irreducible matrix elements [65].

Let A be a complex semisimple algebra with chain of subalgebras

$$A = A_n > A_{n-1} > \cdots > A_1 > A_0 = \mathbb{C},$$

where for ease of notation we let $A_i > A_{i-1}$ denote that A_{i-1} is a subalgebra of A_i. We will see that for any such chain of subalgebras there is a natural isomorphism with a chain of path algebras arising from the Bratteli diagram associated to the chain. We will take advantage of a formulation in terms of *graded quivers*.

Definition 2.4. *A* ***quiver*** *Q is a directed multigraph with vertex set $V(Q)$ and edge set $E(Q)$. For an arrow, $e \in E(Q)$ from vertex β to vertex α, we call α the* ***head*** *of e and β the* ***tail*** *of e.*

Example 2.3. The directed multigraph of Figure 2.1 is an example of a quiver.

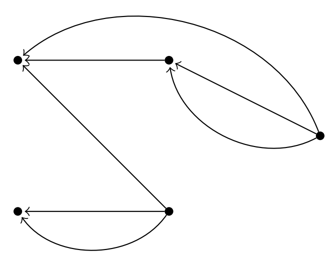


Figure 2.1: A quiver

Let Q be a quiver. For each $e \in E(Q)$, let e^+ denote the head of e and e^- the tail of e.

Definition 2.5. *A quiver Q is* ***graded*** *if there is a function $gr : V(Q) \to \mathbb{N}$ such that $gr(e^+) > gr(e^-)$, for each $e \in E(Q)$.*

Example 2.4. Figure 2.2 is an example of a graded quiver. Each vertex v is labeled by its grading, $gr(v)$.

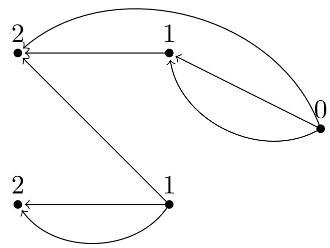


Figure 2.2: A graded quiver

Definition 2.6. *A* ***Bratteli diagram*** *is a finite graded quiver such that:*

(i) there is a unique vertex with grading 0*, called the* ***root****,*

(ii) if $v \in V(Q)$ *is not the root then* v *is the head of at least one arrow,*

(iii) if $v \in V(Q)$ *does not have grading of maximum value then* v *is the tail of at least one arrow,*

(iv) for each $e \in E(Q)$*,* $gr(e^+) = 1 + gr(e^-)$*.*

Example 2.5. Note that the quiver of Figure 2.2 is not a Bratteli diagram. However, a slight modification produces the Bratteli diagram of Figure 2.3.

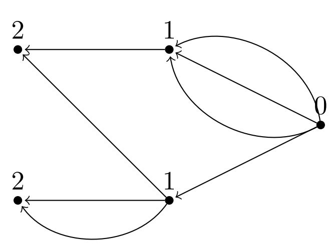


Figure 2.3: A Bratteli Diagram

Let A be a semisimple algebra with chain of subalgebras

$$A = A_n > A_{n-1} > \cdots > A_1 > A_0 = \mathbb{C}.$$

To associate a Bratteli diagram to this chain we follow the language of [77]. Let ρ be an irreducible representation of A_i, i.e., an irreducible A_i-module. Upon restriction to A_{i-1}, $\rho \downarrow_{A_{i-1}}$ decomposes as a direct sum of irreducible A_{i-1}-modules. For γ an irreducible representation of A_{i-1}, let $M(\rho, \gamma)$ denote the multiplicity of γ in $\rho \downarrow_{A_{i-1}}$.

Definition 2.7. *Given the chain of subalgebras,*

$$A = A_n > A_{n-1} > \cdots > A_1 > A_0 = \mathbb{C},$$

the ***(associated) Bratteli diagram*** *is described by*

(i) The vertices of grading i are labeled by the (equivalence classes of) irreducible representations of A_i;

(ii) A vertex labeled by an irreducible representation γ of A_{i-1} is connected to a vertex labeled by an irreducible representation ρ of A_i by $M(\rho, \gamma)$ arrows.

Example 2.6. Let $A = M_N(\mathbb{C})$. The identity map $\mathbf{1}_N$ is an irreducible representation of A with representation space $\mathbb{C}^N$. Similarly, the identity map $\mathbf{1}$ is an irreducible representation of $\mathbb{C}$ with representation space $\mathbb{C}$. Then $M(\mathbb{C}^N, \mathbb{C}) = N$ and so the Bratteli diagram for the chain $M_N(\mathbb{C}) > \mathbb{C}$ is the quiver of Figure 2.4, with N paths between the two vertices.

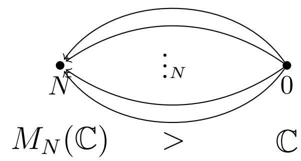


Figure 2.4

Example 2.7. Figure 2.5 shows two examples of Bratteli diagrams, with the gradings listed at the top.

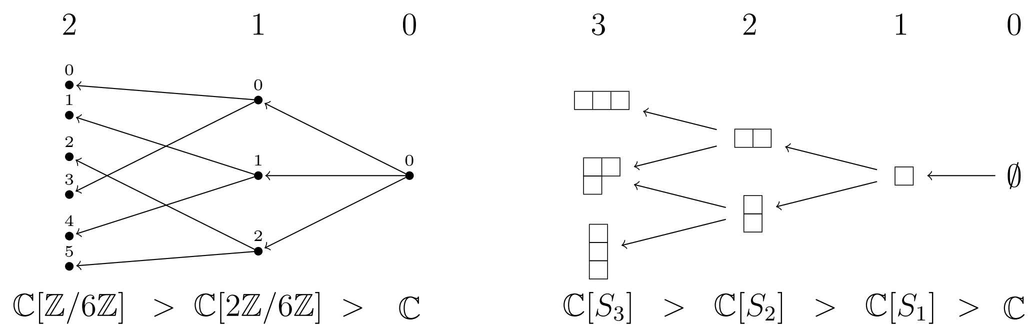


Figure 2.5

On the left we see the Bratteli diagram for a chain of group algebras for the group $\mathbb{Z}/6\mathbb{Z}$:

$$\mathbb{C}[\mathbb{Z}/6\mathbb{Z}] > \mathbb{C}[2\mathbb{Z}/6\mathbb{Z}] > \mathbb{C}.$$

On the right we see the Bratteli diagram for a chain of group algebras for the symmetric group S_3:

$$\mathbb{C}[S_3] > \mathbb{C}[S_2] > \mathbb{C}[S_1] > \mathbb{C}.$$

Note that we distinguish $\mathbb{C}[S_1]$ from $\mathbb{C}$ only so that vertices at level i, $i > 0$, correspond to representations of $\mathbb{C}[S_i]$. For the group algebra $\mathbb{C}[\mathbb{Z}/N\mathbb{Z}]$, irreducible representations are naturally indexed at each level by the integers $0, \ldots, N-1$, while for $\mathbb{C}[S_n]$, the irreducible representations are indexed by partitions of n (as determined by Young in [95]; see [49] for an introduction to the representation theory of S_n). For $N = mn$, $m\mathbb{Z}/N\mathbb{Z} = \{0, m, \ldots, (n-1)m\} < \mathbb{Z}/mn\mathbb{Z} = \mathbb{Z}/N\mathbb{Z}$, while S_i will be the subgroup of S_n fixing the elements $\{i+1, \ldots, n\}$. This gives a natural way to embed $\mathbb{C}[\mathbb{Z}/n\mathbb{Z}]$ as a subalgebra of $\mathbb{C}[\mathbb{Z}/N\mathbb{Z}]$ and $\mathbb{C}[S_i]$ as a subalgebra of $\mathbb{C}[S_n]$.

Both Bratteli diagrams in Figure 2.5 are examples of **multiplicity-free** diagrams in that there is at most one edge from any vertex of grading i to any vertex of grading $i+1$.

Given a chain of subalgebras with Bratteli diagram $\mathcal{B}$, there is a canonical chain of algebras associated to $\mathcal{B}$ called the chain of path algebras.

Definition 2.8. *Let $\mathcal{B}$ be a Bratteli diagram. The **path algebra (at level i)**, denoted $\mathbb{C}[\mathcal{B}_i]$, is the $\mathbb{C}$-vector space with basis given by ordered pairs of paths in $\mathcal{B}$ of length i which start at the root of $\mathcal{B}$ and end at the same vertex at level i in $\mathcal{B}$.*

The path algebra $\mathbb{C}[\mathcal{B}_i]$ is an algebra under the multiplication that linearly by the distributive laws extends $(P, Q) * (P', Q') = \delta_{QP'}(P, Q')$:

$$\sum_{(P,Q)} a_{PQ}(P, Q) * \sum_{(P',Q')} b_{P'Q'}(P', Q') = \sum \left(\sum_{Q} a_{PQ} b_{QQ'} \right) (P, Q').$$

Pictorially, the path algebra multiplication corresponds to Figure 2.6. The first arrow represents gluing the pairs of paths along identical middle paths $Q = P'$ and the second arrow represents summation over all possible gluings.

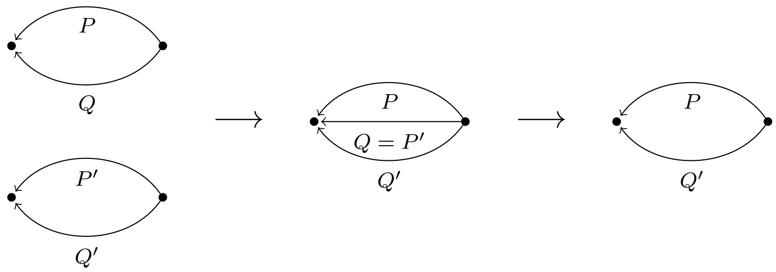


Figure 2.6: Multiplication in the Path Algebra

Example 2.8. By indexing matrix entries by pairs of paths in the Bratteli diagram for $M_n(\mathbb{C}) > \mathbb{C}$ (Figure 2.4), the path algebra multiplication is matrix multiplication.

Example 2.9. For the Bratteli diagram $\mathcal{B}$ of Figure 2.7 associated to the chain $\mathbb{C}[S_3] > \mathbb{C}[S_2] > \mathbb{C}[S_1] > \mathbb{C}$, let P_1, P_2, P_3, P_4 be the paths from the root to level 3 in $\mathcal{B}$, labeled from top to bottom. Then $\mathbb{C}[\mathcal{B}_3]$ has basis

$$\{(P_1, P_1), (P_2, P_2), (P_2, P_3), (P_3, P_2), (P_3, P_3), (P_4, P_4)\}.$$

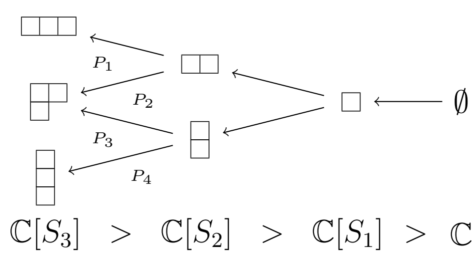


Figure 2.7: Paths P_1, P_2, P_3, P_4, with label on the last arrow of the path.

Note that for a vertex v, labeled by a representation ρ, the dimension of ρ is given by the number of paths from the root to v. Moreover, each path corresponds to a subgroup-equivariant embedding of $\mathbb{C}$ into the representation space of ρ (for more details, see Appendix A.1).

Further, $\mathbb{C}[\mathcal{B}_i]$ embeds into $\mathbb{C}[\mathcal{B}_{i+1}]$ as a subalgebra by mapping any pair of paths $(P, Q) \in \mathbb{C}[\mathcal{B}_i]$ to the sum

$$\sum_e (e \circ P, e \circ Q),$$

over all arrows e such that the tail of e is the head of P (equivalently, of Q), and $\circ$ denotes concatenation of paths.

Example 2.10. Let $\mathcal{B}$ be the Bratteli diagram of Example 2.9 associated to the chain $\mathbb{C}[S_3] > \mathbb{C}[S_2] > \mathbb{C}[S_1] > \mathbb{C}$ with paths P_1, P_2, P_3, P_4 from the root to level 3. Then $\mathbb{C}[\mathcal{B}_2]$ has basis $\{(p_1, p_1), (p_2, p_2)\}$ for p_1 and p_2 the paths of Figure 2.8 from the root to level 2 in $\mathcal{B}$. The basis element (p_1, p_1) sits inside $\mathbb{C}[\mathcal{B}_3]$ as $(P_1, P_1) + (P_2, P_2)$ and similarly p_2 corresponds with $(P_3, P_3) + (P_4, P_4)$. Together this determines an embedding $\mathbb{C}[\mathcal{B}_2] \hookrightarrow \mathbb{C}[\mathcal{B}_3]$.

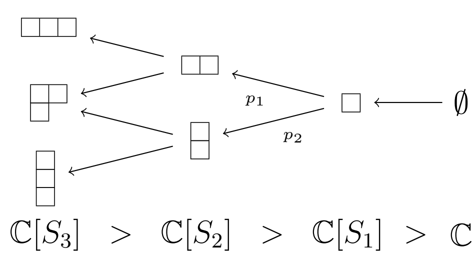


Figure 2.8: Paths p_1 and p_2, with label on the last arrow of path.

For a Bratteli diagram $\mathcal{B}$ with highest grading n, consider the chain of path algebras associated to $\mathcal{B}$:

$$\mathbb{C}[\mathcal{B}_n] > \mathbb{C}[\mathcal{B}_{n-1}] > \cdots > \mathbb{C}[\mathcal{B}_1] > \mathbb{C}[\mathcal{B}_0] = \mathbb{C}.$$

It is not too difficult to see that the Bratteli diagram associated to the chain of path algebras is $\mathcal{B}$.

For example, for $\mathbb{C}[\mathcal{B}_N] > \mathbb{C}[\mathcal{B}_0] = \mathbb{C}$ the path algebra chain of Example 2.6, letting pairs of paths index matrix entries yields a canonical representation $\mathbb{C}[\mathcal{B}_N] \to M_n(\mathbb{C})$, and under this identification the Bratteli diagram for the chain is exactly $\mathcal{B}$.

In fact:

Lemma 2.1. *Any chain of semisimple algebras is uniquely determined up to isomorphism by its Bratteli diagram.*

For further explanation see Appendix A and Section 2.3 of [39].

Remark 2.1. Quivers were first introduced by Gabriel in the study of modular representation theory [36]. Bratteli diagrams were first introduced to classify inductive limits of C^*-algebras [8]. After Elliot's use of Bratteli diagrams in the classification of AF-algebras [32], these ideas motivated a program to classify C^*-algebras in terms of their *K-theory* [82]. In terms of the representation theory of semisimple algebras, Bratteli diagrams have been of interest because those satisfying certain properties correspond to algebras satisfying the *Jones Basic Construction* [45, 46]. Bratteli diagrams have also been used to explicitly construct complete sets of irreducible representations that are analogs of *Young's seminormal form* in the symmetric group, and to describe restriction relations of representations [20, 43, 44, 56].

2.2.2 Adapted Representations and Gel'fand-Tsetlin Bases

As in [63, 65] we use adapted sets of bases to compute Fourier transforms efficiently.

Definition 2.9. *Given a group G with subgroup $H \leq G$, a complete set R of inequivalent irreducible matrix representations of G is* **H-*adapted*** *if there exists a complete set R_H of inequivalent irreducible matrix representations of H such that for all $\rho \in R$, $\rho \downarrow_H = \bigoplus \gamma_s$, for (not neccessarily distinct) representations γ_s in R_H. The set R is* ***adapted to the chain***

$$G = G_n > G_{n-1} > \cdots > G_0$$

if for each $1 \leq i \leq n$ there is a complete set R_i of inequivalent representations of G_i such that R_i is G_{i-1}-adapted and $R_n = R$. A set of bases for the representation spaces that give rise to adapted representations is an ***adapted basis****.*

The notion of an adapted basis coincides with that of a set, for each $1 \leq i \leq n$, of G_i-equivariant maps between the representation spaces of representations in R_i and those in R_{i+1}. For further details, see Appendix A.1.

For the FFT results of the following sections we assume the ability to construct adapted sets of representations. This requirement is not a limitation, as any set of representations is equivalent to an adapted set of representations. One such construction is outlined in [65].

The analogous concept in the path algebra associated to the group algebra chain under the isomorphism of Lemma 2.1 is a *system of Gel'fand-Tsetlin bases*:

Definition 2.10. *For a Bratteli diagram $\mathcal{B}$ associated to a chain of subalgebras of a semisimple algebra A, a* ***system of Gel'fand-Tsetlin bases for*** $\mathcal{B}$ *consists of a collection of bases for the representation spaces $\{V_\alpha |\ \alpha \in V(\mathcal{B})\}$ of the representations corresponding to α indexed by paths from the root to α, along with maps from the* *paths to the basis vectors; i.e., a set of basis vectors along with knowledge of the path corresponding to each vector.*

Example 2.11. Let $\mathcal{B}$ be the Bratteli diagram of Figure 2.5 associated to the chain $\mathbb{C}[S_3] > \mathbb{C}[S_2] > \mathbb{C}[S_1] > \mathbb{C}$. Then for the paths P_1, P_2, P_3, P_4 defined in Example 2.9, a basis $\{w_{P_2}, w_{P_3}\}$ for the two-dimensional representation space $V_{\yng(2,1)}$ is part of a system of Gel'fand-Tsetlin bases for $\mathcal{B}$. Note that the entries of the matrix of this representation are indexed by pairs $\{(w_{P_i}, w_{P_j}) \mid i, j = 1, 2\}$ and so correspond to basis elements of the path algebra $\mathbb{C}[\mathcal{B}_3]$.

Systems of Gel'fand-Tsetlin bases were originally developed by Gel'fand and Tsetlin to calculate the matrix coefficients of compact groups [37]. Clausen was the first to apply them to the efficient computation of Fourier transforms on finite groups [13].

In Remark A.1 of Appendix A.1, we show that systems of Gel'fand-Tsetlin bases for the chain of path algebras corresponding to a group algebra chain are equivalent to adapted bases for the chain of subgroups.

2.2.3 The Separation of Variables Idea

Gel'fand-Tsetlin bases provide a means to better understand the isomorphism of Lemma 2.1 between a chain of group algebras and the corresponding chain of path algebras. To see this, first note that the Fourier transform of a function f on G with respect to a complete set of inequivalent irreducible representations R of G is an algebra isomorphism

$$\mathbb{C}[G] \xrightarrow{\mathcal{F}_R} \bigoplus_{\rho \in R} M_{\dim(\rho)}(\mathbb{C}),$$

where

$$f \longleftrightarrow \sum_{s\in G} f(s)s \xrightarrow{\mathcal{F}_R} \bigoplus_{\rho\in R} \hat{f}(\rho) = \bigoplus_{\rho\in R} \sum_{s\in G} f(s)\rho(s).$$

Now for a chain of subgroups $G = G_n > G_{n-1} > \cdots > G_0 = \{e\}$, let $\mathcal{B}$ be the Bratteli diagram of the chain of subalgebras $\mathbb{C}[G_n] > \mathbb{C}[G_{n-1}] > \cdots > \mathbb{C}[G_0] = \mathbb{C}$, with corresponding chain of path algebras $\mathbb{C}[\mathcal{B}_n] > \mathbb{C}[\mathcal{B}_{n-1}] > \cdots > \mathbb{C}[\mathcal{B}_1] > \mathbb{C}$. Indexing the matrices $\rho(s)$ by paths of length n in the Bratteli diagram corresponding to a Gel'fand-Tsetlin basis for $\mathcal{B}$, the isomorphism above becomes

$$\sum_{s\in G} f(s)s \xrightarrow{\mathcal{F}_R} \bigoplus_{\rho\in R} \sum_{s\in G} f(s)\tilde{\rho}(s),$$

where $\tilde{\rho}$ represents the matrix ρ written with respect to a Gel'fand-Tsetlin basis for $\mathcal{B}$.

Since Gel'fand-Tsetlin bases are indexed by paths in $\mathcal{B}$ and a basis for the path algebra $\mathbb{C}[\mathcal{B}_n]$ consists of pairs of paths, we identify the group algebra $\mathbb{C}[G]$ with its realization in coordinates relative to the Gel'fand-Tsetlin basis, indexed by pairs of paths of length n in $\mathcal{B}$ that share the same endpoint. Then

$$\sum_{s\in G} f(s)\tilde{s} := \sum_{s\in G} f(s) \sum_{(P,Q)\in\mathbb{C}[\mathcal{B}_n]} [s]_{P,Q}(P,Q) \longleftrightarrow \sum_{s\in G} f(s)s,$$

and so

$$\sum_{s\in G} f(s)\tilde{s} \longleftrightarrow \sum_{s\in G} f(s)s \xrightarrow{\mathcal{F}_R} \bigoplus_{\rho\in R} \hat{f}(\rho) = \bigoplus_{\rho\in R} \sum_{s\in G} f(s)\tilde{\rho}(s).$$

In other words, as in [13, 63]:

Lemma 2.2. *For* $\mathbb{C}[G] = \mathbb{C}[G_n] > \mathbb{C}[G_{n-1}] > \cdots > \mathbb{C}[G_1] > \mathbb{C}[G_0] = \mathbb{C}$ *a chain of* *subalgebras for* $\mathbb{C}[G]$ *with Bratteli diagram* $\mathcal{B}$*, and* R *a complete set of inequivalent irreducible representations of* G *adapted to the chain of subgroups* $G_n > G_{n-1} > \cdots > G_0$*, the computation of the Fourier transform of a function* f *on* G *with respect to* R *is equivalent to computation of*

$$\sum_{s \in G} f(s)s$$

in the group algebra, relative to the Gel'fand-Tsetlin basis for $\mathcal{B}$ *associated to* R*. Identifying the group algebra with its realization in coordinates with respect to this basis, this computation is equivalent to computation of*

$$\sum_{s \in G} f(s)\tilde{s}$$

in the path algebra $\mathbb{C}[\mathcal{B}_n]$*.*

Example 2.12. Young's orthogonal form gives an example of a complete set of irreducible matrix representations for S_n adapted to the chain $S_n > S_{n-1} > \cdots > S_1$. Since restriction of representations from S_n to S_{n-1} is multiplicity-free, the basis vectors of a system of Gel'fand-Tsetlin bases for the irreducible representations relative to this chain are determined up to scalar multiplies, and in the case of $n = 3$, the paths are the paths P_1, P_2, P_3, P_4 of Example 2.9. In [63], Maslen gives an efficient algorithm for computation of the Fourier transform of a function on S_n by considering the computation of $\sum_{s \in S_n} f(s)s$ in the group algebra for S_n relative to this Gel'fand-Tsetlin basis.

One motivation for Lemma 2.2 is that translation of matrix sums into path algebra elements takes advantage of the direct sum structure of these matrices in that the matrix elements corresponding to paths ending at different vertices in the Bratteli diagram (and thus zero) are ignored in the path algebra. For matrices with a more structured form, e.g., those lying in subalgebras or centralizer algebras, the nonzero entries are indexed by specific pairs of paths in the Bratteli diagram. Further, recall that the definition of multiplication in the path algebra corresponds to gluing and summing operations on quivers. Then to compute a product of elements in the path algebra, we glue together subquivers corresponding to each element in the product and find that the complexity estimates can be computed in terms of counts of the number of such configurations in the Bratteli diagram. This was done in [63] for the symmetric group case. Our translation to the path algebra and a reframing in terms of quivers and configuration spaces enables an extension to arbitrary groups and semisimple algebras.

Key to this is the introduction in Section 2.3 of a diagrammatic technique of gluing and summing quivers to keep track of the nonzero entries.

The first steps of the SOV approach involve expressing a path algebra element as a factorization over subsets of the Bratteli diagram in such a way as to disentangle the dependencies in the sum $\hat{f}$. To do so we first factor the Fourier transform through the subalgebras $\mathbb{C}[G_i]$. We illustrate with an example before stating the general result.

Example 2.13. To compute $\sum_{s \in S_3} f(s)s$, consider the set $Y = \{e, t_1t_2, t_2\} \subseteq S_3$, for t_i the simple transposition $(i\; i{+}1)$. This is a complete set of left coset representatives for S_3/S_2. In particular, $S_3 = \{ee, et_1, t_1t_2e, t_1t_2t_1, t_2e, t_2t_1\}$. The reason for keeping the identity element e in the factorizations will become apparent. Then

$$\begin{aligned}\sum_{s\in S_3} f(s)s &= f(ee)ee + f(et_1)et_1 + f(t_1t_2e)t_1t_2e + \\ & \quad f(t_1t_2t_1)t_1t_2t_1 + f(t_2e)t_2e + f(t_2t_1)t_2t_1 \\ &= \sum_{y\in Y} y \sum_{s\in S_2} f(ys)s.\end{aligned}$$

Thus, the Fourier transform of f on $\mathbb{C}[S_3]$ becomes a linear combination of Fourier transforms of f on $\mathbb{C}[S_2]$.

For $G = S_n$ a set of coset representatives for S_n/S_{n-1} is

$$Y = \{e, t_1 \cdots t_{n-1}, t_2 \cdots t_{n-1}, \ldots, t_{n-1}\}$$

and the same argument as above can be used in this general case.

We extend this idea to a general group G given a subgroup H and a set of coset representatives Y for G/H.

Recall that $T_G(R)$ is defined to be the minimum number of operations to compute the Fourier transform of a function f on G with respect to a set of representations R of G. Recall also that

$$t_G(R) := \frac{1}{|G|} T_G(R).$$

For $H < G$, let $\mathcal{B}$ be the Bratteli diagram of the group algebra chain $\mathbb{C}[G] > \mathbb{C}[H] > \mathbb{C}$, with corresponding path algebra chain $\mathbb{C}[\mathcal{B}_G] > \mathbb{C}[\mathcal{B}_H] > \mathbb{C}$. For Y a set of coset representatives for G/H and F_y arbitrary elements of $\mathbb{C}[\mathcal{B}_H]$, let

$$m_G(R, Y, H) = \frac{1}{|G|} \times \begin{cases} \text{minimum number of operations required to compute} \\ \sum_{y\in Y} \tilde{y} F_y \text{ in a system of Gel'fand-Tsetlin bases for } \mathcal{B} \end{cases}$$

Lemma 2.3 is a restatement of Lemma 2.10 of [63] and Proposition 1 of [26].

Lemma 2.3. *Let H be a subgroup of G, R a complete H-adapted set of inequivalent irreducible matrix representations of G, and $Y \subseteq G$ a set of coset representatives for G/H. Let $\mathcal{B}$ be the Bratteli diagram of the group algebra chain $\mathbb{C}[G] > \mathbb{C}[H] > \mathbb{C}$, with corresponding path algebra chain $\mathbb{C}[\mathcal{B}_G] > \mathbb{C}[\mathcal{B}_H] > \mathbb{C}$. Then*

$$t_G(R) \leq t_H(R_H) + m_G(R, Y, H).$$

Proof. By Lemma 2.2, computation of the Fourier transform of a function f on G at R is equivalent to computation of $\mathcal{F} := \sum_{s \in G} f(s)\tilde{s}$ in $\mathbb{C}[\mathcal{B}_G]$ by expressing group algebra elements in coordinates relative to a Gel'fand-Tsetlin basis for $\mathcal{B}$. Let H be a subgroup of G and $Y \subseteq G$ a set of coset representatives for G/H. Then

$$\mathcal{F} := \sum_{s \in G} f(s)\tilde{s} = \sum_{y \in Y} \sum_{h \in H} f(yh)\tilde{y}\tilde{h} = \sum_{y \in Y} \tilde{y} \sum_{h \in H} f(yh)\tilde{h}$$

$$= \sum_{y \in Y} \tilde{y} F_y, \tag{2.2}$$

where for each $y \in Y$,

$$F_y = \sum_{h \in H} f_y(h)\tilde{h} \in \mathbb{C}[\mathcal{B}_H]$$

with $f_y(h) := f(yh)$. Then to compute $\mathcal{F}$, first compute $F_y \in \mathbb{C}[\mathcal{B}_{n-1}]$ for all $y \in Y$ relative to a system of Gel'fand-Tsetlin bases for the chain $\mathbb{C}[B_H] > \mathbb{C}$ corresponding to R_H, by means of a Fourier transform on H. This requires at most $\frac{|G|}{|H|} T_H(R_H)$ scalar operations. Next, express the elements F_y in coordinates relative to a system of Gel'fand-Tsetlin bases for the path algebra chain $\mathbb{C}[\mathcal{B}_G] > \mathbb{C}[\mathcal{B}_H] > \mathbb{C}$ corresponding to R, which requires no operations. Finally, compute $\mathcal{F}$ using (2.2), which requires at most $|G|m_G(R, Y, H)$ operations. Thus,

$$T_G(R) \leq \frac{|G|}{|H|} T_H(R_H) + |G| m_G(R, Y, H),$$

and dividing by $|G|$ proves the lemma. □

By Lemma 2.3, to compute the Fourier transform of a complex function at a set of H-adapted representations, we need to compute

$$\mathcal{F}_Y = \sum_{y \in Y} \tilde{y} F_y,$$

for Y a set of coset representatives for G/H. Then to compute the Fourier transform of a complex function at a set of representations R adapted to a chain $G = G_n > G_{n-1} > \cdots > G_0 = e$, let Y_i be a set of coset representatives for G_i/G_{i-1}. Iteration of Lemma 2.3 gives

$$t_G(R) \leq t_{G_0}(R_{G_0}) + \sum_{i=1}^{n} M_{G_i}(R_{G_i}, Y, G_{i-1}). \tag{2.3}$$

In Section 2.3 we detail an approach for computing $\mathcal{F}_{Y_i}$. Applying this approach yields the following theorems, proved in Section 2.4:

Theorem 2.2. *For the Weyl group B_n and R a complete set of irreducible matrix representations of B_n adapted to the subgroup chain $B_n > B_{n-1} > \cdots > B_0 = \{e\}$,*

$$C(B_n) \leq T_{B_n}(R) \leq n(2n-1)|B_n|.$$

Theorem 2.3. *For the Weyl group D_n and R a complete set of irreducible matrix* *representations of D_n adapted to the subgroup chain $D_n > D_{n-1} > \cdots > D_0 = \{e\}$,*

$$C(D_n) \leq T_{D_n}(R) \leq \frac{n(13n-11)}{2}|D_n|.$$

Theorem 2.4. *For the matrix group $Gl_n(\mathbb{F}_q)$ and R a complete set of irreducible matrix representations of $Gl_n(\mathbb{F}_q)$ adapted to the subgroup chain $Gl_n(\mathbb{F}_q) > Gl_{n-1}(\mathbb{F}_q) > \cdots > \{e\}$*

$$C(Gl_n(\mathbb{F}_q)) \leq T_{Gl_n(\mathbb{F}_q)}(R) = O(q^n)|Gl_n(\mathbb{F}_q)|.$$

We also consider the case of Fourier transforms on homogeneous spaces, producing results such as:

Theorem 2.5. *For the homogenous space B_n/B_{n-k} of the Weyl group B_n and R a complete set of irreducible matrix representations of B_n adapted to the subgroup chain $B_n > B_{n-1} > \cdots > \{e\}$,*

$$C(B_n/B_{n-k}) \leq T_{B_n/B_{n-k}}(R) \leq k(4n-2k-1)\frac{|B_n|}{|B_{n-k}|}.$$

2.3 The Separation of Variables Approach

The SOV approach is comprised of three main steps:

1 Use Lemma 2.3 and Equation (2.3) to reduce computation of a Fourier transform to computation of $\mathcal{F}_Y = \sum_{y\in Y} \tilde{y}F_y$ in the path algebra $\mathbb{C}[\mathcal{B}_i]$, for Y a set of coset representatives for G_i/G_{i-1}.

2 Rearrange $\mathcal{F}_Y$ to be a recursively structured summation.

3 Use diagrammatic techniques to keep track of nonzero entries and give complexity counts.

We discussed Step 1 in Section 2.2.3. For Step 2, we develop a method to recursively structure elements of the path algebra of the form $\sum_{z \in Z \subseteq G} \tilde{z}$. Let $\tilde{Z} = \{\tilde{z} \mid z \in Z\}$. We begin with a preliminary algorithm to compute $\sum_{\tilde{z} \in \tilde{Z}} \tilde{z}$.

Algorithm 2.1.

I. Choose $m \in \mathbb{N}$ and a subset $X \subseteq (\mathbb{C}[\mathcal{B}_n])^m = \mathbb{C}[\mathcal{B}_n] \times \cdots \times \mathbb{C}[\mathcal{B}_n]$ such that $|X| = |\tilde{Z}|$ and

$$\tilde{Z} = \{x_1 \cdots x_m | (x_1, \ldots, x_m) \in X\}.$$

Thus, X is a choice of factorization for each element of $\tilde{Z}$.

II. Let

$$X_i = \{(x_{i+1}, \ldots, x_m) | (x_1, \ldots, x_m) \in X\}, \quad 0 \leq i \leq m,$$
$$X_m = \emptyset.$$

III. Define a sequence of functions $L_i : X_i \to \mathbb{C}[\mathcal{B}_n]$ recursively by:

$$L_1(x_2, \ldots, x_m) = \sum_{(x_1, x_2, \ldots, x_m) \in X_0} x_1,$$
$$L_2(x_3, \ldots, x_m) = \sum_{(x_2, x_3, \ldots, x_m) \in X_1} L_1(x_2, \ldots, x_m) * x_2.$$
$$L_i(x_{i+1}, \ldots, x_m) = \sum_{(x_i, x_{i+1}, \ldots, x_m) \in X_{i-1}} L_{i-1}(x_i, \ldots, x_m) * x_i.$$

Induction shows that $L_m(\emptyset) = \sum_{z \in Z} \tilde{z}$.

Example 2.14. Let $G = S_3 = \{e, t_1, t_2, t_1t_2, t_2t_1, t_1t_2t_1\}$ and let $\mathcal{B}$ be the Bratteli diagram corresponding to the chain $\mathbb{C}[S_3] > \mathbb{C}[S_2] > \mathbb{C}[S_1] > \mathbb{C}$, with corresponding path algebra chain $\mathbb{C}[\mathcal{B}_3] > \mathbb{C}[\mathcal{B}_2] > \mathbb{C}[\mathcal{B}_1] > \mathbb{C}$. Factor the elements of S_3 as

$$S_3 = \{eee, t_1ee, et_2e, t_1t_2e, et_2t_1, t_1t_2t_1\}.$$

Let $Z = S_3$. To compute $\sum_{z\in Z} \tilde{z} \in \mathbb{C}[\mathcal{B}_3]$, let

$$X = \{(\tilde{e},\tilde{e},\tilde{e}), (\tilde{t}_1,\tilde{e},\tilde{e}), (\tilde{e},\tilde{t}_2,\tilde{e}), (\tilde{t}_1,\tilde{t}_2,\tilde{e}), (\tilde{e},\tilde{t}_2,\tilde{t}_1), (\tilde{t}_1,\tilde{t}_2,\tilde{t}_1)\}.$$

Then

$$\begin{aligned}
X_1 &= \{(\tilde{e},\tilde{e}), (\tilde{t}_2,\tilde{e}), (\tilde{t}_2,\tilde{t}_1)\},\\
X_2 &= \{(\tilde{e}), (\tilde{t}_1)\},\\
X_3 &= \emptyset,
\end{aligned}$$

and

$$\begin{aligned}
&L_1(\tilde{e},\tilde{e}) = L_1(\tilde{t}_2,\tilde{e}) = L_1(\tilde{t}_2,\tilde{t}_1) = \tilde{e} + \tilde{t}_1,\\
&L_2(\tilde{e}) = L_1(\tilde{e},\tilde{e})\tilde{e} + L_1(\tilde{t}_1,\tilde{e})\tilde{t}_1 = \tilde{e}\tilde{e} + \tilde{t}_1\tilde{e} + \tilde{e}\tilde{t}_2 + \tilde{t}_1\tilde{t}_2,\\
&L_2(\tilde{t}_1) = L_1(\tilde{t}_2,\tilde{t}_1)\tilde{t}_2 = \tilde{e}\tilde{t}_2 + \tilde{t}_1\tilde{t}_2,\\
&L_3(\emptyset) = L_2(\tilde{e})\tilde{e} + L_2(\tilde{t}_1)\tilde{t}_1 = \tilde{e}\tilde{e}\tilde{e} + \tilde{t}_1\tilde{e}\tilde{e} + \tilde{e}\tilde{t}_2\tilde{e} + \tilde{t}_1\tilde{t}_2\tilde{e} + \tilde{e}\tilde{t}_2\tilde{t}_1 + \tilde{t}_1\tilde{t}_2\tilde{t}_1 = \sum_{z\in Z}\tilde{z}.
\end{aligned}$$

Algorithm 2.1 is the basis for Step 2 of the SOV approach. It provides a strategy for recursively structuring the path algebra element $\sum_{z\in Z}\tilde{z}$ as a summation of products, using factorizations of $\tilde{z}$.

We refine this approach further by recalling that the purpose of our translation of matrix sums into path algebra elements was to take advantage of the fact that matrices with a more structured form (e.g., those lying in subalgebras or centralizer algebras) have nonzero entries indexed by specific paths in the Bratteli diagram. By using factorizations of $\tilde{z}$ that take advantage of this structure, we develop an efficient algorithm for the computation. To do so, for a factorization $\tilde{z} = x_1 x_2 \cdots x_m$, we keep track of the space in which each element x_i lies. Further, we translate multiplication into a bilinear map on these vector spaces, the complexity of which is easily computed.

To give the general idea, recall that the definition of multiplication in the path algebra (Section 2.2.1) corresponds to the natural gluing and summing operations of Figure 2.9, where the first arrow represents gluing the pairs of paths along identical middle paths $Q = P'$ and the second arrow represents summation over all possible gluings.

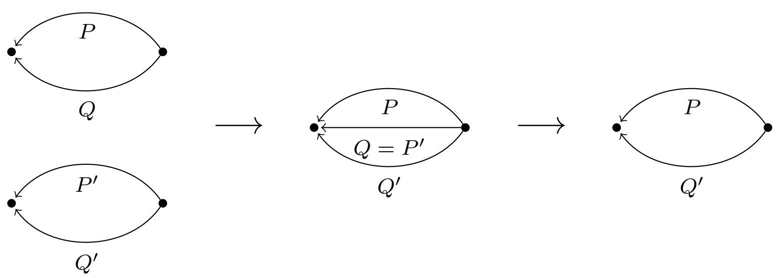


Figure 2.9: Multiplication in the Path Algebra

Then to compute a product of elements in the path algebra, we form a quiver $\mathcal{Q}$ by gluing together quivers that correspond to each element in the product. Summing over all possible gluings amounts to counting the occurrences of $\mathcal{Q}$ in the corresponding Bratteli diagram $\mathcal{B}$. We do so by defining the space of maps from $\mathcal{Q}$ into $\mathcal{B}$, as the dimension of this space is the number of occurrences of $\mathcal{Q}$ in $\mathcal{B}$.

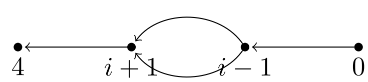
Example 2.15. Let $G = S_4$ and consider the subalgebra chain $\mathbb{C}[S_4] > \mathbb{C}[S_3] >$

$\mathbb{C}[S_2] > \mathbb{C}[S_1] > \mathbb{C}[S_0] = \mathbb{C}$ with Bratteli diagram $\mathcal{B}$ and corresponding path algebra chain $\mathbb{C}[\mathcal{B}_4] > \mathbb{C}[\mathcal{B}_3] > \mathbb{C}[\mathcal{B}_2] > \mathbb{C}[\mathcal{B}_1] > \mathbb{C}[\mathcal{B}_0]$. Let $Y = \{e, s_1s_2s_3, s_2s_3, s_3\}$, a set of coset representatives for S_4/S_3. Then by Lemma 2.2.3 and an extension of Example 2.13, to compute $\sum_{s\in S_4} f(s)\tilde{s}$ we need to compute

$$\sum_{y\in Y} \tilde{y}F_y = \tilde{e}F_e + \tilde{t}_1\tilde{t}_2\tilde{t}_3F_{t_1t_2t_3} + \tilde{t}_2\tilde{t}_3F_{t_2t_3} + \tilde{t}_3F_{t_3}.$$

For the purposes of this example, consider the single term $\tilde{t}_1\tilde{t}_2\tilde{t}_3F_{t_1t_2t_3}$.

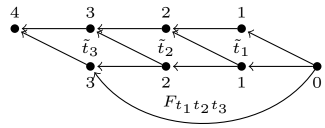
Note that $t_i \in \mathbb{C}[S_{i+1}] \cap \text{Centralizer}\,\mathbb{C}[S_{i-1}]$. Identifing the group algebra with the path algebra by expressing t_i in coordinates, $\tilde{t}_i := \sum_{(P,Q)}[t_i]_{PQ}(P,Q)$, an application of Schur's lemma and standard facts about Gel'fand-Tsetlin bases show that $[t_i]_{P,Q}$ is 0 unless P and Q are paths in $\mathcal{B}$ that agree from level 4 to level $i+1$, and from level $i-1$ to level 0. Thus, $\tilde{t}_i \in \mathbb{C}[\mathcal{B}_{i+1}] \cap \text{Centralizer}\,\mathbb{C}[\mathcal{B}_{i-1}]$, isomorphic to the space of maps from the quiver Q_i of Figure 2.10 into $\mathcal{B}$. For further details, see Lemma 2.4.



Q_i

Figure 2.10

Moreover, multiplication in the path algebra corresponds to gluing paths, and so the product $\tilde{t}_1\tilde{t}_2\tilde{t}_3F_{t_1t_2t_3}$ lies in a space isomorphic to the space of maps from the quiver $\mathcal{Q}$ of Figure 2.11 into $\mathcal{B}$.



$\mathcal{Q}$

Figure 2.11

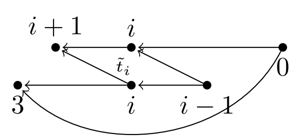
However, determining the maps from $\mathcal{Q}$ into $\mathcal{B}$ (i.e., the number of occurrences of $\mathcal{Q}$ as a subquiver of $\mathcal{B}$) can be complicated. The key is to instead use the recursive sum strategy of Algorithm 2.1 in order to consider each multiplication individually.

We rewrite the product as individual gluing operations

$$(((F_{t_1t_2t_3} * \tilde{t}_1) * \tilde{t}_2) * \tilde{t}_3),$$

where $w_1 * w_2$ represents gluing together the subquivers of $\mathcal{Q}$ corresponding to w_1 and w_2. Note that gluing $F_{t_1t_2t_3}$ to $\tilde{t}_1$ corresponds to the subquiver $\mathcal{Q}_1$ of $\mathcal{Q}$ (see Figure 2.12), while gluing $(F_{t_1t_2t_4} * \tilde{t}_1)$ to $\tilde{t}_2$ corresponds to the subquiver $\mathcal{Q}_2$ of $\mathcal{Q}$, and gluing $((F_{t_1t_2t_4} * \tilde{t}_1) * \tilde{t}_2)$ to $\tilde{t}_3$ corresponds to the subquiver $\mathcal{Q}_3$.

The complexity of the algorithm will result from counting the number of occurrences of the subquivers $\mathcal{Q}_i$ in the Bratteli diagram $\mathcal{B}$.



$\mathcal{Q}_i$

Figure 2.12

In general, to compute $\sum_{\tilde{z}\in\tilde{Z}} \tilde{z}$, we proceed as in Example 2.15 by factoring each $\tilde{z} = x_1 \cdots x_m$, determining the corresponding spaces W_i such that $x_i \in W_i$, and defining bilinear maps $*$ corresponding to gluing.

Algorithm 2.2.

I. *Choose* $m \in \mathbb{N}$ *and a subset* $X \subseteq (\mathbb{C}[\mathcal{B}_n])^m = \mathbb{C}[\mathcal{B}_n] \times \cdots \mathbb{C}[\mathcal{B}_n]$ *such that* $|X| = |\tilde{Z}|$ *and*

$$\tilde{Z} = \{x_1 \cdots x_m | (x_1, \ldots, x_m) \in X\}.$$

Thus, X *can be thought of as a choice of factorization into at most* m *elements of each element of* $\tilde{Z}$ *.*

II. *For* $\sigma \in S_m$*, let* $w_i = x_{\sigma(i)}$*. For* $1 \leq i \leq m$ *let* W_i *be a vector space containing* w_i *and for* $2 \leq j \leq m$ *let* $*_j$ *be a bilinear map such that*

$$x_1 \cdots x_m = (((w_1 *_2 w_2) *_3 w_3) \cdots *_m w_m).$$

III. *Let*

$$X_i = \{(w_{i+1}, \ldots, w_m) \mid (x_1, \ldots, x_m) \in X\} \subseteq W_{i+1} \times \cdots \times W_m, \quad 0 \leq i \leq m,$$
$$X_m = \emptyset.$$

IV. *Define a sequence of functions* L_i *recursively by:*

$$L_1(w_2, \ldots, w_m) = \sum_{(w_1, w_2, \ldots, w_m) \in X_0} w_1,$$
$$L_i(w_{i+1}, \ldots, w_m) = \sum_{(w_i, w_{i+1}, \ldots, w_m) \in X_{i-1}} (L_{i-1}(w_i, \ldots, w_m) *_i w_i).$$

Theorem 2.6. *For L_i as defined above,*

$$L_m := L_m(\emptyset) = \sum_{(w_1,\ldots,w_m)\in X_0} (((w_1 *_2 w_2) *_3 w_3) \cdots *_m w_m) = \sum_{\tilde{z}\in\tilde{Z}} z$$

Proof. First note that

$$\sum_{\tilde{z}\in\tilde{Z}} \tilde{z} = \sum_{(x_1,\ldots,x_m)\in X} x_1 \cdots x_m = \sum_{(w_1,\ldots,w_m)\in X_0} (((w_1 *_2 w_2) *_3 w_3) \cdots *_m w_m).$$

Then induction gives that

$$\begin{aligned} &\sum_{(w_1,\ldots,w_m)\in X_0} (((w_1 *_2 w_2) *_3 w_3) \cdots *_m w_m) \\ = &\sum_{(w_i,\ldots,w_m)\in X_{i-1}} (((L_{i-1}(w_i,\ldots,w_m) *_i w_i) *_{i+1} w_{i+1}) \cdots *_m w_m), \end{aligned}$$

so for $i = m$ we have the theorem statement. □

Note 2.1. When $\sigma = e$, i.e. $w_i = x_i$, and the bilinear map is multiplication, i.e. $(w_{i-1} *_i w_i) = w_{i-1}w_i$, Theorem 2.6 is exactly the conclusion of Algorithm 2.1, illustrated by Example 2.14. The bilinear maps we define are essentially multiplication of matrix entries, but will take advantage of sparse matrix structure.

In the remaining parts of Section 2.3 we flesh out Algorithm 2.2 by constructing the required spaces W_i and bilinear maps $*_j$. As elements of subalgebras and centralizer algebras have nonzero entries indexed by specific paths in the Bratteli diagram (see Example 2.15), these spaces will be intersections of such algebras.

2.3.1 The Spaces W_i

Let $\mathcal{B}$ be a Bratteli diagram with highest grading at least n and for $x_1, \ldots, x_m \in \mathbb{C}[\mathcal{B}_n]$, consider the product $x_1 \cdots x_m$.

Definition 2.11. *For a path algebra product $x_1 \cdots x_m$, Let i^+ denote the smallest integer such that $x_i \in \mathbb{C}[\mathcal{B}_{i^+}]$ and let i^- denote the largest integer less than or equal to i^+ such that $x_i \in \text{Centralizer}(\mathbb{C}[\mathcal{B}_{i^-}])$. Then for $1 \leq i \leq m$ define*

$$W_i := \mathbb{C}[\mathcal{B}_{i^+}] \cap \text{Centralizer}(\mathbb{C}[\mathcal{B}_{i^-}]).$$

Note that $x_i \in W_i$ and the product $x_1 \cdots x_m$ is a multilinear map

$$W_1 \times \cdots \times W_m \to \mathbb{C}[\mathcal{B}_n].$$

Example 2.16. Let $\mathcal{B}$ be the Bratteli diagram for the chain of symmetric group algebras $\mathbb{C}[S_4] > \mathbb{C}[S_3] > \mathbb{C}[S_2] > \mathbb{C}[S_1] > \mathbb{C}$ and let $\mathbb{C}[\mathcal{B}_4] > \mathbb{C}[\mathcal{B}_3] > \mathbb{C}[\mathcal{B}_2] > \mathbb{C}[\mathcal{B}_1] > \mathbb{C}$ be the corresponding chain of path algebras.

Consider the product $x_1x_2x_3x_4 = \tilde{t}_1\tilde{t}_2\tilde{t}_3F_{t_1t_2t_3}$ of Example 2.15, where t_i is the transposition $(i\ i+1)$ and $F_{t_1t_2t_3} \in \mathbb{C}[\mathcal{B}_3]$. Since $\tilde{t}_i \in \mathbb{C}[\mathcal{B}_{i+1}] \cap \text{Centralizer}(\mathbb{C}[\mathcal{B}_{i-1}])$, we have that

$$1^+ = 2, \quad 1^- = 0,$$
$$2^+ = 3, \quad 2^- = 1,$$
$$3^+ = 4, \quad 3^- = 2.$$

Finally, we know that $x_4 = F_{t_1t_2t_3}$ is an arbitrary element of $\mathbb{C}[\mathcal{B}_3]$, implying that $4^+ = 3$ and $4^- = 0$.

We will show (Lemma 2.4 below), that each space W_i is isomorphic to the *config-*

uration space of a specific quiver.

For an arrow e of a quiver Q, let $e^+ \in V(Q)$ denote the head of e and e^- the tail.

Definition 2.12. *For graded quivers Q and B, a **morphism** $\phi : Q \to B$ is a mapping from arrows in Q to paths in B, along with a grading-preserving mapping between vertices so that $\phi(e^+) = \phi(e)^+$ and $\phi(e^-) = \phi(e)^-$ for all arrows $e \in E(Q)$.*

Example 2.17. For Q, B as in Figure 2.13, let $\phi : Q \to B$ send the arrow e_1 to the path $f_3 \circ f_2 \circ f_1$.

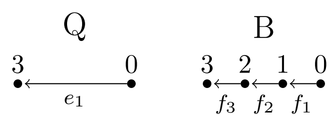


Figure 2.13

For two graded quivers Q and B, let $\mathrm{Hom}(Q; B)$ denote the set of morphisms from Q to B. For Q, R, and B graded quivers such that Q is a subquiver of R, let $\mathrm{Hom}(Q \uparrow R; B)$ denote the set of morphisms from Q to B that extend to R.

Definition 2.13. *The **configuration space** associated to Q and R relative to B, denoted $A(Q \uparrow R; B)$, is the space of finitely supported formal $\mathbb{C}$-linear combinations of morphisms in $\mathrm{Hom}(Q \uparrow R; B)$.*

Note 2.2.

1. When $Q = R$, we simplify notation by writing $A(Q; B)$.

2. If Q is a finite subquiver of R and B is locally finite, i.e. each vertex has finitely many neighbors, then $\#\,\mathrm{Hom}(Q \uparrow R; B) = \dim A(Q \uparrow R; B)$.

Certain configuration spaces are naturally isomorphic to the spaces W_i of definition 2.11:

Lemma 2.4. *Let $\{A_i\}$ be a chain of subalgebras of a semisimple algebra A with corresponding Bratteli diagram $\mathcal{B}$ of highest grading at least n. Consider the quivers Q_{n0} and Q^n_{ji} of Figure 2.14, along with the subquiver Q_{ji} of Q^n_{ji} consisting of the two vertices at level i and level j, along with the two paths from level i to level j:*

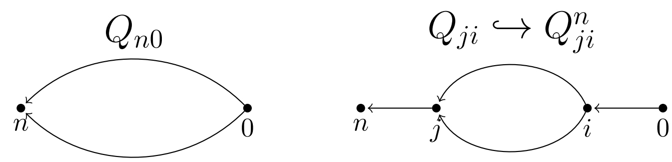


Figure 2.14

Then as vector spaces,

(i) $A(Q_{n0};\mathcal{B}) \cong \mathbb{C}[\mathcal{B}_n] \cong A_n$,

(ii) $A(Q_{ji};\mathcal{B}) = A(Q_{ji} \uparrow Q^n_{ji};\mathcal{B}) \cong \mathbb{C}[\mathcal{B}_j] \cap \text{Centralizer}(\mathbb{C}[\mathcal{B}_i]) \cong A_j \cap \text{Centralizer}(A_i)$.

Proof. (i) follows from the proof of (ii). Alternatively, note that morphisms $\phi : Q_{n0} \to \mathcal{B}$ correspond to pairs of paths of length n in $\mathcal{B}$ starting at the root, $\hat{0}$, and ending at the same vertex, i.e., basis elements of the path algebra $\mathbb{C}[\mathcal{B}_n]$.

To prove (ii), note that morphisms $\phi : Q_{ji} \to B$ correspond to pairs of paths in $\mathcal{B}$ starting at the same vertex at level i and ending at the same vertex at level j.

Let $w_{ji} \in W_{ji} := A_j \cap \text{Centralizer}(A_i)$. Recall that a pair of paths in $\mathcal{B}$,

$$P = p_n \leftarrow p_{n-1} \leftarrow \cdots \leftarrow p_1 \leftarrow \hat{0},$$

$$Q = q_n \leftarrow q_{n-1} \leftarrow \cdots \leftarrow q_1 \leftarrow \hat{0},$$

with $p_n = q_n$ determines an element of the Gel'fand-Tsetlin basis for $\mathbb{C}[\mathcal{B}_n]$ corresponding to the path algebra chain. Represent w_{ji} in its coordinates relative to the Gel'fand-Tsetlin basis by $[w_{ji}]_{PQ}$. Using standard facts about Gel'fand-Tsetlin bases (see eg. [63, Lemma 4.1], [39, Proposition 2.3.12]),

$$[w_{ji}]_{PQ} = \delta_{p_{n-1}q_{n-1}} \cdots \delta_{p_j q_j} P^{ji}_{w_{ji}} \delta_{p_i q_i} \cdots \delta_{p_1 q_1},$$

for $P^{ji}_{w_{ji}}$ a complex function on the vertices $p_j, \ldots p_i$, $q_j, \ldots, q_i$ in $\mathcal{B}$. Thus, under the map

$$w_{ji} \to P^{ji}_{w_{ji}},$$

the space W_{ji} is isomorphic to the space of complex functions on vertices $p_j, \ldots p_i$, $q_j, \ldots, q_i$ of $\mathcal{B}$ that form paths in $\mathcal{B}$ with $p_j = q_j$ and $p_i = q_i$.

□

By Lemma 2.4, for the spaces $W_i := \mathbb{C}[\mathcal{B}_{i^+}] \cap \text{Centralizer}(\mathbb{C}[\mathcal{B}_{i^-}])$ defined above,

$$W_i \cong A(Q_{i^+i^-}; \mathcal{B}).$$

2.3.2 The Bilinear Maps $*$

To understand multiplication of elements $w_i \in W_i$, we first determine (Theorem 2.7) that the bilinear map on $A(Q_{i^+i^-}; \mathcal{B})$ corresponding to path algebra multiplication is $\iota^* \circ \phi_* \circ \bigotimes$, for ι and ϕ natural inclusions, and ι^*, ϕ_*, and $\bigotimes$ defined below.

For B a graded quiver, $\iota : Q_1 \to Q_2$ an inclusion of graded quivers, and $\mu \in$

$\mathrm{Hom}(Q_2; B)$, let $\mu\downarrow_{Q_1}$ denote $\mu \circ \iota \in \mathrm{Hom}(Q_1; B)$.

Definition 2.14. *For B a graded quiver and $\iota : Q_1 \to Q_2$ an inclusion of graded quivers, define $\iota^* : A(Q_2; B) \to A(Q_1; B)$ by linearly extending the map that sends $\mu \in \mathrm{Hom}(Q_2; B)$ to $\mu\downarrow_{Q_1}$.*

For Q_1 a subquiver of R_1, Q_2 a subquiver of R_2, and $\iota : Q_1 \to Q_2$ an inclusion that extends to an inclusion from R_1 to R_2, analogously define $\iota^* : A(Q_2 \uparrow R_2; B) \to A(Q_1 \uparrow R_1; B)$.

Definition 2.15. *For B a locally finite quiver, Q_2 a finite quiver, and $\iota : Q_1 \to Q_2$ an inclusion of graded quivers, define $\iota_* : A(Q_1; B) \to A(Q_2; B)$ by linearly extending the map that sends $\tau \in \mathrm{Hom}(Q_1; B)$ to $\displaystyle\sum_{\mu\downarrow_{Q_1}=\tau} \mu$.*

As above, this definition may be extended to define $\iota_* : A(Q_1 \uparrow R_1; B) \to A(Q_2 \uparrow R_2; B)$.

Example 2.18. Consider the quivers Q_1, Q_2, B of Figure 2.15 and the morphism $\phi : Q_1 \to Q_2$ that sends q_1 to p_1. Note that $\mathrm{Hom}(Q_1; B) = \{G_1, G_2\}$, where $G_1(q_1) = s_1$ and $G_2(q_1) = s_2$. Further, $\mathrm{Hom}(Q_2; B) = \{H_1, H_2, H_3, H_4\}$, for

$$\begin{aligned} H_1(p_1) &= s_1, \quad H_1(p_2) = s_3, \\ H_2(p_1) &= s_1, \quad H_2(p_2) = s_4, \\ H_3(p_1) &= s_2, \quad H_3(p_2) = s_3, \\ H_4(p_1) &= s_2, \quad H_4(p_2) = s_4. \end{aligned}$$

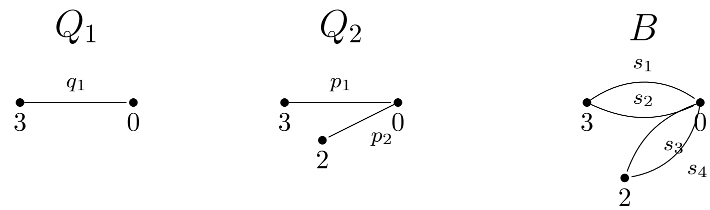


Figure 2.15

Then for $f = \sum_{H_i} f|_{H_i} H_i \in A(Q_2; B)$,

$$\phi^*(f) = f|_{H_1} G_1 + f|_{H_2} G_1 + f|_{H_3} G_2 + f|_{H_4} G_2.$$

Similarly, for $g = \sum_{G_i} f|_{G_i} G_i \in A(Q_1; B)$,

$$\phi_*(g) = g|_{G_1} H_1 + g|_{G_1} H_2 + g|_{G_2} H_3 + g|_{G_2} H_4.$$

Denote the *disjoint union* of two directed graded graphs (quivers) Q_1 and Q_2 by $Q_1 \sqcup Q_2$. Then $Q_1 \sqcup Q_2$ is the quiver with vertex set $V(Q_1) \sqcup V(Q_2)$, arrows $E(Q_1) \sqcup E(Q_2)$, and grading $gr_1 \sqcup gr_2$, where for $v \in Q_1 \sqcup Q_2$,

$$gr_1 \sqcup gr_2(v) = \begin{cases} gr_1(v) & \text{if } v \in Q_1, \\ gr_2(v) & \text{if } v \in Q_2. \end{cases}$$

Let ι_j denote the natural inclusions $Q_j \hookrightarrow Q_1 \sqcup Q_2$, for $j = 1, 2$.

Definition 2.16. *For R_1 and R_2 graded quivers with subquivers Q_1 and Q_2, respectively, define*

$$\bigotimes : A(Q_1 \uparrow R_1; B) \times A(Q_2 \uparrow R_2; B) \to A(Q_1 \sqcup Q_2 \uparrow R_1 \sqcup R_2; B),$$

as follows: for $(f,g) \in A(Q_1 \uparrow R_1; B) \times A(Q_2 \uparrow R_2; B)$,

$$\bigotimes(f,g) = \sum_{\eta \in \mathrm{Hom}(Q_1 \sqcup Q_2 \uparrow R_1 \sqcup R_2; B)} f|_{\eta \circ \iota_1} g|_{\eta \circ \iota_2} \eta.$$

It is easily checked that $\bigotimes$ *is bilinear.*

Recall the pictorial representation (Figure 2.6) of multiplication in the path algebra. Theorem 2.7 translates this multiplication into a composition of maps on configuration spaces, using the surjection $\phi := \iota_1 \sqcup \iota_2$ and inclusion $\iota := \iota_\triangle$ of Figure 2.16, where $\triangle$ denotes the *symmetric difference*, defined below.

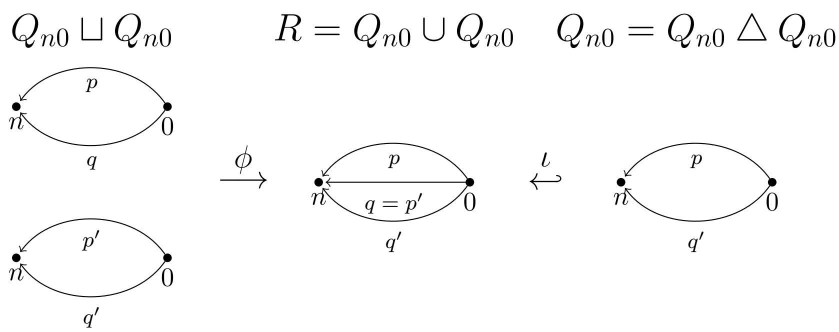


Figure 2.16

Definition 2.17. *For a graded quiver* R *with subquivers* Q_1, Q_2*, the* ***symmetric difference*** *of* Q_1 *and* Q_2 *is*

$$Q_1 \triangle Q_2 = (Q_1 \setminus Q_2) \cup (Q_2 \setminus Q_1) \cup \{\textit{isolated vertices of } Q_1 \textit{ and } Q_2\},$$

where $(Q_i \setminus Q_j) := Q_i \setminus (Q_i \cap Q_j)$*, i.e., the smallest subquiver of* Q_i *that contains all the vertices and arrows of* Q_i *not in* $Q_i \cap Q_j$.

Theorem 2.7. *For $\mathcal{B}$ a Bratteli diagram of highest grading at least n, $\phi = \iota_1 \sqcup \iota_2$, and $\iota = \iota_\triangle$ as in Figure 2.16, the bilinear map*

$$\iota^* \circ \phi_* \circ \bigotimes : A(Q_{n0}; \mathcal{B}) \times A(Q_{n0}; \mathcal{B}) \to A(Q_{n0}; \mathcal{B})$$

corresponds to the algebra product on $\mathbb{C}[\mathcal{B}_n]$ under the isomorphism $A(Q_{n0}; \mathcal{B}) \cong \mathbb{C}[\mathcal{B}_n]$ of Lemma 2.4.

Proof. As the proof is little more than applying the definitions of ι^* and ϕ_*, we have deferred it to Appendix A.2. □

Note 2.3. By Lemma 2.4,

$$\mathbb{C}[\mathcal{B}_i] \cap \text{Centralizer}\, \mathbb{C}[\mathcal{B}_j] \cong A(Q_{ji}; \mathcal{B})$$

and so by Theorem 2.7, for

$$x \in \mathbb{C}[\mathcal{B}_j] \cap \text{Centralizer}\, \mathbb{C}[\mathcal{B}_i], \quad y \in \mathbb{C}[\mathcal{B}_l] \cap \text{Centralizer}\, \mathbb{C}[\mathcal{B}_k],$$

the product xy corresponds to the bilinear product

$$\iota^* \circ \phi_* \circ \bigotimes : A(Q_{ji}; \mathcal{B}) \times A(Q_{lk}; \mathcal{B}) \to A(Q_{ji} \triangle Q_{lk}; \mathcal{B}).$$

As an example see Figure 2.17.

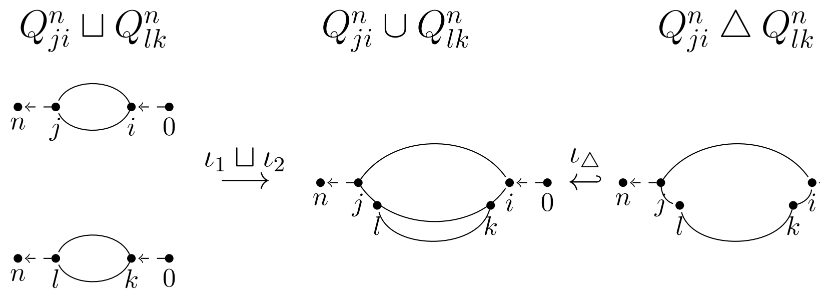


Figure 2.17: $i \leq k \leq l \leq j$

A product of two elements in the path algebra thus corresponds to bilinear maps arising from natural injections and surjections on the configuration spaces of the corresponding quivers.

Definition 2.18. *For Q_1 and Q_2 graded quivers, $\iota_\triangle$ the natural injection of $Q_1 \triangle Q_2$ into $Q_1 \cup Q_2$ and $\iota_1 \sqcup \iota_2$ the natural surjection of $Q_1 \sqcup Q_2$ into $Q_1 \cup Q_2$, define the* ***restricted product*** $* : A(Q_1; B) \times A(Q_2; B) \longrightarrow A(Q_1 \triangle Q_2; B)$ *as follows: for* $f_i \in A(Q_i; B)$,

$$*(f_1, f_2) = (\iota_\triangle)^* \circ (\iota_1 \sqcup \iota_2)_* \circ \bigotimes(f_1, f_2).$$

We may think of the restricted product as gluing Q_1 to Q_2, embedding this into $Q_1 \cup Q_2$, and deleting arrows in $Q_1 \cap Q_2$; hence, 'restricting' the product.

In Appendix A.2 we prove that the restricted product is associative and commutative. Further, the complexity is easily determined:

Lemma 2.5. *For B a locally finite graded quiver, R a graded quiver with finite subquivers Q_1 and Q_2, and $f_i \in A(Q_i; B)$, the restricted product $f_1 * f_2$ requires at most*

$$\# \operatorname{Hom}((Q_1 \cup Q_2) \uparrow R; B)$$

scalar multiplications and

$$\#\operatorname{Hom}((Q_1 \cup Q_2) \uparrow R; B) - \#\operatorname{Hom}((Q_1 \triangle Q_2) \uparrow R; B)$$

scalar additions.

Proof. See appendix A.2 for proof. □

By Theorem 2.7, for

$$x_j \in W_j = \mathbb{C}[\mathcal{B}_{j^+}] \cap \operatorname{Centralizer}(\mathbb{C}[\mathcal{B}_{j^-}]) \cong A(Q_{j^+j^-}; \mathcal{B}),$$

the product $x_i x_{i+1}$ corresponds to a restricted product

$$* : A(Q_{i^+i^-}; \mathcal{B}) \times A(Q_{(i+1)^+(i+1)^-}; \mathcal{B}) \to A(Q_{i^+i^-} \triangle Q_{(i+1)^+(i+1)^-}; \mathcal{B}),$$

where we identify edges of $Q_{i^+i^-}$ with edges of $Q_{(i+1)^+(i+1)^-}$ as in Figure 2.16. Then multiplying this product by x_{i+2} corresponds to a restricted product where we identify edges of $Q_{i^+i^-} \triangle Q_{(i+1)^+(i+1)^-}$ with edges of $Q_{(i+2)^+(i+2)^-}$.

Then to compute $x_1 \cdots x_m$, we must make these identifications for all $Q_{i^+i^-}$. To be explicit, we do so all at once by associating a quiver $\mathcal{Q}$. Each $Q_{i^+i^-} \triangle Q_{(i+1)^+(i+1)^-}$ is a subquiver of $\mathcal{Q}$ so to compute the product $x_1 \cdots x_m$ we consider restricted products of elements in $A(\mathcal{Q}; \mathcal{B})$.

2.3.3 The Quiver $\mathcal{Q}$

To construct $\mathcal{Q}$, for each $i, 1 \leq i \leq m$, consider the quiver $Q^n_{i^+i^-}$ with arrows L_i and R_i as in Figure 2.18.

$$Q_{i^+i^-} \hookrightarrow Q^n_{i^+i^-}$$

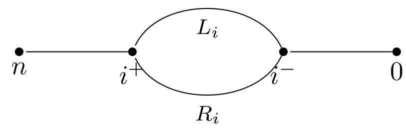


Figure 2.18

First, 'unsmooth' each quiver by adding in vertices that allow R_i to be glued to L_{i+1}: for each $1 \leq i \leq m$, add vertices to L_i and R_i at levels j^+ and j^- for all $1 \leq j \leq m$. Note that it is not always necessary to add in all of these vertices to be able to glue R_i to L_{i+1} (see Example 2.19).

Let $\mathcal{Q}$ be the quiver formed by gluing R_i to L_{i+1}, $1 \leq i < m$. Note that each quiver $Q^n_{i^+i^-}$ and $Q_{i^+i^-}$ injects naturally into $\mathcal{Q}$, via the inclusion map, ι. Let

$$\mathcal{Q}_i := \iota(Q^n_{i^+i^-}), \quad F_i := \iota(Q_{i^+i^-}), \quad F := \cup_{i=1}^m F_i,$$

$$\hat{L}_i := \iota(L_i), \text{ for } 1 \leq i \leq m, \quad \hat{L}_{m+1} = \iota(R_m).$$

Example 2.19. Suppose $1^+ = 7, 1^- = 4, 2^+ = 3, 2^- = 1, 3^+ = 5, 3^- = 2$. Then we have the quivers $Q^n_{i^+i^-}$ of Figure 2.19. We unsmooth Q^n_{74} by adding vertices at level 5, 3, and 1 to R_1; we unsmooth Q^n_{31} by adding a vertex at level 2 to R_2 and vertices at level 4, 5, and 7 to L_2; we unsmooth Q^n_{52} by adding vertices at level 7, 4, 3, and 1 to L_3.

We then glue together Q^n_{74} and Q^n_{31} by gluing R_1 to L_2 at the vertices at level $1, 3, 4$ and 7. We then glue this quiver to Q^n_{52} by gluing R_2 to L_3 along the vertices at levels $1, 2, 3, 5$. This gluing process creates the quivers $\mathcal{Q}$ and F, with subgraphs $\mathcal{Q}_i$, and F_i as in Figure 2.20.

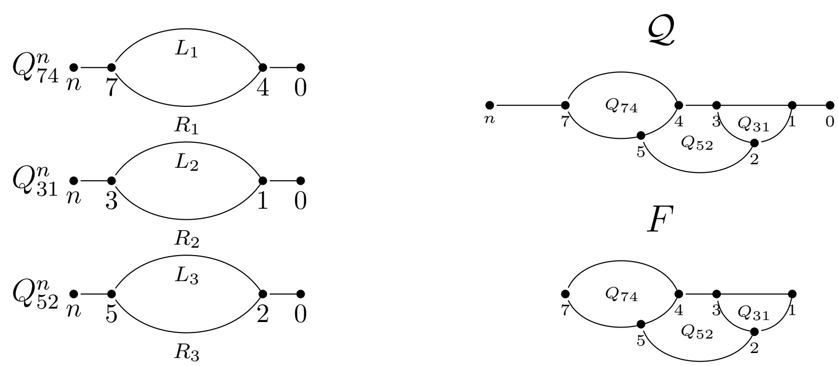


Figure 2.19

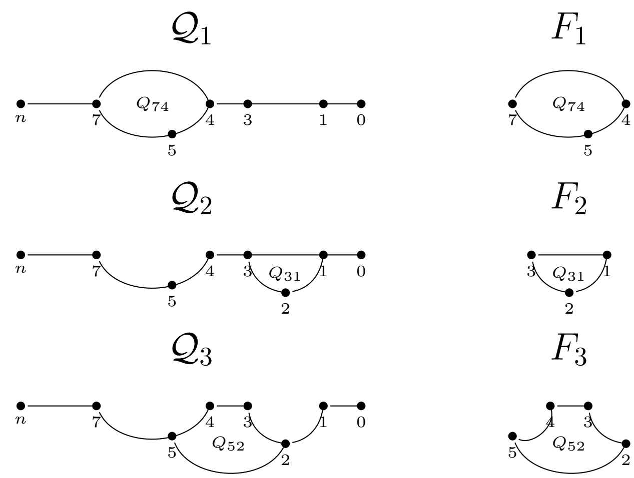


Figure 2.20

Example 2.20. Consider the product $\tilde{t}_1\tilde{t}_2\tilde{t}_3F_{t_1t_2t_3}$ of Example 2.15 and recall from Example 2.16 that

$$1^+ = 2, \quad 1^- = 0,$$

$$2^+ = 3, \quad 2^- = 1,$$

$$3^+ = 4, \quad 3^- = 2,$$

$$4^+ = 3, \quad 4^- = 0.$$

Then gluing together the quivers Q^4_{20}, Q^4_{31}, Q^4_{42}, and Q^4_{30} (i.e. the quivers Q_1, Q_2, Q_3, Q_F of Example 2.15) yields the quiver $\mathcal{Q}$ of Example 2.15.

Recall from Lemma 2.4 that

$$A(Q_{i^+i^-};\mathcal{B}) \cong \mathbb{C}[\mathcal{B}_{i^+}] \cap \text{Centralizer}(\mathbb{C}[\mathcal{B}_{i^-}]) = W_i.$$

Let $\zeta_i : W_i \to A(Q_{i^+i^-};\mathcal{B})$ be the isomorphism of Lemma 2.4. By Theorem 2.7, for $w_i \in W_i$, the algebra product $w_1 \cdots w_m$ corresponds to the restricted product $\zeta_1(w_1) * \cdots * \zeta_m(w_m)$. To be explicit, we have a map

$$\zeta^{-1} : A(Q_{1^+1^-} \bigtriangleup \cdots \bigtriangleup Q_{m^+m^-};\mathcal{B}) \to \mathbb{C}[\mathcal{B}_n]$$

such that

$$\zeta^{-1}(\zeta_1(w_1) * \cdots * \zeta_m(w_m)) = w_1 \cdots w_m.$$

After smoothing F_i, Corollary A.2 in Appendix A.3.1 gives an isomorphism

$$\chi_i : A(Q_{i^+i^-};\mathcal{B}) \to A(F_i;\mathcal{B}),$$

which in turn allows for definition of a map:

$$\chi^{-1} : A(F_1 \bigtriangleup \cdots \bigtriangleup F_m;\mathcal{B}) \to A(Q_{1^+1^-} \bigtriangleup \cdots \bigtriangleup Q_{m^+m^-};\mathcal{B})$$

such that for $f_i \in A(Q_{i^+i^-};\mathcal{B})$,

$$\chi^{-1}(\chi_1(f_1) * \cdots * \chi_m(f_m)) = f_1 * \cdots * f_m.$$

Thus, multiplication in the path algebra becomes a restricted product on configuration spaces corresponding to subquivers of $\mathcal{Q}$.

Theorem 2.8. *For $w_i \in W_i = \mathbb{C}[\mathcal{B}_{i^+}] \cap \mathrm{Centralizer}(\mathbb{C}[\mathcal{B}_{i^-}])$, $1 \leq i \leq m$, and quivers $\mathcal{Q}, \mathcal{Q}_i, F, F_i$ and maps $\chi^{-1}, \chi_i, \zeta^{-1}, \zeta_i$ as defined above, let*

$$\theta = \zeta^{-1} \circ \chi^{-1} : A(F_1 \triangle \cdots \triangle F_m; \mathcal{B}) \to \mathbb{C}[\mathcal{B}_n],$$

and let

$$\xi_i = \chi_i \circ \zeta_i : W_i \to A(F_i; \mathcal{B}).$$

Then

$$w_1 \cdots w_m = \theta(\xi_1(w_1) * \cdots * \xi_m(w_m)).$$

2.3.4 The SOV Algorithm

To apply Theorem 2.6 we define vector spaces W_i and V_i along with bilinear maps $*_j : V_{j-1} \times W_j \to V_j$ satisfying the conditions of Algorithm 2.2.

For $\sigma \in S_m$ and $1 \leq i \leq m$, let $R_i^\sigma := F_{\sigma(1)} \triangle \cdots \triangle F_{\sigma(i)} \subseteq \mathcal{Q}$. Let

$$W_i := W_{\sigma(i)} = \mathbb{C}[\mathcal{B}_{\sigma(i)^+}] \cap \mathrm{Centralizer}(\mathbb{C}[\mathcal{B}_{\sigma(i)^-}]),$$

$$V_1 = W_1, \quad V_i = A(R_i^\sigma; \mathcal{B}), \text{ for } 2 \leq i < m, \quad \text{and } V_m = \mathbb{C}[\mathcal{B}_n].$$

Define $*_j : V_{j-1} \times W_j \to V_j$ by

$$*_2(v_1, w_2) = v_1 *_2 w_2 := (\xi_{\sigma(1)} v_1) * (\xi_{\sigma(2)} w_2),$$
$$*_j(v_{j-1}, w_j) = v_{j-1} *_j w_j := v_{j-1} * (\xi_{\sigma(j)} w_j), \text{ for } 2 < j < m,$$
$$*_m(v_{m-1}, w_m) = v_{m-1} *_m w_m := \theta(v_{m-1} * (\xi_{\sigma(m)} w_m)).$$

Theorem 2.9. *Let $m, n \in \mathbb{Z}^+$ and for $1 \leq i \leq m$ let $i^+, i^- \in \mathbb{N}$ such that $i^+, i^- \leq n$. For $x_i \in \mathbb{C}[\mathcal{B}_{i^+}] \cap \text{Centralizer}(\mathbb{C}[\mathcal{B}_{i^-}])$, $\sigma \in S_m$, and $w_i := x_{\sigma(i)} \in W_i$,*

1. $x_1 \cdots x_m = (((w_1 *_2 w_2) *_3 w_3) \cdots *_m w_m)$
2. *This may be computed in at most*

$$\sum_{i=2}^{m} \dim A(R^\sigma_{i-1} \cup F_{\sigma(i)} \uparrow \mathcal{Q})$$

scalar multiplications, and

$$\sum_{i=2}^{m} (\dim A(R^\sigma_{i-1} \cup F_{\sigma(i)} \uparrow \mathcal{Q}) - \dim A(R^\sigma_i \uparrow \mathcal{Q}))$$

scalar additions.

Proof. To prove Part 1, it follows by definition of $*_i$ and induction that for $2 \leq i < m$,

$$(((w_1 *_2 w_2) *_3 w_3) \cdots *_i w_i) = \xi_{\sigma(1)}(w_1) * \cdots * \xi_{\sigma(i)}(w_i).$$

Then by commutativity of $*$ and Theorem 2.8,

$$\begin{aligned}(((w_1 *_2 w_2) *_3 w_3) \cdots *_m w_m) &= *_m(\xi_{\sigma(1)}(w_1) * \cdots * \xi_{\sigma(m-1)}(w_{m-1}), w_m) \\ &= \theta(\xi_{\sigma(1)}(w_1) * \cdots * \xi_{\sigma(m)}(w_m)) \\ &= \theta(\xi_1(x_1) * \cdots * \xi_m(x_m)) = x_1 \cdots x_m.\end{aligned}$$

Part 2 follows from Lemma 2.5 and the fact that the maps θ and ξ_i require no arithmetic operations to compute with respect to the natural bases.

□

With this framework, for $\mathcal{B}$ a Bratteli diagram and $\tilde{Z}$ a subset of the path algebra $\mathbb{C}[\mathcal{B}_n]$, to compute the sum $\sum_{\tilde{z} \in \tilde{Z}} \tilde{z}$,

SOV Algorithm 2.1.

I. Choose $m \in \mathbb{N}$ and a subset $X \subseteq (\mathbb{C}[\mathcal{B}_n])^m$ such that $|X| = |\tilde{Z}|$ and

$$\tilde{Z} = \{x_1 \cdots x_m | (x_1, \ldots, x_m) \in X\}.$$

Thus, X can be thought of as a choice of factorization into at most m elements of each element of $\tilde{Z}$.

II. For $x \in \mathbb{C}[\mathcal{B}_n]$, let $c^+(x)$ denote the smallest integer such that $x \in \mathbb{C}[\mathcal{B}_{c^+(x)}]$ and let $c^-(x)$ denote the largest integer less than or equal to $c^+(x)$ such that $x \in \mathbb{C}[\mathcal{B}_{c^-(x)}]$. Let

$$i^+ = \max\{c^+(x_i)\}, \quad i^- = \min\{c^-(x_i)\},$$

over all ith coordinates x_i of elements in X.

III. For each $1 \leq i \leq m$*, use quivers* $Q^n_{i^+i^-}$ *as in Section 2.3.3 to build a quiver* $\mathcal{Q}$ *with subquivers* $\mathcal{Q}_i$*,* F_i*, and* F*.*

IV. Let $\sigma \in S_m$ *and let* $W_i = \mathbb{C}[\mathcal{B}_{\sigma(i)^+}] \cap \mathrm{Centralizer}(\mathbb{C}[\mathcal{B}_{\sigma(i)^-}])$*. Define*

$$X_0 = \{(x_{\sigma(1)}, \ldots, x_{\sigma(m)}) | (x_1, \ldots, x_m) \in X\} \subseteq W_1 \times \cdots \times W_m.$$

For $1 \leq i < m$ *let*

$$X_i = \{(w_{i+1}, \ldots, w_m) | (w_1, \ldots, w_m) \in X_0 \ \textit{for some} \ w_1, \ldots, w_i\},$$

and let $X_m = \emptyset$*.*

V Define $L_i : X_i \to A(R^\sigma_i \uparrow \mathcal{Q}; \mathcal{B})$ *recursively as in Step IV preceeding Theorem 2.6.*

Theorem 2.10. *For* $\mathcal{B}$ *a Bratteli diagram,* $\tilde{Z}$ *a subset of the path algebra* $\mathbb{C}[\mathcal{B}_n]$*,*

$$L_m := L_m(\emptyset) = \sum_{\tilde{z} \in \tilde{Z}} \tilde{z}.$$

Proof. This follows directly from Theorem 2.6 and Theorem 2.9. □

Theorem 2.11. *For* $\mathcal{B}$ *a Bratteli diagram,* $\tilde{Z}$ *a subset of the path algebra* $\mathbb{C}[\mathcal{B}_n]$*, we may compute* $\sum_{\tilde{z} \in \tilde{Z}} \tilde{z}$ *in at most*

$$\sum_{i=2}^{m} |X_{i-1}| \dim A(R^\sigma_{i-1} \cup F_{\sigma(i)} \uparrow \mathcal{Q}; \mathcal{B})$$

multiplications and

$$(|X_0| - |X_1|) + \sum_{i=2}^{m}(|X_{i-1}| - |X_i|)(\dim A(R^\sigma_{i-1} \cup F_{\sigma(i)} \uparrow \mathcal{Q}; \mathcal{B}) - \dim A(R^\sigma_i \uparrow \mathcal{Q}))$$

additions.

Proof. To compute $\sum \tilde{z}$:

Stage 0: Find X_0 by reordering X.

Stage 1: Compute L_1 for all $(w_2, \ldots, w_m)$ in X_1.

Stage i: Compute L_i given X_{i-1} and L_{i-1}.

Stage 0 requires no operations, while stage 1 requires $(|X_0| - |X_1|)$ additions. For $2 \leq i \leq m$, we see from combining Lemma 2.5 with Theorem 2.9 that stage i requires $|X_{i-1}| \dim A(R^\sigma_{i-1} \cup F_{\sigma(i)} \uparrow \mathcal{Q}; \mathcal{B})$ multiplications, while the number of additions required is $(|X_{i-1}| - |X_i|)(\dim A(R^\sigma_{i-1} \cup F_{\sigma(i)} \uparrow \mathcal{Q}; \mathcal{B}) - \dim A(R^\sigma_i \uparrow \mathcal{Q}))$. □

2.3.5 General Morphism Counts

The SOV algorithm 2.1 computes path algebra sums by first factoring each element and then translating multiplication into maps on configuration spaces. The complexity is determined by the size of the factorization sets and the dimension of the configuration spaces.

In this section, we present methods to determine the dimension of a configuration space. Recall from Note 2.2 that if Q is a finite subquiver of R and B a locally finite quiver, $\dim A(Q \uparrow R; B) = \# \operatorname{Hom}(Q \uparrow R; B)$. In the SOV approach, B is the Bratteli diagram associated to a chain of semisimple algebras and hence locally finite, so in this section we give results to count $\#\operatorname{Hom}(Q \uparrow R; B)$.

For B a locally finite graded quiver, $\alpha, \beta \in V(B)$, let $M_B(\alpha, \beta)$ denote the number of paths from β to α in B. Note that for $\mathcal{B}$ a Bratteli diagram, $\alpha, \beta \in V(\mathcal{B})$ correspond to irreducible representations γ, ρ and

$$M_{\mathcal{B}}(\alpha, \beta) = M(\gamma, \rho),$$

as in Definition 2.7.

Theorem 2.12. *Let Q, R, B be graded quivers with Q a finite subquiver of R and B locally finite. Then*

$$\#\operatorname{Hom}(Q \uparrow R; B) = \sum_{\phi \in \operatorname{Hom}(V(Q) \uparrow R; B)} \prod_{\substack{\text{arrows} \\ \beta \to \alpha \\ \text{in } Q}} M_B(\phi\alpha, \phi\beta)$$

Proof. A morphism specifies the image of each vertex and each arrow. This may be counted by first fixing the image of each vertex and counting all possible arrow images, then varying over all possible images of $V(Q)$. □

Theorem 2.12 gives a procedure for computing $\#\operatorname{Hom}(Q \uparrow R; B)$. For a quiver Q, let Q^i denote the vertices of Q at level i. Then:

1. label each vertex $\alpha_i \in Q^i$ with a vertex $\alpha_i' \in B^i$ such that this labeling could extend to a map from R into B;

2. label each edge of Q from β to α by $M_B(\alpha', \beta')$;

3. multiply the labels and sum over all possible labellings.

Example 2.21. Let $Q_1 = R_1$ be as in Figure 2.21. Steps 1 and 2 then give the labelling of the figure and by Theorem 2.12,

$$\#\,\mathrm{Hom}(Q \uparrow R; B) = \sum_{\alpha'_i \in B^i} M_B(\alpha'_5, \alpha'_0) M_B(\alpha'_5, \alpha'_3) M_B(\alpha'_4, \alpha'_3) M_B(\alpha'_4, \alpha'_0).$$

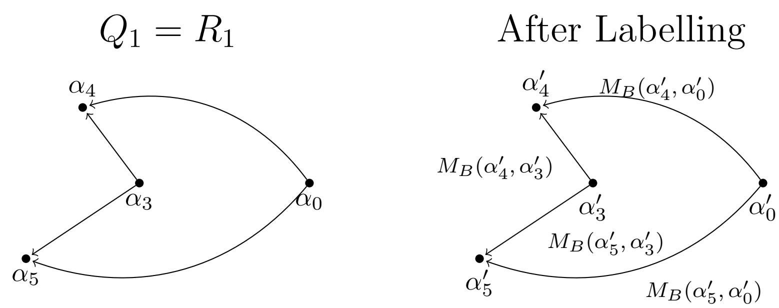


Figure 2.21

To further simplify counts, we first 'smooth' quivers before counting morphisms, i.e. we remove superfluous vertices (see Corollary A.2 in Appendix A.3.1).

Example 2.22. Let $Q_2 = R_2$ be as in Figure 2.22. Then for Q_1, R_1 as in Figure 2.21, Corollary A.2 gives the isomorphism

$$A(Q_2 \uparrow R_2; B) \cong A(Q_1 \uparrow R_1; B).$$

To compute $\#\,\mathrm{Hom}(Q_2 \uparrow R_2; B)$, remove vertices α_2 and α_1, then use the labelling of Figure 2.21.

$$Q_2 = R_2$$

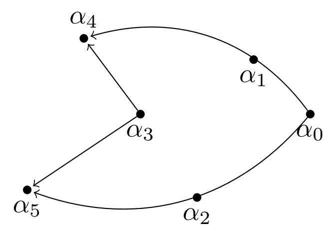


Figure 2.22

2.3.6 Morphisms into Locally Free Bratteli Diagrams

In Section 2.3.5 we obtained general quiver morphism counting results. For *locally free* Bratteli diagrams we rewrite these results in terms of the dimensions of the corresponding subalgebras. This generalizes Stanley's work on differential posets [87].

Definition 2.19. *A Bratteli diagram $\mathcal{B}$ is* ***locally free*** *if for each $i \geq 1$, $\mathbb{C}[\mathcal{B}_i]$ is free as a module over $\mathbb{C}[\mathcal{B}_{i-1}]$.*

Example 2.23. For G a finite group, the Bratteli diagram associated to the chain of group algebras $\mathbb{C}[G] = \mathbb{C}[G_n] > \cdots > \mathbb{C}[G_0] = \mathbb{C}$ is locally free.

Let $\mathbb{C}[V(\mathcal{B})]$ denote the space of finitely supported linear combinations of vertices of $\mathcal{B}$, let $\mathcal{B}^i$ denote the vertices $\alpha \in V(\mathcal{B})$ with $gr(\alpha) = i$, and let $\mathbb{C}[\mathcal{B}^i]$ denote the space of finitely supported linear combinations of vertices at level i in $\mathcal{B}$. Define an inner product $\langle\ ,\ \rangle$ on $\mathbb{C}[V(\mathcal{B})]$ making the vertices orthonormal. As in [87] define linear operators U and D on $\mathbb{C}[V(\mathcal{B})]$ by linearly extending the action on $\alpha \in \mathcal{B}^i$:

$$U\alpha = \sum_{\gamma \in \mathcal{B}^{i+1}} M_{\mathcal{B}}(\gamma, \alpha)\gamma,$$

$$D\alpha = \sum_{\beta \in \mathcal{B}^{i-1}} M_{\mathcal{B}}(\alpha, \beta)\beta,$$

where, by convention, if $\mathcal{B}$ has highest grading n,

$$\mathcal{B}^{-1} = \emptyset = \mathcal{B}^{n+1} = \mathcal{B}^{n+2} = \cdots .$$

Note 2.4. As the vertices of $\mathcal{B}$ are labeled by the irreducible representations of $\mathbb{C}[\mathcal{B}_i]$, elements of $\mathbb{C}[\mathcal{B}^i]$ correspond to representations of the path algebra $\mathbb{C}[\mathcal{B}_i]$. In this context, U is induction and D restriction (see [39] Proposition 2.3.1).

We can now rewrite the sums of Theorem 2.12 as words in U and D, as in Example 2.24 below.

Example 2.24. For Q_1 as in Example 2.21, we can trace each arrow on the quiver by starting at the root, moving up four levels to vertex α_4, down to vertex α_3, up two levels to vertex α_5, and back down to the root. It is then easily checked that

$$\begin{aligned} \langle D^5U^2DU^4\hat{0}, \hat{0}\rangle &= \textstyle\sum_{\alpha_i' \in B^i} M_B(\alpha_5', \alpha_0')M_B(\alpha_5', \alpha_3')M_B(\alpha_4', \alpha_3')M_B(\alpha_4', \alpha_0') \\ &= \#\operatorname{Hom}(Q \uparrow R; B). \end{aligned}$$

In Corollary 2.2 we give explicit formulas for these inner products.

For $\alpha \in V(B)$ and $\hat{0}$ the root of $\mathcal{B}$, let $d_\alpha = M_{\mathcal{B}}(\alpha, 0)$ and let $d_i = \sum_{\alpha \in \mathcal{B}^i} d_\alpha \alpha$.

Lemma 2.6.

(i) For $\alpha \in \mathcal{B}^i$, $\langle d_i, \alpha \rangle = d_\alpha$.

(ii) $d_i = U^i\hat{0}$,

Proof. Clear from definition and induction. □

Proposition 2.1. *Let $\mathcal{B}$ be a Bratteli diagram. Then the following properties are equivalent:*

(i) *$\mathcal{B}$ is locally free.*

(ii) *For each i and all $\beta \in \mathcal{B}^{i-1}$, there exists $\lambda_i \in \mathbb{C}$ such that*

$$\sum_{\alpha \in \mathcal{B}^i} M_{\mathcal{B}}(\alpha, \beta) d_\alpha = \lambda_i d_\beta.$$

(iii) *For each i, d_i is an eigenvector of DU.*

(iv) *For each i there exists $\lambda_i \in \mathbb{C}$ with $DU^i\hat{0} = \lambda_i U^{i-1}\hat{0}$.*

Proof. As this proof comes down to definitions and the fact that D is restriction (cf. Note 2.4), we defer it to Appendix A.3.2. □

Corollary 2.1. *Let $\mathcal{B}$ be a locally free Bratteli diagram and λ_i the eigenvalue of DU associated to d_{i-1}. Then λ_i is integral and*

(i) $\lambda_i = \dfrac{\dim_{\mathbb{C}} \mathbb{C}[\mathcal{B}_i]}{\dim_{\mathbb{C}} \mathbb{C}[\mathcal{B}_{i-1}]}$,

(ii) $\dim_{\mathbb{C}} \mathbb{C}[\mathcal{B}_i] = \prod_{j=1}^{i} \lambda_j$.

Example 2.25. For a group algebra chain $\mathbb{C}[G_n] > \cdots > \mathbb{C}[G_0]$, the corresponding Bratteli diagram $\mathcal{B}$ is locally free and

$$\lambda_i = \frac{\dim_{\mathbb{C}} \mathbb{C}[\mathcal{B}_i]}{\dim_{\mathbb{C}} \mathbb{C}[\mathcal{B}_{i-1}]} = \frac{\dim_{\mathbb{C}} \mathbb{C}[G_i]}{\dim_{\mathbb{C}} \mathbb{C}[G_{i-1}]} = |G_i/G_{i-1}|.$$

Example 2.26. A Bratteli diagram is **r-differential** [87] if it is multiplicity-free and $DU - UD = rI$. Then by induction,

$$DU^i\hat{0} = UDU^{i-1}\hat{0} + rU^{i-1}\hat{0} = irU^{i-1}\hat{0},$$

hence, an r-differential poset is locally free and $\lambda_i = ir$.

Example 2.27. For $\vec{r} = (r_0, r_1, \dots)$ an infinite sequence of numbers, a Bratteli diagram is **$\vec{r}$-differential** if it is multiplicity-free and at the ith level, $DU - UD = r_i I$. Then by induction,

$$DU^i\hat{0} = UDU^{i-1}\hat{0} + r_{i-1}U^{i-1}\hat{0} = \sum_{j=0}^{i-1} r_j U^{i-1}\hat{0},$$

hence, an $\vec{r}$-differential poset is locally free and $\lambda_i = \sum_{j=0}^{i-1} r_j$.

In light of Examples 2.26 and 2.27, Theorem 2.13 below generalizes Theorem 3.7 of [87] and Theorem 2.3 of [88].

Definition 2.20. *Let $w = w_l \cdots w_1$ be a word in U and D and let $\mathcal{S} = \{i \mid w_i = D\}$. For each $i \in \mathcal{S}$, let $a_i = \#\{D\text{'s in } w \text{ to the right of } w_i\}$, and similarly let $b_i = \#\{U\text{'s in } w \text{ to the right of } w_i\}$. If $b_i - a_i \geq 0$ for all $i \in \mathcal{S}$, we call w an* ***admissible word****.*

Theorem 2.13. *Let $\mathcal{B}$ be a locally free Bratteli diagram and $w = D^{d_n}U^{u_n} \cdots D^{d_1}U^{u_1}$ an admissible word in U and D. Then for $s = \sum_{i=1}^{n} u_i - d_i$ and $\alpha \in \mathcal{B}^s$,*

$$\langle w\hat{0}, \alpha \rangle = d_\alpha \prod_{i \in \mathcal{S}} \lambda_{b_i - a_i}.$$

Proof. The proof comes down to inductively showing:

$$w_k\hat{0} = \prod_{i\in\mathcal{S}_k} \lambda_{b_i-a_i} \sum_{\alpha\in\mathcal{B}^{s_k}} d_\alpha \alpha.$$

For full details, see Appendix A.3.2 ☐

For $\mathcal{B}$ a locally free Bratteli diagram and Q an *n-toothed* quiver, Theorem 2.13 allows us to determine $\#\operatorname{Hom}(Q;\mathcal{B})$.

Definition 2.21. *A quiver Q is **n-toothed** if it consists of $2n+1$ (not necessarily distinct) vertices $\gamma_0,\ldots,\gamma_n,\beta_1,\ldots,\beta_n$ and distinct arrows connecting γ_{i-1} to β_i and γ_i to β_i.*

Example 2.28. The quiver of Figure 2.23 is an example of a 3-toothed quiver.

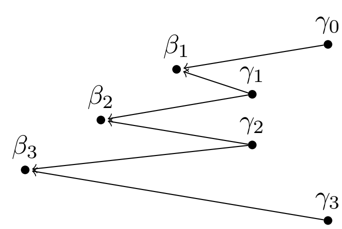


Figure 2.23

Example 2.29. The quiver Q_1 of Example 2.21 is 2-toothed, with $\gamma_0 = \alpha_0, \gamma_1 = \alpha_3, \gamma_2 = \alpha_0, \beta_1 = \alpha_4, \beta_2 = \alpha_5$.

Theorem 2.14. *Let $\mathcal{B}$ be a locally free Bratteli diagram, Q an n-toothed quiver with vertices γ_i at level l_i, β_i at level m_i. Then for*

$$w = D^{m_n-l_n}U^{m_n-l_{n-1}}\cdots D^{m_1-l_1}U^{m_1-l_0},$$

we have

$$\sum_{\alpha \in \mathcal{B}^{l_n - l_0}} \langle w\hat{0}, \alpha \rangle = \#\,\mathrm{Hom}(Q; \mathcal{B}).$$

Proof. Follows from Theorem 2.12 and induction. □

Corollary 2.2. *Let $\mathcal{B}$ be a locally free Bratteli diagram, Q an n-toothed quiver with vertices γ_i at level l_i, β_i at level m_i. Then for*

$$w = D^{m_n - l_n} U^{m_n - l_{n-1}} \cdots D^{m_1 - l_1} U^{m_1 - l_0},$$

we have

$$\#\,\mathrm{Hom}(Q; \mathcal{B}) = \sum_{\alpha \in \mathcal{B}^{l_n - l_0}} \langle w\hat{0}, \alpha \rangle = \prod_{i \in \mathcal{S}} \lambda_{b_i - a_i} \sum_{\alpha \in \mathcal{B}^{l_n - l_0}} d_\alpha.$$

Example 2.30. For Q_1 as in Example 2.21, we see that

$$l_o = l_2 = 0, \quad l_1 = 3, \quad m_1 = 4, \quad m_2 = 5.$$

Then $\mathcal{B}^{l_2 - l_0} = \mathcal{B}^0 = \hat{0}$ and by Corollary 2.2, for $w = D^5 U^2 D U^4$,

$$\#\,\mathrm{Hom}(Q_1; \mathcal{B}) = \langle w\hat{0}, \hat{0} \rangle = \lambda_4 \lambda_5 \lambda_4 \lambda_3 \lambda_2 \lambda_1.$$

Note that this is the inner product of Example 2.24.

We will use Corollary 2.2 for many of our complexity results in Section 2.4.

2.4 The Complexity of Fourier Transforms on Finite Groups

2.4.1 The Weyl Groups B_n and D_n

For our first application of SOV algorithm 2.1 we consider the Fourier transform of functions on the Weyl groups of type B_n and D_n, improving upon the results of [65].

Theorem 2.15 (Theorem 2.2)**.** *For the Weyl group B_n and R a complete set of irreducible matrix representations of B_n adapted to the subgroup chain $B_n > B_{n-1} > \cdots > B_0 = \{e\}$,*

$$C(B_n) \leq T_{B_n}(R) \leq n(2n-1)|B_n|.$$

Proof. Let $s_1, \ldots, s_n$ denote the simple reflections for B_n, labeled according to Figure 2.24.

Figure 2.24

Recall from [65] that elements in a set of minimal coset representatives for B_n/B_{n-1} have the following factorizations:

$$e,\ s_n,\ s_{n-1}s_n,\ \ldots\ ,\ s_1 \cdots s_n,\ s_2 s_1 \cdots s_n,\ \ldots\ ,\ s_n \cdots s_1 \cdots s_n,$$

so for $A_i = \{e, s_i\} = A'_i$, a complete set of coset representatives is contained in $Y = \{a_n \cdots a_2 a_1 a'_2 \cdots a'_n |\ a_i, a'_i \in A_i\}$.

For $\mathcal{B}$ the Bratteli diagram associated to the chain

$$\mathbb{C}[B_n] > \mathbb{C}[B_{n-1}] > \cdots > \mathbb{C}$$

let $\{\mathbb{C}[\mathcal{B}_i]\}$ be the chain of path algebras associated to $\{\mathbb{C}[B_i]\}$. Let $\tilde{Y} = \{\tilde{y} \mid y \in Y\}$, and similarly define $\tilde{A}_i, \tilde{A}'_i$. Note that $\tilde{A}_i, \tilde{A}'_i \subseteq \mathbb{C}[\mathcal{B}_i] \cap \text{Centralizer}(\mathbb{C}[\mathcal{B}_{i-2}])$. By Lemma 2.3, computation of the Fourier transform of a complex function f on B_n is equivalent to computation of

$$\sum_{y \in \tilde{Y}} y F_y = \sum_{\substack{a_i \in \tilde{A}_i \\ a'_i \in \tilde{A}'_i}} a_n \cdots a_2 a_1 a'_2 \cdots a'_n F_{a_n \cdots a_2 a_1 a'_2 \cdots a'_n}$$

for $F_y = F_{a_n \cdots a_2 a_1 a'_2 \cdots a'_n} \in \mathbb{C}[\mathcal{B}_{n-1}]$. We now use SOV algorithm 2.1:

I. Let $X = \{(a_n, \ldots, a_2, a_1, a'_2, \ldots, a'_n, F_{a_n \cdots a_2 a_1 a'_2 \cdots a'_n}) | \, a_i \in \tilde{A}_i, a'_i \in \tilde{A}'_i\}$.

II. Note that

$$\begin{aligned}
&i^+ = \max_{a_{n-i+1} \in \tilde{A}_{n-i+1}} \{c(a_{n-i+1})^+\} = n - i + 1, && 1 \le i \le n, \\
&i^+ = \max_{a'_{i-n+1} \in \tilde{A}'_{i-n+1}} \{c(a'_{i-n+1})^+\} = i - n + 1, && n < i < 2n, \\
&i^- = i^+ - 2, && 1 \le i < 2n, \\
&2n^+ = \max\{c(F_{a_n \cdots a_2 a_1 a'_2 \cdots a'_n})^+\} = n - 1, \\
&2n^- = 0.
\end{aligned}$$

III. Glue together quivers $Q^n_{i^+ i^-}$ in Figure 2.25 (labeled by $A_j, \tilde{A}_j, F_y$) to build the quiver $\mathcal{Q}$ of Figure 2.26. The left column of Figure 2.25 shows the quivers $Q^n_{i^+ i^-}$ for $1 \le i \le n$ and the right column shows the quivers $Q^n_{i^+ i^-}$ for $n + 1 \le i \le 2n$

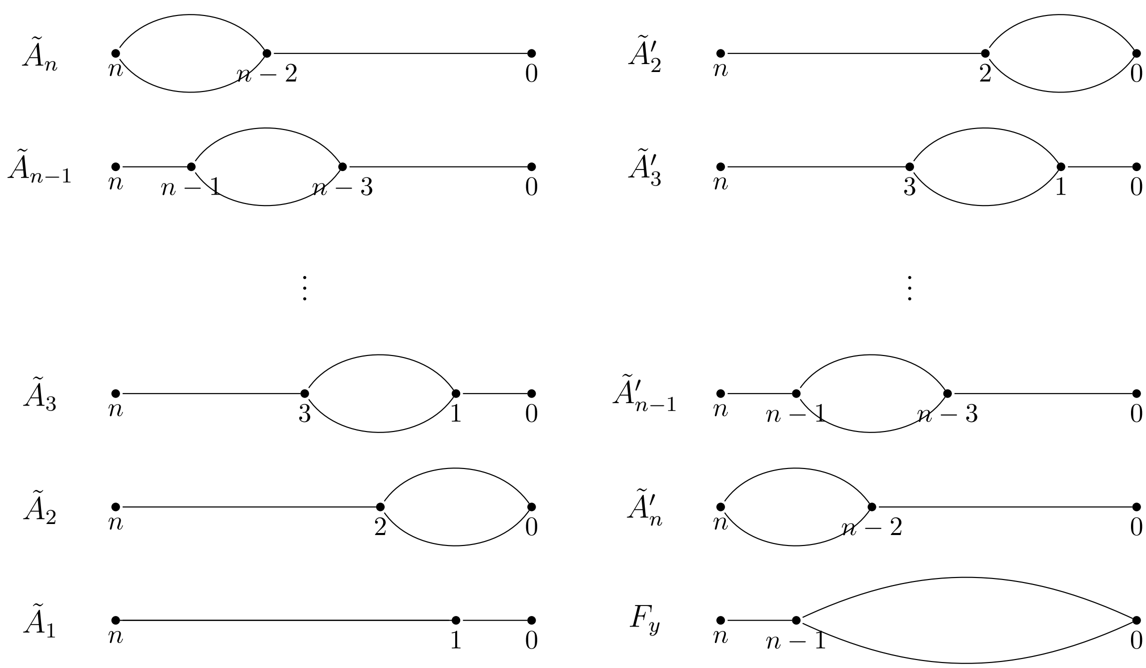


Figure 2.25

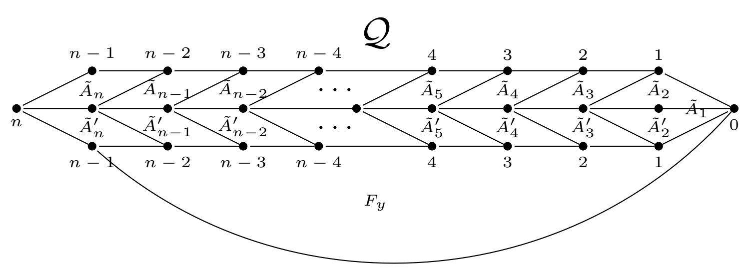


Figure 2.26

IV. Let $\sigma \in S_{2n}$ be the permutation reordering X so that

$$\begin{aligned} X_0 &= \{(x_{\sigma(1)}, \ldots, x_{\sigma(2n)}) | \, (x_1, \ldots, x_{2n}) \in X\} \\ &= \{(F_{a_n \cdots a_2 a_1 a_2 \cdots a_n}, a'_2, a'_3, \ldots a'_n, a_1, a_2, a_3, \ldots, a_n)\}. \end{aligned}$$

Then

$$X_1 = \{(a'_2, a'_3, \dots a'_n, a_1, a_2, a_3, \dots, a_n) |\ a_i \in \tilde{A}_i, a'_i \in \tilde{A}'_i\},$$
$$X_2 = \{(a'_3, \dots a'_n, a_1, a_2, a_3, \dots, a_n) |\ a_i \in \tilde{A}_i, a'_i \in \tilde{A}'_i\},$$
$$\vdots$$
$$X_{2n-1} = \{(a_n) |\ a_n \in \tilde{A}_n\}.$$

Note that

$$|X_{i-1}| = |\tilde{A}'_i| \cdots |\tilde{A}'_n||\tilde{A}_1| \cdots |\tilde{A}_n|, \quad 2 \le i \le n,$$
$$|X_{i-1}| = |\tilde{A}_{i-n}| \cdots |\tilde{A}_n|, \qquad n < i \le 2n.$$

By Theorem 2.11, we may compute $\sum yF_y$ in at most

$$\sum_{i=2}^{n} |X_{i-1}| \dim A(R^\sigma_{i-1} \cup F_{\sigma(i)} \uparrow \mathcal{Q}; \mathcal{B}) \tag{2.4}$$

multiplications, with $R^\sigma_{i-1} \cup F_{\sigma(i)}$ as in Figure 2.27.

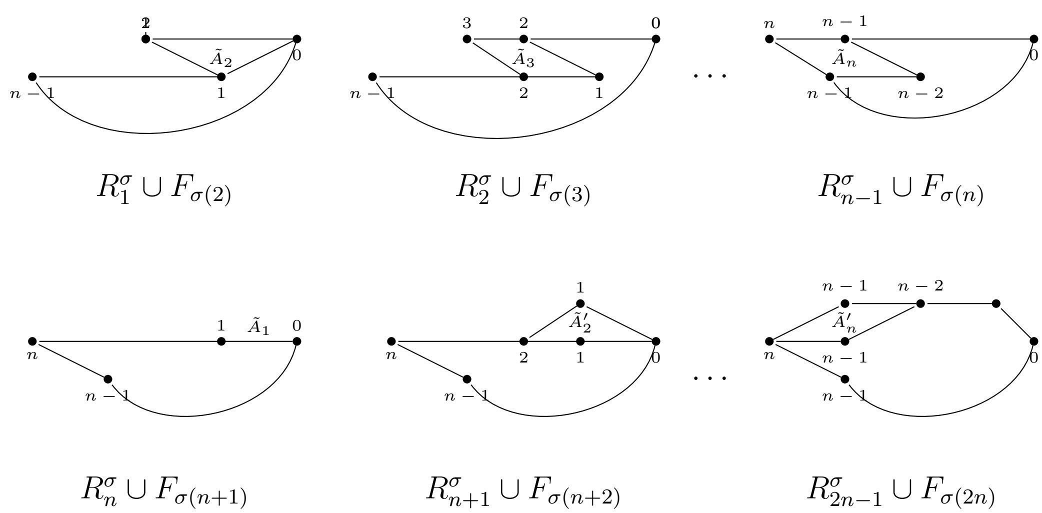


Figure 2.27

Then for $\mathcal{H}_i^n, \mathcal{J}_i^n, \mathcal{K}^n$ the quivers of Figure 2.28,

$$\begin{aligned}
&\dim A(R^\sigma_{i-1} \cup F_{\sigma(i)} \uparrow \mathcal{Q}; \mathcal{B}) = \#\operatorname{Hom}(\mathcal{H}_i^n \uparrow \mathcal{Q}; \mathcal{B}), \qquad 1 \le i \le n,\\
&\dim A(R^\sigma_{n} \cup F_{\sigma(n+1)} \uparrow \mathcal{Q}, ; \mathcal{B}) = \#\operatorname{Hom}(\mathcal{K}^n \uparrow \mathcal{Q}; \mathcal{B}),\\
&\dim A(R^\sigma_{i-1} \cup F_{\sigma(i)} \uparrow \mathcal{Q}; \mathcal{B}) = \#\operatorname{Hom}(\mathcal{J}_{i-n}^n \uparrow \mathcal{Q}; \mathcal{B}), \quad n+2 \le i \le 2n.
\end{aligned}$$

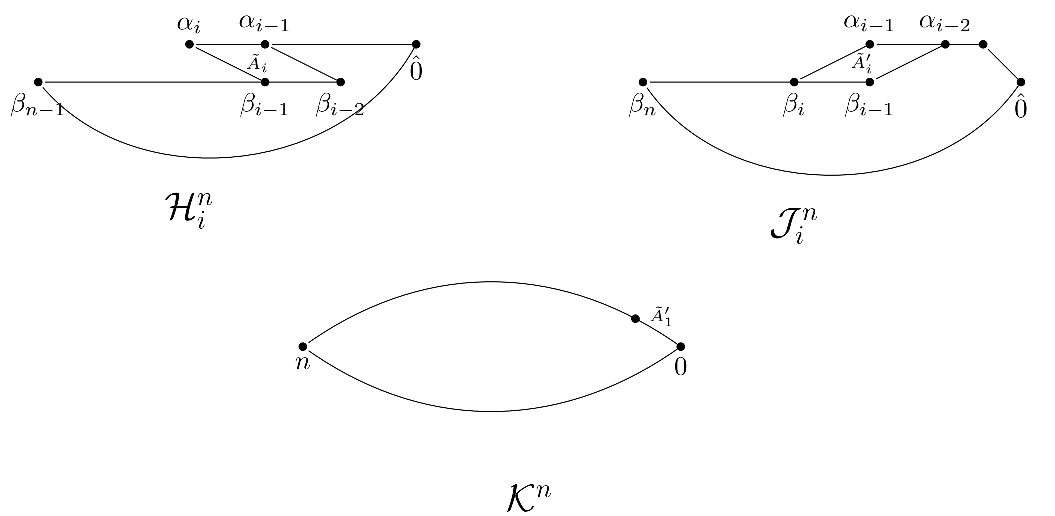


Figure 2.28

By Lemma 2.4, $A(\mathcal{K}^n; \mathcal{B}) \cong \mathbb{C}[\mathcal{B}_n]$, so

$$\dim A(\mathcal{K}^n \uparrow \mathcal{Q}; \mathcal{B}) = \#\operatorname{Hom}(\mathcal{K}^n \uparrow \mathcal{Q}; \mathcal{B}) = |B_n|$$

By Theorem 2.12,

$$\#\operatorname{Hom}(\mathcal{J}_i^n \uparrow \mathcal{Q}; \mathcal{B}) =$$

$$\sum_{\alpha_j, \beta_j \in \mathcal{B}^j} M_\mathcal{B}(\beta_n, \beta_i) M_\mathcal{B}(\beta_i, \beta_{i-1}) M_\mathcal{B}(\beta_{i-1}, \alpha_{i-2}) M_\mathcal{B}(\beta_i, \alpha_{i-1}) M_\mathcal{B}(\alpha_{i-1}, \alpha_{i-2}) d_{\alpha_{i-2}} d_{\beta_n}$$

$$= \sum_{\alpha_j,\beta_j \in \mathcal{B}^j} M_{\mathcal{B}}(\beta_n,\beta_i)M_{\mathcal{B}}(\beta_i,\alpha_{i-2})M_{\mathcal{B}}(\beta_i,\alpha_{i-1})M_{\mathcal{B}}(\alpha_{i-1},\alpha_{i-2})d_{\alpha_{i-2}}d_{\beta_n}$$

$$\leq M_{\mathcal{B}}(B_i,B_{i-2}) \sum_{\alpha_j,\beta_j \in \mathcal{B}^j} M_{\mathcal{B}}(\beta_n,\beta_i)M_{\mathcal{B}}(\beta_i,\alpha_{i-1})M_{\mathcal{B}}(\alpha_{i-1},\alpha_{i-2})d_{\alpha_{i-2}}d_{\beta_n}.$$

Then by Corollary 2.2,

$$\begin{aligned} \#\,\mathrm{Hom}(\mathcal{J}_i^n \uparrow \mathcal{Q};\mathcal{B}) &= M_{\mathcal{B}}(B_i,B_{i-2})\langle D^n U^n \hat{0},\hat{0}\rangle \\ &= M_{\mathcal{B}}(B_i,B_{i-2})|B_n|. \end{aligned}$$

Simple counting in the Bratteli diagram for B_n (Lemma A.6 of Appendix A.4) shows that $M_{\mathcal{B}}(B_i,B_{i-2}) \leq 2$. Thus

$$\#\,\mathrm{Hom}(\mathcal{J}_i^n \uparrow \mathcal{Q};\mathcal{B}) \leq 2|B_n|.$$

Similarly, (using Corollary A.3 of Appendix A.4),

$$\#\,\mathrm{Hom}(\mathcal{H}_i^n \uparrow \mathcal{Q};\mathcal{B}) \leq \frac{4(i-1)}{n}|B_n|.$$

Finally, note that since a restricted product with the identity element e requires no operations to compute (see Corollary A.1 of Appendix A.2), for all $1 \leq i \leq n$,

$$|\tilde{A}_i| = |\tilde{A}_i'| = 1.$$

Plugging in to (2.4), we may compute $\sum_{y \in \tilde{Y}} yF_y$ in at most

$$\#\,\mathrm{Hom}(\mathcal{K}^n \uparrow \mathcal{Q};\mathcal{B}) + \sum_{i=2}^{n} \#\,\mathrm{Hom}(\mathcal{H}_i^n \uparrow \mathcal{Q};\mathcal{B}) + \sum_{j=2}^{n} \#\,\mathrm{Hom}(\mathcal{J}_j^n \uparrow \mathcal{Q};\mathcal{B}) = (4n-3)|B_n|$$

multiplications (and fewer additions). By Lemma 2.3,

$$t_{B_n}(R) \leq t_{B_{n-1}}(R_{B_{n-1}}) + 4n - 3,$$

and so

$$t_{B_n}(R) \leq n(2n-1).$$

□

Analogous arguments give the following result for Weyl groups of type D_n.

Theorem 2.16 (Theorem 2.3)**.** *For the Weyl group D_n and R a complete set of irreducible matrix representations of D_n adapted to the subgroup chain $D_n > D_{n-1} > \cdots > D_0 = \{e\}$,*

$$C(D_n) \leq T_{D_n}(R) \leq \frac{n(13n-11)}{2}|D_n|.$$

Proof. Let $s_1, \ldots, s_n$ denote the simple reflections for D_n, labeled according to Figure 2.29.

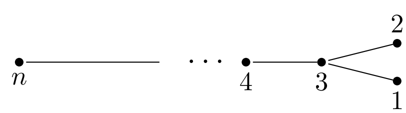


Figure 2.29

Recall from [65] that elements in a set of minimal coset representatives for D_n/D_{n-1} have the following factorizations:

$$e,\ s_n,\ s_{n-1}s_n,\ \ldots\ ,\ s_3\cdots s_n,\ s_2s_3\cdots s_n,\ s_1s_3\cdots s_n,$$

$$s_1s_2s_3\cdots s_n,\ s_3s_1s_2s_3\cdots s_n,\ \ldots\ ,\ s_n\cdots s_3s_2s_1s_3\cdots s_n.$$

Then for $A_i = \{e, s_i\} = A_i'$, the proof of Theorem 2.2 shows we need only determine $\#\operatorname{Hom}(\mathcal{H}_i^n \uparrow \mathcal{Q}; \mathcal{B})$, $\#\operatorname{Hom}(\mathcal{J}_i^n \uparrow \mathcal{Q}; \mathcal{B})$, and $\#\operatorname{Hom}(\mathcal{K}^n \uparrow \mathcal{Q}; \mathcal{B})$, for $\mathcal{H}_i^n, \mathcal{J}_i^n, \mathcal{K}^n$ the quivers of Figure 2.28. As before,

$$\#\operatorname{Hom}(\mathcal{K}^n \uparrow \mathcal{Q}; \mathcal{B}) = |D_n|.$$

By Lemma A.9 and Corollary A.4 of Appendix A.4,

$$\begin{aligned} &\#\operatorname{Hom}(\mathcal{J}_i^n \uparrow \mathcal{Q}; \mathcal{B}) \leq 3|D_n|, \\ &\#\operatorname{Hom}(\mathcal{H}_i^n \uparrow \mathcal{Q}; \mathcal{B}) \leq \tfrac{20(i-1)}{n}|D_n|, \end{aligned}$$

so by Theorem 2.11 we may compute $\sum yF_y$ in at most

$$\begin{aligned} &\#\operatorname{Hom}(\mathcal{K}^n \uparrow \mathcal{Q}; \mathcal{B}) + \textstyle\sum_{i=2}^n \#\operatorname{Hom}(\mathcal{H}_i^n \uparrow \mathcal{Q}; \mathcal{B}) + \textstyle\sum_{j=2}^n \#\operatorname{Hom}(\mathcal{J}_j^n \uparrow \mathcal{Q}; \mathcal{B}) \\ &= (13n - 12)|D_n| \end{aligned}$$

multiplications (and fewer additions). Then by Lemma 2.3,

$$t_{D_n}(R) \leq t_{D_{n-1}}(R_{D_{n-1}}) + 13n - 12,$$

and so

$$t_{D_n}(R) \leq \frac{n(13n - 11)}{2}.$$

□

2.4.2 The General Linear Group

Let $\mathbb{F}_q := \mathbb{F}_{p^k}$ be a finite field of characteristic p and order $q = p^k$. Let $Gl_n(q)$ denote the matrix group $Gl_n(\mathbb{F}_q)$ and consider $Gl_{n-1}(q)$ as a subgroup of $Gl_n(q)$ under the embedding

$$A \to \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix},$$

for $A \in Gl_{n-1}(q)$. In this section we show:

Theorem 2.17 (Theorem 2.4)**.** *For the matrix group $Gl_n(q)$ and R a complete set of irreducible matrix representations of $Gl_n(q)$ adapted to the subgroup chain $Gl_n(q) > Gl_{n-1}(q) > \cdots > \{e\}$*

$$C(Gl_n(q)) \leq T_{Gl_n(q)}(R) = O(q^n)|Gl_n(q)|.$$

Proof. Let P be the set of permutation matrices of $Gl_n(q)$. By Proposition A.2 in Appendix A.5, for $p \neq 2$,

$$Y = \{\pi s_i |\ 1 \leq i \leq n, (i-1) \text{ divisible by } p, \pi \in P\}$$

contains a complete set of coset representatives for $GL_n(q)/Gl_{n-1}(q)$, where s_i has form

$$u_2 \cdots u_{p-1} u'_{p+1} t_p u_{p+1} \cdots u_{2p-1} u'_{2p+1} t_{2p} u_{2p+1} \cdots u_i v_{i+1} \cdots v_n \epsilon,$$

for ϵ a scalar matrix, t_j the permutation matrix corresponding to $(j\ j-1)$, and

$$u_j, u'_j, v_j \in Gl_j(q) \cap \text{Centralizer}(Gl_{j-2}(q)),$$

with $(q-1)$ possible matrices for u_j and u'_j, and q^2 possible matrices for v_j.

Let U_j (respectively U'_j, V_j, E) be the set of matrices u_j (respectively u'_j, v_j, ϵ), and let $T_j = \{t_j\}$. Let $\tilde{Y} = \{\tilde{y} \mid y \in Y\}$, and similarly define $\tilde{U}_j, \tilde{U}'_j$, $\tilde{V}_j$, $\tilde{T}_j, \tilde{E}, \tilde{P}$. Note that

$$\tilde{U}_j, \tilde{U}'_j, \tilde{V}_j, \tilde{T}_j \in \mathbb{C}[\mathcal{B}_j] \cap \text{Centralizer}(\mathbb{C}[\mathcal{B}_{j-2}]),$$
$$E \in \text{Centralizer}(\mathbb{C}[\mathcal{B}_n]).$$

By Lemma 2.3 computation of the Fourier transform of a complex function f on $GL_n(q)$ is equivalent to computation of:

$$\sum_{y \in \tilde{Y}} y F_y \tag{2.5}$$

for $F_y \in \mathbb{C}[\mathcal{B}_{n-1}]$. The number of operations to compute (2.5) is bounded by the number of operations to compute

$$\sum_{\pi} \sum_{\substack{1 \le i \le n \\ p \mid (i-1)}} \sum_{\substack{u_j, u'_j, \\ v_j, t_j, \epsilon}} \pi u_2 \cdots v_{i+1} \cdots v_n \epsilon F_{\pi u_2 \cdots v_{i+1} \cdots v_n \epsilon}$$

$$= \sum_{\pi} \sum_{\epsilon} \pi\epsilon \sum_{\substack{1 \le i \le n \\ p \mid (i-1)}} \sum_{\substack{u_j, u'_j \\ v_j, t_j}} u_2 \cdots v_{i+1} \cdots v_n F_{\pi u_2 \cdots v_{i+1} \cdots v_n \epsilon}. \tag{2.6}$$

To compute (2.6), fix π and ϵ and compute:

$$\sum_{\substack{1 \le i \le n \\ p \mid (i-1)}} \sum_{\substack{u_j, u'_j \\ v_j, t_j}} u_2 \cdots v_{i+1} \cdots v_n F_{\pi u_2 \cdots v_{i+1} \cdots v_n \epsilon}, \tag{2.7}$$

then multiply by $\pi\epsilon$ and sum.

To compute sums of the form (2.7):

I. Let

$$X = \{(u_2, \ldots, u_{p-1}, u'_{p+1}, t_p, u_{p+1}, \ldots, u_i, v_{i+1}, \ldots, v_n, F_{u_2 \cdots v_n}) \mid i \leq n, i \mid (p-1)\},$$

for $u_j \in \tilde{U}_j, u'_j \in \tilde{U}'_j, v_j \in \tilde{V}_j, t_j \in \tilde{T}_j$.

III. Glue together the quivers $Q_{i^+i^-}$ (labeled by $\tilde{U}_j, \tilde{U}'_j, \tilde{V}_j, \tilde{T}_j, F_{u_2 \cdots v_n}$) of Figures 2.30 and 2.31 to build the quiver $\mathcal{Q}$ of Figure 2.32.

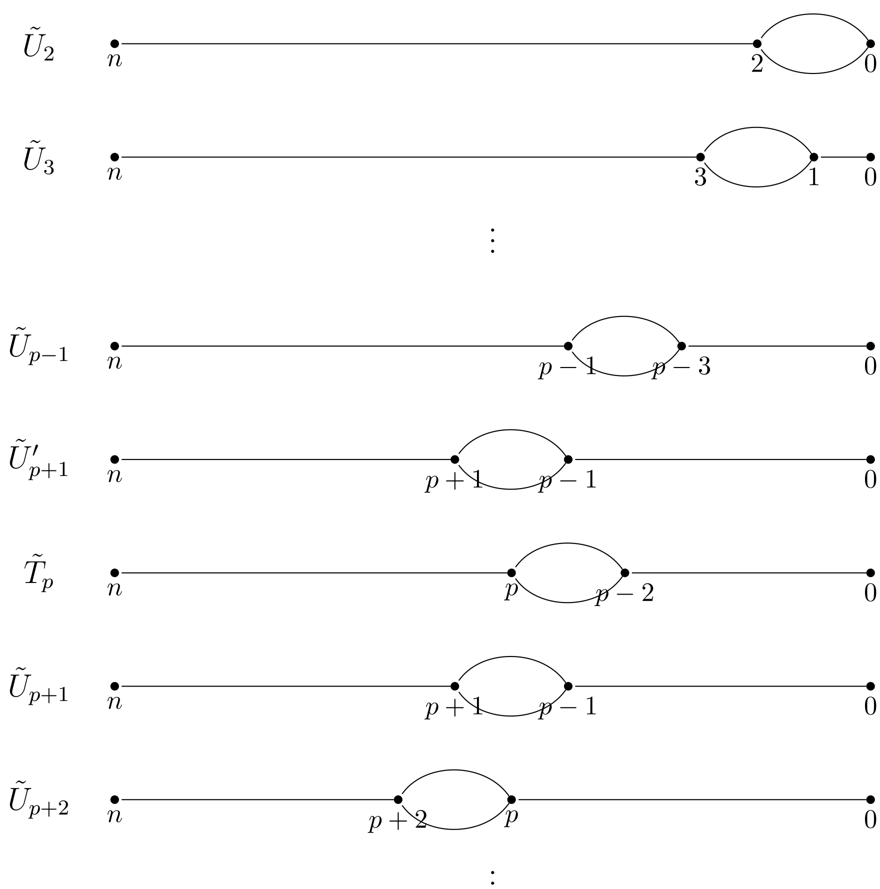


Figure 2.30

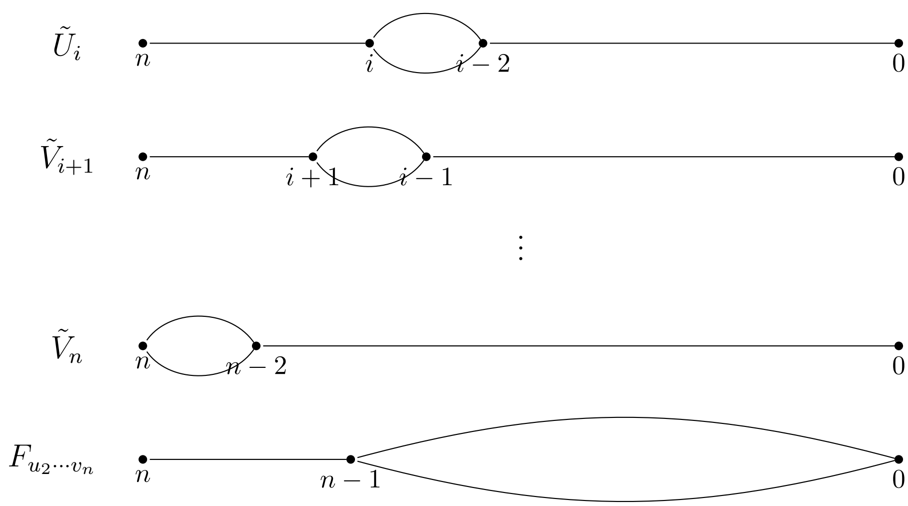


Figure 2.31

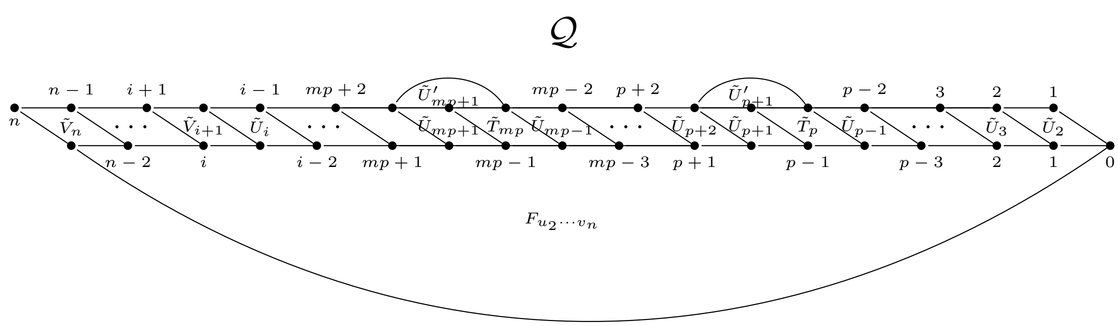


Figure 2.32

IV. Let $\sigma \in S_{n+m-1}$ be the permutation reordering X so that

$$\begin{aligned} X_0 &=: \{(x_{\sigma(1)}, x_{\sigma(2),\ldots,x_{\sigma(n+m-1)}})|\,(x_1, \ldots, x_{n+m-1}) \in X\} \\ &= \{(F_{u_2 \cdots v_n}, u_2, u_3, \ldots, u_{p-1}, t_p, u_{p+1}, u'_{p+1}, u_{p+2}, ..., u_i, v_{i+1}, \ldots, v_n)\}. \end{aligned}$$

Then

$$X_1 = \{(u_2, u_3, \ldots, u_{p-1}, t_p, u_{p+1}, u'_{p+1}, u_{p+2}, ..., u_i, v_{i+1}, \ldots, v_n)\}$$
$$X_2 = \{(u_3, \ldots, u_{p-1}, t_p, u_{p+1}, u'_{p+1}, u_{p+2}, ..., u_i, v_{i+1}, \ldots, v_n)\}$$
$$\vdots$$
$$X_{m+n-2} = \{(v_n)\}.$$

By Theorem 2.11, we may compute (2.7) in at most

$$\sum_{k=2}^{n} |X_{k-1}| \dim A(R^{\sigma}_{k-1} \cup F_{\sigma(k)} \uparrow \mathcal{Q}; \mathcal{B})$$

multiplications, with $R^{\sigma}_{k-1} \cup F_{\sigma(k)}$ as in Figure 2.33.

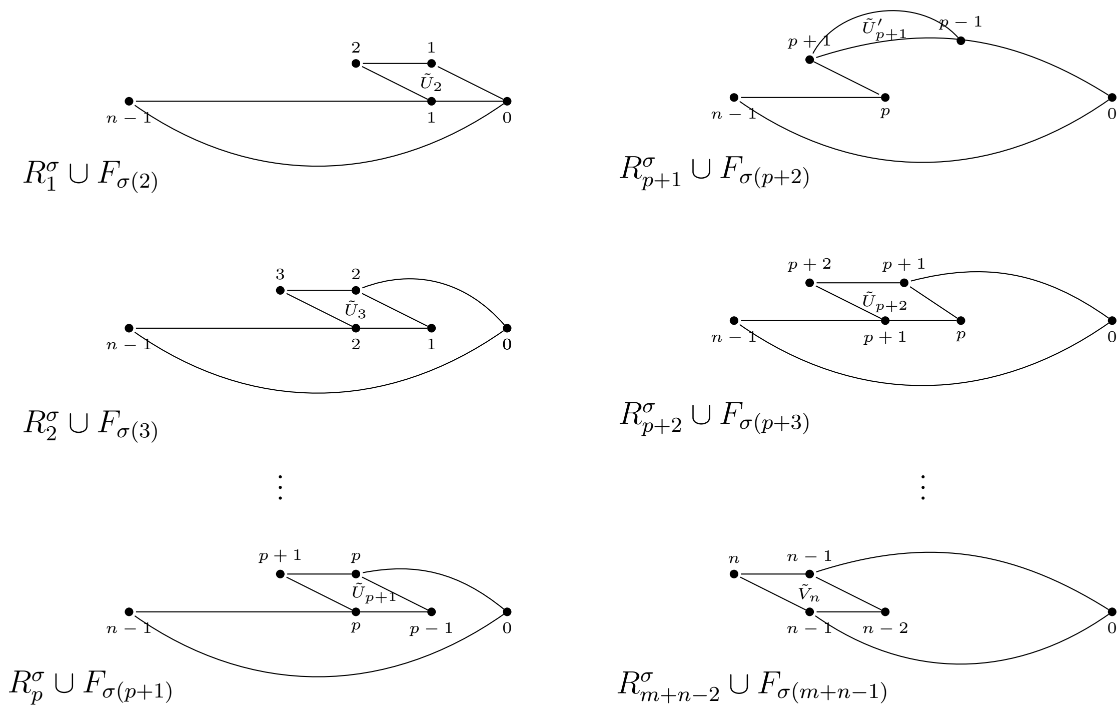


Figure 2.33

Then for $\mathcal{H}_j^n, \mathcal{J}_j^n$ the quivers of Figure 2.34,

$$\dim A(R_{k-1}^{\sigma} \cup F_{\sigma(k)} \uparrow \mathcal{Q}; \mathcal{B}) = \dim(\mathcal{H}_j^n; \mathcal{B}) \text{ or } \dim(\mathcal{J}_j^n; \mathcal{B}).$$

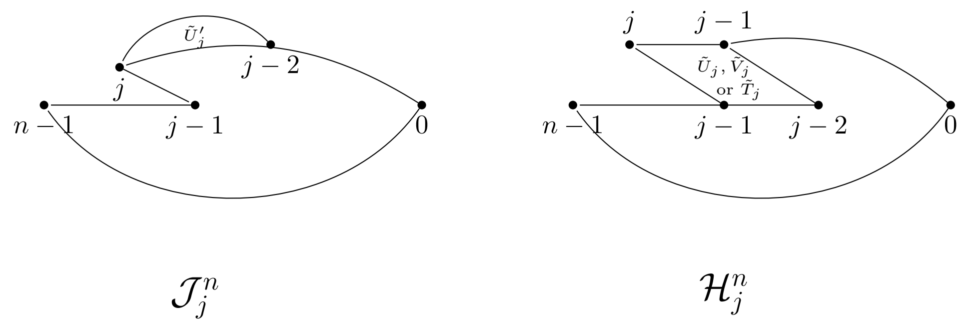


Figure 2.34

First consider the quiver $\mathcal{H}_j^n$ of Figure 2.34, which corresponds to:

$$\tilde{U}_j, \text{ when } 1 \leq j \leq i,\ p \nmid j$$
$$\tilde{T}_j, \text{ when } 1 \leq j \leq i,\ p \mid j,$$
$$\tilde{V}_j, \text{ when } i < j \leq n.$$

By Theorem 2.12,

$$\begin{aligned}\mathrm{Hom}(\mathcal{H}_j^n \uparrow \mathcal{Q}; \mathcal{B}) &= \sum_{\alpha_i, \beta_i \in \mathcal{B}^i} \Big(M_{\mathcal{B}}(\beta_{n-1}, \beta_{j-1}) M_{\mathcal{B}}(\beta_{j-1}, \beta_{j-2}) M_{\mathcal{B}}(\alpha_j, \beta_{j-1}) \\ &\qquad M_{\mathcal{B}}(\alpha_j, \alpha_{j-1}) M_{\mathcal{B}}(\alpha_{j-1}, \beta_{j-2}) d_{\alpha_{j-1}} d_{\beta_{n-1}} \Big) \\ &\leq M_{\mathcal{B}}(Gl_{j-1}(q), Gl_{j-2}(q))^2 |\hat{G}l_{j-2}(q)| \sum_{\alpha_i, \beta_i \in \mathcal{B}^i} \Big(M_{\mathcal{B}}(\beta_{n-1}, \beta_{j-1}) \\ &\qquad M_{\mathcal{B}}(\alpha_j, \beta_{j-1}) M_{\mathcal{B}}(\alpha_j, \alpha_{j-1}) d_{\alpha_{j-1}} d_{\beta_{n-1}} \Big).\end{aligned}$$

By Corollary 2.2,

$$\begin{aligned}
\#\,\mathrm{Hom}(\mathcal{H}_j^n \uparrow \mathcal{Q}; \mathcal{B}) &\leq M_{\mathcal{B}}(Gl_{j-1}(q), Gl_{j-2}(q))^2 |\hat{G}l_{j-2}(q)| \langle D^{n-1} U^{n-j} D U^j \hat{0}, \hat{0} \rangle \\
&= M_{\mathcal{B}}(Gl_{j-1}(q), Gl_{j-2}(q))^2 |\hat{G}l_{j-2}(q)| \lambda_j \lambda_{n-1} \lambda_{n-2} \cdots \lambda_1 \\
&= M_{\mathcal{B}}(Gl_{j-1}(q), Gl_{j-2}(q))^2 |\hat{G}l_{j-2}(q)| \frac{|Gl_j(q)|}{|Gl_{j-1}(q)|} |Gl_{n-1}(q)|.
\end{aligned}$$

By [65, Lemma 5.9], $M(Gl_j(q), Gl_{j-1}(q)) \leq 2^{j-1}$ and $|\hat{G}l_j(q)| \leq q^j$. Thus, since $\dim A(\mathcal{H}_j^n \uparrow \mathcal{Q}; \mathcal{B}) = \#\,\mathrm{Hom}(\mathcal{H}_j^n \uparrow \mathcal{Q}; \mathcal{B})$,

$$\begin{aligned}
\dim A(\mathcal{H}_j^n \uparrow \mathcal{Q}; \mathcal{B}) &\leq 2^{2j-4} q^{j-2} q^{j-1} (q^j - 1) |Gl_{n-1}(q)| \\
&= 2^{2j-4} q^{j-2} \frac{q^{j-1}(q^j - 1)}{q^{n-1}(q^n - 1)} |Gl_n(q)| \\
&= O(q^{3j-2n-2}) |Gl_n(q)|.
\end{aligned}$$

Further,

$$|\tilde{X}_j| := \begin{cases} \#(u_j, \ldots, u_i, v_{i+1}, \ldots, v_n) & 1 \leq j \leq i,\ p \nmid j, \\ \#(t_j, \ldots, u_i, v_{i+1}, \ldots, v_n) & 1 \leq j \leq i,\ p \mid j, \\ \#(v_j, \ldots, v_n) & i < j \leq n. \end{cases}$$

In particular,

$$|\tilde{X}_j| \leq \begin{cases} (q-1)^{i-j+1+\frac{i-1}{p}} (q^2)^{n-i} & 1 \leq j \leq i, \\ (q^2)^{n-j+1} & i < j \leq n, \end{cases}$$

and so for all quivers $R^{\sigma}_{k-1} \cup F_{\sigma(k)}$ of form $\mathcal{H}^n_j$,

$$|X_{k-1}| \dim A(R^{\sigma}_{k-1} \cup F_{\sigma(k)} \uparrow \mathcal{Q}; \mathcal{B}) = |\tilde{X}_j| \dim A(\mathcal{H}^n_j, \mathcal{B}) \leq O(q^n)|Gl_n(q)|.$$

Now consider the quiver $\mathcal{J}^n_j$ of Figure 2.34, which corresponds to $\tilde{U}_j$ for all $j \leq i$ with $p \mid (j-1)$:

$$\dim A(\mathcal{J}_j \uparrow \mathcal{Q}; \mathcal{B}) = \sum_{\alpha_i, \beta_i \in \mathcal{B}^i} M_{\mathcal{B}}(\beta_{n-1}, \beta_{j-1}) M_{\mathcal{B}}(\alpha_j, \beta_{j-1}) M_{\mathcal{B}}(\alpha_j, \alpha_{j-2})^2 d_{\alpha_{j-2}} d_{\beta_{n-1}}$$

$$\leq M_{\mathcal{B}}(Gl_j(q), Gl_{j-2}(q)) \sum_{\alpha_i, \beta_i \in \mathcal{B}^i} M_{\mathcal{B}}(\beta_{n-1}, \beta_{j-1}) M_{\mathcal{B}}(\alpha_j, \beta_{j-1}) M_{\mathcal{B}}(\alpha_j, \alpha_{j-2}) d_{\alpha_{j-2}} d_{\beta_{n-1}}$$

$$= M_{\mathcal{B}}(Gl_j(q), Gl_{j-2}(q)) \langle D^{n-1} U^{n-j} D U^j \hat{0}, \hat{0} \rangle$$
$$= M_{\mathcal{B}}(Gl_j(q), Gl_{j-2}(q)) \frac{|Gl_j(q)|}{|Gl_{j-1}(q)|} |Gl_{n-1}(q)|.$$

By [65, Lemma 5.9], $M(Gl_j(q), Gl_{j-2}(q)) \leq 2^{2j-3} q^{j-1}$. Thus,

$$\dim A(\mathcal{J}_j \uparrow \mathcal{Q}; \mathcal{B}) \leq 2^{2j-3} q^{j-1} \frac{q^{j-1}(q^j - 1)}{q^{n-1}(q^n - 1)} |Gl_n(q)| = O(q^{3j-2n-1})|Gl_n(q)|.$$

Further,

$$|\tilde{X}_j| := \#(u'_j, \ldots, u_i, v_{i+1}, \ldots, v_n) \leq \begin{cases} (q-1)^{i-j+1+\frac{i-1}{p}} (q^2)^{n-i} & j \neq n \\ (q-1) & j = i = n \end{cases}$$

and so for all quivers $R^{\sigma}_{k-1} \cup F_{\sigma(k)}$ of form $\mathcal{J}^n_j$,

$$|X_{k-1}| \dim A(R^{\sigma}_{k-1} \cup F_{\sigma(k)} \uparrow \mathcal{Q}; \mathcal{B}) = |\tilde{X}_j| \dim A(\mathcal{J}_j \uparrow \mathcal{Q}; \mathcal{B}) \leq O(q^n)|Gl_n(q)|.$$

Thus, by Theorem 2.11, we may compute (2.7) in at most $O(q^n)|Gl_n(q)|$ operations.

In order to compute (2.5) we must consider multiplication by ϵ and by π. Let $F_\epsilon \in \mathbb{C}[\mathcal{B}_n]$. To compute $\sum_\epsilon \epsilon F_\epsilon$, note that ϵ is a scalar matrix, and so a single multiplication ϵF_ϵ requires at most n^2 multiplications. Then $\sum_\epsilon \epsilon F_\epsilon$ requires at most $(q-1)n^2 < O(q^n)|Gl_n(q)|$ operations.

Finally, consider multiplication by $\pi \in \tilde{P}$. Let $F_\pi \in \mathbb{C}[\mathcal{B}_n]$. To compute $\sum_\pi \pi F_\pi$, note that π is a permutation matrix, and so every row and column contains exactly one nonzero entry, and that entry is 1. Then a single multiplication πF_π requires no multiplications, and so $\sum_\pi \pi F_\pi$ does not add to the complexity.

Now suppose $p = 2$. By Theorem A.5 in Appendix A.5,

$$Y = \{\pi s_i |\ 1 \leq i \leq n, (i-1) \text{ divisible by } p, \pi \in P\}$$

contains a complete set of coset representatives for $Gl_n(q)/Gl_{n-1}(q)$, where s_i is of form

$$a_3 b_2 c_3 a_5 b_4 c_5 \cdots a_i b_{i-1} c_i v_{i+1} \cdots v_n,$$

for $a_j, b_j, c_j \in Gl_j(q) \cap \text{Centralizer}(Gl_{j-2}(q))$ with $(q-1)$ possible matrices for a_j and b_j, q^2 possible matrices for v_j, and c_j completely determined by a_j and b_{j-1}. The same arguments as in the $p \neq 2$ case then yield the quiver $\mathcal{Q}$ of Figure 2.35, from which it is clear that analogous arguments show that the Fourier transform may be computed in $O(q^n)|Gl_n(q)|$ operations.

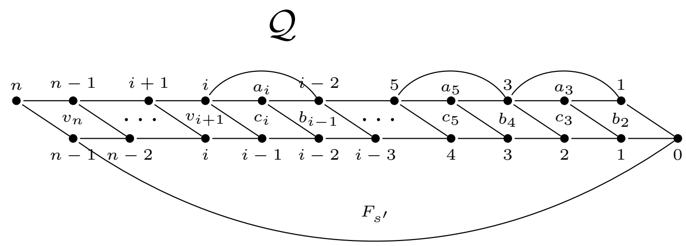


Figure 2.35

□

2.4.3 Generalized Symmetric Group Case

We next give a general result (Theorem 2.18) to find efficient Fourier transforms on groups with special subgroup structure. As the proof follows the same structure of the proofs of Theorems 2.24, 2.29, and 2.4, we leave it as an exercise.

Suppose

$$G_n > G_{n-1} > \cdots > G_0 = e,$$

is a chain of subgroups with subsets $A_i \subseteq G_i$ such that

(1) $A_1 = G_1$

(2) $G_i = A_2 \cdots A_i G_{i-1}$ for $2 \leq i \leq n$.

(3) A_i commutes with G_{i-2}.

Let $\mathcal{B}$ be the Bratteli diagram associated to the chain

$$\mathbb{C}[G_n] > \mathbb{C}[G_{n-1}] > \cdots > \mathbb{C},$$

and let $\{\mathbb{C}[\mathcal{B}_i]\}$ be the chain of path algebras associated to $\{\mathbb{C}[G_i]\}$. Let

$$M_{\mathcal{B}}(G_i, G_j) := \max M_{\mathcal{B}}(\alpha_i, \alpha_j)$$

over all $\alpha_i \in \mathcal{B}^i, \alpha_j \in \mathcal{B}^j$ and let $|\hat{G}_i|$ denote the number of conjugacy classes of G_i; equivalently, the number of irreducible representations in a complete set of inequivalent irreducible representations of G_i.

Theorem 2.18. *Let G_i, A_i be as described above. Then the Fourier transform of a complex function on G_n may be computed at a complete set R of irreducible representations of G_n adapted to the chain*

$$G_n > G_{n-1} > \cdots > G_0 = e$$

in at most

$$|G_n| \sum_{k=1}^{n} \sum_{i=2}^{k} M_{\mathcal{B}}(G_{i-1}, G_{i-2})^2 |\hat{G}_{i-2}| \frac{|G_i|}{|G_{i-1}|} \frac{|G_{k-1}|}{|G_k|} \prod_{j=i}^{k} |A_j|$$

operations.

Note 2.5. Theorem 2.18 is a refinement of Theorem 3.1 of [64]: rather than considering the maximum length of an element in a set of coset representatives, we build the coset representatives up using sets A_i of smaller size.

Note 2.6. For $G_i = S_i$, this theorem gives an efficient algorithm for the computation of the Fourier transform of a function on the symmetric group by letting $A_1 = \{e\}$ and $A_i = \{e, t_{i-1}\}$ for $2 \leq i \leq n$.

2.4.4 The Complexity of Fourier Transforms on Homogeneous Spaces

We next consider the Fourier transform of a function on a homogeneous space, a special case of harmonic analysis on groups. This can be viewed as a coset space G/K, so a Fourier transform on a homogeneous space is a Fourier transform of the space of functions on G/K or, equivalently, of the space of associated right-K invariant functions on G. See [63, 65] for further background on Fourier transforms on homogeneous spaces and some of their applications.

Definition 2.22. *Let G be a finite group with subgroup K and let f be a complex-valued function on G/K. The* ***Fourier transform of f*** *at a K-adapted representation ρ of G, denoted $\hat{f}(\rho)^K$ or a K-adapted set R of matrix representations of G, denoted $\mathcal{F}_R^K f$ is the Fourier transform of the right K-invariant function $\tilde{f} : G \to \mathbb{C}$ defined by*

$$\tilde{f}(g) = \frac{1}{|K|} f(gK).$$

Note that $\hat{f}(\rho)$ is zero unless the representation space, V_ρ, contains a nontrivial K-invariant vector. Such a representation is said to be **class 1 relative to K**, and we could restrict to class 1 representations if desired.

Definition 2.23. *Let G be a finite group with subgroup K and let R be a set of representations of G.*

(i) The ***arithmetic complexity*** *of a Fourier transform on R, denoted $T_{G/K}(R)$, is the minimum number of arithmetic operations needed to compute the Fourier transform of f on R via a straight-line program for an arbitrary complex-valued function f defined on G/K.*

(ii) The ***reduced complexity****, denoted* $t_{G/K}(R)$*, is defined by*

$$t_G(R) = \frac{1}{|G/K|} T_{G/K}(R).$$

Note that the complexity always satisfies the inequalities

$$|G/K| - 1 \leq T_{G/K}(R) \leq |G/K|^2.$$

Further, the proof of Lemma 2.3 gives an analogous result for the case of homogenous spaces: for H a subgroup of G, R a complete H-adapted set of inequivalent irreducible representations of G, $Y \subseteq G$ a set of coset representatives, and $\tilde{Y}$ the corresponding subset of the path algebra under the isomorphism of Lemma 2.1,

$$t_{G/K}(R) \leq t_{H/K}(R_H) + m_{G/K}(R, \tilde{Y}, H).$$

With this result, we obtain complexity results for homogeneous spaces as in the group case.

Let G be a group with chain of subgroups $G = G_n > G_{n-1} > \cdots > G_0$. For f a function on G_n that is right G_{n-k}-invariant, the corresponding element $\sum_{s \in G_n} f(s)s$ in $\mathbb{C}[G_n]$ is invariant under right multiplication by elements of $\mathbb{C}[G_{n-k}]$. In particular, the elements F_y in the proofs of Section 2.4 are $\mathbb{C}[\mathcal{B}_{n-k}]$-invariant, so as in [65, Theorem 6.2] the nonzero coefficients of F_y correspond to paths passing through 1_{n-k}. Using the SOV approach as in the proofs of Section 2.4, the final quiver in the construction of $\mathcal{Q}$, $Q_{i^+i^-}$, corresponding to F_y now has form as in Figure 2.36, with $\mathrm{Hom}(Q_{i^+i^-}; \mathcal{B})$ consisting only of morphisms sending α_{n-k} to 1_{n-k}.

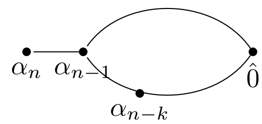


Figure 2.36

Then the proofs of Section 2.4 extend to the following results for homogenous spaces:

Theorem 2.19 (Theorem 2.5). *For the homogenous space B_n/B_{n-k} of the Weyl group B_n and R a complete set of irreducible matrix representations of B_n adapted to the subgroup chain $B_n > B_{n-1} > \cdots > \{e\}$,*

$$C(B_n/B_{n-k}) \leq T_{B_n/B_{n-k}}(R) \leq k(4n-2k-1)\frac{|B_n|}{|B_{n-k}|}.$$

Theorem 2.20. *For the homogenous space D_n/D_{n-k} of the Weyl group D_n and R a complete set of irreducible matrix representations of D_n adapted to the subgroup chain $D_n > D_{n-1} > \cdots > \{e\}$,*

$$C(D_n/D_{n-k}) \leq T_{D_n/D_{n-k}}(R) \leq \frac{k(26n-13k-11)}{2}\frac{|D_n|}{|D_{n-k}|}.$$

Theorem 2.21. *For the homogenous space $Gl_n(q)/Gl_{n-k}(q)$ of the general linear group $Gl_n(q)$ and R a complete set of irreducible matrix representations of $Gl_n(q)$ adapted to the subgroup chain $Gl_n(q) > Gl_{n-1}(q) > \cdots > \{e\}$,*

$$C(Gl_n(q)/Gl_{n-k}(q)) \leq T_{Gl_n(q)/Gl_{n-k}(q)}(R) \leq O(q^n)\frac{|G_n|}{|G_{n-k}|}.$$

As in Section 2.4.3, suppose

$$G_n > G_{n-1} > \cdots > G_1 = e,$$

is a chain of groups with subsets $A_i \subseteq G_i$ such that

(1) $A_1 = G_1$

(2) $G_i = A_2 \cdots A_i G_{i-1}$ for $2 \leq i \leq n$.

(3) A_i commutes with G_{i-2}.

Theorem 2.22. *Let G_i, A_i be as above. For the homogeneous space G_n/G_{n-k} and R a complete set of irreducible matrix representations of G_n adapted to the chain*

$$G_n > G_{n-1} > \cdots > G_1 = e$$

$$C(G_n/G_{n-k}) \leq T_{G_n/G_{n-k}}(R) \leq \sum_{j=n-k+1}^{n} \sum_{i=2}^{j} M_{\mathcal{B}}(G_{i-1}, G_{i-2})^2 |\hat{G}_{i-2}| \frac{|G_i|}{|G_{i-1}|} \frac{|G_{j-1}|}{|G_j|} \prod_{l=i}^{j} |A_l|$$

operations.

2.5 Extension to Semisimple Algebras

Recall that an algebra A is semisimple if A decomposes as a direct sum of simple algebras:

$$A \cong \bigoplus_{\lambda \in \Lambda} M_{d_\lambda}(\mathbb{C}),$$

for Λ a finite index set.

In this section we extend the SOV approach to general semisimple algebras and apply it to the Brauer and BMW algebras. Note that although we considered complex group algebras in Sections 2.1-2.4, for any field F, we may define the group algebra of a group G over F to be the F-linear span of elements in G. Then for a parameter q, the group algebra of the symmetric group over $\mathbb{C}(q)$ is a subalgebra of the Brauer algebra: $\mathbb{C}(q)[S_n] < \mathcal{B}r_n$, while the BMW algebra is a "deformation" of the Brauer algebra. As such, these results are natural extensions of Fourier transforms of functions on the symmetric group.

2.5.1 Background: The Brauer Algebra

First note that an element in the symmetric group S_n can be realized as a diagram on $2n$ points, consisting of two rows of n points each, with each point in the top row connected by an edge to exactly one point in the bottom row (see Figure 2.37). For two elements x, y in S_n, the product xy is the concatenation of the two diagrams: to compute the product xy, place the diagram for x on top of the one for y and trace the edges from top to bottom (note that we consider multiplication from left to right).

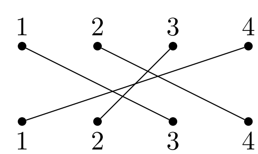


Figure 2.37: (1324)

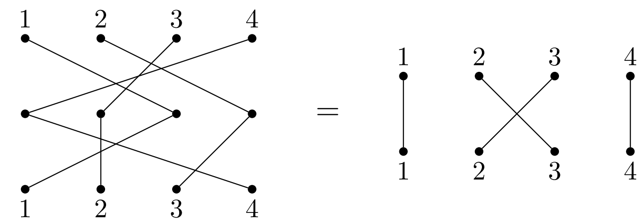


Figure 2.38: (1324)*(143)=(23)

The simple transpositions $\{r_i = (i\ i+1) \mid 1 \leq i \leq n-1\}$ form a generating set for the symmetric group.

Elements of the Brauer monoid, Br_n, are realized by generalizing symmetric group diagrams: consider diagrams on 2 rows of n points each, with edges connecting pairs of points regardless of row and each point part of exactly one edge. Multiplication may again be realized as concatenation of diagrams. Note that in some cases, this introduces a closed loop. A parameter q is used to keep track of the number of closed loops: for two diagrams $x, y \in Br_n$, let c denote the number of closed loops in the multiplication xy and let z be the diagram of this product with the closed loops removed. Then $xy = q^c z$.

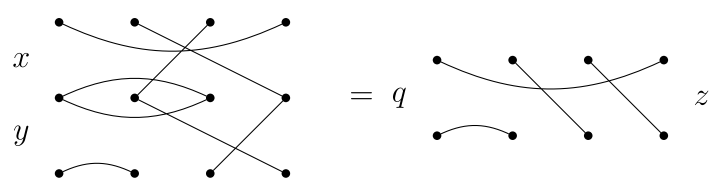
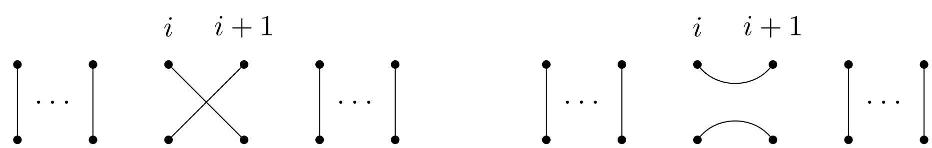


Figure 2.39: $xy = q^1 z$

Two Brauer diagrams d_1 and d_2 are *equivalent* if they differ only in the number of closed loops, i.e., if when $q = 1$, $d_1 = d_2$. For example, for x, y, z as in Figure 2.39, the product xy is equivalent to z. The Brauer monoid, Br_n consists of the set of equivalence classes of such diagrams and is generated by the set of elements $\{r_i, e_i \mid 1 \leq i \leq n-1\}$ (see Figure 2.40). Note that the symmetric group S_n is generated by the transpositions $\{r_i \mid 1 \leq i \leq n-1\}$ and so $\mathbb{C}(q)[S_n] < Br_n$.



r_i $\qquad$ e_i

Figure 2.40: $r_i, e_i \in Br_n$

The Brauer algebra, $\mathcal{B}r_n$, is the $\mathbb{C}(q)$-algebra with basis Br_n. Equivalently (see, e.g., [5]), $\mathcal{B}r_n$ has algebraic presentation given by generating set

$$\{r_i, e_i \mid 1 \leq i \leq n-1\},$$

and relations:

(1) $r_i^2 = 1,$ (2) $r_i r_j = r_j r_i, \quad r_i e_j = e_j r_i, \quad e_i e_j = e_j e_i, \ |i-j| > 1$

(3) $e_i^2 = q e_i,$ (4) $e_i r_i = r_i e_i = e_i,$

(5) $r_i r_{i+1} r_i = r_{i+1} r_i r_{i+1},$ (6) $e_i e_{i+1} e_i = e_i, \quad e_{i+1} e_i e_{i+1} = e_{i+1},$

(7) $r_i e_{i+1} e_i = r_{i+1} e_i,$ (8) $e_{i+1} e_i r_{i+1} = e_{i+1} r_i.$

In [93], Wenzl showed that the Brauer algebra, $\mathcal{B}r_n(q)$, is a semisimple algebra over $\mathbb{C}(q)$. In fact, replacing q by $\alpha \in \mathbb{C}$, $\mathcal{B}r_n(\alpha)$ is semisimple for all but finitely many integers α [83].

2.5.2 Generalized SOV Approach

For A a finite-dimensional semisimple algebra with basis D, recall that to compute the Fourier transform of $f = \sum_{d \in D} f(d)d$ we consider computation of the sums $\sum_{d \in D} f(d)\rho(d)$ for a complete set of inequivalent irreducible representations ρ of A. By Lemma 2.2, for a chain of subalgebras $A = A_n > \cdots > A_0$ this is equivalent to the computation of

$$\sum f(d)\tilde{d}$$

in the path algebra corresponding to the chain, by expressing d in coordinates with respect to a system of Gel'fand-Tsetlin bases.

Recall from Lemma 2.3 that for a finite group G with subgroup H, computation of a Fourier transform on G is equivalent to computation of

$$\sum_{y \in Y} \tilde{y} F_y,$$

for Y a set of coset representatives for G/H.

While there is no notion of coset representatives for a general chain of semisimple algebras, factorization through the chain still gives the result of Lemma 2.3 and so the SOV approach extends to this general case.

Let A be a finite-dimensional semisimple algebra. Recall that $T_A(R)$ is defined to be the minimum number of operations to compute the Fourier transform of f with respect to a set of representations R of A. Recall also that

$$t_A(R) := \frac{1}{\dim(A)} T_A(R).$$

Let $C(A) = \min_R\{T_A(R)\}$, where R varies over all complete sets of inequivalent irreducible matrix representations of A. As in the group case, direct computation of a Fourier transform on A gives $C(A) \leq \dim(A)^2$.

For A_1 a subalgebra of A let $\mathcal{B}$ be the Bratteli diagram of the chain $A > A_1 > A_0$, with path algebra chain $\mathbb{C}[\mathcal{B}_A] > \mathbb{C}[\mathcal{B}_{A_1}] > \mathbb{C}[\mathcal{B}_{A_0}]$. For Y a set of elements of A and F_y arbitrary elements of $C[\mathcal{B}_{A_1}]$, let

$$m_A(R, Y, A_{n-1}) = \frac{1}{\dim(A)} \times \begin{cases} \text{minimum number of operations required to compute} \\ \displaystyle\sum_{y \in \tilde{Y}} yF_y \text{ in a system of Gel'fand-Tsetlin bases for } \mathcal{B}. \end{cases}$$

Lemma 2.7. *Let A_1 be a subalgebra of A, R a complete A_1-adapted set of inequivalent irreducible matrix representations of A, D a basis for A, and D' a basis for A_1. Let $Y \subseteq D$ be a set of elements of A such that for all $d \in D$, $d = yd'$ for $y \in Y$, $d' \in D'$. Let $\mathcal{B}$ be the Bratteli diagram of the algebra chain $A > A_1 > A_0$ with corresponding path algebra chain $\mathbb{C}[\mathcal{B}_A] > \mathbb{C}[\mathcal{B}_{A_1}] > \mathbb{C}[\mathcal{B}_{A_0}]$. Then*

$$t_A(R) \leq t_A(R_{A_{n-1}}) + m_A(R, Y, A_{n-1}).$$

With Lemma 2.7, we use SOV algorithm 2.1 to compute the Fourier transform of functions on semisimple algebras.

2.5.3 The Brauer Algebra

Theorem 2.23. *The Fourier transform of an element f in the Brauer algebra $\mathcal{B}r_n$ may be computed at a complete set R of irreducible matrix representations of $\mathcal{B}r_n$*

adapted to the chain of algebras

$$\mathcal{B}r_n > \mathcal{B}r_{n-1} > \cdots > \mathcal{B}r_0 = \mathbb{C}(q)$$

in at most $(4n^2 - n + 4)\dim(\mathcal{B}r_n) = (4n^2 - n + 4)(2n - 1)!!$ *operations.*

Proof. A basis for the Brauer algebra consists of Brauer monoid elements: diagrams $d \in Br_n$ on $2n$ points as described in Section 2.5.1, generated by the elements e_i and r_i, $1 \leq i \leq n-1$. Consider the sets

$$R = \{1, r_1 \cdots r_{n-1}, r_2 \cdots r_{n-1}, \ldots, r_{n-1}\},$$
$$ER = \{r_j \cdots r_{i-1} e_i \cdots e_{n-1} \mid 1 \leq i \leq n-1, 1 \leq j \leq i-1\}$$

Then for $Y = R \cup ER$, Y satisfies the conditions of Lemma 2.7. To see this, first note that $R \subseteq S_n$ forms a complete set of coset representatives for S_n/S_{n-1} [63] and so we need only show that for any $d \in Br_n - S_n$, $d = yd'$, for $y \in Y$, $d' \in Br_{n-1}$. We view diagrams in Br_{n-1} as elements of Br_n by adding a point to the end of the top and bottom rows and connecting these two points with an edge.

Each element of ER has exactly one horizontal edge in the bottom row, and this edge connects the last two points in this row. Further, each element of ER has exactly one horizontal edge in the top row, and each possible such edge corresponds to an element of ER. As an example, see Figure 2.41.

Let $d \in Br_n - S_n$. Then d has at least one horizontal edge, e, in the top row. Choose the element, y, of ER with edge e. Then this determines an element d', in Br_{n-1} with $d = yd'$. For an example, see Figure 2.42.

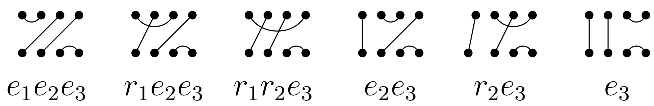


Figure 2.41: ER in Br_4

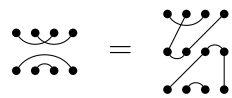


Figure 2.42: $d = r_1 e_2 e_3 d'$

Thus, Y satisfies the conditions of Lemma 2.7 and for $A_i = \{id, r_i, e_i\}$,

$$Y \subseteq \{a_1 a_2 \cdots a_{n-1} \mid a_i \in A_i\}.$$

Note that $r_i, e_i \in \mathcal{B}r_i \cap \text{Centralizer}\, \mathcal{B}r_{i-2}$.

For $\mathcal{B}$ the Bratteli diagram associated to the chain

$$\mathcal{B}r_n > \mathcal{B}r_{n-1} > \cdots > \mathcal{B}r_0 = \mathbb{C}(q),$$

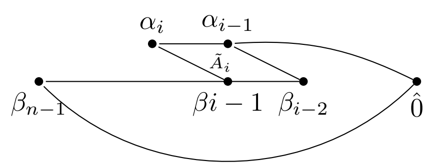
let $\{\mathbb{C}[\mathcal{B}_i]\}$ be the chain of path algebras associated to $\{\mathcal{B}r_i\}$. Let $\tilde{Y} = \{\tilde{y} \mid y \in Y\}$ and similarly define $\tilde{A}_i$. Note that $\tilde{A}_i \subseteq \mathbb{C}[\mathcal{B}_i] \cap \text{Centralizer}(\mathbb{C}[\mathcal{B}_{i-2}])$. By Lemma 2.7, the complexity of the computation of a Fourier transform of a complex function f on $\mathcal{B}r_n$ is bounded by the complexity of computation of:

$$\sum_{a_i \in \tilde{A}_i} a_1 \cdots a_{n-1} F_{a_1 \cdots a_{n-1}}.$$

Using the SOV approach, we may compute this sum in at most

$$\sum_{i=2}^{n} |X_{i-1}| \dim A(R_{i-1}^{\sigma} \cup F_{\sigma(i)} \uparrow \mathcal{Q}; \mathcal{B}) \tag{2.8}$$

multiplications, where for $\mathcal{H}_i^n$ the quiver of Figure 2.43, $\dim A(R_{i-1}^{\sigma} \cup F_{\sigma(i)} \uparrow \mathcal{Q}; \mathcal{B})=$ $\#\operatorname{Hom}(\mathcal{H}_i^n \uparrow \mathcal{Q}; \mathcal{B})$.



$\mathcal{H}_i^n$

Figure 2.43

By Corollary A.5 in Appendix A.4.3,

$$\#\operatorname{Hom}(\mathcal{H}_i^n \uparrow \mathcal{Q}; \mathcal{B}) \leq \frac{16i-17}{2n-1} \dim(\mathcal{B}r_n).$$

Then by Lemma 2.7 and Equation 2.8,

$$\begin{aligned} t_{\mathcal{B}r_n}(R) &\leq t_{\mathcal{B}r_{n-1}}(R_{\mathcal{B}r_{n-1}}) + 2\sum_{i=2}^{n} \frac{16i-17}{2n-1} \\ &= t_{\mathcal{B}r_{n-1}}(R_{\mathcal{B}r_{n-1}}) + 2\frac{(n-1)(8n-1)}{(2n-1)} \\ &\leq t_{\mathcal{B}r_1}(R_{\mathcal{B}r_1}) + 2\sum_{i=2}^{n} \frac{(n-1)(8n-1)}{(2n-1)} \\ &\leq t_{\mathcal{B}r_1}(R_{\mathcal{B}r_1}) + (4n^2 - n + 3) \\ &= 4n^2 - n + 4. \end{aligned} \tag{2.9}$$

□

2.5.4 The BMW Algebra

The BMW algebra is defined in a similar manner to the Brauer algebra (see Section 3.3.3 of Chapter 3). The Bratteli diagram for the BMW algebra is identical to that of the Brauer algebra. Further, a natural basis, $\mathcal{B}_n = \{T_d \mid d \in Br_n\}$ for the BMW algebra is indexed by Brauer monoid elements. As such, Theorem 2.23 extends to the BMW algebra:

Theorem 2.24. *The Fourier transform of an element f in the BMW algebra $\mathcal{BMW}_n$ may be computed at a complete set R of irreducible matrix representations of $\mathcal{BMW}_n$ adapted to the chain of algebras*

$$\mathcal{BMW}_n > \mathcal{BMW}_{n-1} > \cdots > \mathcal{BMW}_0$$

in at most $(4n^2 - n + 4)\dim(\mathcal{BMW}_n) = (4n^2 - n + 4)(2n-1)!!$ *operations.*

2.5.5 General Result

We next give a general result (Theorem 2.25) to find efficient Fourier transforms on a finite dimensional semisimple algebra A with special subalgebra structure. This is a generalization of Theorem 2.18.

Suppose

$$A = A_n > A_{n-1} > \cdots > A_0,$$

is a chain of subalgebras of A with subsets $B_i \subseteq A_i$ such that

(1) $B_1 = A_1$

(2) $A_i = B_2 \cdots B_i A_{i-1}$ for $2 \leq i \leq n$.

(3) B_i commutes with A_{i-2}.

Let $\mathcal{B}$ be the Bratteli diagram associated to the chain

$$A_n > A_{n-1} > \cdots > A_0,$$

and let $\{\mathbb{C}[\mathcal{B}_i]\}$ be the associated chain of path algebras. Let

$$M_{\mathcal{B}}(A_i, A_j) := \max M_{\mathcal{B}}(\alpha_i, \alpha_j)$$

over all $\alpha_i \in \mathcal{B}^i, \alpha_j \in \mathcal{B}^j$ and let $|\hat{A}_i|$ denote the number of irreducible representations in a complete set of inequivalent irreducible representations of A_i.

Theorem 2.25. *Let A_i, B_i be as described above. Then the Fourier transform of an element $f \in A$ may be computed at a complete set R of irreducible representations of A_n adapted to the chain*

$$A_n > A_{n-1} > \cdots > A_0$$

in at most

$$\dim(A_n) \sum_{k=1}^{n} \sum_{i=2}^{k} M_{\mathcal{B}}(A_{i-1}, A_{i-2})^2 |\hat{A}_{i-2}| \frac{\dim(A_i)}{\dim(A_{i-1})} \frac{\dim(A_{k-1})}{\dim(A_k)} \prod_{j=i}^{k} |B_j|$$

operations.

Chapter 3

Random Walks on the Birman-Murakami-Wenzl Algebra

3.1 Introduction

Markov Chain Monte Carlo (MCMC) methods are used widely to simulate and study stochastic systems. While the idea of using probabilistic simulation to understand the behavior of a system predates the existence of computers, it was not until after World War II that these methods began to take hold, as a focus on the development of nuclear weapons combined with access to a device that could carry out large-scale simulations enabled researchers to estimate the behavior of large collections of atomic particles through sampling (see [79] for a history of MCMC methods).

The main idea behind MCMC methods is that they provide algorithms for sampling from a probability distribution π by simulating some suitable random dynamics that converge to π. These dynamics are phrased in terms of *Markov chains*, stochastic processes that describe transitions between states. We describe Markov chains in more detail in Section 3.2.1, and further background can be found in many standard probability texts (see e.g. [33, 57]).

One of the earliest and most widely used MCMC method is the *Metropolis algorithm*. First introduced by Metropolis, Rosenbluth, Rosenbluth, Teller, and Teller to analyze the *hard spheres model* [68], a model of nonoverlapping molecules, the Metropolis algorithm gives a method for sampling from a probability distribution π by modifying an existing Markov chain $(K(x,y))_{x,y\in X}$ to produce a new Markov chain $(M(x,y))_{x,y\in X}$ with stationary distribution π. This proves particularly useful for simulating configurations of particles with an associated energy (e.g., the influence that neighboring particles exert on each other). Metropolis et al. noted that sampling from the uniform distribution for such systems is highly unrealistic, as physical laws dictate that low-energy configurations are far likelier to occur in practice. The Metropolis algorithm instead provided a way to sample from the *Boltzmann distribution*, which weights configurations by their likelihood of being observed. Later applications of the Metropolis algorithm include the simulation of Ising models, initially developed to model a ferromagnet but (surprisingly) also of use in image analysis, and Gibbs sampling [10, 34]. See [58] for additional applications.

The Metropolis algorithm has the advantage of being straightforward to construct and implement, despite the often complicated nature of the system under study. However, in analyzing the rate of convergence to π (the *mixing time*) rigorous bounds are often dependent on the specific situation (see [74] for a review of the existing literature for spin systems alone). Further, these methods are most often examples of *random update Markov chains* in that the process involved is that of selecting a site or set of sites to update at random. In other words, if the process of updating a particular site i can be characterized by the application of a Markov chain M_i, then a random scan has form $M_{i_l} \dots M_{i_2} M_{i_1}$ for a sequence $\{i_l\}$ chosen at random.

A more intuitively appealing and often more frequently used method in experimental work is that of a *systematic scan Markov chain*, a method to cycle through and update the sites in a deterministic order, i.e., a deterministic choice of the sequence $\{i_l\}$. For example, consider the toy problem presented in [25]: for $0 < \theta \leq 1$ try to fill n empty spaces $x_1, \dots, x_n$ with 0's and 1's so that the probability of a configuration with m 1's is $\theta^{n-m}(1-\theta)^m$. A random scan would pick spaces to fill at random, and take on the order of $\frac{1}{4} n \log n$ steps to converge, while a systematic approach would work from left to right, filling each space x_i with a θ-coin toss, and require order n steps. For examples of 'real' applications of systematic scans, see [10, 30, 52, 74].

Systematic scans often prove harder to analyze than random update chains due to the need to study the entire scan's effect and so understand all intermediary steps. Recently, results have begun to bring the upper bounds on mixing time for systematic scans closer to those on random scans [48, 73, 75], or in some cases have shown these bounds to be the same [25].

In [25], Diaconis and Ram consider the problem of comparing systematic scanning techniques with random scanning techniques in the context of generating elements of a finite Coxeter group W. They use the Metropolis algorithm to produce systematic scans for generating elements of W. To analyze these scans, they translate the chains into left multiplication operators in the *Iwahori-Hecke Algebra* corresponding to W, an algebra which can be thought of as a deformation of W. For a finite Coxeter group W generated by simple reflections $s_1, \dots, s_n$, the corresponding Iwahori-Hecke algebra is the algebra with basis $\{T_w \mid w \in W\}$ and multiplication

$$T_{s_i}T_w = \begin{cases} T_{s_i w} & l(s_i w) = l(w) + 1, \\ (q-1)T_w + qT_{s_i w}, & l(s_i w) = l(w) - 1, \end{cases}$$

for q an indeterminate and $l(w)$ the minimum number of reflections needed to express w [18].

Hecke algebras arise naturally in the extension of *Schur-Weyl duality* to general centralizer algebras. For $V \cong \mathbb{C}^n$, there exist natural actions of the groups $G = Gl_n(\mathbb{C})$ and $S = S_k$ on the k-fold tensor space $V^{\otimes k}$. In [85] Schur found that these actions are full centralizers of each other and so $V^{\otimes k}$ decomposes as a direct sum of simple $\mathbb{C}[G]$-modules tensored with simple $\mathbb{C}[S]$-modules.

This extends to more general algebras through the Double Centralizer Theorem (see e.g. [42] for a statement). In particular, for $G = Gl_n(\mathbb{F}_q)$ and B the subgroup of upper triangular matrices, let $L(G/B)$ be the space of right B-invariant functions on G. Then for H the Hecke algebra of the symmetric group, $L(G/B)$ decomposes as a direct sum of simple G-modules tensored with simple H-modules [18]. Even more generally, this statement holds for G a finite *Chevalley group*, B its *Borel subgroup*, and H the Hecke algebra corresponding to a finite Coxeter group W. A restatement shows that the Hecke algebra is in Schur-Weyl duality with the *quantum universal enveloping algebra* of the general linear group [50]. This duality has motivated study of the representation theory of the Hecke algebra, and more generalized versions of these algebras (see e.g. [1, 29, 41, 92, 78]).

More relevant for this thesis is an alternative definition of the Hecke algebra in terms of *braids*. The thesis [38] gives a thorough introduction to braids and their relationship with the Hecke algebra. Let $b_1, \ldots, b_n \in \mathbb{R}$ with $b_1 < \cdots < b_n$. An n-strand braid is a disjoint union of n smooth curves in $\mathbb{R}^3$ connecting the points $\{(b_1, 1, 0), (b_2, 1, 0), \ldots, (b_n, 1, 0)\}$ with $\{(b_1, 0, 0), (b_2, 0, 0), \ldots, (b_n, 0, 0)\}$ so that they intersect each parallel plane $y = t$ as t ranges between 0 and 1 only once. A braid can be represented by its 2-dimensional projection, its *braid diagram*, and connecting the top strands to the bottom strands of a braid diagram gives rise to a *link*. Two links are *isotopic* if they are related by a sequence of Reidemeister moves (defined in Section 3.3.3), and, in fact, every isotopic oriented link can be represented by the closure of a braid [38]. The braid group has a natural presentation in terms of generators $T_{r_1}, \ldots, T_{r_{n-1}}$ corresponding to certain braid diagrams. Remarkably, adding a quadratic relation to this presentation yields the Hecke algebra, and in this case the elements T_{r_i} are exactly those of Figure 3.4 in Setion 3.3.3.

Under this definition of the Hecke algebra there is a natural generalization to the BMW algebra. By generalizing the notion of an n-strand braid to allow any two points in $\{(b_1, 1, 0), (b_2, 1, 0), \ldots, (b_n, 1, 0)\} \cup \{(b_1, 0, 0), (b_2, 0, 0), \ldots, (b_n, 0, 0)\}$ to be connected, we have the definition of an *n-tangle*, which gives rise to the idea of a *tangle diagram* by considering its two-dimensional projection. We define tangle diagrams in detail in Section 3.3.3. As with the algebra associated to braid diagrams, an algebra is naturally associated to these tangle diagrams. Defined independently as the *Kauffman tangle algebra* by Murakami [70] and algebraically by Birman and Wenzl [7], it was shown in an unpublished paper by Wasserman [69] that these two notions are equivalent, giving rise to the single BMW algebra.

Further, just as the Hecke algebra is in Schur-Weyl duality with the quantum universal enveloping algebra of the general linear group, the BMW algebra is in

Schur-Weyl duality with the quantum universal enveloping algebra of the orthogonal group [45]. This duality has motivated the study of the representation theory of the BMW algebra and its generalizations [16, 21, 22, 45, 56].

A third perspective would note that just as the Hecke algebra is a q-analogue of the symmetric group, the BMW algebra is a q-analogue of the Brauer algebra, as defined in 2.5.1.

As discussed above, in [25], Diaconis and Ram consider the problem of systematically generating elements of a finite Coxeter group W. In terms of the group algebra $\mathbb{C}[W]$, this problem is equivalent to generating elements of the basis W of $\mathbb{C}[W]$. We extend these ideas to the BMW algebra. The Metropolis algorithm in this context gives rise to systematic scanning strategies for generating basis elements via multiplication of generators. As the diagrams forming the BMW monoid basis of the BMW algebra are tangles, scanning strategies for generating BMW monoid elements have applications arising in physics: random generation of links and tangles has been of use in [28, 60, 96]. As in [25], our algorithm gives rise to a natural random walk, in this case on the *BMW* and *Brauer monoids*, defined in Sections 3.3.2 and 3.3.3. We translate the random walk into a multiplication operator in the BMW algebra and develop a trace form to study it. This enables the use of tools from representation theory and Fourier analysis to analyze the time to stationarity of such walks.

3.2 Preliminaries: Probability Theory

Background on Markov chains can be found in many standard probability texts (see eg [33]). The book of Levin, Peres, and Wilmer [57] gives a particularly thorough introduction to Markov chains, including classification of states and the Metropolis algorithm, while [25] gives a concise introduction to the probabilistic background needed for this chapter. We will follow the notation and outline of [25].

3.2.1 Markov Chains

A finite Markov chain with state space X is a process that moves among states in X such that the conditional probability of moving from state x to state y is independent of the preceding sequence of states. More formally:

Definition 3.1. *A* ***Markov chain*** *on a finite set X is a matrix $K = (K(x,y))_{x,y\in X}$ such that $K(x,y) \in [0,1]$ and for all $x \in X$,*

$$\sum_{y\in X} K(x,y) = 1.$$

We call X the ***state space****.*

Note that $K(x,y)$ gives the probability of moving from x to y in one step, while $K^m(x,y)$ gives the probability of moving from x to y in m steps.

Definition 3.2. *A Markov chain K is* ***irreducible*** *if for each $x, y \in X$, there exists an integer m such that $K^m(x,y) > 0$. Let $T(x)$ denote the minimum number of steps for the chain to return to x, i.e. the minimum t such that $K^t(x,x) > 0$. Then K is* ***aperiodic*** *if*

$$\gcd_x(T(x)) = 1.$$

Note that if K is irreducible and aperiodic, there exists an integer r such that $K^r(x,y) > 0$ for all $x, y \in X$ [57, Proposition 1.7].

Definition 3.3. *A Markov chain is **reversible** if there exists a probability distribution $\pi : X \to [0,1]$ such that for all $x, y \in X$,*

$$\pi(x)K(x,y) = \pi(y)K(y,x).$$

*We call π the **stationary distribution** of K.*

An irreducible, aperiodic, reversible Markov chain K converges to its stationary distribution:

$$\lim_{l\to\infty} K^m(x,y) = \pi(y).$$

The Metropolis construction introduced in Section 3.2.2 produces a reversible Markov chain with a chosen stationary distribution. Our interest is in the time to stationarity of such chains.

Definition 3.4. *Let K_x^m denote the probability distribution $K^m(x,\cdot)$. The **total variation distance** from K_x^m to π is*

$$|K_x^m - \pi|_{TV} := \max_{A\subseteq X} |\sum_{y\in A} K^m(x,y) - \pi(y)|.$$

For $L^2(\pi)$ the space of functions $f : X \to \mathbb{R}$, equipped with the inner product

$$\langle f, g\rangle_2 = \sum f(x)g(x)\pi(x),$$

the total variation distance is bounded by the $L^2(\pi)$ norm:

Lemma 3.1. *[25, Lemma 2.3] For* $f \in L^2(\pi)$*,*

$$|f|_{TV}^2 \leq \frac{1}{4}\|f/\pi\|_2^2,$$

where $f/\pi(x) = 0$ *if* $\pi(x) = 0$*.*

3.2.2 The Metropolis Algorithm

Given a symmetric Markov chain P and a probability distribution π, the Metropolis algorithm modifies P to produce a reversible Markov chain M with stationary distribution π:

$$M(x,y) = \begin{cases} P(x,y) & \text{if } x \neq y \text{ and } \pi(y) \geq \pi(x), \\ P(x,y)\dfrac{\pi(y)}{\pi(x)} & \text{if } x \neq y \text{ and } \pi(y) < \pi(x), \\ P(x,x) + \displaystyle\sum_{\pi(z)<\pi(x)} P(x,z)\left(1 - \frac{\pi(z)}{\pi(x)}\right) & \text{if } x = y. \end{cases}$$

As an example, in [25], Diaconis and Ram consider the Markov chain that arises from applying the Metropolis algorithm to the usual random walk on the symmetric group based on a generating set. We consider this chain in further detail in Section 3.4.

The Metropolis algorithm yields a reversible Markov chain with stationary distribution π; however, irreducibility and aperiodicity is not guaranteed. In particular, the Markov chains we consider in Section 3.4 are aperiodic but not irreducible. To analyze these chains we consider their *closed communication classes*.

Definition 3.5. *Let* K *be a Markov chain with state space* X*. For* $x, y \in X$*,* y ***is***

accessible *from x, denoted $x \to y$, if x can reach y in finitely many steps. We say x* ***communicates with*** *y, denoted $x \leftrightarrow y$, if $x \to y$ and $y \to x$. The equivalence classes under the relation $\leftrightarrow$ are the* ***communication classes*** *of K. A communication class C is* ***closed*** *if for $x \in C$ and for all $y \notin C$, y is not accessible from x.*

Note that studying the time to stationarity of a reversible, aperiodic Markov chain K reduces to studying the time to stationarity of the closed communication classes of K.

3.2.3 Systematic Scans

The Metropolis algorithm, in the context of generating elements of a group, provides systematic and random scanning strategies. For example, for each generator $r_i = (i\; i+1)$ of S_n, let

$$P_i(x,y) = \begin{cases} 1 & \text{if } y = r_i x, \\ 0 & \text{else.} \end{cases}$$

Then for l_S the length function on words in S_n, let π be the probability distribution

$$\pi(x) = \frac{\theta^{-l_S(x)}}{\sum_{w \in S_n} \theta^{-l_S(w)}}.$$

The Metropolis algorithm construction then produces Markov chains $M_1, M_2, \ldots, M_{n-1}$ corresponding to multiplication by the generators $r_1, \cdots, r_{n-1}$. For an explicit description see Section 3.4.

A choice of infinite sequence $\{i_l\}_{l=1}^{\infty}$ gives a scanning strategy:

$$\cdots M_{i_l} M_{i_{l-1}} \cdots M_{i_1}.$$

For M_i reversible, each with stationary distribution π, the following systematic scans produce reversible Markov chains with stationary distribution π (see, eg [25]):

$$
\begin{aligned}
&\frac{1}{n-1}\sum_{i=1}^{n-1} M_i && \text{(random scan)},\\
&M_1M_2\cdots M_{n-1}M_{n-1}\cdots M_2M_1 && \text{(short systematic scan)},\\
&(M_1\cdots M_{n-1}M_{n-1}\cdots M_1)\cdots(M_1M_2M_2M_1)(M_1M_1) && \text{(long systematic scan)}.
\end{aligned}
$$

While such scanning strategies may seem intuitive for sampling from π, they have proven difficult to analyze in many situations. In the context of generation of Coxeter group elements, Diaconis and Ram [25] show that convergence of the short systematic scan for the distribution π above, with l_S replaced by the length function on the Coxeter group coming from writing words as a product of simple reflections, occurs in the same number of steps as that of a random scan, i.e., choosing a random sequence of indices $\{i_\ell\}_{\ell=1}^{\infty}$. However, results for different scanning techniques or probability distributions remain open. In the context of graph colorings, Dyer et al. compare systematic scans with random scans for sampling proper q-colorings of paths for $q \geq 4$, in which a vertex is assigned a new color c only if none of its neighbors are colored by c [30]. However, results for more general graphs have resisted analysis.

Fishman [34] gives an overview of scanning strategies, while Diaconis and Saloff-Coste's survey [27] provides further applications of the Metropolis algorithm.

3.3 Preliminaries: Semisimple Algebras

3.3.1 Fourier Inversion and Plancherel

Recall from Chapter 2 that for A a semisimple algebra with basis $\{a_i\}_{i \in I}$, $f = \sum_{i \in I} f(a_i)a_i \in A$, and $\rho : A \to M_d(\mathbb{C})$ a matrix representation of A, the Fourier transform of f at ρ is the matrix sum:

$$\hat{f}(\rho) = \sum_{i \in I} f(a_i)\rho(a_i).$$

Random walks on groups are frequently studied using Fourier analysis. For example:

Theorem 3.1 (Diaconis, [24])**.** *For G a group, Q a probability distribution on G, and U the uniform distribution on G,*

$$|Q - U|_{TV}^2 \leq \frac{1}{4} \sum_{\rho} d_\rho \operatorname{Tr}(\hat{Q}(\rho)\hat{Q}(\rho)^*),$$

where $*$ denotes conjugate transpose and the sum is over all nontrivial irreducible representations ρ of G.

Diaconis and Ram find similar such bounds in their analysis of the Metropolis algorithm applied to the symmetric group [25]. Both Theorem 3.1 and the results of [25] require the notion of Fourier inversion and and Plancherel's Theorem (see Theorem 3.2 below).

Definition 3.6. *For A a semisimple algebra, a **trace** function on A is a $\mathbb{C}$-linear* *function* $\tau : A \to \mathbb{C}$ *such that for all* $a, b \in A$,

$$\tau(ab) = \tau(ba).$$

Note that by linearity the usual trace function on $M_d(\mathbb{C})$ is in fact unique up to multiplication by a constant. Hence, for any trace τ on A and set R of inequivalent irreducible representations of A, there exist constants $t_\rho \in \mathbb{C}$ such that:

$$\tau = \sum_{\rho \in R} t_\rho T_\rho,$$

where for $a \in A$, $T_\rho(a) = \mathrm{Tr}(\rho(a))$.

A trace function τ gives rise to a symmetric bilinear form $\langle \cdot, \cdot \rangle_\tau : A \times A \to \mathbb{C}$ by letting

$$\langle a, b \rangle_\tau = \tau(ab),$$

for $a, b \in A$.

Theorem 3.2 (Diaconis, Ram [25])**.** *Let* A *be a semisimple algebra with basis* $\{a_i\}$ *and* τ *a nondegenerate trace on* A*. Let* $\{a_i^*\}$ *be the dual basis to* $\{a_i\}$ *with respect to the trace form* $\langle \cdot, \cdot \rangle_\tau$*. Then for* f, f_1, f_2 *complex-valued functions on* A*,*

$$f(a_i) = \sum_\rho t_\rho \, \mathrm{Tr}(\hat{f}(\rho)\rho(a_i^*)), \tag{3.1}$$

$$\langle f_1, f_2 \rangle_\tau = \sum_\rho t_\rho \, \mathrm{Tr}(\hat{f}_1(\rho)\hat{f}_2(\rho)). \tag{3.2}$$

3.3.2 The Brauer Algebra

Recall from Chapter 2, Section 2.5.1, that an element in the symmetric group S_n can be realized as a diagram on $2n$ points, consisting of two rows of n points each, with each point in the top row connected by an edge to one in the bottom row. Multiplication in the symmetric group is realized as concatenation of diagrams. The simple transpositions $\{r_i = (i\ i+1) \mid 1 \leq i \leq n-1\}$ form a generating set for the symmetric group, allowing us to define a natural length function $l_S : S_n \to \mathbb{N}$ on the symmetric group: for $w \in S_n$, let $l_S(w)$ be the minimum number of generators needed to express w.

Elements of the Brauer monoid, Br_n, are realized as generalized symmetric group diagrams: consider diagrams on 2 rows of n points each, with edges connecting pairs of points regardless of row and each point part of exactly one edge. Multiplication may again be realized as concatenation of diagrams. Note that in some cases, this introduces a closed loop. A parameter q is used to keep track of the number of closed loops: for two diagrams $x, y \in Br_n$, let c denote the number of closed loops in the multiplication xy and let z be the diagram of this product with the closed loops removed. Then $xy = q^c z$.

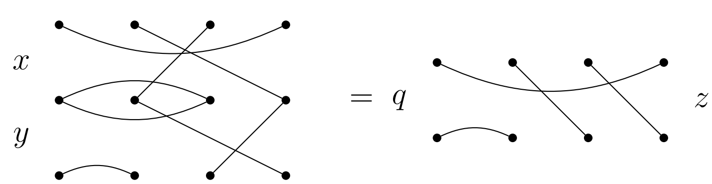


Figure 3.1: $xy = q^1 z$

Two Brauer diagrams d_1 and d_2 are equivalent if they differ only in the number of closed loops, i.e., if when $q = 1$, $d_1 = d_2$. For example, for x, y, z as in Figure

3.1, the product xy is equivalent to z. The Brauer monoid, Br_n consists of the set of equivalence classes of such diagrams and is generated by $\{r_i, e_i \mid 1 \leq i \leq n-1\}$ (see Figure 2.40). The symmetric group S_n, generated by the transpositions $\{r_i \mid 1 \leq i \leq n-1\}$, sits inside of Br_n. As in the symmetric group, a natural length function $l_{Br} : Br_n \longrightarrow \mathbb{N}$ exists for the Brauer monoid: for $w \in Br_n$, define $l_{Br}(w)$ to be the minimum number of generators ($\{r_i, e_i\}$) needed to express w.

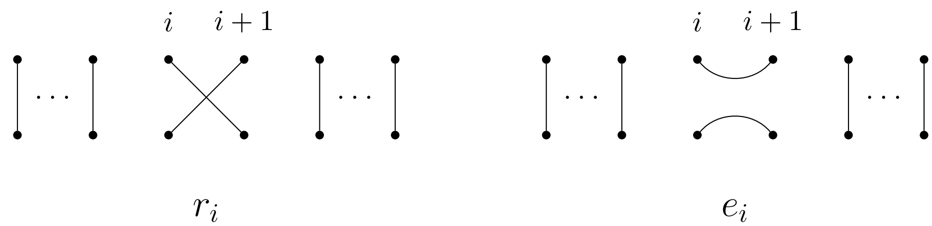


Figure 3.2: $r_i, e_i \in Br_n$

The Brauer algebra, $\mathcal{B}r_n$, is the $\mathbb{C}(q)$-algebra with basis Br_n. Equivalently (see, for example [5]), $\mathcal{B}r_n$ has algebraic presentation given by generating set

$$\{r_i, e_i \mid 1 \leq i \leq n-1\},$$

along with relations:

(1) $r_i^2 = 1$,	(2) $r_i r_j = r_j r_i, \quad r_i e_j = e_j r_i, \quad e_i e_j = e_j e_i, \;\; \lvert i-j\rvert > 1$
(3) $e_i^2 = q e_i$,	(4) $e_i r_i = r_i e_i = e_i$,
(5) $r_i r_{i+1} r_i = r_{i+1} r_i r_{i+1}$,	(6) $e_i e_{i+1} e_i = e_i, \quad e_{i+1} e_i e_{i+1} = e_{i+1}$,
(7) $r_i e_{i+1} e_i = r_{i+1} e_i$,	(8) $e_{i+1} e_i r_{i+1} = e_{i+1} r_i$.

3.3.3 The BMW Algebra

Elements of the BMW monoid are realized as generalized Brauer diagrams called **tangles**. A tangle is again a diagram on 2 rows of n points each with edges connecting pairs of points regardless of row with each point part of exactly one edge. At each crossing of two edges we distinguish which edge passes above and which passes below (see Figure 3.3). As in the Brauer monoid, multiplication is concatenation of diagrams and two tangles are equivalent if they differ only in their number of closed loops.

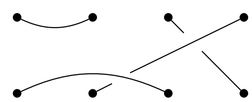
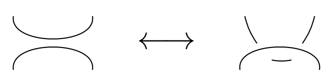
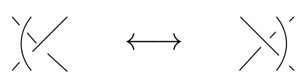


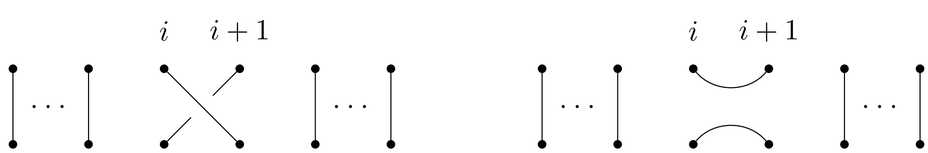
Figure 3.3: A Tangle

Further, two tangles are equivalent if they are related by a sequence of Reidemeister moves of type II and III:

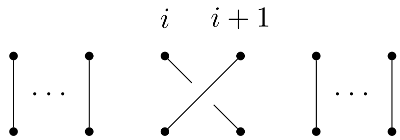
R_{II}: 

R_{III}: 

Consider the elements T_{r_i}, $T_{r_i}^{-1}$, and T_{e_i} of Figure 3.4.



T_{r_i} $\qquad$ T_{e_i}

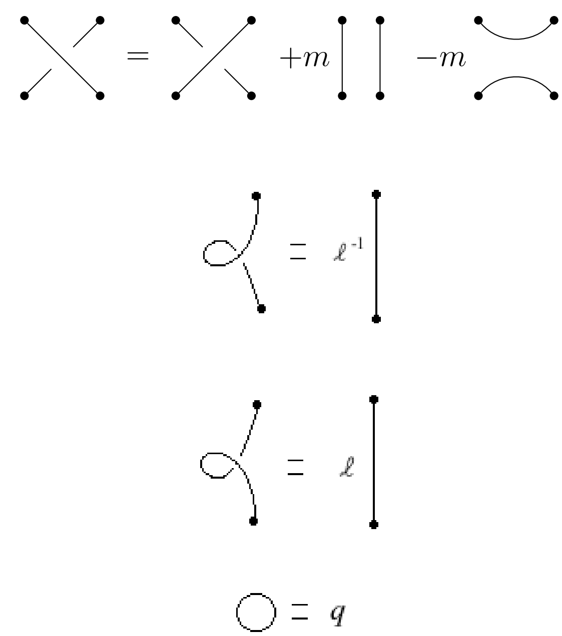


$T_{r_i}^{-1}$

Figure 3.4: $T_{r_i}, T_{e_i}, T_{r_i}^{-1}$

A tangle is **reachable** if it can be obtained as a finite product of elements from $\{T_{r_i}, T_{e_i}, T_{r_i}^{-1} \mid 1 \leq i \leq n-1\}$. The BMW monoid, BMW_n, consists of the set of equivalence classes of reachable tangles on $2n$ points.

For m, ℓ, q parameters satisfying $q = (\ell - \ell^{-1})(m - m^{-1})^{-1} + 1$, the BMW algebra, $\mathcal{BMW}_n$, is the $\mathbb{C}(q, m, \ell)$-algebra with basis BMW_n and the following untangling relations:



Equivalently (see, for example [40]), the BMW algebra has algebraic presentation given by generating set $\{T_{e_i}, T_{r_i}, T_{r_i}^{-1} \mid 1 \leq i \leq n-1\}$, along with relations:

$$(A1) \quad T_{e_i}^2 = qT_{e_i}, \qquad (A2) \quad T_{e_i}T_{r_i} = T_{r_i}T_{e_i} = \ell^{-1}T_{e_i}$$

$$(A3) \quad T_{e_i}T_{e_{i\pm1}}T_{e_i} = T_{e_i}, \qquad (A4) \quad T_{e_i}T_{r_{i\pm1}}T_{e_i} = \ell T_{e_i},$$

$$(A5) \quad T_{r_i}T_{r_{i+1}}T_{r_i} = T_{r_{i+1}}T_{r_i}T_{r_{i+1}}, \qquad (A6) \quad T_{r_i}T_{r_{i\pm1}}T_{e_i} = T_{e_{i\pm1}}T_{e_i} = T_{e_{i\pm1}}T_{r_i}T_{r_{i\pm1}},$$

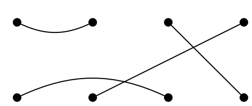
$$(A7) \quad T_{r_i} = T_{r_i}^{-1} + mT_{id} - mT_{e_i} \qquad (A8) \quad T_{r_i}T_{r_j} = T_{r_j}T_{r_i}, \quad T_{r_i}T_{e_j} = T_{e_j}T_{r_i},$$

$$T_{e_i}T_{e_j} = T_{e_j}T_{e_i}, \quad |i-j| > 1,$$

for $q = (\ell - \ell^{-1})(m - m^{-1})^{-1} + 1$ and T_{id} the identity element.

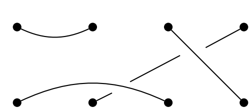
We map an element of the BMW monoid to the Brauer monoid by 'forgetting' the crossing information. Denote this map by $\phi : BMW_n \longrightarrow Br_n$.

Example 3.1. For x the tangle of Figure 3.3, $\phi(x)$ has form:



Further, each element of the Brauer monoid lifts to the BMW algebra: for $d \in Br_n$, the **BMW image** of d, T_d, realizes d as a tangle by redrawing the edges of d from right to left across the first $\lceil \frac{n}{2} \rceil$ points in the bottom row, lifting the pen when crossing an edge that has already been drawn, then moving to the top row of points and drawing all horizontal edges in this row, again lifting the pen when crossing an edge that has already been drawn, and finally drawing the remaining edges of d from right to left across the bottom row of points.

Example 3.2. For d the Brauer diagram of Example 3.1, the BMW image of d is:



Note that when $\ell = 1$ the BMW image of d has a simple algebraic description: for $d \in Br_n$ and $s_i \in \{r_i, e_i\}$, define a **reduced expression** for d to be an expression $d = s_{i_1} s_{i_2} \cdots s_{i_k}$ for d that has each occurrence of $e_{i\pm1} r_i$ replaced by $e_{i\pm1} e_i r_{i\pm1}$ (whenever this occurrence does not respect the method of drawing above) and has minimal length over all such expressions for d.

Then the **BMW image** of d, T_d, realizes d as a tangle by setting

$$T_d := T_{s_{i_1}} T_{s_{i_2}} \cdots T_{s_{i_k}},$$

for $d = s_{i_1} s_{i_2} \cdots s_{i_k}$ a reduced expression that contains the maximum number of e terms over all reduced expressions of d. This allows us to define a natural length function $L : \mathcal{T}_n \longrightarrow \mathbb{N}$ as

$$L(T_d) = l'_{Br}(d) + \max_s(e(s))$$

over all reduced expressions s of d, where $e(s)$ gives the number of e terms in s and $l'_{Br}(d)$ gives the minimum number of generators needed for a reduced expression of d.

Example 3.3. Let $d = r_3 e_2 e_1 r_3$, so

$$T_d = T_{r_3} T_{e_2} T_{e_1} T_{r_3},$$

and $L(T_d) = l'_{Br}(d) + 2 = 6$. An alternate reduced expression for d with the same number of e terms is $d = r_3 e_2 r_3 e_1$, which has the same BMW image by BMW relation (A8):

$$T_{r_3} T_{e_2} T_{r_3} T_{e_1} = T_{r_3} T_{e_2} T_{e_1} T_{r_3}.$$

An additional reduced expression for d with the same number of e terms is $d = r_2e_3r_2e_1$, but we must replace e_3r_2 using Brauer relation (8) as follows:

$$d = r_2e_3r_2e_1 = r_2e_3e_2r_3e_1,$$

but then using Brauer relation (7),

$$d = r_3e_2r_3e_1,$$

as before.

Theorem 3.12 of [45] shows that the BMW images of the Brauer monoid elements form a basis for $\mathcal{BMW}_n$. We denote this basis by $\mathcal{T}_n := \{T_d \mid d \in Br_n\}$.

Define functions $\mathscr{T}_{r_i}, \mathscr{T}_{e_i} : \mathcal{T}_n \longrightarrow \mathcal{BMW}_n$ as follows: for $x \in \mathcal{T}_n$,

$$\mathscr{T}_{r_i}(x) = T_{r_i}x$$

$$\mathscr{T}_{e_i}(x) = \begin{cases} T_{e_i}x & \text{if } T_{r_i}x \notin \mathcal{T}_n, \\ T_{r_i}x & \text{else.} \end{cases}$$

We consider generation of elements in $\mathcal{T}_n$ via random walks on $\mathcal{T}_n$ and translate these walks into left multiplication by $\mathscr{T}_{r_i}$ and $\mathscr{T}_{e_i}$.

3.4 The Random Walk

In the finite group case, left multiplication by a generating set gives rise to a random walk on the group. For example, for each generator r_i of S_n, consider the probability distribution

$$P_i(x,y) = \begin{cases} 1 & \text{if } y = r_i x, \\ 0 & \text{else.} \end{cases}$$

Then setting π to be the probability distribution

$$\pi(x) = \frac{\theta^{-l_S(x)}}{\sum_{w \in S_n} \theta^{-l_S(w)}},$$

the Metropolis algorithm construction yields

$$M_i(x,y) = \begin{cases} 1 & \text{if } y = r_i x \text{ and } l_S(y) > l_S(x), \\ \theta & \text{if } y = r_i x \text{ and } l_S(y) < l_S(x), \\ 1-\theta & \text{if } y = x, \end{cases}$$

where l_S represents the length function on words in S_n.

Interpreted as a random walk on S_n, the chain M_i describes the process:

> From $x \in S_n$ multiply by r_i. If the length increases, move to $r_i x$. If the length decreases, flip a θ-coin and if heads move to $r_i x$. If tails, remain at x. $(*)$

We generalize this walk to the basis of tangles $\mathcal{T}_n$ of the BMW algebra. For the remainder of this chapter, let $\ell = 1$ in the BMW algebra. For $T_d \in \mathcal{T}_n$ and L the length function on $\mathcal{T}_n$ defined in Section 3.2, let

$$\pi(T_d) = \frac{\theta^{-L(T_d)}}{\sum_{w \in \mathcal{T}_n} \theta^{-L(w)}},$$

and for $y \in \mathcal{T}_n$ let

$$P_i'(T_d, y) = \begin{cases} 1 & y = T_{r_i d} \\ 0 & \text{else.} \end{cases}$$

Then the Metropolis algorithm applied to P' with probability distribution π yields:

$$K_i(T_d, y) = \begin{cases} 1 & \text{if } y = T_{r_i d} \text{ and } L(y) \geq L(T_d), \\ \theta & \text{if } y = T_{r_i d} \text{ and } L(y) < L(T_d), \\ 1 - \theta & \text{if } y = T_d. \end{cases}$$

Remark 3.1. Recall that $S_n \subseteq Br_n$ and note that for $d \in S_n$, $L(T_d) = l'_{Br}(d) = l_{Br}(d) = l_S(d)$, where L, l'_{Br}, l_{Br}, and l_S denote the length functions on $\mathcal{T}_n, Br_n$, and S_n as defined in Section 3.2. Thus, the submatrix of K_i corresponding to states $\{T_d \mid d \in S_n\}$ is exactly the chain M_i of [25].

Interpreted as a random walk on $\mathcal{T}_n$, the chain K_i describes the process:

> From $T_d \in \mathcal{T}_n$ consider $d \in Br_n$ and multiply by r_i. If the length of the BMW image, $T_{r_i d}$, increases, move to it. If the length decreases, flip a θ-coin and if heads move to $T_{r_i d}$. If tails, remain at T_d. (†)

In light of Proposition 3.1 below, Walk (†) can be rephrased as:

> From $T_d \in \mathcal{T}_n$ multiply by T_{r_i}. If the result is an element of $\mathcal{T}_n$, move to $T_{r_i}T_d$. Else, flip a θ-coin and if heads move to $T_{r_i}^{-1}T_d$. If tails, remain at T_d. (††)

Rephrasing in this way yields the equivalent corresponding Markov chain:

$$K_i(x,y) = \begin{cases} 1 & \text{if } y = T_{r_i}x, \\ \theta & \text{if } y = T_{r_i}^{-1}x, \\ 1-\theta & \text{if } y = x. \end{cases}$$

An example of K_i can be found in Appendix B.1.

Proposition 3.1. *For $T_d \in \mathcal{T}_n$,*

$$L(T_{r_i d}) < L(T_d) \iff T_{r_i}T_d \notin \mathcal{T}_n.$$

Further, if $T_{r_i}T_d \notin \mathcal{T}_n$, then $T_{r_i}^{-1}T_d = T_{r_i d} \in \mathcal{T}_n$, while if $T_{r_i}T_d \in \mathcal{T}_n$, then $T_{r_i}T_d = T_{r_i d}$.

Proof. First write $T_d = T_{s_{i_1}}T_{s_{i_2}}\cdots T_{s_{i_k}}$, for $s_{i_1}\cdots s_{i_k}$ a reduced expression for d with maximum number of e terms. Then

$$T_{r_i}T_d = T_{r_i}T_{s_{i_1}}T_{s_{i_2}}\cdots T_{s_{i_k}},$$

which, after possibly rearranging using BMW relations (A5) and (A8), has one of the following forms, for some $1 \le j \le k-2$:

(a) $T_{r_i}T_d = T_{s_{i_1}}T_{s_{i_2}}\cdots T_{s_{i_j}}T_{r_i}T_{s_i}T_{s_{i\pm1}}T_{s_{i_{j+3}}}\cdots T_{s_{i_k}}$

(b) $T_{r_i}T_d = T_{s_{i_1}}T_{s_{i_2}}\cdots T_{s_{i_j}}T_{r_i}T_{s_{i\pm1}}T_{s_i}T_{s_{i_{j+3}}}\cdots T_{s_{i_k}}$,

(c) $T_{r_i}T_d = T_{r_i}T_{s_{i_1}}T_{s_{i_2}}\cdots T_{s_{i_k}}, |i_1 - i| > 1$.

The proof then reduces to checking each possible case. For example, if in case (1) with

$$T_{r_i}T_d = T_{s_{i_1}}T_{s_{i_2}}\cdots T_{s_{i_j}}T_{r_i}T_{r_i}T_{e_{i+1}}T_{s_{i_{j+3}}}\cdots T_{s_{i_k}},$$

then since $T_{r_i}T_{r_i} \notin \mathcal{T}_n$, we see that $T_{r_i}T_d \notin \mathcal{T}_n$. Further, since BMW relations (A5) and (A8) hold in the Brauer monoid,

$$r_i d = r_i s_{i_1}\cdots s_{i_k} = s_{i_1}s_{i_2}\cdots s_{i_j}r_i r_i e_{i+1}s_{i_{j+3}}\cdots s_{i_k},$$

which by Brauer relation (1) gives

$$r_i d = s_{i_1}\cdots s_{i_j}e_{i+1}s_{i_{j+3}}\cdots s_{i_k},$$

and so $l'_{Br}(r_i d) = l'_{Br}(d) - 1 = k - 1$.

Now let $d' = r_i d$ and recall for $w \in Br_n$, $e(w)$ gives the maximum number of e terms over all reduced expressions for w. If $e(d') > e(d)$, then there is an expression for d' with length $k-1$ and at least one more additional e term than in $s_{i_1}\cdots s_{i_j}e_{i+1}s_{i_{j+3}}\cdots s_{i_k}$. However, by Brauer relations (1)-(8), the only way to increase the number of e terms without changing the length of an expression is through the relation

$$e_{i+1}r_i r_{i+2}e_{i+1} = e_{i+1}e_i e_{i+2}e_{i+1}.$$

This would require either $s_{i_{j-2}}s_{i_{j-1}}s_{i_j} = e_{i+1}r_i r_{i+2}$ or $s_{i_{j+3}}s_{i_{j+4}}s_{i_{j+5}} = r_i r_{i+2}e_{i+1}$, which contradicts the fact that $s_{i_1}\cdots s_{i_k}$ is a reduced expression for d with maximal number of e terms. Hence, $e(d') \leq e(d)$, and so

$$L(T_{r_i d}) < L(d).$$

For the second statement, note that

$$T_{r_i}^{-1} T_d = T_{s_{i_1}} T_{s_{i_2}} \cdots T_{s_{i_j}} T_{r_i}^{-1} T_{r_i} T_{e_{i\pm 1}} T_{s_{i_{j+3}}} \cdots T_{s_{i_k}} = T_{r_i d}.$$

The remaining cases are checked similarly. □

In [25], Diaconis and Ram translate the Markov chain M_i into left multiplication by Hecke algebra elements on a suitably chosen basis. Similarly, we translate the chains K_i arising from our Metropolis construction into left multiplication by BMW algebra elements on the basis $\mathcal{T}_n$. Recall the functions $\mathscr{T}_{r_i}, \mathscr{T}_{e_i} : \mathcal{T}_n \longrightarrow \mathcal{T}_n$ defined at the end of Section 3.2.

Theorem 3.3. *Let Br_n be the Brauer monoid and $\mathcal{BMW}_n$ the BMW algebra with basis $\mathcal{T}_n = \{T_d \mid d \in Br_n\}$ as defined in Section 3.2. Let $m = (1-\theta)(\theta)^{-1}$ and $\ell = 1$. Then the chain K_i is the same as the matrix of left multiplication by*

$$\theta \mathscr{T}_{r_i} + (1-\theta)\mathscr{T}_{e_i},$$

with respect to the basis $\mathcal{T}_n$ of $\mathcal{BMW}_n$.

Proof. Let $x \in \mathcal{T}_n$ and consider left multiplication by T_{r_i}. If $T_{r_i} x \in \mathcal{T}_n$,

$$(\theta \mathscr{T}_{r_i} + (1-\theta)\mathscr{T}_{e_i})x = \theta T_{r_i} x + (1-\theta) T_{r_i} x = T_{r_i} x.$$

If $T_{r_i}x \notin \ mathcalT_n$ then by BMW Relation (A7),

$$T_{r_i}x = (T_{r_i}^{-1} + mT_{id} - ml^{-1}T_{e_i})x = T_{r_i}^{-1}x + (1-\theta)(\theta)^{-1}x - (1-\theta)(\theta)^{-1}T_{e_i}x.$$

By Proposition 3.1, $T_{r_i}^{-1}x \in \mathcal{T}_n$, and

$$\begin{aligned}(\theta\mathscr{T}_{r_i} + (1-\theta)\mathscr{T}_{e_i})x &= \theta T_{r_i}^{-1}x + (1-\theta)x - (1-\theta)T_{e_i}x + (1-\theta)T_{e_i}x. \\ &= \theta T_{r_i}^{-1}x + (1-\theta)x,\end{aligned}$$

□

The chains K_i provide scanning strategies for generating elements of the BMW and Brauer monoids:

$$\begin{aligned}&\frac{1}{n-1}\sum_{i=1}^{n-1} K_i && \text{(random scan)}, \\ &K_1K_2\cdots K_{n-1}K_{n-1}\cdots K_2K_1 && \text{(short systematic scan)}, \\ &(K_1\cdots K_{n-1}K_{n-1}\cdots K_1)\cdots(K_1K_2K_2K_1)(K_1K_1) && \text{(long systematic scan)}.\end{aligned}$$

Theorem 3.3, coupled with the results of Section 3.5, allows for the study of the rate of convergence of the systematic scans arising from the chains K_i in terms of the representation theory and Fourier analysis of the BMW algebra.

3.5 Analysis of the Walk

We first discuss the stationary distribution of the random scan, short systematic scan, and long systematic scan described in Section 3.4. For the remainder of the section, we use K to refer to the matrix corresponding to any of these scans, as the results hold true for all three scans.

Note that K is Markov and recall that a communication class C of a Markov chain is closed if for each state $x \in C$ and for all $y \notin C$, y is not accessible from x. We determine the closed communication classes of K and analyze the stationary distribution of each closed communication class.

The communication classes of K depend on the number of *lower horizontal edges* in the tangle diagrams for the states.

Definition 3.7. *Let $x \in \mathcal{T}_n$. An edge of x is* ***lower (respectively, upper) horizontal*** *if it connects two points that are both on the bottom (respectively, top) row of the diagram of x.*

Example 3.4. In Figure 3.5, E_3 is the only lower horizontal edge and E_1 is the only upper horizontal edge.

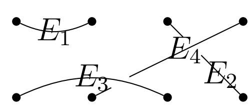


Figure 3.5

K is determined by left multiplication by T_{r_i} and $T_{r_i}^{-1}$ only, and left multiplication by these elements cannot affect existing lower horizontal edges in a tangle diagram, nor can it create new ones. Thus, the communication classes of K consist of those states with the same lower horizontal edges. For $x_i \in \mathcal{T}_n$, let $\mathbf{X}_i$ denote its communication class:

$$\mathbf{X}_i := \{y \in \mathcal{T}_n \mid \text{lower horizontal edges of } y \text{ the same as those of } x_i\}.$$

For each communication class $\mathbf{X}_i$, let $[K]_i$ denote the corresponding submatrix of K. Note that the communication class for $x_0 := T_{id}$ consists of the states $\{T_d \mid d \in S_n\}$. Then by Remark 3.1, $[K_i]_0 = M_i$, and so $[K]_0$ can be analyzed using the methods of [25]. For the remainder of the chapter we consider the remaining communication classes of K:

$$\{\mathbf{X}_i \mid i \neq 0\}.$$

To analyze the time to stationarity of the submatrix $[K]_1$ corresponding to a communication class $\mathbf{X}_i$, we pair $\mathbf{X}_1$ with another communication class, $\mathbf{X}_2$, whose states have the same number of lower horizontal edges as those in $\mathbf{X}_1$. For $w \in \mathbf{X}_1$, let w^* denote the element of $\mathbf{X}_2$ with the same upper configuration as w. Define the matrix:

$$\tilde{K}(x,y) = \begin{cases} K(x,y) & \text{if } x, y \in \mathbf{X}_1 \text{ or if } x = w^*, y = z^* \text{ for } w, z \in \mathbf{X}_1, \\ 1 & \text{if } x = y, x \notin \mathbf{X}_1 \cup \mathbf{X}_2, \\ 0 & \text{else.} \end{cases}$$

Note with Example 3.5 that often $K(x,y) = K(x^*, y^*)$.

Example 3.5. For $\mathcal{T}_3$, let $x_1 = T_{e_1}$ and $x_2 = T_{e_1}T_{r_2}$, so $\mathbf{X}_1 = \{T_{e_1}, T_{r_2}T_{e_1}, T_{e_2}T_{e_1}\}$, $\mathbf{X}_2 = \{T_{e_1}T_{r_2}, T_{r_2}T_{e_1}T_{r_2}, T_{e_2}T_{e_1}T_{r_2}\}$, $T_{e_1}^* = T_{e_1}T_{r_2}$, $T_{r_2}T_{e_1}^* = T_{r_2}T_{e_1}T_{r_2}$, and $T_{e_2}T_{e_1}^* = T_{e_2}T_{e_1}T_{r_2}$.

Then for $K = \frac{1}{2}(K_1 + K_2)$,

$$2[K]_1 = \begin{array}{c} \begin{matrix} T_{e_1} & T_{r_2}T_{e_1} & T_{e_2}T_{e_1} \end{matrix} \\ \begin{pmatrix} 1 & \theta & 0 \\ 1 & 1-\theta & \theta \\ 0 & 1 & 2-\theta \end{pmatrix} \end{array},$$

$$2[K]_2 = \begin{array}{c} \begin{matrix} T_{e_1}T_{r_2} & T_{r_2}T_{e_1}T_{r_2} & T_{e_2}T_{e_1}T_{r_2} \end{matrix} \\ \begin{pmatrix} 1 & \theta & 0 \\ 1 & 1-\theta & \theta \\ 0 & 1 & 2-\theta \end{pmatrix} \end{array},$$

and $\tilde{K} = [K]_1 \bigoplus [K]_2 \bigoplus I_9$, for I_9 the 9×9 identity matrix.

Let π denote the stationary distribution of $\tilde{K}$. For $T_x \in \mathcal{T}_n$ let $[\pi]_x$ denote the column of π corresponding to T_x:

$$[\pi]_x := \begin{pmatrix} \pi_x(x_1) \\ \pi_x(x_2) \\ \vdots \\ \pi_x(x_k) \end{pmatrix} = \sum_{T_y \in \mathcal{T}_n} \pi_x(y) T_y.$$

Note that $\pi_x(y)$ represents the probability of ending at state T_y after starting at T_x. Recall from Section 3.2.1 that to analyze the time to stationarity of $\tilde{K}$ we consider:

$$|\tilde{K}_x^m - \pi|_{TV}. \tag{3.3}$$

By Lemma 3.1, the total variation norm is bounded by the $L^2(\pi)$-norm, so we translate the $L^2(\pi)$-norm into a trace norm on $\mathcal{BMW}_n$ in order to use the representation theory of the BMW algebra to analyze (3.3).

Definition 3.8. *Define $\tilde{\tau} : \mathcal{T}_n \to \mathbb{C}$ as follows: for $x \in \mathcal{T}_n$,*

$$\tilde{\tau}(x) = \begin{cases} 1 & \text{if } x = T_{id}, \\ 0 & \text{else}, \end{cases}$$

*The **restricted trace**, $\tau : \mathcal{BMW}_n \to \mathbb{C}$, is the linear extension of $\tilde{\tau}$ to $\mathcal{BMW}_n$.*

Recall from Section 3.2 that $\mathcal{T}_n = \{T_d \mid d \in Br_n\}$.

Proposition 3.2. *For $T_x, T_y \in \mathcal{T}_n$, $\tau(T_xT_y) = \begin{cases} 1 & \text{if } x = y^{-1}, \\ 0 & \text{else}. \end{cases}$*

Corollary 3.1. *τ is a trace function on $\mathcal{BMW}_n$.*

Proof of Proposition 3.2. Let $T_x, T_y \in \mathcal{T}_n$. By definition, $T_x = T_{s_{i_1}} \cdots T_{s_{i_k}}$, where for each $1 \leq j \leq k$, $T_{s_{i_j}} \in \{T_{r_i}, T_{e_i} \mid 1 \leq i \leq n-1\}$. First note that the LHS of any of the BMW relations (A1)-(A8) has a T_e factor if and only if the RHS also has at least one T_e factor. In particular, if for some $1 \leq j \leq n-1$, T_{e_j} is a factor of T_x, then each term of the product T_xT_y has at least one T_e factor. Hence, no term in the product T_xT_y is the identity, so $\tau(T_xT_y) = 0$. Similarly, $\tau(T_yT_x) = 0$.

Thus, if $T_{s_{i_l}} = T_{e_j}$ for some $1 \leq l \leq k$, $1 \leq j \leq n-1$, then $\tau(T_xT_y) = \tau(T_yT_x) = 0$ for all $T_y \in \mathcal{T}_n$. Equivalently, $\tau(T_xT_y) = 0$ for all $x \in Br_n - S_n$, $y \in Br_n$.

Next note that $T_x \in \mathcal{T}_n$ has an inverse iff $x \in S_n \subset Br_n$, because the diagram for any $y \in Br_n - S_n$ has at least one lower horizontal edge, and left multiplication cannot affect this edge. Hence we need show for $x, y \in S_n$ that

$$\tau(T_xT_y) = \begin{cases} 1 & \text{if } x = y^{-1}, \\ 0 & \text{else.} \end{cases}$$

As in Section 3.2 let $l_S : S_n \to \mathbb{N}$ denote the length function of S_n. We proceed by induction on the length $l_S(x) + l_S(y)$.

If $l_S(x) + l_S(y) = 0$ then $l_S(x) = l_S(y) = 0$, so $x = y = id$. Clearly $\tau(T_xT_y) = 1$ and $x = y^{-1}$.

Now suppose the result holds true for all $w_1, w_2 \in S_n$ with $l_S(w_1)+l_S(w_2) \leq n-1$. Let $x, y \in S_n$ with $r_{i_1} \cdots r_{i_k}$ a reduced expression for x, $r_{j_1} \cdots r_{j_l}$ a reduced expression for y, and $k + l = n$. Then $T_x = T_{r_{i_1}} \cdots T_{r_{i_k}}$, $T_y = T_{r_{j_1}} \cdots T_{r_{j_l}}$, and

$$T_xT_y = T_{r_{i_1}} \cdots T_{r_{i_k}} T_{r_{j_1}} \cdots T_{r_{j_l}}.$$

If $T_xT_y \in \mathcal{T}_n$ then no matter the rearrangement using BMW Relations (A5) and (A8), $T_{r_i}^2$ must not occur for any i. Then r_i^2 does not occur in xy, so $l_S(xy) = n$. In particular, $x \neq y^{-1}$ and $T_{id} \neq T_xT_y = T_{xy} \in \mathcal{T}_n$, so $\tau(T_xT_y) = 0$.

Suppose now that $T_xT_y \notin \mathcal{T}_n$. Then, after applying Relations (A5) and (A8) and possibly reindexing, the product T_xT_y has form:

$$T_xT_y = (T_{r_{i_1}} \cdots T_{r_{i_k}})(T_{r_{i_k}} T_{r_{j_2}} \cdots T_{r_{j_l}}) = T_{x'} T_{r_{i_k}} T_{r_{i_k}} T_{y'},$$

for $x' = r_{i_1} \cdots r_{i_{k-1}}$ and $y' = r_{j_2} \cdots r_{j_l}$. Then by BMW relation (A7),

$$\begin{aligned} T_x T_y &= T_{x'}(T_{id} - mT_{r_{i_k}} + ml^{-1}T_{e_{i_k}})T_{y'} \\ &= T_{x'}T_{y'} - mT_{x'}T_{r_{i_k}}T_{y'} + ml^{-1}T_{x'}T_{e_{i_k}}T_{y'}. \end{aligned} \tag{3.4}$$

Since $T_{x'}T_{e_{i_k}}T_{y'} \in Br_n - S_n$, $\tau(T_{x'}\tilde{T_{e_{i_k}}}T_{y'}) = 0$.

Now consider $\tau(T_{x'}T_{r_{i_k}}T_{y'}) = \tau(T_x T_{y'})$. Since $l_S(x) + l_S(y') = n - 1$, by the induction hypothesis,

$$\tau(T_x T_{y'}) = 1 \iff y' = x^{-1},$$

and is zero otherwise.

But if $y' = x^{-1}$, then $l_S(y') = l_S(x) = k$ and $y' = r_{i_k} \cdots r_{i_1}$. Since $l_S(y') = l_S(y) - 1$, this implies that

$$l_S(y) = k + 1.$$

However, $y = r_{i_k} y' = r_{i_k} r_{i_k} \cdots r_{i_1}$, and so

$$l_S(y) = k - 1.$$

With this contradiction, we have $\tau(T_x T_{y'}) = 0$.

Finally, consider $\tau(T_{x'}T_{y'})$. Since $l_S(x') + l_S(y') = n - 2$, then by the induction hypothesis,

$$\begin{aligned} &\tau(T_{x'}T_{y'}) = 1 \\ \iff& x' = (y')^{-1} \\ \iff& y' = r_{i_{k-1}} \cdots r_{i_1} \\ \iff& y = r_{i_k} r_{i_{k-1}} \cdots r_{i_1} \\ \iff& y = x^{-1}, \end{aligned} \tag{3.5}$$

and $\tau(T_{x'}T_{y'}) = 0$ otherwise.

Thus,

$$\tau(T_xT_y) = \begin{cases} 1 & \text{if } x = y^{-1}, \\ 0 & \text{else.} \end{cases}$$

□

Thus τ is a trace function on $\mathcal{BMW}_n$ with $\tau(T_xT_y) = 0$ for all $x, y \in Br_n - S_n$. In fact, τ extends the natural trace function of the Hecke algebra, $\mathcal{H}_n$, viewing $\mathcal{H}_n$ as a subalgebra of $\mathcal{BMW}_n$. We analyze $\tilde{K}$ using the bilinear form arising from τ, which reformulates questions about the time to stationarity in terms of the representation theory of the underlying Hecke subalgebra of $\mathcal{BMW}_n$.

Recall that $\tilde{K}$ is comprised of two submatrices corresponding to two communication classes $\mathbf{X}_1$ and $\mathbf{X}_2$ of K. Recall further that these communication classes consist of elements of $\mathcal{T}_n$ with at least one lower horizontal edge. In particular, for each $T_x \in \mathbf{X}_1 \cup \mathbf{X}_2$, $\tau(T_xT_y) = 0$ for all $T_y \in \mathcal{T}_n$. In order for τ to be nontrivial on the communication classes of $\tilde{K}$, we rewrite $\tilde{K}$ with respect to a shifted basis for $\mathcal{BMW}_n$.

Definition 3.9. *Let π denote the stationary distribution of $\tilde{K}$. To each $T_x \in \mathbf{X}_1$,* *associate a distinct* $s_x \in S_n$ *such that* $s_x \neq s_y^{-1}$ *for all* $T_y \in \mathbf{X_1}$ *and* s_x *has order greater than 2. For* $T_x \in \mathbf{X}_1$ *and for* $T_y \notin \mathbf{X}_1 \cup \mathbf{X}_2$*, let*

$$\begin{aligned}
\hat{T}_x &:= T_x + \pi_x(x)^{-\frac{1}{2}} T_{s_x}, \\
\hat{T}_{x^*} &:= T_{x^*} + \pi_x(x)^{-\frac{1}{2}} T_{s_x^{-1}} = T_{x^*} + \pi_{x^*}(x^*)^{-\frac{1}{2}} T_{s_x^{-1}}, \\
\hat{T}_y &:= T_y.
\end{aligned} \tag{3.6}$$

We note in Appendix B.2 that for all communication classes corresponding to elements with at least two lower horizontal edges, S_n contains enough distinct elements to make the associations of Definition 3.9. For the remainder of this section let $\mathbf{X}_1$ be a communication class whose elements contain at least two lower horizontal edges. (The remaining communication classes are analyzed separately through techniques discussed in Appendix B.2.)

Lemma 3.2. $\hat{\mathcal{T}}_n := \{\hat{T}_x \mid x \in Br_n\}$ *is a basis for* $\mathcal{BMW}_n$*.*

Proof. This follows from the fact that $\mathcal{T}_n$ is a basis for $\mathcal{BMW}_n$. □

Now let $\langle\ ,\ \rangle_{\mathcal{BMW}}$ denote the trace form of Section 3.3.1

Lemma 3.3. *For* $T_x \in \mathbf{X}_1 \cup \mathbf{X}_2$ *and* $y \in Br_n$*,*

$$\langle \hat{T}_x, \hat{T}_y \rangle_{\mathcal{BMW}} = \begin{cases} \pi_x(x)^{-1} & \textit{if } y = x^*, \\ \pi_x(x)^{-\frac{1}{2}} & \textit{if } y = (s_x)^{-1}, \\ 0 & \textit{else}, \end{cases}$$

while

$$\langle \hat{T}_{s_x}, \hat{T}_y \rangle_{\mathcal{BMW}} = \begin{cases} \pi_x(x)^{-\frac{1}{2}} & \text{if } y = x^*, \\ 1 & \text{if } y = (s_x)^{-1}, \\ 0 & \text{else.} \end{cases}$$

Proof. Follows from Proposition 3.2 and the linearity of trace. □

Let $\hat{K}$ be the matrix of $\tilde{K}$ with respect to $\hat{\mathcal{T}}_n$. Note that time to stationarity is invariant under change of basis.

Lemma 3.4. *For $\hat{T}_x \in \hat{\mathcal{T}}_n$,*

(a) If $\hat{T}_x \in \hat{\mathbf{X}}_1$,

$$\hat{K}(\hat{T}_x, \hat{T}_y) = \begin{cases} K(T_x, T_y) & \text{if } \hat{T}_y \in \hat{\mathbf{X}}_1 \cup \hat{\mathbf{X}}_2, \\ (1 - K(T_x, T_x))\pi_x(x)^{-\frac{1}{2}} & \text{if } y = s_x, \\ -K(T_x, T_y)\pi_y(y)^{-\frac{1}{2}} & \text{if } y = s_z, z \neq x, z \in \mathbf{X}_1 \\ 0 & \text{else,} \end{cases}$$

and similarly for $\hat{T}_x \in \hat{\mathbf{X}}_2$.

(b) If $\hat{T}_x \notin \hat{\mathbf{X}}_1 \cup \hat{\mathbf{X}}_2$,

$$\hat{K}(\hat{T}_x, \hat{T}_y) = \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{else.} \end{cases}$$

Proof. Follows from definition of $\hat{K}$ and $\hat{\mathcal{T}}_n$. □

Recall that $\tilde{K}$ is the direct sum $K_1 \bigoplus K_2 \bigoplus I_m$, where K_1 corresponds to the elements in $\mathbf{X}_1$, K_2 to those in $\mathbf{X}_2$, and $m = |\mathcal{T}_n| - 2|\mathbf{X}_1|$. Lemma 3.4 shows that $\hat{K}$ is also a direct sum $\hat{K}_1 \bigoplus \hat{K}_2 \bigoplus I_{\hat{m}}$, where for $i = 1, 2$, the matrix $\hat{K}_i$ corresponds to

$\{\hat{T}_x, \hat{T}_{s_x} \mid T_x \in \mathbf{X}_i\}$, and $\hat{m} = |\mathcal{T}_n| - 4|\mathbf{X}_1|$. Further, for all $T_x, T_y \in \mathbf{X}_1 \cup \mathbf{X}_2$,

$$\hat{K}(\hat{T}_x, \hat{T}_y) = \tilde{K}(T_x, T_y) = K(T_x, T_y).$$

Recall that π denotes the stationary distribution of $\tilde{K}$ and $[\pi]_x$ the stationary distribution of $\tilde{K}$ corresponding to column T_x:

$$[\pi]_x = \sum_{T_y \in \mathcal{T}_n} \pi_x(y) T_y.$$

Note that for $T_x \notin \mathbf{X}_1 \cup \mathbf{X}_2$, $\pi_x(y) = 0$ for all $T_y \neq T_x$, and so

$$[\pi]_x = T_x.$$

Further, for $T_x \in \mathbf{X}_1$, $\pi_x(y) = 0$ for all $T_y \notin \mathbf{X}_1$, and so

$$[\pi]_x = \sum_{T_y \in \mathbf{X}_1} \pi_x(y) T_y,$$

and similarly for $\mathbf{X}_2$. Finally, note that $\pi_x(y) = \pi_{x'}(y)$ for all $x, x', y \in \mathbf{X}_1$, and similarly for $\mathbf{X}_2$.

Let $\hat{\pi}$ denote the stationary distribution of $\hat{K}$ and $[\hat{\pi}]_x$ the stationary distribution of $\hat{K}$ corresponding to column $\hat{T}_x$. Let $\hat{\mathbf{X}}_i = \{\hat{T}_x \mid T_x \in \mathbf{X}_i\}$.

Lemma 3.5. *Let π be the stationary distribution of $\tilde{K}$ and $\hat{\pi}$ the stationary distribution of $\hat{K}$.*

(a) For $\hat{T}_x \in \hat{\mathbf{X}}_1$*,*

$$[\hat{\pi}]_x = \sum_{\hat{T}_y \in \mathbf{X}_1} (\pi_x(y)\hat{T}_y - \pi_x(y)^{\frac{1}{2}}\hat{T}_{s_y}) + \pi_x(x)^{-\frac{1}{2}}\hat{T}_{s_x},$$

and similarly for $\hat{T}_x \in \hat{\mathbf{X}}_2$*.*

(b) If $\hat{T}_y \notin \hat{\mathbf{X}}_1 \cup \hat{\mathbf{X}}_2$*,* $[\hat{\pi}]_y = \hat{T}_y$

Proof. Part (2) follows from Lemma 3.4. To prove (1), note that for $\hat{T}_x \in \hat{\mathbf{X}}_1$

$$\begin{aligned}
[\hat{\pi}]_x &= [\pi]_x + \pi_x(x)^{-\frac{1}{2}}[\pi]_{s_x} \\
&= \sum_{T_y \in \mathbf{X}_1} \pi_x(y)T_y + \pi_x(x)^{-\frac{1}{2}}T_{s_x} \\
&= \sum_{T_y \in \mathbf{X}_1} \Big(\pi_x(y)(T_y + \pi_y(y)^{-\frac{1}{2}}T_{s_y}) - \pi_x(y)\pi_y(y)^{-\frac{1}{2}}T_{s_y}\Big) + \pi_x(x)^{-\frac{1}{2}}T_{s_x}. \\
&= \sum_{\hat{T}_y \in \hat{\mathbf{X}}_1} \Big(\pi_x(y)\hat{T}_y - \pi_x(y)^{\frac{1}{2}}\hat{T}_{s_y}\Big) + \pi_x(x)^{-\frac{1}{2}}\hat{T}_{s_x}.
\end{aligned}$$

□

For $T_x \in \mathbf{X}_1 \cup \mathbf{X}_2$, Lemma 3.5 shows that $\pi_x(y) = \hat{\pi}_x(y)$ for all $T_y \in \mathbf{X}_1 \cup \mathbf{X}_2$. However, $\pi_x(s_y) = 0$, but $\hat{\pi}_x(s_x) = \pi_x(x)^{-\frac{1}{2}} - \pi_x(x)^{\frac{1}{2}}$ and $\hat{\pi}_x(s_y) = -\pi_x(y)^{\frac{1}{2}}$, for $y \neq x$.

Let $\hat{\mathcal{S}} := \{\hat{T}_{s_x} \mid s_x \in \mathcal{S}\}$. Consider the $L^2(\hat{\pi})$-norm restricted to the subspace generated by $\hat{\mathbf{X}}_1 \cup \hat{\mathbf{X}}_2 \cup \hat{\mathcal{S}}$:

Definition 3.10. *For functions* $f, g : \hat{\mathbf{X}}_1 \cup \hat{\mathbf{X}}_2 \cup \hat{\mathcal{S}} \to \mathbb{C}$*, let*

$$\langle f, g \rangle_2 := \sum_{\hat{T}_x \in \hat{\mathbf{X}}_1 \cup \hat{\mathbf{X}}_2 \cup \hat{\mathcal{S}}} f(x)g(x)\hat{\pi}_x(x).$$

For $m \in \mathbb{N}$, let $[\hat{K}^m]_x$ denote the column of $\hat{K}^m$ corresponding to $\hat{T}_x$:

$$[\hat{K}^n]_x = \sum_{\hat{T}_{x_i} \in \hat{\mathbf{X}}_1 \cup \hat{\mathbf{X}}_2 \cup \hat{\mathcal{S}}} K_x^n(x_i)\hat{T}_{x_i}.$$

To find the time to stationarity of $\hat{K}$ (and hence $\tilde{K}$ and K), we analyze $\|[\hat{K}^m]_x - [\hat{\pi}]_x\|_2$.

Lemma 3.6. *Let f, g be complex-valued functions on $\hat{\mathbf{X}}_1 \cup \hat{\mathbf{X}}_2 \cup \hat{\mathcal{S}}$ and let $* : \hat{\mathbf{X}}_1 \cup \hat{\mathbf{X}}_2 \cup \hat{\mathcal{S}} \to \hat{\mathbf{X}}_1 \cup \hat{\mathbf{X}}_2 \cup \hat{\mathcal{S}}$ be the involution that sends $\hat{T}_x$ to $\hat{T}_{x^*}$ for $\hat{T}_x \in \hat{\mathbf{X}}_1 \cup \hat{\mathbf{X}}_2$, and $\hat{T}_{s_x}$ to $\hat{T}_{s_x^{-1}}$ for $\hat{T}_{s_x} \in \hat{\mathcal{S}}$. Then for $\langle\ ,\ \rangle_{BMW}$ the bilinear form arising from the trace τ,*

$$\langle f/\hat{\pi}, g/\hat{\pi}\rangle_2 = \langle f, g^*\rangle_{BMW} - \sum_{\hat{T}_{s_x}} \frac{f(x)g(s_x^{-1}) + f(s_x^{-1})g(x^*)}{\hat{\pi}_x(x)^{\frac{1}{2}}}.$$

Proof. By Lemma 3.3,

$$\begin{aligned}
\langle f/\hat{\pi}, g/\hat{\pi}\rangle_2 &= \sum \frac{f(x)g(x)}{\hat{\pi}_x(x)} \\
&= \sum_{\hat{T}_x \in \hat{\mathbf{X}}_1 \cup \hat{\mathbf{X}}_2 \cup \hat{\mathcal{S}}} f(x)g(x)\langle \hat{T}_x, (\hat{T}_x)^*\rangle_{BMW} \\
&= \sum_{\hat{T}_x, \hat{T}_y \in \hat{\mathbf{X}}_1 \cup \hat{\mathbf{X}}_2 \cup \hat{\mathcal{S}}} f(x)g(y)\langle \hat{T}_x, (\hat{T}_y)^*\rangle_{BMW} - \sum_{\hat{T}_{s_x} \in \hat{\mathcal{S}}} \frac{f(x)g(s_x^{-1})}{\hat{\pi}_x(x)^{\frac{1}{2}}} \\
&\quad - \sum_{\hat{T}_{s_x} \in \hat{\mathcal{S}}} \frac{f(s_x^{-1})g(x^*)}{\hat{\pi}_x(x)^{\frac{1}{2}}} \\
&= \langle f, g^*\rangle_{BMW} - \sum_{\hat{T}_{s_x}} \frac{f(x)g(s_x^{-1}) + f(s_x^{-1})g(x^*)}{\hat{\pi}_x(x)^{\frac{1}{2}}}.
\end{aligned}$$

□

Corollary 3.2. *For* $\hat{T}_x \in \hat{\mathbf{X}}_1$,

$$\langle [\hat{K}^m/\hat{\pi}]_x, [\hat{K}^m/\hat{\pi}]_x \rangle_2 = \langle [\hat{K}^m]_x, [\hat{K}^m]_x \rangle_{BMW} - \sum_{\hat{T}_y \in \mathbf{X}_2} \frac{\hat{K}_x^m(s_y^{-1})\hat{K}_x^m(y^*)}{\hat{\pi}_y(y)^{\frac{1}{2}}}.$$

Proof. Follows from Lemma 3.6 and the fact that $([\hat{K}^m]_x)^* = [\hat{K}^m]_x$, $K_x^m(s_y^{-1}) = 0$ if $y \in \mathbf{X}_1$, and $K_x^m(y) = 0$ if $y \in \mathbf{X}_2$. □

K is Markov, so there exists $N \in \mathbb{N}$ with $\hat{K}_x^m \geq 0$ for all $m > N$. Further, $\hat{\pi}$ is the stationary distribution of a Markov chain, so $\hat{\pi}_y(y) \geq 0$. We have thus shown:

Theorem 3.4. *For* $\hat{T}_x \in \hat{\mathbf{X}}_1 \cup \hat{\mathbf{X}}_2$,

$$\langle [\hat{K}^n/\hat{\pi}]_x, [\hat{K}^n/\hat{\pi}]_x \rangle_2 \leq \langle [\hat{K}^n]_x, [\hat{K}^n]_x \rangle_{BMW}.$$

Hence,

$$\|[\hat{K}^n/\hat{\pi}]_x - 1\|_2^2 \leq \|[\hat{K}^n]_x - 1\|_{BMW}^2.$$

Thus, studying the time to stationarity of $\hat{K}$ can be achieved by studying

$$\|[\hat{K}^n]_x - 1\|_{BMW}^2.$$

Chapter 4

Future Directions

4.1 Further Directions

Our work exploits the interplay between algebra, combinatorics, and probability theory, and we look forward to extending this research program by continuing to explore such links. In particular, we wish to consider the following ideas:

- **Extension of the results of the SOV approach to other chains of subalgebras for the Weyl groups, the general linear groups of finite fields, and the Brauer algebra.**

  The efficiency of the SOV approach of Chapter 2 in part depends on the choice of subgroup (subalgebra) chain. While we provide efficiency counts that improve upon previous bounds [63, 65], we do not consider maximally dense subgroup (subalgebra) chains. We wish to investigate the change in efficiency counts upon considering alternate chains. For example, for the Brauer algebra we wish to consider the algebra chain:

$$\mathcal{B}r_n > \mathcal{B}r_2 \times \mathcal{B}r_{n-2} > \mathcal{B}r_{n-2} > \cdots > \mathcal{B}r_0.$$

- **Extension of the results of the SOV approach to other semisimple algebras, and applications of such transforms.**

  While our algorithm in Chapter 2 allows for the computation of generalized Fourier transforms by factorization through a subgroup (subalgebra) chain, the efficiency of the algorithm depends on the choice of this subgroup (subalgebra) chain as well as knowledge of the irreducible representations and associated Bratteli diagram. There are many particular semisimple algebras to explore in this manner. In [43], Grood constructs the irreducible representations of the rook partition algebra and the associated Bratteli diagram, while Halverson et al. [35, 44] determine analogues of the seminormal representations of S_n for the rook-Brauer algebra and planar-rook algebra. Motivated by these results, we plan to extend our algorithm to these algebras by developing an understanding of the centralizers and irreducible representations and to explore the resulting combinatorial path-counting questions to provide efficient counts.

- **Further analysis of the random walk on the BMW algebra and extension to other semisimple algebras.**

  In Chapter 3 we develop a systematic scanning strategy for generating elements of the BMW monoid basis of the BMW algebra, then view the associated matrix as a left multiplication operator in the BMW algebra. We further develop a trace norm on the BMW algebra and prove that bounds on the time to stationarity of the walk can be determined by bounding this trace norm. Using similar techniques in the Coxeter group setting, Diaconis and Ram [25] provide explicit bounds determined by the dimensions and traces of the irreducible representations of the Hecke algebra. We plan to exploit the representation theory of the BMW algebra to provide explicit bounds in terms of the traces of irreducible representations of the BMW algebra. We wish also to extend these ideas to other diagram algebras such as the rook partition algebra, planar-rook algebra, and rook-Brauer algebra.

- **Extension of generalized Fourier transforms to buildings and application to random walks on p-adic groups.**

  We would like to extend our work on Fourier transforms to the setting of Bruhat-Tits buildings. Motivated by an interest in random walks on buildings, which begins with Sawyer's treatment of the binary tree $Gl_2(\mathbb{Q}_2)$ [84] and continues with the work of Cartwright et al. [12] and Parkinson [72], we wish to determine the appropriate generalization of the Fourier transform in the buildings context in order to apply Fourier theory to questions of random walks on buildings and p-adic groups.

Appendix A

Appendix

A.1 Gel'fand-Tsetlin Bases and Adapted Representations

Definition A.1. *Given a Bratteli diagram $\mathcal{B}$, a* ***representation of*** $\mathcal{B}$ *assigns to each $\alpha \in V(\mathcal{B})$ a linear space V_α and to each edge $e \in E(\mathcal{B})$ a linear map $T_e : V_{e^-} \to V_{e^+}$. Given two representations $(\{V_\alpha\}_{\alpha \in V(\mathcal{B})}, \{T_e\}_{e \in E(\mathcal{B})})$, $(\{W_\alpha\}_{\alpha \in V(\mathcal{B})}, \{S_e\}_{e \in E(\mathcal{B})})$, of $\mathcal{B}$, a* ***morphism*** *$m : V \to W$ is a family of linear maps $\{m_\alpha : V_\alpha \to W_\alpha\}_{\alpha \in V(\mathcal{B})}$ such that the diagram*

$$\begin{array}{ccc} V_{e^-} & \xrightarrow{T_e} & V_{e^+} \\ m_{e^-} \downarrow & & \downarrow m_{e^+} \\ W_{e^-} & \xrightarrow{S_e} & W_{e^+} \end{array}$$

commutes for all $e \in E(\mathcal{B})$.

A ***model representation of*** $\mathcal{B}$ *is a representation of $\mathcal{B}$ such that for all $e \in$*

$E(\mathcal{B})$, T_e is injective, and for all nonroot vertices $\alpha \in V(\mathcal{B})$,

$$V_\alpha = \bigoplus_{e^+=\alpha} Im(T_e).$$

Definition A.2. *Given a chain of subalgebras $\{A_i\}$, a* ***model representation for*** *$\{A_i\}$ is a model representation of the corresponding Bratteli diagram $\mathcal{B}$ such that:*

(i) for each $\alpha \in V(\mathcal{B})$ at level i, V_α is the representation space of the representation of A_i corresponding to α,

(ii) for each $e \in E(\mathcal{B})$ from level i to level $i+1$, T_e is A_i-equivariant, i.e., for ρ_{e^-} the representation of A_i corresponding to e^- and ρ_{e^+} the representation of A_{i+1} corresponding to e^+, the diagram

$$\begin{array}{ccccc} & V_{e^-} & \xrightarrow{T_e} & V_{e^+} & \\ \rho_{e^-} & \downarrow & & \downarrow & \rho_{e^+} \\ & V_{e^-} & \xrightarrow{T_e} & V_{e^+} & \end{array}$$

commutes for all $e \in E(\mathcal{B})$.

A model representation of an algebra chain has a natural basis of paths:

Lemma A.1. *Given a model representation of a chain of subalgebras with Bratteli diagram $\mathcal{B}$, the collection of distinct paths in $\mathcal{B}$ from the root to a vertex $\alpha \in V(\mathcal{B})$ corresponds to a choice of basis for V_α.*

Proof. Consider the space V_β corresponding to the root β, i.e., V_β is the representation space of $\mathbb{C}[\mathcal{B}_0] = \mathbb{C}$, so V_β is one-dimensional. Now let α be a vertex in $\mathcal{B}$ with

$gr(\alpha) = 1$. Then $V_\alpha = \bigoplus_{e^+=\alpha} Im(T_e) \cong \bigoplus_{e^+=\alpha} V_\beta$ since T_e is injective. Induction gives the result. □

Thus, given an irreducible representation ρ of A_i corresponding to a vertex α in the Bratteli diagram associated to the chain of subalgebras $\{A_i\}$, there is a basis for the representation space of ρ indexed by the paths from the root to α. We call such a basis a Gel'fand-Tsetlin basis, as in Definition 2.10. Given a model representation of a Bratteli diagram $\mathcal{B}$, Lemma A.1 gives a system of Gel'fand-Tsetlin bases for $\mathcal{B}$. In fact, these are equivalent concepts:

Theorem A.1. *A system of Gel'fand-Tsetlin bases for a Bratteli diagram $\mathcal{B}$ uniquely determines a model representation for $\mathcal{B}$. Conversely, a model representation uniquely determines a system of Gel'fand-Tsetlin bases for $\mathcal{B}$.*

Proof. Both require a choice of vector space for each vertex of $\mathcal{B}$, so we need only show how a choice of basis corresponds with linear maps T_e for each edge e.

Given a system of bases and an edge $e \in \mathcal{B}$, a basis vector for V_{e^-} corresponds to a path P from the root to e^-. Then $e \circ P$ is a path from the root to e^+, which corresponds to a basis vector for V_{e^+}. In other words, we have an injection of V_{e^-} into V_{e^+}.

Conversely, given a model representation and a vertex α, every path from the root to α corresponds to an injection of $\mathbb{C}$ into V_α. Since

$$V_\alpha = \bigoplus_{e^+=\alpha} Im(T_e),$$

the union of the distinct images of $1 \in \mathbb{C}$ over the collection of injections gives a basis for V_α as we vary over all possible paths from the root to α. □

Remark A.1. The equivalent definitions of Gel'fand-Tsetlin bases and model representations coincide with the notion of a complete set of adapted representations for chains of groups. Clearly a model representation for the group algebra chain gives rise to an adapted basis since the isomorphism

$$V_\alpha = \bigoplus_{e^+=\alpha} Im(T_e) \cong \bigoplus_{\substack{e \in E(\mathcal{B}),\\ e^+=\alpha}} V_{e^-}$$

describes how the representation space V_α decomposes at level $i-1$. Equivariance of the maps T_e then gives the decomposition of the representation ρ_α.

Further, a complete set R of inequivalent irreducible representations adapted to a chain of subgroups $G_n > G_{n-1} > \cdots > G_0$ determines the paths in the Bratteli diagram $\mathcal{B}$ of the group algebra chain by drawing $M(\rho, \gamma)$ arrows from a representation $\gamma \in R$ of G_i to a representation $\rho \in R$ of G_{i+1}. Then a set of bases for the representation spaces of the representations in R determines a system of Gel-fand Tsetlin bases for the group algebra chain, and so by Theorem A.1 a model representation.

A.2 Restricted Product Lemmas

We first prove Theorem 2.7 of Section 2.3.

Theorem A.2 (Theorem 2.7)**.** *For $\mathcal{B}$ a Bratteli diagram of highest grading at least n, $\phi = \iota_1 \sqcup \iota_2$, and $\iota = \iota_\triangle$ as in Figure 2.16, the bilinear map*

$$\iota^* \circ \phi_* \circ \bigotimes : A(Q_{n0}; \mathcal{B}) \times A(Q_{n0}; \mathcal{B}) \to A(Q_{n0}; \mathcal{B})$$

corresponds to the algebra product on $\mathbb{C}[\mathcal{B}_n]$ under the isomorphism $A(Q_{n0}; \mathcal{B}) \cong \mathbb{C}[\mathcal{B}_n]$

of Lemma 2.4.

Proof. For P, Q paths of length n in $\mathcal{B}$ starting at the root and ending at the same vertex, let $\tau_{PQ} \in \mathrm{Hom}(Q_{n0}; \mathcal{B})$ such that $\tau_{PQ}(p) = P$ and $\tau_{PQ}(q) = Q$.

For $f = \sum f|_{PQ}(P, Q) \in \mathbb{C}[\mathcal{B}_n]$, the isomorphism of Lemma 2.4 maps f to

$$\sum_{\tau_{PQ} \in \mathrm{Hom}(Q_{n0}; \mathcal{B})} f|_{PQ} \tau_{PQ}.$$

Now consider multiplication in the path algebra: for f as above and for $g \in \mathbb{C}[\mathcal{B}_n]$,

$$f * g = \sum_{P,Q'} \left(\sum_{Q} f|_{PQ} g|_{QQ'} \right) (P, Q') \longrightarrow \sum_{P,Q'} \left(\sum_{Q} f|_{PQ} g|_{QQ'} \right) \tau_{PQ'}.$$

We show

$$\iota^* \phi_* \bigotimes \left(\sum_{P,Q} f|_{PQ} \tau_{PQ}, \sum_{P',Q'} g|_{P'Q'} \tau_{P'Q'} \right) = \sum_{P,Q'} \left(\sum_{Q} f|_{PQ} g|_{QQ'} \right) \tau_{PQ'}.$$

First note that morphisms $\eta \in \mathrm{Hom}(Q_{n0} \sqcup Q_{n0}; \mathcal{B})$ have form $\tau_{PQ} \sqcup \tau_{P'Q'}$. Thus,

$$\iota^* \phi_* \bigotimes \left(\sum_{P,Q} f|_{PQ} \tau_{PQ}, \sum_{P',Q'} g|_{P'Q'} \tau_{P'Q'} \right) = \iota^* \phi_* \left(\sum_{P,Q,P',Q'} f|_{PQ} g|_{P'Q'} (\tau_{PQ} \sqcup \tau_{P'Q'}) \right).$$

Let $(fg)|_{\tau_{PQ} \sqcup \tau_{P'Q'}} := f|_{PQ} g|_{P'Q'}$ and $\mu_{PQQ'} \in \mathrm{Hom}(R; \mathcal{B})$ such that $\mu_{PQQ'}(p) = P$,

$\mu_{PQQ'}(q) = Q$, and $\mu_{PQQ'}(q') = Q'$. Then

$$\begin{aligned}
\iota^*\phi_*\left(\sum (fg)|_{\tau_{PQ}\sqcup\tau_{P'Q'}}(\tau_{PQ}\sqcup\tau_{P'Q'})\right) &= \iota^*\left(\sum_{\mu_{PQQ'}}(fg)|_{\mu_{PQQ'}\circ\phi}\mu_{PQQ'}\right)\\
&= \iota^*\left(\sum_{\mu_{PQQ'}}(fg)|_{\tau_{PQ}\sqcup\tau_{QQ'}}\mu_{PQQ'}\right)\\
&= \sum_{\mu_{PQQ'}}(fg)|_{\tau_{PQ}\sqcup\tau_{QQ'}}(\mu_{PQQ'}\circ\iota)\\
&= \sum_{P,Q'}\left(\sum_{Q}(fg)|_{\tau_{PQ}\sqcup\tau_{QQ'}}\right)\tau_{PQ'}\\
&= \sum_{P,Q'}\left(\sum_{Q} f|_{PQ}g|_{QQ'}\right)\tau_{PQ'}.
\end{aligned}$$

□

Definition A.3. *Let B be a locally finite graded quiver, R a graded quiver with finite subquivers Q_1 and Q_2, and ι_j the natural inclusion $Q_j \hookrightarrow R$, for $j = 1, 2$. Then define the* ***general restricted product relative to R****,*

$$* : A(Q_1 \uparrow R; B) \times A(Q_2 \uparrow R; B) \to A(Q_1 \triangle Q_2 \uparrow R; B),$$

as follows: for $f \in A(Q_1 \uparrow R; B), g \in A(Q_2 \uparrow R; B)$,

$$*(f, g) = f * g := \sum_{\tau\in \mathrm{Hom}(Q_1\triangle Q_2\uparrow R;B)}\ \sum_{\substack{\eta\in\mathrm{Hom}(Q_1\cup Q_2\uparrow R;B)\\ \eta\downarrow_{Q_1\triangle Q_2}=\tau}} f|_{\eta\circ\iota_1}g|_{\eta\circ\iota_2}\tau.$$

Remark A.2. For $\iota_\triangle$ the natural injection $Q_1 \triangle Q_2 \hookrightarrow Q_1 \cup Q_2$, it is easy to check that this operation is equivalent to $(\iota_\triangle)^* \circ (\iota_1 \sqcup \iota_2)_* \circ \bigotimes$.

A.2 Restricted Product Lemmas

Lemma A.2. *For B a locally finite graded quiver, R a graded quiver with finite subquivers Q_1 and Q_2, and $f_i \in A(Q_i; B)$, the restricted product $f_1 * f_2$ requires no more than*

$$\# \operatorname{Hom}((Q_1 \cup Q_2) \uparrow R; B)$$

scalar multiplications and

$$\# \operatorname{Hom}((Q_1 \cup Q_2) \uparrow R; B) - \# \operatorname{Hom}((Q_1 \triangle Q_2) \uparrow R; B)$$

scalar additions.

Proof. To compute $f_1 * f_2$, first compute $(f_1|_{\eta \circ \iota_1})(f_2|_{\eta \circ \iota_2})$ for each $\eta \in \operatorname{Hom}(Q_1 \cup Q_2 \uparrow R; B)$. This requires $\# \operatorname{Hom}(Q_1 \cup Q_2 \uparrow R; B)$ scalar multiplications.

Next note that a scalar addition comes from each pair $\eta_i, \eta_j \in \operatorname{Hom}(Q_1 \cup Q_2 \uparrow R; B)$ with

$$\eta_i \downarrow_{Q_1 \triangle Q_2} = \eta_j \downarrow_{Q_1 \triangle Q_2} = \tau \in \operatorname{Hom}(Q_1 \triangle Q_2 \uparrow R; B);$$

in total, $\# \operatorname{Hom}((Q_1 \cup Q_2) \uparrow R; B) - \# \operatorname{Hom}((Q_1 \triangle Q_2) \uparrow R; B)$ scalar additions. $\square$

Corollary A.1. *For $w \in W_i = \mathbb{C}[\mathcal{B}_{i^+}] \cap \operatorname{Centralizer}(\mathbb{C}[\mathcal{B}_{i^-}])$, e the identity element of $\mathbb{C}[\mathcal{B}_n]$, and quivers G, G_i, F, F_i and maps ξ_i as in Theorem 2.8,*

$$\xi_i(w) * \xi_j(e)$$

requires no arithmetic operations to compute.

Proof. First note that the value of the identity element at any pair of paths is either 0 or 1. Thus, taking a restricted product with the element of a configuration space corresponding to the identity will require no multiplications.

To find the number of additions, recall that ξ_i maps W_i to $A(F_i;\mathcal{B})$, for F_i a quiver with vertices at levels i^+ and i^-. Since for all $1 \leq k \leq n$,

$$e \in \mathbb{C}[\mathcal{B}_k] \cap \mathrm{Centralizer}(\mathbb{C}[\mathcal{B}_k]),$$

$j^+ = j^- = n$, so F_j consists of a single vertex at level n. Then by Lemma 2.5, the restricted product $\xi_i(w) * \xi_j(e)$ requires no more than

$$\#\,\mathrm{Hom}((F_i \cup F_j \uparrow G;\mathcal{B}) - \#\,\mathrm{Hom}(F_i \triangle F_j \uparrow G;\mathcal{B}) = 0$$

additions. $\square$

Lemma A.3. *For B a locally finite graded quiver, R a graded quiver with finite subquivers Q_1 and Q_2, and $f_i \in A(Q_i;B)$,*

$$f_1 * f_2 = f_2 * f_1$$

Proof. Clear since $Q_1 \cup Q_2 = Q_2 \cup Q_1$, $Q_1 \triangle Q_2 = Q_2 \triangle Q_1$, and the coefficients of f_1 and f_2 lie in $\mathbb{C}$. $\square$

Lemma A.4. *Let B be a locally finite graded quiver and R a graded quiver with finite subquivers $Q_1, Q_2, \ldots, Q_m$ such that $Q_i \cap Q_j \cap Q_k$ has no edges for all $i \neq j \neq k$. Let $Q_i^{\triangle}$ denote the quiver $Q_1 \triangle \cdots \triangle Q_i$ and let $Q_i^{\cup}$ denote the quiver $Q_1 \cup \cdots \cup Q_i$. Then for $f_i \in A(Q_i;B)$,*

$$f_1 * f_2 * \cdots * f_m$$

is independent of bracketing. Moreover, for $\tau \in \mathrm{Hom}(Q_m^{\triangle};B)$ and ι_k the natural *injection* $Q_k \hookrightarrow R$,

$$(f_1 * f_2 * \cdots * f_m)|_\tau = \sum_{\substack{\eta \in \mathrm{Hom}(Q_m^{\cup} \uparrow R; B),\\ \eta\downarrow_{Q_m^{\triangle}} = \tau}} \prod_{k=1}^{m} f_k|_{\eta \circ \iota_k}. \tag{A.1}$$

Proof. We first prove (A.1) inductively, as associativity clearly follows.

For $n = 2$, (A.1) is the definition of the restricted product $f_1 * f_2$.

Now suppose (A.1) holds for $n-1$. Since $Q_i \cap Q_j \cap Q_k = \emptyset$,

$$[(Q_1 \triangle \cdots \triangle Q_{n-1}) \cup Q_n] \cap [Q_1 \cup \cdots \cup Q_{n-1}] = Q_1 \triangle \cdots \triangle Q_{n-1}, \tag{A.2}$$

and

$$[(Q_1 \triangle \cdots \triangle Q_{n-1}) \cup Q_n] \cup [Q_1 \cup \cdots \cup Q_{n-1}] = Q_1 \cup \cdots \cup Q_n. \tag{A.3}$$

By the induction hypothesis,

$$(f_1 * \cdots * f_{n-1} * f_n)|_\tau = \sum_{\substack{\eta \in \mathrm{Hom}(Q_{n-1}^{\triangle} \cup Q_n \uparrow R; B)\\ \eta\downarrow_{Q_n^{\triangle}} = \tau}} \sum_{\substack{\mu \in \mathrm{Hom}(Q_{n-1}^{\cup} \uparrow R; B)\\ \mu\downarrow_{Q_{n-1}^{\triangle}} = \eta\downarrow_{Q_{n-1}^{\triangle}}}} \left(\prod_{k=1}^{n-1} f_k|_{\mu \circ \iota_k} \cdot f_n|_{\eta \circ \iota_n} \right).$$

By (A.2) and (A.3), each choice of μ and η which agree on their intersection, the subquiver $Q_{n-1}^{\triangle}$, uniquely determines a morphism $\gamma \in \mathrm{Hom}(Q_n^{\cup} \uparrow R; B)$ such that

$$\begin{aligned} \gamma \downarrow_{Q_1 \triangle \cdots \triangle Q_n} = &\quad \eta \downarrow_{Q_1 \triangle \cdots \triangle Q_n} = \tau, \\ \gamma \circ \iota_k = \mu \circ \iota_k, &\quad \text{for } 1 \le k \le n-1, \\ \gamma \circ \iota_n = \eta \circ \iota_n. & \end{aligned}$$

□

A.3 Quiver Counts

A.3.1 Smoothing Quivers

Definition A.4. *A quiver B **factors at level** i if there are no arrows from a vertex $\alpha \in V(B)$ with $gr(\alpha) < i$ to a vertex $\beta \in V(B)$ with $gr(\beta) > i$.*

Example A.1. Let $\mathcal{B}$ be a Bratteli diagram with highest grading n. Then for all $0 \leq i \leq n$, $\mathcal{B}$ factors at level i.

Definition A.5. *Let Q be a quiver with a vertex v that is the head of exactly one arrow, e_1, and the tail of exactly one arrow, e_2. To **smooth Q at v**, remove v and replace e_1 and e_2 with an arrow from the tail of e_1 to the head of e_2. To **smooth Q**, smooth Q at all possible v.*

Example A.2. The quiver Q' of Figure A.1 results from smoothing the quiver Q.

Figure A.1

Lemma A.5. *Let B be a graded quiver that factors at level i, R a graded quiver with subquiver Q, and v a vertex of Q at level i such that Q can be smoothed at v. Let Q' (respectively R') be the quiver obtained by smoothing Q (respectively R) at v. Then*

$$\#\operatorname{Hom}(Q \uparrow R; B) = \#\operatorname{Hom}(Q' \uparrow R'; B).$$

Proof. Let $\phi \in \operatorname{Hom}(Q' \uparrow R'; B)$ and let f be the arrow in Q' resulting from smoothing Q at v. Then f replaced two arrows, e_1, e_2 in Q, with $e_1^+ = v, e_2^- = v$. Further,

$e_1^- = f^-, e_2^+ = f^+$, so $\phi(f)$ is a path in B from a vertex α with $gr(\alpha) < i$ to a vertex β with $gr(\beta) > i$. Since B factors at level i, this path contains a vertex, v', with $gr(v') = i$. Let e'_1 be the subpath of f starting at the tail of f and ending at v'. Similarly, let e'_2 be the subpath of f starting at v' and ending at the head of f.

Denote by $\tilde{\phi}$ the morphism in $\mathrm{Hom}(Q \uparrow R; B)$ such that:

$$\begin{aligned} \tilde{\phi}(e_1) &= e'_1, \qquad \tilde{\phi}(e_2) = e'_2 \\ \tilde{\phi}(e_i) &= \phi(e_i), \quad \text{for } i \neq 1, 2, \\ \tilde{\phi}(v_j) &= \phi(v_j), \quad \text{for } v_j \neq \alpha, \beta, v'. \end{aligned}$$

Clearly $\phi \to \tilde{\phi}$ is a bijection. □

Corollary A.2. *Let $\mathcal{B}$ be a Bratteli diagram, R a graded quiver with subquiver Q, and Q' (respectively R') the quiver obtained by smoothing Q (respectively R). Then*

$$\#\,\mathrm{Hom}(Q \uparrow R; \mathcal{B}) = \#\,\mathrm{Hom}(Q' \uparrow R'; \mathcal{B}).$$

A.3.2 Properties of Locally Free Quivers

Proposition A.1 (Theore 2.1)**.** *Let $\mathcal{B}$ be a Bratteli diagram. Then the following properties are equivalent:*

(i) $\mathcal{B}$ is locally free.

(ii) For each i and all $\beta \in \mathcal{B}^{i-1}$, there exists $\lambda_i \in \mathbb{C}$ such that

$$\sum_{\alpha \in \mathcal{B}^i} M_{\mathcal{B}}(\alpha, \beta) d_\alpha = \lambda_i d_\beta.$$

(iii) For each i, d_i is an eigenvector of DU.

(iv) For each i there exists $\lambda_i \in \mathbb{C}$ with $DU^i\hat{0} = \lambda_i U^{i-1}\hat{0}$.

Proof. Statements (ii), (iii), and (iv) are equivalent by definition and Lemma 2.6. For example:

(iii)⇒ (iv) $DU^i\hat{0} = DUd_{i-1} = \lambda_i d_{i-1} = \lambda_i U^{i-1}\hat{0}$

(iv)⇒ (iii) $DUd_i = DU^{i+1}\hat{0} = \lambda_{i+1}U^i\hat{0} = \lambda_{i+1}d_i$

We leave the remaining equivalences of (ii), (iii), and (iv) to the reader.

To show the equivalence of (i) and (iv), recall from Note 2.4 that elements of $\mathbb{C}[\mathcal{B}^i]$ correspond to representations of $\mathbb{C}[\mathcal{B}_i]$, i.e. $\mathbb{C}[\mathcal{B}_i]$-modules. Under this identification, the regular representation of $\mathbb{C}[\mathcal{B}_i]$ corresponds to the sum

$$\sum_{\alpha \in \mathcal{B}^i} d_\alpha \alpha = d_i \in \mathbb{C}[\mathcal{B}^i].$$

Since D is restriction ([39][Proposition 2.3.1]), the restriction of the regular representation of $\mathbb{C}[\mathcal{B}_i]$ to $\mathbb{C}[\mathcal{B}_{i-1}]$ corresponds to $Dd_i = DUd_{i-1}$.

(i)⇒(iv) For $\mathcal{B}$ locally free, $\mathbb{C}[\mathcal{B}_i]$ is free as a module over $\mathbb{C}[\mathcal{B}_{i-1}]$ with rank $\lambda_i \in \mathbb{C}$. Then the regular representation of $\mathbb{C}[\mathcal{B}_i]$ decomposes in $\mathbb{C}[\mathcal{B}_{i-1}]$ as λ_i copies of the regular representation of $\mathbb{C}[\mathcal{B}_{i-1}]$. Thus,

$$DU^i\hat{0} = DUd_{i-1} = Dd_i = \lambda_i d_{i-1} = \lambda_i U^{i-1}\hat{0}.$$

(iv)⇒(i) $DUd_{i-1} = Dd_i = \lambda_i d_{i-1}$ and so $\dim_{\mathbb{C}} \mathbb{C}[\mathcal{B}_i] = \lambda_i \dim_{\mathbb{C}} \mathbb{C}[\mathcal{B}_{i-1}]$; hence λ_i is rational and positive. To show λ_i integral, let

$$\lambda_i = \tfrac{p}{q}, \quad \gcd(p,q) = 1, \quad p, q > 0.$$

Let $m = \gcd(d_\beta)$ over all $\beta \in \mathcal{B}^{i-1}$, so $\frac{d_\beta}{m}$ an integer for all $\beta \in \mathcal{B}^{i-1}$. Then

$$\begin{aligned}
\frac{1}{m}\frac{p}{q}\sum_{\beta \in \mathcal{B}^{i-1}} d_\beta \beta \quad &= \frac{1}{m}\lambda_i d_{i-1} \\
&= \frac{1}{m} DU d_{i-1} \\
&= \frac{1}{m}\sum_{\alpha \in \mathcal{B}^i}\sum_{\beta \in \mathcal{B}^{i-1}} M_{\mathcal{B}}(\alpha,\beta) d_\alpha \beta \\
&= \sum_{\alpha \in \mathcal{B}^i}\sum_{\beta \in \mathcal{B}^{i-1}} M_{\mathcal{B}}(\alpha,\beta)^2 \frac{d_\beta}{m}\beta.
\end{aligned}$$

Then the coefficient of β is an integer and thus $q | \frac{d_\beta}{m}$ for all $\beta \in \mathcal{B}^i$. But $m = \gcd(d_\beta)$, and thus $q = 1$, making λ_i an integer.

□

Theorem A.3 (Theorem 2.13)**.** *Let $\mathcal{B}$ be a locally free Bratteli diagram and $w = D^{d_n}U^{u_n} \cdots D^{d_1}U^{u_1}$ an admissible word in U and D. Then for $s = \sum_{i=1}^n u_i - d_i$ and $\alpha \in \mathcal{B}^s$,*

$$\langle w\hat{0}, \alpha \rangle = d_\alpha \prod_{i \in \mathcal{S}} \lambda_{b_i - a_i}.$$

Proof. To be admissible, $d_i, u_i > 0$ for all $1 \leq i \leq n$ and

$$\sum_{j=1}^{i} d_j \leq \sum_{j=1}^{i} u_j.$$

Let

$$\mathcal{S}_k = \{i \in \mathcal{S} | i \leq \sum_{j=1}^{k} (d_j + u_j)\},$$

let $s_k = \sum_{i=1}^{k} u_i - d_i$, and let $w_k = D^{d_k} U^{u_k} \cdots D^{d_1} U^{u_1}$. We prove inductively that

$$w_k \hat{0} = \prod_{i \in \mathcal{S}_k} \lambda_{b_i - a_i} \sum_{\alpha \in \mathcal{B}^{s_k}} d_\alpha \alpha.$$

For $k = 1$, consider $w_1 = D^{d_1} U^{u_1}$. Then

$$\mathcal{S}_1 = \{u_1 + 1, u_1 + 2 \ldots, u_1 + d_1\}$$

and for all $i \in \mathcal{S}_1$, $b_i = u_1$ and $a_i = i - u_1 - 1$. By Proposition 2.1, Lemma 2.6, and induction,

$$\begin{aligned} w_1 \hat{0} &= D^{d_1} U^{u_1} \hat{0} \\ &= D^{d_1 - 1} \lambda_{u_1} U^{u_1 - 1} \hat{0} \\ &= \lambda_{u_1} D^{d_1 - 1} U^{u_1 - 1} \hat{0} \\ &\ \ \vdots \\ &= \lambda_{u_1} \lambda_{u_1 - 1} \cdots \lambda_{u_1 - d_1 + 1} U^{u_1 - d_1} \hat{0} \\ &= \prod_{i \in \mathcal{S}_1} \lambda_{b_i - a_i} \sum_{\alpha \in \mathcal{B}^{s_1}} d_\alpha \alpha. \end{aligned}$$

Now suppose true for $n-1$. Then

$$\begin{aligned} w\hat{0} &= w_n\hat{0} \\ &= D^{d_n}U^{u_n}w_{n-1}\hat{0} \\ &= D^{d_n}U^{u_n}\prod_{i\in\mathcal{S}_{n-1}}\lambda_{b_i-a_i}U^{s_{n-1}}\hat{0} \\ &= \prod_{i\in\mathcal{S}_{n-1}}\lambda_{b_i-a_i}D^{d_n}U^{u_n+s_{n-1}}\hat{0}, \end{aligned}$$

and the same argument as in the base case gives the result. □

A.4 Combinatorial Lemmas for the Weyl Groups and Brauer Algebra

In this section we consider the Bratteli diagrams associated to the Weyl groups B_n and D_n to provide the bounds used in Theorems 2.2 and 2.3. Note that Lemmas A.6, A.7, A.9, A.10, A.11, and Corollaries A.3, A.4, and A.5 all hold for $n \geq 2$, $i \geq 2$.

A.4.1 The Weyl Group B_n

The Bratteli diagram $\mathcal{B}$ associated to the chain

$$\mathbb{C}[B_n] > \mathbb{C}[B_{n-1}] > \cdots > \mathbb{C}$$

is a generalization of Young's diagram — inequivalent irreducible representations of $\mathbb{C}[B_i]$ are indexed by pairs of partitions (λ_1, λ_2), of k and l, respectively, with $k+l=i$. Pairs (λ_1, λ_2), (μ_1, μ_2) are connected by an edge if either λ_1 may be obtained from μ_1

by adding a box, or if λ_2 may be obtained from μ_2 by adding a box [77] (see Figure A.2). Note that this is a multiplicity-free diagram.

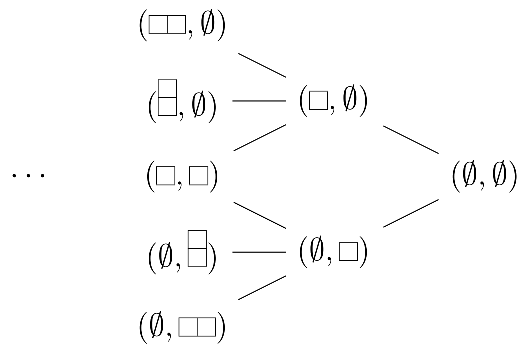


Figure A.2: $\mathbb{C}[B_2] > \mathbb{C}[B_1] > \mathbb{C}$

Lemma A.6. $M_{\mathcal{B}}(B_i, B_{i-2}) \leq 2$

Proof. Suppose not. Then since $\mathcal{B}$ is multiplicity-free, we must have distinct pairs of partitions

$$\kappa = (\kappa_1, \kappa_2), \quad \rho = (\rho_1, \rho_2), \quad \gamma = (\gamma_1, \gamma_2), \quad \eta = (\eta_1, \eta_2), \quad \lambda = (\lambda_1, \lambda_2),$$

as in Figure A.3.

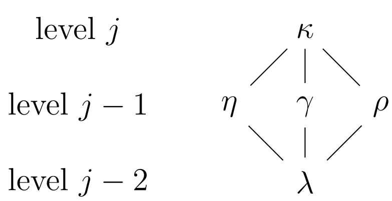


Figure A.3

Pairs of partitions are adjacent in $\mathcal{B}$ if one is acquired from the other by adding a single box; hence either $\rho_1 = \lambda_1$ or $\rho_2 = \lambda_2$. The same holds for η, γ. Similarly,

$\kappa_1 = \rho_1$ or $\kappa_2 = \rho_2$ and the same holds for η, γ.

Without loss of generality, we need only consider the following two cases:

Case 1: $\rho_1, \eta_1, \gamma_1 = \lambda_1$.

If $\kappa_1 \neq \lambda_1$, then $\kappa_2 = \rho_2, \gamma_2, \eta_2$, but then $\rho = \eta = \gamma$, a contradiction.

If $\kappa_1 = \lambda_1$ then κ_2 is obtained from λ_2 by adding two boxes, which may be done in at most two ways, so η, γ, ρ are not all distinct, a contradiction.

Case 2: $\rho_1 = \lambda_1 = \eta_1$ and $\gamma_2 = \lambda_2$.

If $\kappa_1 = \lambda_1$ then since $\gamma_1 \neq \lambda_1$, we see that $\kappa_1 \neq \gamma_1$. Thus, $\kappa_2 = \gamma_2$. But then $(\kappa_1, \kappa_2) = (\lambda_1, \lambda_2)$, a contradiction.

Now if $\kappa_1 \neq \lambda_1$, then $\kappa_2 = \rho_2, \eta_2$, but then $\rho = \eta$, a contradiction.

□

Lemma A.6 is used in the proof of Theorem 2.2 to give a bound on $\dim A(\mathcal{J}_i^n \uparrow G; \mathcal{B})$ for $\mathcal{J}_i^n$ as in Figure 2.28. The following two lemmas provide a bound for $\dim A(\mathcal{H}_i^n \uparrow G; \mathcal{B})$, for $\mathcal{H}_i^n$ as in Figure 2.28.

Lemma A.7.

(1) $\#\operatorname{Hom}(\mathcal{H}_i^n \uparrow G; \mathcal{B}) = \dfrac{|B_{n-1}|}{|B_{i-1}|} \#\operatorname{Hom}(\mathcal{H}_i^i \uparrow G; \mathcal{B})$

(2) $\#\operatorname{Hom}(\mathcal{H}_i^i \uparrow G; \mathcal{B}) =$

$$2(i-1)|B_{i-1}| + \sum_{\substack{(\beta_{i-1}^1, \beta_{i-1}^2) = \\ \beta_{i-1} \in \mathcal{B}^{i-1}}} (\operatorname{jmp}(\beta_{i-1}^1) + \operatorname{jmp}(\beta_{i-1}^2))(\operatorname{jmp}(\beta_{i-1}^1) + \operatorname{jmp}(\beta_{i-1}^2) + 1) d_{\beta_{i-1}}^2,$$

where jmp *denotes the jump of a partition, i.e, the number of ways to remove a single box to form a new partition.*

Proof. To prove (1), first note by Theorem 2.12,

$$\#\operatorname{Hom}(\mathcal{H}_i^n \uparrow G; \mathcal{B}) = \sum_{\alpha_j,\beta_j\in\mathcal{B}^j} \Big(M_{\mathcal{B}}(\beta_{n-1},\beta_{i-1}) M_{\mathcal{B}}(\beta_{i-1},\beta_{i-2}) M_{\mathcal{B}}(\alpha_i,\alpha_{i-1}) \\ M_{\mathcal{B}}(\alpha_i,\beta_{i-1}) M_{\mathcal{B}}(\alpha_{i-1},\beta_{i-2}) d_{\alpha_{i-1}} d_{\beta_{n-1}} \Big).$$

By Corollary 2.2,

$$\begin{aligned} \sum_{\beta_{n-1}\in\mathcal{B}^{n-1}} M_{\mathcal{B}}(\beta_{n-1},\beta_{i-1}) d_{\beta_{n-1}} &= \langle D^{n-i} U^{n-1} \hat{0}, \beta_{i-1} \rangle \\ &= \lambda_{n-1}\lambda_{n-2}\cdots\lambda_i d_{\beta_{i-1}} \\ &= \frac{|B_{n-1}|}{|B_{i-1}|} d_{\beta_{i-1}}. \end{aligned}$$

Then

$$\begin{aligned} \#\operatorname{Hom}(\mathcal{H}_i^n \uparrow G; \mathcal{B}) &= \frac{|B_{n-1}|}{|B_{i-1}|} \sum_{\alpha_j,\beta_j\in\mathcal{B}^j} \Big(M_{\mathcal{B}}(\beta_{i-1},\beta_{i-2}) M_{\mathcal{B}}(\alpha_i,\alpha_{i-1}) \\ &\qquad M_{\mathcal{B}}(\alpha_i,\beta_{i-1}) M_{\mathcal{B}}(\alpha_{i-1},\beta_{i-2}) d_{\beta_{i-1}} d_{\alpha_{i-1}} \Big) \\ &= \frac{|B_{n-1}|}{|B_{i-1}|} \#\operatorname{Hom}(\mathcal{H}_i^i \uparrow G; \mathcal{B}). \end{aligned}$$

To prove (2),

$$\begin{aligned} &\#\operatorname{Hom}(\mathcal{H}_i^i \uparrow G; \mathcal{B}) = \\ &= \sum_{\alpha_j,\beta_j\in\mathcal{B}^j} M_{\mathcal{B}}(\beta_{i-1},\beta_{i-2}) M_{\mathcal{B}}(\alpha_i,\alpha_{i-1}) M_{\mathcal{B}}(\alpha_i,\beta_{i-1}) M_{\mathcal{B}}(\alpha_{i-1},\beta_{i-2}) d_{\beta_{i-1}} d_{\alpha_{i-1}} \\ &= \sum_{\alpha_{i-1}\neq\beta_{i-1}} + \sum_{\alpha_{i-1}=\beta_{i-1}}, \end{aligned}$$

for $\sum_{\alpha_{i-1}\neq\beta_{i-1}}$ the sum

$$\sum_{\substack{\alpha_j,\beta_j\in\mathcal{B}^j\\ \alpha_{i-1}\neq\beta_{i-1}}} M_{\mathcal{B}}(\beta_{i-1},\beta_{i-2})M_{\mathcal{B}}(\alpha_i,\alpha_{i-1})M_{\mathcal{B}}(\alpha_i,\beta_{i-1})M_{\mathcal{B}}(\alpha_{i-1},\beta_{i-2})d_{\beta_{i-1}}d_{\alpha_{i-1}}$$

and $\sum_{\alpha_{i-1}=\beta_{i-1}}$ the sum

$$\sum_{\substack{\alpha_j,\beta_j\in\mathcal{B}^j\\ \alpha_{i-1}=\beta_{i-1}}} M_{\mathcal{B}}(\beta_{i-1},\beta_{i-2})^2 M_{\mathcal{B}}(\alpha_i,\beta_{i-1})^2(d_{\beta_{i-1}})^2.$$

First suppose $\alpha_{i-1}=(\alpha^1_{i-1},\alpha^2_{i-1}), \beta_{i-1}=(\beta^1_{i-1},\beta^2_{i-1})$ are distinct pairs of partitions. Then they jointly determine $\alpha_i=(\alpha^1_i,\alpha^2_i)$. Thus,

$$\sum_{\alpha_{i-1}\neq\beta_{i-1}} = \sum_{\substack{\beta_{i-2}\in\mathcal{B}^{i-2}\\ \alpha_{i-1}\neq\beta_{i-1}\in\mathcal{B}^{i-1}}} M_{\mathcal{B}}(\beta_{i-1},\beta_{i-2})M_{\mathcal{B}}(\alpha_{i-1},\beta_{i-2})d_{\alpha_{i-1}}d_{\beta_{i-1}}$$

$$= \sum_{\substack{\beta_{i-2}\in\mathcal{B}^{i-2}\\ \alpha_{i-1},\beta_{i-1}\in\mathcal{B}^{i-1}}} M_{\mathcal{B}}(\beta_{i-1},\beta_{i-2})M_{\mathcal{B}}(\alpha_{i-1},\beta_{i-2})d_{\beta_{i-1}}d_{\alpha_{i-1}} - \sum_{\substack{\beta_j\in\mathcal{B}\\ \alpha_{i-1}=\beta_{i-1}}} M_{\mathcal{B}}(\beta_{i-1},\beta_{i-2})^2(d_{\beta_{i-1}})^2$$

$$= \frac{|B_{i-1}|}{|B_{i-2}|} \sum_{\beta_j, \alpha_j \in \mathcal{B}^j} M_{\mathcal{B}}(\alpha_{i-1}, \beta_{i-2}) d_{\beta_{i-2}} d_{\alpha_{i-1}} - \sum_{\substack{\beta_j \in \mathcal{B}^j \\ \alpha_{i-1} = \beta_{i-1}}} M_{\mathcal{B}}(\beta_{i-1}, \beta_{i-2})^2 (d_{\beta_{i-1}})^2$$

$$= \frac{|B_{i-1}|}{|B_{i-2}|} \sum_{\alpha_{i-1} \in \mathcal{B}^{i-1}} (d_{\alpha_{i-1}})^2 - \sum_{\substack{\beta_j \in \mathcal{B}^j \\ \alpha_{i-1} = \beta_{i-1}}} M_{\mathcal{B}}(\beta_{i-1}, \beta_{i-2})^2 (d_{\beta_{i-1}})^2$$

$$= \frac{|B_{i-1}|^2}{|B_{i-2}|} - \sum_{\beta_{i-1} \in \mathcal{B}^{i-1}} (\mathrm{jmp}(\beta^1_{i-1}) + \mathrm{jmp}(\beta^2_{i-1}))(d_{\beta_{i-1}})^2$$

$$= 2(i-1)|B_{i-1}| - \sum_{\beta_{i-1} \in \mathcal{B}^{i-1}} (\mathrm{jmp}(\beta^1_{i-1}) + \mathrm{jmp}(\beta^2_{i-1}))(d_{\beta_{i-1}})^2. \tag{A.4}$$

Now suppose $\alpha_{i-1} = \beta_{i-1}$. Then α_i is obtained from β_{i-1} by adding a box to β^1_{i-1} or β^2_{i-1}, while β_{i-2} is obtained from β_{i-1} by removing a box from β^1_{i-1} or β^2_{i-1}. Thus,

$$\sum_{\alpha_{i-1} = \beta_{i-1}} = \sum_{\beta_{i-1} \in \mathcal{B}^{i-1}} (\mathrm{jmp}(\beta^1_{i-1}) + \mathrm{jmp}(\beta^1_{i-2}))(\mathrm{jmp}(\beta^1_{i-1}) + \mathrm{jmp}(\beta^1_{i-2}) + 2)(d_{\beta_{i-1}})^2. \tag{A.5}$$

Summing equations (A.4) and (A.5), we have

$$\# \operatorname{Hom}(\mathcal{H}^i_i \uparrow G; \mathcal{B}) = 2(i-1)|B_{i-1}| + \sum_{\substack{(\beta^1_{i-1}, \beta^2_{i-1}) = \\ \beta_{i-1} \in \mathcal{B}^{i-1}}} (\mathrm{jmp}(\beta^1_{i-1}) + \mathrm{jmp}(\beta^2_{i-1}))(\mathrm{jmp}(\beta^1_{i-1}) + \mathrm{jmp}(\beta^2_{i-1}) + 1) d^2_{\beta_{i-1}}.$$

□

Lemma A.8. *For any pair of partitions* (β_i^1, β_i^2) *with* $|\beta_i^1| + |\beta_i^2| = i$*,*

$$(\mathrm{jmp}(\beta_i^1) + \mathrm{jmp}(\beta_i^2))(\mathrm{jmp}(\beta_i^1) + \mathrm{jmp}(\beta_i^2) + 1) \leq 6i$$

Proof. Let $k = |\beta_i^1|$, $l = |\beta_i^2|$, $a_k = \mathrm{jmp}(\beta_i^1)$, and $a_l = \mathrm{jmp}(\beta_i^2)$. Then $k + l = i$ and by [63, Lemma 5.3],

$$a_k(a_k + 1) \leq 2k, \quad a_l(a_l + 1) \leq 2l.$$

Then

$$\begin{aligned}
(\mathrm{jmp}(\beta_i^1) + \mathrm{jmp}(\beta_i^2))(\mathrm{jmp}(\beta_i^1) + \mathrm{jmp}(\beta_i^2) + 1) &= (a_k + a_l)(a_k + a_l + 1) \\
&= a_k(a_k + 1) + a_l(a_l + 1) + 2a_k a_l \\
&\leq 2k + 2l + 2(2i) \\
&\leq 6i.
\end{aligned}$$

□

Combining Lemmas A.7 and A.8 gives the following bound:

Corollary A.3. $\#\,\mathrm{Hom}(\mathcal{H}_i^n \uparrow G; \mathcal{B}) \leq \frac{4(i-1)}{n}|B_n|$.

A.4.2 The Weyl Group D_n

The Bratteli diagram $\mathcal{B}$ associated to the chain

$$\mathbb{C}[D_n] > \mathbb{C}[D_{n-1}] > \cdots > \mathbb{C}$$

is similar to the Bratteli diagram associated to the Weyl group B_n in that irreducible representations of $\mathbb{C}[D_i]$ are indexed by pairs of partitions, (λ_1, λ_2) of k and l, respectively, with $k + l = i$. However, if $\lambda_1 \neq \lambda_2$, the irreducible representation indexed by (λ_1, λ_2) is the same as that indexed by (λ_2, λ_1). If $\lambda_1 = \lambda_2 = \lambda$ then two distinct irreducible representations are indexed by the pair (λ, λ), and denoted by $(\lambda, \lambda)^+$ and $(\lambda, \lambda)^-$ [77] (see Figure A.4). Note that this is a multiplicity-free diagram.

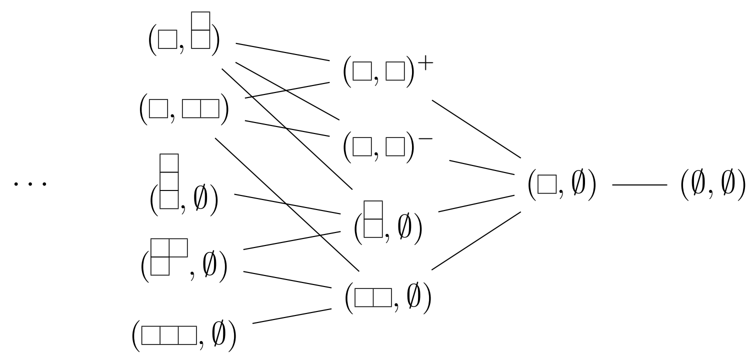


Figure A.4: $\mathbb{C}[D_3] > \mathbb{C}[D_2] > \mathbb{C}[D_1] > \mathbb{C}$

Lemma A.9. *$M_{\mathcal{B}}(D_i, D_{i-2}) \leq 3$.*

Proof. Suppose not. Then since $\mathcal{B}$ is multiplicity-free, there exist pairs of partitions

$$\kappa = (\kappa_1, \kappa_2), \quad \rho = (\rho_1, \rho_2), \quad \gamma = (\gamma_1, \gamma_2), \quad \eta = (\eta_1, \eta_2), \quad \mu = (\mu_1, \mu_2), \quad \lambda = (\lambda_1, \lambda_2),$$

connected in $\mathcal{B}$ as in Figure A.5.

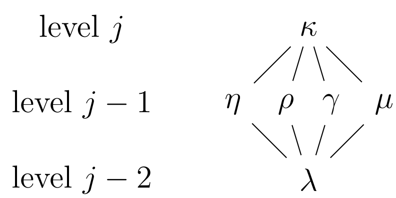


Figure A.5

However, the proof of Lemma A.6 dictates that no three of η, μ, γ, ρ are distinct pairs of partitions. Thus, without loss of generality,

$$(\eta_1, \eta_2) = (\alpha, \alpha)^+, \quad (\mu_1, \mu_2) = (\alpha, \alpha)^-, \quad (\gamma_1, \gamma_2) = (\beta, \beta)^+, \quad (\rho_1, \rho_2) = (\beta, \beta)^-,$$

for α, β distinct partitions of $\frac{j-1}{2}$. Then as in the proof of Lemma A.6, either $\lambda_1 = \eta_1 = \alpha$ or $\lambda_2 = \eta_2 = \alpha$. Without loss, suppose $\lambda_1 = \alpha$. Then since $\alpha \neq \beta$, λ_2 must be β. However, $|\lambda_1| + |\lambda_2| = |\alpha| + |\beta| > j - 2$, a contradiction. □

Lemma A.9 is used in the proof of Theorem 2.3 to give a bound on $\dim A(\mathcal{J}_i^n \uparrow G; \mathcal{B})$, for $\mathcal{J}_i^n$ as in Figure 2.28. The following two lemmas provide a bound for $\dim A(\mathcal{H}_i^n \uparrow G; \mathcal{B})$, for $\mathcal{H}_i^n$ as in Figure 2.28.

Lemma A.10.

(1) $\# \operatorname{Hom}(\mathcal{H}_i^n \uparrow G; \mathcal{B}) = \dfrac{|D_{n-1}|}{|D_{i-1}|} \# \operatorname{Hom}(\mathcal{H}_i^i \uparrow G; \mathcal{B})$,

(2) for i odd,

$$\# \operatorname{Hom}(\mathcal{H}_i^i \uparrow G; \mathcal{B}) \leq$$

$$\frac{|D_{i-1}|^2}{|D_{i-2}|} + \sum_{(\alpha,\alpha)^\pm \in \mathcal{B}^{i-1}} (\operatorname{jmp}(\alpha))(\operatorname{jmp}(\alpha)+1)(d_{(\alpha,\alpha)^+})^2 +$$
$$\sum_{\substack{(\beta^1_{i-1}, \beta^2_{i-1}) = \\ \beta_{i-1} \in \mathcal{B}^{i-1}}} (\operatorname{jmp}(\beta^1_{i-1}) + \operatorname{jmp}(\beta^2_{i-1}))(\operatorname{jmp}(\beta^1_{i-1}) + \operatorname{jmp}(\beta^2_{i-1}) + 1) d^2_{\beta_{i-1}},$$

(3) for i even,

$$\# \operatorname{Hom}(\mathcal{H}_i^i \uparrow G; \mathcal{B}) \leq 2\frac{|D_{i-1}|^2}{|D_{i-2}|}$$
$$+2 \sum_{\substack{(\beta_{i-1}^1,\beta_{i-1}^2)= \\ \beta_{i-1}\in\mathcal{B}^{i-1}}} (\operatorname{jmp}(\beta_{i-1}^1) + \operatorname{jmp}(\beta_{i-1}^2))(2\operatorname{jmp}(\beta_{i-1}^1) + 2\operatorname{jmp}(\beta_{i-1}^2) + 3)d_{\beta_{i-1}}^2,$$

where jmp *denotes the jump of a partition, i.e., the number of ways to remove a single box to form a new partition.*

Proof. Part (1) follows from the proof of Lemma A.7.

To prove (2), consider

$$\# \operatorname{Hom}(\mathcal{H}_i^i \uparrow G; \mathcal{B}) =$$

$$= \sum_{\alpha_j,\beta_j\in\mathcal{B}^j} M_{\mathcal{B}}(\beta_{i-1},\beta_{i-2})M_{\mathcal{B}}(\alpha_i,\alpha_{i-1})M_{\mathcal{B}}(\alpha_i,\beta_{i-1})M_{\mathcal{B}}(\alpha_{i-1},\beta_{i-2})d_{\beta_{i-1}}d_{\alpha_{i-1}}$$
$$= \sum_{\alpha_{i-1}\neq\beta_{i-1}} + \sum_{\substack{\alpha_{i-1}\neq\beta_{i-1} \\ \alpha_{i-1}=(\alpha,\alpha)^{\pm}=\beta_{i-1}}} + \sum_{\alpha_{i-1}=\beta_{i-1}},$$

For

$$\sum_{\alpha_{i-1}\neq\beta_{i-1}} :=$$
$$\sum_{\substack{\alpha_j,\beta_j\in\mathcal{B}^j \\ \alpha_{i-1}\neq\beta_{i-1}}} M_{\mathcal{B}}(\beta_{i-1},\beta_{i-2})M_{\mathcal{B}}(\alpha_i,\alpha_{i-1})M_{\mathcal{B}}(\alpha_i,\beta_{i-1})M_{\mathcal{B}}(\alpha_{i-1},\beta_{i-2})d_{\beta_{i-1}}d_{\alpha_{i-1}},$$

over partitions $\alpha_{i-1} \neq \beta_{i-1}$ such that if $\alpha_{i-1} = (\alpha, \alpha)^{\pm}$ then $\beta_{i-1} \neq (\alpha, \alpha)^{\pm}$,

$$\sum_{\substack{\alpha_{i-1}\neq\beta_{i-1}\\ \alpha_{i-1}:=(\alpha,\alpha)^{\pm}\\ =\beta_{i-1}}} := \sum_{\substack{\alpha_j,\beta_j\in\mathcal{B}^j\\ \alpha_{i-1}\neq\beta_{i-1}\\ \alpha_{i-1}=(\alpha,\alpha)^{\pm}=\beta_{i-1}}} M_{\mathcal{B}}(\beta_{i-1}, \beta_{i-2}) M_{\mathcal{B}}(\alpha_i, \alpha_{i-1}) M_{\mathcal{B}}(\alpha_i, \beta_{i-1}) M_{\mathcal{B}}(\alpha_{i-1}, \beta_{i-2}) d_{\beta_{i-1}} d_{\alpha_{i-1}},$$

and

$$\sum_{\alpha_{i-1}:=\beta_{i-1}} := \sum_{\substack{\alpha_j,\beta_j\in\mathcal{B}^j\\ \alpha_{i-1}=\beta_{i-1}}} M_{\mathcal{B}}(\beta_{i-1}, \beta_{i-2})^2 M_{\mathcal{B}}(\alpha_i, \beta_{i-1})^2 (d_{\beta_{i-1}})^2.$$

As in the proof of Lemma A.7,

$$\sum_{\alpha_{i-1}\neq\beta_{i-1}} \leq \frac{|D_{i-1}|^2}{|D_{i-2}|} - \sum_{\beta_{i-1}\in\mathcal{B}^{i-1}} (\mathrm{jmp}(\beta^1_{i-1}) + \mathrm{jmp}(\beta^2_{i-1}))(d_{\beta_{i-1}})^2, \tag{A.6}$$

the inequality appearing because if $\beta_{i-1} = (\alpha, \alpha)$, $\mathrm{jmp}(\alpha)+\mathrm{jmp}(\alpha)$ is an overestimate since (α, β) represents the same representation as (β, α) in $\mathcal{B}$. Similarly, the proof of Lemma A.7 gives

$$\sum_{\alpha_{i-1}=\beta_{i-1}} \leq \sum_{\beta_{i-1}\in\mathcal{B}^{i-1}} (\mathrm{jmp}(\beta^1_{i-1})+\mathrm{jmp}(\beta^2_{i-1}))(\mathrm{jmp}(\beta^1_{i-1})+\mathrm{jmp}(\beta^2_{i-1})+2)(d_{\beta_{i-1}})^2. \tag{A.7}$$

Now suppose $\alpha_{i-1} \neq \beta_{i-1}$ and $\alpha_{i-1} = (\alpha, \alpha)^{\pm} = \beta_{i-1}$. Then

$$\sum_{\substack{\alpha_{i-1}\neq\beta_{i-1}\\ \alpha_{i-1}=(\alpha,\alpha)^{\pm}=\beta_{i-1}}} = \sum_{\substack{\alpha_j,\beta_j\in\mathcal{B}^j\\ (\alpha,\alpha)^{\pm}}} M_{\mathcal{B}}((\alpha,\alpha)^{\pm}, \beta_{i-2})^2 M_{\mathcal{B}}(\alpha_i, (\alpha,\alpha)^{\pm})^2 (d_{(\alpha,\alpha)^{\pm}})^2$$

$$\leq \sum_{(\alpha,\alpha)^{\pm}\in\mathcal{B}^{i-1}} \mathrm{jmp}(\alpha)(\mathrm{jmp}(\alpha)+1)(d_{(\alpha,\alpha)^{\pm}})^2. \tag{A.8}$$

Summing equations (A.6), (A.7), and (A.8) gives part (2).

To prove (3), note that in this case

$$\#\operatorname{Hom}(\mathcal{H}_i^i \uparrow G; \mathcal{B}) = \sum_{\alpha_{i-1} \neq \beta_{i-1}} + \sum_{\alpha_{i-1} = \beta_{i-1}},$$

since $i-1$ is odd so $(\alpha, \alpha)^{\pm} \notin \mathcal{B}^{i-1}$. However, pairs of partitions of this form may be found at levels i and $i-2$.

First suppose $\alpha_{i-1} \neq \beta_{i-1}$. Then as in the proof of Lemma A.7 they jointly determine $\alpha_i = (\alpha_i^1, \alpha_i^2)$. This means that they jointly determine at most two pairs of partitions (if $\alpha_i^1 = \alpha_i^2$). Thus

$$\sum_{\alpha_{i-1} \neq \beta_{i-1}} \leq 2 \sum_{\substack{\beta_{i-2} \in \mathcal{B}^{i-2} \\ \alpha_{i-1} \neq \beta_{i-1} \in \mathcal{B}^{i-1}}} M_{\mathcal{B}}(\beta_{i-1}, \beta_{i-2}) M_{\mathcal{B}}(\alpha_{i-1}, \beta_{i-2}) d_{\alpha_{i-1}} d_{\beta_{i-1}}$$

$$= 2\frac{|D_{i-1}|^2}{|D_{i-2}|} - 2 \sum_{\beta_{i-1} \in \mathcal{B}^{i-1}} (\operatorname{jmp}(\beta_{i-1}^1) + \operatorname{jmp}(\beta_{i-1}^2))(d_{\beta_{i-1}})^2, \tag{A.9}$$

as in the proof of Lemma A.7.

Now suppose $\alpha_{i-1} = \beta_{i-1}$. As before there are $\operatorname{jmp}(\beta_{i-1}^1)$ ways to obtain β_i^1 and $\operatorname{jmp}(\beta_{i-1}^2)$ ways to obtain β_i^2, but to account for when $\beta_i^1 = \beta_i^2$, we overcount by multiplying by 2. The same holds for the number of ways to obtain α_{i-2} from β_{i-1}. Thus,

$$\sum_{\alpha_{i-1} \beta_{i-1}} \leq \sum_{\beta_{i-1} \in \mathcal{B}^{i-1}} 2(\operatorname{jmp}(\beta_{i-1}^1) + \operatorname{jmp}(\beta_{i-1}^2)) 2(\operatorname{jmp}(\beta_{i-1}^1) + \operatorname{jmp}(\beta_{i-1}^2) + 2)(d_{\beta_{i-1}})^2. \tag{A.10}$$

Summing equations (A.9) and (A.10) gives part 3. □

Combining Lemma A.10 with Lemma A.8 gives the following bound:

Corollary A.4. $\#\operatorname{Hom}(\mathcal{H}_i^n \uparrow G; \mathcal{B}) \leq \frac{20(i-1)}{n}|D_n|$.

A.4.3 The Brauer Algebra $\mathcal{B}r_n$

In the Bratteli diagram $\mathcal{B}$ associated to the chain

$$\mathcal{B}r_n > \mathcal{B}r_{n-1} > \cdots > \mathbb{C}(q)$$

irreducible representations of $\mathcal{B}r_i$ are indexed by partitions of $i - 2k$, $0 \leq k \leq i/2$, with an edge between $\rho \in \mathcal{B}^i$ and $\lambda \in \mathcal{B}^{i-1}$ if ρ is obtained from λ by adding or removing a box [56] (see Figure A.6). Note that this is a multiplicity-free diagram.

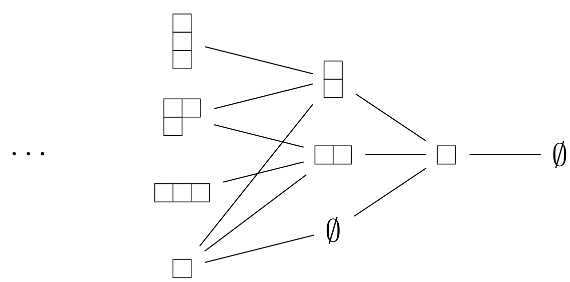


Figure A.6: $\mathcal{B}r_3 > \mathcal{B}r_2 > \mathcal{B}r_1 > \mathbb{C}(q)$

The following two lemmas provide a bound for $\dim A(\mathcal{H}_i^n \uparrow G; \mathcal{B})$, for $\mathcal{H}_i^n$ as in Figure 2.43.

Lemma A.11.

(1) $\#\operatorname{Hom}(\mathcal{H}_i^n \uparrow G; \mathcal{B}) = \frac{\dim(\mathcal{B}r_{n-1})}{\dim(\mathcal{B}r_{i-1})}\#\operatorname{Hom}(\mathcal{H}_i^i \uparrow G; \mathcal{B})$,

(2) $\# \operatorname{Hom}(\mathcal{H}_i^i \uparrow G; \mathcal{B})$

$$\leq 2\frac{\dim(\mathcal{B}r_{i-1})^2}{\dim(\mathcal{B}r_{i-2})} + \sum_{\beta_{i-1} \in \mathcal{B}^{i-1}} (4\operatorname{jmp}(\beta_{i-1})^2 + 2\operatorname{jmp}(\beta_{i-1}) + 1)(d_{\beta_{i-1}})^2,$$

where jmp *denotes the jump of a partition, i.e, the number of ways to remove a single box to form a new partition.*

Proof. Part (1) follows from the proof of Lemma A.7.

To prove (2), consider

$$\begin{aligned}
&\# \operatorname{Hom}(\mathcal{H}_i^i \uparrow G; \mathcal{B}) \\
&= \sum_{\alpha_j, \beta_j \in \mathcal{B}^j} M_{\mathcal{B}}(\beta_{i-1}, \beta_{i-2}) M_{\mathcal{B}}(\alpha_i, \alpha_{i-1}) M_{\mathcal{B}}(\alpha_i, \beta_{i-1}) M_{\mathcal{B}}(\alpha_{i-1}, \beta_{i-2}) d_{\beta_{i-1}} d_{\alpha_{i-1}} \\
&= \sum_{\alpha_{i-1} \neq \beta_{i-1}} + \sum_{\alpha_{i-1} = \beta_{i-1}},
\end{aligned}$$

for $\sum_{\alpha_{i-1} \neq \beta_{i-1}}$ the sum

$$\sum_{\substack{\alpha_j, \beta_j \in \mathcal{B}^j \\ \alpha_{i-1} \neq \beta_{i-1}}} M_{\mathcal{B}}(\beta_{i-1}, \beta_{i-2}) M_{\mathcal{B}}(\alpha_i, \alpha_{i-1}) M_{\mathcal{B}}(\alpha_i, \beta_{i-1}) M_{\mathcal{B}}(\alpha_{i-1}, \beta_{i-2}) d_{\beta_{i-1}} d_{\alpha_{i-1}}$$

and $\sum_{\alpha_{i-1} = \beta_{i-1}}$ the sum

$$\sum_{\substack{\alpha_j, \beta_j \in \mathcal{B}^j \\ \alpha_{i-1} = \beta_{i-1}}} M_{\mathcal{B}}(\beta_{i-1}, \beta_{i-2})^2 M_{\mathcal{B}}(\alpha_i, \beta_{i-1})^2 (d_{\beta_{i-1}})^2.$$

First suppose α_{i-1} and β_{i-1} are distinct partitions. Then they jointly determine α_i up to two choices. This is clear if α_{i-1} and β_{i-1} both partition k, as they then they jointly determine exactly one partition of $k+1$ and one partition of $k-1$. Now suppose, without loss of generality, that α_{i-1} is a partition of k while β_{i-1} is a partition of $k-2$. Then to both be connected to a vertex, α_i, at level i, β_{i-1} must be obtained from α_{i-1} by removing two boxes, which can only be done in two ways.

Then as in the proof of Lemma A.7,

$$\sum_{\alpha_{i-1}\neq\beta_{i-1}} \leq 2\left(\frac{\dim(\mathcal{B}r_{i-1})^2}{\dim(\mathcal{B}r_{i-2})} - \sum \operatorname{jmp}(\beta_{i-1})(d_{\beta_{i-1}})^2\right). \tag{A.11}$$

Now suppose $\alpha_{i-1} = \beta_{i-1}$. Then α_i is obtained from β_{i-1} by either adding or removing a box, and similarly for β_{i-2}. Thus,

$$\sum_{\alpha_{i-1}=\beta_{i-1}} = \sum_{\beta_{i-1}\in\mathcal{B}^{i-1}} (2\operatorname{jmp}(\beta_{i-1})+1)(2\operatorname{jmp}(\beta_{i-1})+1)(d_{\beta_{i-1}})^2. \tag{A.12}$$

Summing equations (A.11) and (A.12) gives part (2).

□

Combining Lemma A.11 with the proof of Lemma A.8 gives the following bound:

Corollary A.5. $\#\operatorname{Hom}(\mathcal{H}_i^n \uparrow G; \mathcal{B}) \leq \frac{16i-17}{2n-1}\dim(\mathcal{B}r_n)$.

A.5 Factoring Coset Representatives of $GL_n(\mathbb{F}_q)$

In this section we provide the set of coset representatives and their factorizations used in the proof of Theorem 2.4 by developing a correspondence between $Gl_n(q)/Gl_{n-1}(q)$ and the set

$$X_n = \{(\mathbf{y},\mathbf{x}) \mid \mathbf{x},\mathbf{y} \in (\mathbb{F}_q)^n,\ y * x^T = 1\}.$$

A.5 Factoring Coset Representatives of $GL_n(\mathbb{F}_q)$

Define an action of $Gl_n(q)$ on X_n via $A.(\mathbf{y}, \mathbf{x}) = (\mathbf{y}A^{-1}, A\mathbf{x}^T)$. Note that this action preserves the product $\mathbf{y} * \mathbf{x}^T$. For $\mathbf{1} = ((0, \ldots, 0, 1), (0, \ldots, 0, 1))$, we show (Theorem A.4) that

$$X_n = \text{Orb}(\mathbf{1}).$$

Further, note that $Gl_{n-1}(q)$, viewed as a subgroup of $Gl_n(q)$, stabilizes $\mathbf{1}$, so the orbit-stabilizer theorem gives a bijection between $X_n = \text{Orb}(\mathbf{1})$ and $Gl_n(q)/Gl_{n-1}(q)$ through the correspondence

$$g.\mathbf{1} \longleftrightarrow gGl_{n-1}(q). \tag{A.13}$$

Thus, writing $(\mathbf{x}, \mathbf{y}) = A_1 \cdots A_m.\mathbf{1}$ for each $(\mathbf{x}, \mathbf{y}) \in X_n$ gives a factorization of the corresponding coset representative. We find a factorization in which each matrix $A_i = A \bigoplus I_{n-2}$ for $A \in Gl_2(q)$.

Lemma A.12. *Suppose* $\mathbf{x}, \mathbf{y} \in (\mathbb{F}_q)^2$ *with* $x_1y_1 + x_2y_2 \neq 0$. *Then there exists a matrix* $A \in Gl_2(q)$ *and* $y_2', x_2' \in \mathbb{F}_q^\times$ *such that*

$$A.(\mathbf{y}, \mathbf{x}) = (\begin{pmatrix} 0, y_2' \end{pmatrix}, \begin{pmatrix} 0, x_2' \end{pmatrix}).$$

Proof. Since the action of A preserves the product $\mathbf{y} * \mathbf{x}^T$,

$$y_2'x_2' = x_1y_1 + x_2y_2 \neq 0.$$

Case 1: $x_1 = 0$. Let $A = \begin{pmatrix} 1 & 0 \\ \frac{y_1}{y_2} & 1 \end{pmatrix}$. Note there are q possibilities for A.

Case 2: $x_1 \neq 0$, $y_1 = 0$. Let $A = \begin{pmatrix} 1 & \frac{-x_1}{x_2} \\ 0 & 1 \end{pmatrix}$. Note there are $q-1$ possibilities for A.

Case 3: $x_1 \neq 0$, $y_1 \neq 0$. Let $A = \begin{pmatrix} \frac{-x_2}{x_1} & 1 \\ 1 & \frac{y_2}{y_1} \end{pmatrix}$. Note there are q^2 possibilities for A. Note further that for $z_1 := x_1 y_1$ and $z_2 := x_2 y_2$ fixed and nonzero,

$$A = \begin{pmatrix} \frac{-x_2}{x_1} & 1 \\ 1 & \frac{z_2}{z_1}\frac{x_1}{x_2} \end{pmatrix},$$

and there are $q-1$ possibilities for A.

□

We use Lemma A.12 to systematically write $(\tilde{\mathbf{y}}, \tilde{\mathbf{x}}) \in X_n$ in form

$$(\tilde{\mathbf{y}}, \tilde{\mathbf{x}}) = A_1 A_2 \cdots A_k.(\left(0, \ldots, 0, \tilde{y}'_n\right), \left(0, \ldots, 0, \tilde{x}'_n\right)),$$

with $A_i \in Gl_n(q)$ and $\tilde{y}'_n \tilde{x}'_n = 1$. To do so we first simplify $(\tilde{\mathbf{y}}, \tilde{\mathbf{x}})$. Recall that p is the characteristic of $\mathbb{F}_q$.

Proposition A.2. *Let $(\tilde{\mathbf{y}}, \tilde{\mathbf{x}}) \in X_n$. Then there is a permutation matrix $\pi \in Gl_n(q)$, $b \in \mathbb{F}_q^\times$, and $i \geq 1$ such that $\pi.(\tilde{\mathbf{y}}, \tilde{\mathbf{x}}) = (\mathbf{y}, \mathbf{x})$ with:*

(i) $z_1 + \cdots + z_j \neq 0$ *for all* $i \leq j \leq n$, *where* $z_i := x_i y_i$,

(ii) $z_1 = \cdots = z_i = b$,

(iii) $p|(i-1)$.

Proof. Let $\tilde{\mathbf{z}} = (\tilde{z}_1, \ldots, \tilde{z}_n) := (\tilde{x}_1\tilde{y}_1 \ldots, \tilde{x}_n\tilde{y}_n)$. Note that $\tilde{\mathbf{y}} * \tilde{\mathbf{x}}^T = \tilde{z}_1 + \cdots + \tilde{z}_n = 1 \neq 0$. Let j be an index (if it exists) such that $\tilde{\mathbf{y}} * \tilde{\mathbf{x}}^T - \tilde{z}_j \neq 0$. Note that for a permutation matrix π, $(\pi^T)^{-1} = \pi$, so the action of π on $\tilde{\mathbf{y}}, \tilde{\mathbf{x}})$ permutes $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ in the same way and we may equivalently consider the action of π on $\tilde{\mathbf{z}}$. Permute $\tilde{\mathbf{z}}$ to make $\tilde{z}_j$ the last entry, then delete $\tilde{z}_j$ to produce a vector of length $n-1$. Repeat until no such index exists, and let i be the length of the resultant vector, $\mathbf{z}$. Then clearly $z_1 + \cdots + z_j \neq 0$ for all $i \leq j \leq n$. Further, $z_1 + \cdots + z_i - z_k = 0$ for all $1 \leq k \leq i$; in particular, $z_1 = \cdots = z_i = b \in \mathbb{F}_q^\times$. Finally note that $z_1 + \cdots + z_{i-1} = (i-1)b = 0$ and so $p|(i-1)$. ☐

In light of Proposition A.2, let

$$S_i(n) = \{(\mathbf{y}, \mathbf{x}) \in X_n |\ (\mathbf{y}, \mathbf{x}) \text{ satisfies (i) and (ii) of Proposition A.2}\}.$$

Note that if $p|i$, $S_i(n) = \emptyset$.

Theorem A.4. *For $p \neq 2$ and $(\mathbf{y}, \mathbf{x}) \in S_i(n)$, there exist invertible matrices*

$$u_j, u_j', v_j, t_j \in Gl_j(q) \cap \text{Centralizer}(Gl_{j-2}(q)),$$

and a scalar matrix ϵ such that

$$(\mathbf{y}, \mathbf{x}) = (u_2 \cdots u_{p-1} u_{p+1}' t_p u_{p+1} \cdots u_{2p-1} u_{2p+1}' t_{2p} u_{2p+1} \cdots u_i v_{i+1} \cdots v_n \epsilon).\mathbf{1}.$$

Proof. Let $(\mathbf{y}, \mathbf{x}) \in S_i(n)$ and let $i = mp + l$, for $m \geq 1$. For $\mathbf{z} = (z_1, \ldots, z_n) := (x_1y_1, \ldots x_ny_n)$,

$$\mathbf{z} = \left(b, \ldots, b, z_{i+1}, \ldots, z_n\right).$$

Note that $z_1 + z_2 = 2b \neq 0$ and so by Lemma A.12, there is a matrix $A \in Gl_2(q)$ such that $A.(\left(y_1, y_2\right), \left(x_1, x_2\right)) = (\left(0, y_2'\right), \left(0, x_2'\right))$ with $y_2'x_2' = 2b$. Let

$$u_2^{-1} = \begin{pmatrix} A & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \in Gl_n(q).$$

Then $u_2^{-1}.(\mathbf{y}, \mathbf{x}) = (\left(0, y_2', y_3, \ldots, y_n\right), \left(0, x_2', x_3, \ldots, x_n\right))$ with new $\mathbf{z}$-vector

$$\left(0, 2b, b, \ldots, b, z_{i+1}, \ldots, z_n\right).$$

Provided $z_j + z_{j+1} \neq 0$, we repeat this process, defining matrices $u_3^{-1}, u_4^{-1}, \ldots, u_j^{-1}$. This affects the $\mathbf{z}$-vector as follows:

$$\begin{gathered} \left(b, b, b, b, \ldots, b, z_{i+1}, \ldots, z_n\right) \\ \downarrow u_2^{-1} \\ \left(0, 2b, b, b, \ldots, b, z_{i+1}, \ldots, z_n\right) \\ \downarrow u_3^{-1} \\ \left(0, 0, 3b, b, \ldots, b, z_{i+1}, \ldots, z_n\right) \\ \downarrow u_4^{-1} \\ \vdots \\ \downarrow u_{p-1^{-1}} \\ \left(0, \ldots, 0, (p-1)b, b, b, b, \ldots, b, z_{i+1}, \ldots, z_n\right). \end{gathered}$$

Since $z_{p-1} + z_p = pb = 0$, instead, define $(u'_{p+1})^{-1}$ as above and consider the permutation matrix t_p^{-1} corresponding to the transposition $(p-1\ p)$:

$$\downarrow (u'_{p+1})$$
$$\Big(0, \dots, 0, (p-1)b, 0, 2b, b, \dots, b, z_{i+1}, \dots, z_n\Big)$$
$$\downarrow t_p^{-1}$$
$$\Big(0, \dots, 0, (p-1)b, 2b, b, b, \dots, b, z_{i+1}, \dots, z_n\Big)$$
$$\downarrow u_{p+1}^{-1}$$
$$\Big(0, \dots, 0, 0, (p+1)b = b, b, b, \dots, b, z_{i+1}, \dots, z_n\Big)$$
$$\downarrow u_{p+2}^{-1}$$
$$\vdots$$

Repeat this process until the matrix $u_{mp+l}^{-1} = u_i^{-1}$ is defined:

$$\vdots$$
$$\downarrow u_{mp+l}^{-1}$$
$$\Big(0, \dots, 0, z_1 + \cdots + z_i, z_{i+1}, \dots, z_n\Big).$$

Since $z_1 + \cdots + z_j \neq 0$ for all $i \leq j \leq n$, use Lemma A.12 to define:

$$v_j^{-1} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & A & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix},$$

with A in the $(j-1,j)$x$(j-1,j)$th subblock to continue the process:

$$\begin{array}{c} \downarrow v_{i+1}^{-1} \\ \left(0, \ldots, 0, z_1 + \cdots + z_{i+1}, z_{i+2}, \ldots, z_n\right) \\ \downarrow v_{i+2}^{-1} \\ \vdots \\ \downarrow v_n^{-1} \\ \left(0, \ldots, 0, z_1 + \cdots + z_n\right) \end{array}.$$

Note that $z_1 + \cdots + z_n = 1$. Thus,

$$v_n^{-1} \cdots v_{i+1}^{-1} u_i^{-1} \cdots u_{2p+1}^{-1} t_{2p}^{-1} (u'_{2p+1})^{-1} u_{2p-1}^{-1} \cdots u_{p+1}^{-1} t_p^{-1} (u'_{p+1})^{-1} u_{p-1}^{-1} \cdots u_2^{-1}.(\mathbf{y}, \mathbf{x})$$

$$= (\left(0, \ldots, 0, y'_n\right), \left(0, \ldots, 0, x'_n\right)),$$

with $y'_n x'_n = 1$.

For $i < p$ analogous arguments apply without needing the matrices t_p.

Multiplying by a scalar matrix, ϵ^{-1}, and noting that each matrix is invertible and that $v_j, u_j, u'_j, t_j \in Gl_j(q) \cap \text{Centralizer}(Gl_{j-2}(q))$ gives the result. $\square$

Remark A.3. By Lemma A.12, there are $(q-1)$ possibilities for each u_j and q^2 possibilities for each v_j. Since ϵ is a scalar matrix, there are $(q-1)$ possibilities for ϵ.

By Proposition A.2,

$$X_n = \bigcup_{\pi \in S_n} \bigcup_{\substack{1 \leq i \leq n \\ p|(i-1)}} \pi S_i(n)$$

and so by Correspondence A.13 a complete set of coset representatives for $Gl_n(q)/Gl_{n-1}(q)$ is contained in $\{\pi s_i | \ 1 \leq i \leq n, p \mid (i-1), s_i \in S_i(n)\}$, with each s_i of form:

$$s_i = u_2 \cdots u_{p-1} u'_{p+1} t_p u_{p+1} \cdots u_i v_{i+1} \cdots v_n.$$

Finally, consider the $p = 2$ case.

Theorem A.5. *For $p = 2$, $i \geq 3$ odd, $(\mathbf{y}, \mathbf{x}) \in S_i(n)$, there exist invertible matrices*

$$a_j, b_j, c_j, v_j \in Gl_j(q) \cap \text{Centralizer}(Gl_{j-2}(q))$$

such that

$$(\mathbf{y}, \mathbf{x}) = a_3 b_2 c_3 \cdots a_i b_{i-1} c_i v_{i+1} \cdots v_n.\mathbf{1}.$$

Proof. Let $(\mathbf{y}, \mathbf{x}) \in S_i(n)$. For $\mathbf{z} = (z_1, \ldots, z_n) := (x_1 y_1, \ldots x_n y_n)$, Proposition A.2 gives

$$\mathbf{z} = \Big(1, \ldots, 1, z_{i+1}, \ldots, z_n\Big).$$

First note that since $p \nmid i$, i is odd, and for $i = 1$, Lemma A.12 applies as in the proof of Theorem A.4, so assume $i \geq 3$. Also note that for $j \leq i$, $z_j = 1$ and so neither x_j nor y_j are zero. Then for

$$A_j = \begin{pmatrix} 1 & \frac{y_j}{y_{j-1}} \\ 0 & 1 \end{pmatrix}, \quad B_j = \begin{pmatrix} 0 & 1 \\ 1 & \frac{y_j}{y_{j-1}} \end{pmatrix}, \quad C_j = \begin{pmatrix} \frac{y_{j-2}}{y_j} & 1 \\ 1 & 0 \end{pmatrix},$$

let a_j^{-1} (respectively b_j^{-1}, c_j^{-1})$\in Gl_n(q)$ have 1's along the diagonal and A_j (respectively B_j, C_j) in the $(j-1, j)$x$(j-1, j)$th subblock. Then for $j \leq i$,

$$c_j^{-1} b_{j-1}^{-1} a_j^{-1}.(\mathbf{x}, \mathbf{y}) =$$
$$(\left(y_1, \ldots, y_{j-3}, 0, 0, y_{j-2}, y_{j+1}, \ldots, y_n\right), \left(x_1, \ldots, x_{j-3}, 0, 0, x_{j-2}, x_{j+1}, \ldots, x_n\right)).$$

Further, note that $a_j^{-1}, b_j^{-1}, c_j^{-1}$ are invertible for each $1 \leq j \leq i$. Then for v_j^{-1} as in the proof of Theorem A.4,

$$v_n^{-1} \cdots v_{i+1}^{-1} c_i^{-1} b_{i-1}^{-1} a_i^{-1} c_{i-2}^{-1} b_{i-3}^{-1} a_{i-2}^{-1} \cdots c_3^{-1} b_2^{-1} a_3^{-1}.(\mathbf{y}, \mathbf{x}) = \mathbf{1}$$

which proves the theorem. □

Note that there are $(q-1)$ choices for a_j and b_j, that c_j is completely determined by a_j and b_j, and that there are q^2 choices for v_j.

Appendix B

Appendix

B.1 Example of Walk in $\mathcal{BMW}_3$

Example B.1. In $\mathcal{BMW}_3$,

$$\mathcal{B}_3 = \mathbf{R} \cup \mathbf{E}_1 \cup \mathbf{E}_2 \cup \mathbf{E}_3,$$

for

$$\begin{aligned} \mathbf{R} &= \{T_{id}, T_{r_1}, T_{r_2}, T_{r_1}T_{r_2}, T_{r_2}T_{r_1}, T_{r_1}T_{r_2}T_{r_1}\}, & \mathbf{E}_1 &= \{T_{e_1}, T_{r_2}T_{e_1}, T_{e_2}T_{e_1}\}, \\ \mathbf{E}_2 &= \{T_{e_2}, T_{r_1}T_{e_2}, T_{e_1}T_{e_2}\}, & \mathbf{E}_3 &= \{T_{e_1}T_{r_2}, T_{r_2}T_{e_1}T_{r_2}, T_{e_2}T_{e_1}T_{r_2}\}. \end{aligned}$$

The Markov chain K_1 has form

$$R \bigoplus E_1 \bigoplus E_2 \bigoplus E_3,$$

for

$$R = \begin{array}{c} \\ id \\ r_1 \\ r_2 \\ r_1r_2 \\ r_2r_1 \\ r_1r_2r_1 \end{array} \begin{array}{c} \begin{array}{cccccc} id & r_1 & r_2 & r_1r_2 & r_2r_1 & r_1r_2r_1 \end{array} \\ \begin{pmatrix} 0 & \theta & 0 & 0 & 0 & 0 \\ 1 & 1-\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta & 0 & 0 \\ 0 & 0 & 1 & 1-\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \theta \\ 0 & 0 & 0 & 0 & 1 & 1-\theta \end{pmatrix} \end{array},$$

$$E_1 = \begin{array}{c} \\ e_1 \\ r_2e_1 \\ e_2e_1 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & r_2e_1 & e_2e_1 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \theta \\ 0 & 1 & 1-\theta \end{pmatrix} \end{array}, \quad E_2 = \begin{array}{c} \\ e_2 \\ r_1e_2 \\ e_1e_2 \end{array} \begin{array}{c} \begin{array}{ccc} e_2 & r_1e_2 & e_1e_2 \end{array} \\ \begin{pmatrix} 0 & \theta & 0 \\ 1 & 1-\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array},$$

$$E_3 = \begin{array}{c} \\ e_1r_2 \\ r_2e_1r_2 \\ e_2e_1r_2 \end{array} \begin{array}{c} \begin{array}{ccc} e_1r_2 & r_2e_1r_2 & e_2e_1r_2 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \theta \\ 0 & 1 & 1-\theta \end{pmatrix} \end{array}.$$

B.2 Symmetric Group Elements

Lemma B.1. *Let $\mathbf{X}_1$ be a communication class of K whose elements have at least two lower horizontal edges. Then there exist enough $s_x \in S_n$ with $s_x^2 \neq id$ to associate a distinct s_x to each $x \in \mathbf{X}_1$ such that $s_x \neq s_y^{-1}$ for any $y \in \mathbf{X}_1$.*

Proof. The size of a communication class is determined by the number of lower horizontal edges of its elements. Let $\mathbf{X}_i$ be the communication class of an element x_i

with m lower horizontal edges. Then a simple counting argument gives:

$$|\mathbf{X}_i| = (n-2m)! \prod_{j=0}^{m-1} \binom{n-2j}{k}.$$

In particular, for $x_i, x_j \in \mathcal{B}_n$,

$$|\mathbf{X}_i| > |\mathbf{X}_j| \iff x_i \text{ has fewer lower horizontal edges than } x_j.$$

Note that if x_i has exactly one lower horizontal edge,

$$|\mathbf{X}_i| = \frac{n!}{2} = \frac{|S_n|}{2},$$

and so S_n cannot contain enough elements of order greater than 2 to make the associations required by the lemma, as we need $2|\mathbf{X}_i|$ elements of order greater than 2.

Now let x_i have exactly two lower horizontal edges. Then for all $x_j \in \mathcal{B}_n$ with at least two lower horizontal edges,

$$|\mathbf{X}_j| \leq |\mathbf{X}_j| = (n-4)! \binom{n}{2} \binom{n-2}{2} = \frac{n!}{8} = \frac{|S_n|}{8},$$

and so

$$2|\mathbf{X}_j| \leq \frac{|S_n|}{2}. \tag{B.1}$$

Let T_n be the set of elements of S_n of order 2. Then by Equation B.1 we need show

$$\frac{|S_n|}{2} \leq |S_n| - |T_n|,$$

in other words, that $|T_n| \leq \frac{|S_n|}{2}$.

But

$$|T_n| = \begin{cases} \displaystyle\sum_{k=1}^{\frac{n}{2}} \frac{n!}{(n-2k)!k!2^k} & \text{if } n \text{ even },\\ \\ \displaystyle\sum_{k=1}^{\frac{n-1}{2}} \frac{n!}{(n-2k)!k!2^k} & \text{if } n \text{ odd },\end{cases}$$

and so $|T_n| < \frac{|S_n|}{2}$ for $n > 4$. As the only communication classes when $n < 4$ correspond to elements with fewer than 2 lower horizontal edges, this proves the lemma. □

Finally, for x_i with exactly one lower horizontal edge, while $\mathbf{X}_i$ contains too many elements to make the associations of Lemma B.1, note that each $y \in \mathbf{X}_i$ can be viewed as an element, y' of $\mathcal{B}_{n+1}$ by adding a verticle edge to the end of the diagram. Then to analyze $[K]_i$, let $\mathbf{X}'_i = \{y' \mid y \in \mathbf{X}_i\}$ and let K' be the matrix of K with respect to $\mathcal{B}_{n+1}$. Then since $[K']_i = [K]_i$, we can analyze this case by considering K'.

Bibliography

[1] S. Ariki, N. Jacon, and C. Lecouvey, *The modular branching rule for affine Hecke algebras of type A*, Adv. Math. **228** (2011), no. 1, 481–526.

[2] L. Auslander and R. Tolimieri, *Is computing with the finite Fourier transform pure or applied mathematics?*, Bull. Amer. Math. Soc. (N.S.) **1** (1979), no. 6, 847–897.

[3] D. Barros, S. Wilson, and J. Kahn, *Comparison of orthogonal frequency-division multiplexing and pulse-amplitude modulation in indoor optical wireless links*, IEEE Trans. Commun. **60** (2012), no. 1, 153–163.

[4] L. Beckett and P. Diaconis, *Spectral analysis for discrete longitudinal data*, Adv. Math. **103** (1994), no. 1, 107–128.

[5] G. Benkart, A. Ram, and C.L. Shader, *Tensor product representations for orthosymplectic Lie superalgebras*, J. Pure Appl. Algebra **130** (1998), no. 1, 1–48.

[6] T. Beth, *On the computational complexity of the general discrete Fourier transform*, Theoret. Comput. Sci. **51** (1987), no. 3, 331–339.

[7] J. Birman and H. Wenzl, *Braids, link polynomials and a new algebra*, Trans. Amer. Math. Soc. **313** (1989), no. 1, 249–273.

[8] O. Bratteli, *Inductive limits of finite dimensional C^*-algebras*, Trans. Amer. Math. Soc. . **171** (1972), 195–234.

[9] P. Bürgisser, M. Clausen, and M. Shokrollahi, *Algebraic Complexity Theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 315, Springer-Verlag, Berlin, 1997, With the collaboration of Thomas Lickteig.

[10] Y. Cai, *How rates of convergence for Gibbs fields depend on the interaction and the kind of scanning used*, Markov processes and controlled Markov chains (Changsha, 1999), Kluwer Acad. Publ., Dordrecht, 2002, pp. 489–498.

[11] K. Cannon, R. Cariou, A. Chapman, M. Crispin-Ortuzar, N. Fotopoulos, M. Frei, C. Hanna, E. Kara, D. Keppel, L. Liao, S. Privitera, A. Searle, L. Singer, and A. Weinstein, *Toward early-warning detection of gravitational waves from compact binary coalescence*, The Astrophysical Journal **748** (2012), no. 2, 136.

[12] D. Cartwright and W. Woess, *Isotropic random walks in a building of type $\tilde{A}_d$*, Math. Z. **247** (2004), no. 1, 101–135.

[13] M. Clausen, *Fast generalized Fourier transforms*, Theoret. Comput. Sci. **67** (1989), no. 1, 55–63.

[14] M. Clausen and U. Baum, *Fast Fourier Transforms*, Bibliographisches Institut, Mannheim, 1993.

[15] M. Clausen and R. Kakarala, *Computing Fourier transforms and convolutions of S_{n-1}-invariant signals on S_n in time linear in n*, Appl. Math. Lett. **23** (2010), no. 2, 183–187.

[16] A. Cohen, D. Gijsbers, and D. Wales, *BMW algebras of simply laced type*, J. Algebra **286** (2005), no. 1, 107–153.

[17] J. Cooley and J. Tukey, *An algorithm for the machine calculation of complex Fourier series*, Math. Comp. **19** (1965), 297–301.

[18] C. Curtis and I. Reiner, *Methods of Representation Theory: with Applications to Finite Groups and Orders. Vols. I and II*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1990, Reprint of the 1981 original.

[19] A. Danelakis, M. Mitrouli, and D. Triantafyllou, *Blind image deconvolution using a banded matrix method*, Numer. Algorithms **64** (2013), no. 1, 43–72.

[20] Z. Daugherty and R. Orellana, *The quasi-partition algebra*, J. Algebra **404** (2014), 124–151.

[21] Z. Daugherty, A. Ram, and R. Virk, *Affine and degenerate affine BMW algebras: actions on tensor space*, Selecta Math. (N.S.) **19** (2013), no. 2, 611–653.

[22] ______, *Affine and degenerate affine BMW algebras: the center*, Osaka J. Math. **51** (2014), no. 1, 257–283.

[23] P. Diaconis, *Average running time of the fast Fourier transform*, J. Algorithms **1** (1980), 187–208.

[24] ______, *Group Representations in Probability and Statistics*, Institute of Mathematical Statistics Lecture Notes—Monograph Series, 11, Institute of Mathematical Statistics, Hayward, CA, 1988.

[25] P. Diaconis and A. Ram, *Analysis of systematic scan Metropolis algorithms using Iwahori-Hecke algebra techniques*, Michigan Math. J. **48** (2000), 157–190.

[26] P. Diaconis and D. Rockmore, *Efficient computation of the Fourier transform on finite groups*, J. Amer. Math. Soc. **3** (1990), no. 2, 297–332.

[27] P. Diaconis and L. Saloff-Coste, *What do we know about the Metropolis algorithm?*, J. Comput. System Sci. **57** (1998), no. 1, 20–36, 27th Annual ACM Symposium on the Theory of Computing (STOC'95) (Las Vegas, NV).

[28] Y. Diao, C. Ernst, and U. Ziegler, *Generating large random knot projections*, Physical and Numerical Models in Knot Theory, Ser. Knots Everything, vol. 36, World Sci. Publ., Singapore, 2005, pp. 473–494.

[29] R. Dipper and G. James, *Blocks and idempotents of Hecke algebras of general linear groups*, Proc. London Math. Soc. (3) **54** (1987), no. 1, 57–82.

[30] M. Dyer, L. Goldberg, and M. Jerrum, *Systematic scan for sampling colorings*, Ann. Appl. Probab. **16** (2006), no. 1, 185–230.

[31] D. Elliott and K. Rao, *Fast Transforms: Algorithms, Analyses, Applications*, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1982.

[32] G. Elliott, *On the classification of inductive limits of sequences of semisimple finite-dimensional algebras*, J. Algebra **38** (1976), no. 1, 29–44.

[33] W. Feller, *An Introduction to Probability Theory and its Applications. Vol. I*, Third edition, John Wiley & Sons, Inc., New York-London-Sydney, 1968.

[34] G. Fishman, *Coordinate selection rules for Gibbs sampling*, Ann. Appl. Probab. **6** (1996), no. 2, 444–465.

[35] D. Flath, T. Halverson, and K. Herbig, *The planar rook algebra and Pascal's triangle*, Enseign. Math. (2) **55** (2009), no. 1-2, 77–92.

[36] P. Gabriel, *Unzerlegbare darstellungen I*, Manuscripta Math. **6** (1972), no. 1, 71–103 (English).

[37] I. Gel′fand and M. Cetlin, *Finite-dimensional representations of the group of unimodular matrices*, Doklady Akad. Nauk SSSR (N.S.) **71** (1950), 825–828.

[38] D. Gijsbers, *BMW Algebras of Simply Laced Type*, Universiteitsdrukkerij Technische Universiteit Eindhoven, 2005, Thesis (Ph.D.)–Technische Universiteit Eindhoven.

[39] F. Goodman, P. de la Harpe, and V. Jones, *Coxeter Graphs and Towers of Algebras*, Mathematical Sciences Research Institute Publications, vol. 14, Springer-Verlag, New York, 1989.

[40] F. Goodman and H. Hauschild, *Affine Birman-Wenzl-Murakami algebras and tangles in the solid torus*, Fund. Math. **190** (2006), 77–137.

[41] F. Goodman and H. Wenzl, *Iwahori-Hecke algebras of type A at roots of unity*, J. Algebra **215** (1999), no. 2, 694–734.

[42] R. Goodman and N. Wallach, *Representations and Invariants of the Classical Groups*, Encyclopedia of Mathematics and its Applications, vol. 68, Cambridge University Press, Cambridge, 1998.

[43] C. Grood, *The rook partition algebra*, J. Combin. Theory Ser. A **113** (2006), no. 2, 325–351.

[44] T. Halverson and E. delMas, *Representations of the Rook-Brauer algebra*, Comm. Algebra **42** (2014), no. 1, 423–443.

[45] T. Halverson and A. Ram, *Characters of algebras containing a Jones basic construction: the Temperley-Lieb, Okada, Brauer, and Birman-Wenzl algebras*, Adv. Math. **116** (1995), no. 2, 263–321.

[46] T. Halverson and M. Reeks, *Gelfand models for diagram algebras*, J. Algebraic Combin. **41** (2015), no. 2, 229–255.

[47] M. Heideman, D. Johnson, and C. Burrus, *Gauss and the history of the fast Fourier transform*, Arch. Hist. Exact Sci. **34** (1985), no. 6, 265–277.

[48] M. Jalsenius and K. Pedersen, *A systematic scan for* 7*-colourings of the grid*, Internat. J. Found. Comput. Sci. **19** (2008), no. 6, 1461–1477.

[49] G. James and A. Kerber, *The Representation Theory of the Symmetric Group*, Encyclopedia of Mathematics and its Applications, vol. 16, Addison-Wesley Publishing Co., Reading, Mass., 1981.

[50] M. Jimbo, *A q-analogue of $U(\mathfrak{gl}(N+1))$, Hecke algebra, and the Yang-Baxter equation*, Lett. Math. Phys. **11** (1986), no. 3, 247–252.

[51] S. Johnson and M. Frigo, *A modified split-radix FFT with fewer arithmetic operations*, IEEE Trans. Signal Process. **55** (2007), no. 1, 111–119.

[52] H. Kfir and I. Kanter, *Parallel versus sequential updating for belief propagation decoding*, Ann. Appl. Probab. **16** (2006), no. 1, 185–230.

[53] I. Kondor, *Group Theoretical Methods in Machine Learning*, 2008, Thesis (Ph.D.)–Columbia University.

[54] P. Kostelec and D. Rockmore, *FFTs on the rotation group*, J. Fourier Anal. Appl. **14** (2008), no. 2, 145–179.

[55] G. Lebanon and J. Lafferty, *Cranking: Combining rankings using conditional probability models on permutations*, Proceedings of the 19th International Conference on Machine Learning, 2002, pp. 363–370.

[56] R. Leduc and A. Ram, *A ribbon Hopf algebra approach to the irreducible representations of centralizer algebras: The Brauer,Birman-Wenzl, and type A Iwahori-Hecke algebras*, Adv. Math. **125** (1997), 1–94.

[57] D. Levin, Y. Peres, and E. Wilmer, *Markov Chains and Mixing Times*, American Mathematical Society, Providence, RI, 2009.

[58] J. Liu, *Monte Carlo Strategies in Scientific Computing*, Springer Series in Statistics, Springer, New York, 2008.

[59] T. Lundy and J. Van Buskirk, *A new matrix approach to real FFTs and convolutions of length* 2^k, Computing **80** (2007), no. 1, 23–45.

[60] J. Ma, *Components of random links*, J. Knot Theory Ramifications **22** (2013), no. 8, 1350043, 11.

[61] M. Malandro, *Fast Fourier transforms for finite inverse semigroups*, J. Algebra **324** (2010), no. 2, 282–312.

[62] D. Maslen, *Efficient computation of Fourier transforms on compact groups*, J. Fourier Anal. Appl. **4** (1998), no. 1, 19–52.

[63] ______, *The efficient computation of Fourier transforms on the symmetric group*, Math. Comp. **67** (1998), no. 223, 1121–1147.

[64] D. Maslen and D. Rockmore, *Adapted diameters and FFTs on groups*, Proc. 6th ACM-SIAM SODA, ACM, 1995, pp. 253–262.

[65] ______, *Separation of variables and the computation of Fourier transforms on finite groups. I*, J. Amer. Math. Soc. **10** (1997), no. 1, 169–214.

[66] ______, *Double coset decompositions and computational harmonic analysis on groups*, Journal of Fourier Analysis and Applications **6** (2000), no. 4, 349–388.

[67] ______, *The Cooley-Tukey FFT and group theory*, Notices of the Amer. Math. Soc. **48** (2001), no. 10, 1151–1160.

[68] N. Metropolis, A. Rosenbluth, M. Rosenbluth, A. Teller, and E. Teller, *Equation of state calculations by fast computing machines*, J. Chem. Phys. **21** (1953), no. 1087, 1087–1092.

[69] H. Morton and A. Wasserman, *A basis for the Birman-Wenzl algebra*, (1989, revised 2000), 29 pp., unpublished manuscript, arXiv:1012.3116.

[70] J. Murakami, *The Kauffman polynomial of links and representation theory*, Osaka J. Math. **24** (1987), no. 4, 745–758.

[71] A. Oppenheim, R. Schafer, and J. Buck, *Discrete-Time Signal Processing (2nd ed.)*, Prentice-Hall, Inc., Upper Saddle River, NJ, USA, 1999.

[72] J. Parkinson, *Isotropic random walks on affine buildings*, Ann. Inst. Fourier (Grenoble) **57** (2007), no. 2, 379–419.

[73] K. Pedersen, *Dobrushin conditions for systematic scan with block dynamics*, Mathematical foundations of computer science 2007, Lecture Notes in Comput. Sci., vol. 4708, Springer, Berlin, 2007, pp. 264–275.

[74] ______, *On Systematic Scan*, 2008, Thesis (Ph.D.)–The University of Liverpool.

[75] ______, *On systematic scan for sampling H-colorings of the path*, Random Struct. Algor. **35** (2009), no. 1, 15–41.

[76] A. Ram, *Representation Theory and Character Theory of Centralizer Algebras*, ProQuest LLC, Ann Arbor, MI, 1991, Thesis (Ph.D.)–University of California, San Diego.

[77] ______, *Seminormal representations of Weyl groups and Iwahori-Hecke algebras*, Proc. London Math. Soc. (3) **75** (1997), no. 1, 99–133.

[78] A. Ram and J. Ramagge, *Affine Hecke algebras, cyclotomic Hecke algebras and Clifford theory*, A tribute to C. S. Seshadri (Chennai, 2002), Trends Math., Birkhäuser, Basel, 2003, pp. 428–466.

[79] M. Richey, *The evolution of Markov chain Monte Carlo methods*, Amer. Math. Monthly **117** (2010), no. 5, 383–413.

[80] D. Rockmore, *Some applications of generalized FFTs*, Groups and computation, II (New Brunswick, NJ, 1995), DIMACS Ser. Discrete Math. Theoret. Comput. Sci., vol. 28, Amer. Math. Soc., Providence, RI, 1997, pp. 329–369.

[81] ———, *The FFT: An algorithm the whole family can use*, Computing in Science and Eng. **2** (2000), no. 1, 60–64.

[82] M. Rørdam and E. Størmer, *Classification of Nuclear C^*-algebras. Entropy in Operator Algebras*, Encyclopaedia of Mathematical Sciences, vol. 126, Springer-Verlag, Berlin, 2002.

[83] H. Rui, *A criterion on the semisimple Brauer algebras*, J. Combin. Theory Ser. A **111** (2005), no. 1, 78–88.

[84] S. Sawyer, *Isotropic random walks in a tree*, Z. Wahrsch. Verw. Gebiete **42** (1978), no. 4, 279–292.

[85] I. Schur, *Uber die rationalen darstellungen der allgemeinen linearen gruppe*, Reprinted in I. Schur, Gesammelte Abhandlungen, Vol. III, pp. 68-85, Springer-Verlag, Berlin, 1973.

[86] J. Serre, *Linear Representations of Finite Groups*, Springer-Verlag, New York, 1977, Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42.

[87] R. Stanley, *Differential posets*, J. Amer. Math. Soc. **1** (1988), no. 4, 919–961.

[88] ———, *Variations on differential posets*, Invariant theory and tableaux (Minneapolis, MN, 1988), IMA Vol. Math. Appl., vol. 19, Springer, New York, 1990, pp. 145–165.

[89] R. Tolimieri, M. An, and C. Lu, *Algorithms for Discrete Fourier Transform and Convolution*, second ed., Signal Processing and Digital Filtering, Springer-Verlag, New York, 1997.

[90] E. Trachtenberg and M. Karpovsky, *Filtering in a communication channel by Fourier transforms over finite groups*, Spectral techniques and fault detection, Notes Rep. Comput. Sci. Appl. Math., vol. 11, Academic Press, Orlando, FL, 1985, pp. 179–216.

[91] C. Van Loan, *Computational Frameworks for the Fast Fourier Transform*, Frontiers in Applied Mathematics, vol. 10, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.

[92] H. Wenzl, *Hecke algebras of type A_n and subfactors*, Invent. Math. **92** (1988), no. 2, 349–383.

[93] ______, *On the structure of Brauer's centralizer algebras*, Ann. of Math. (2) **128** (1988), no. 1, 173–193.

[94] A. Willsky, *On the algebraic structure of certain partially observable finite-state Markov processes*, Inform. and Control **38** (1978), no. 2, 179–212.

[95] A. Young, *On quantitative substitutional analysis*, Proc. London Math. Soc. **31** (1929), no. 2, 273–288.

[96] P. Zinn-Justin, *Conjectures on the enumeration of alternating links*, Physical and numerical models in knot theory, Ser. Knots Everything, vol. 36, World Sci. Publ., Singapore, 2005, pp. 597–606.

[97] P. Zizler, *Some remarks on the non-abelian Fourier transform in crossover designs in clinical trials*, Appl. Math. **5** (2014), 917–927.