Fault Tolerance in Belief Formation Networks

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Abstract. The paper investigates the formation of beliefs in multi-agent systems with a fixed topology of the communication channels. Specifically, it considers the relation "beliefs formed by agents in set A are not influenced by faulty or malicious behavior of agents in set B". This relation has a non-trivial Shield Wall property that has no equivalent in other settings in which information flow over a fixed network of communication channels has been previously studied.
A new logical system based on the Shield Wall property is proposed and is proven to be sound and complete with respect to the fault tolerance semantics.

1 Introduction

Belief Networks. In this paper we study dependencies between beliefs in multi-agent systems. An example of a set of such dependencies in a four-agent system is given in Figure 1. In this figure, the accused (*Eva*), two witnesses (*Alice* and *Bob*), *Jury*, and *Public* are agents and *EvaMurdered* and *EvaIsGuilty* are two beliefs. The two Horn clauses given in the figure are *belief formation rules*. These rules express how different agents form new beliefs based on existing beliefs of the other agents. For example, according to the first cause, if both witnesses believe that Eva is a murderer, then the jury will form a belief that Eva is guilty. According to the second rule, once the jury decides that Eva is guilty, this belief could propagate to the public. We will refer to settings such as the one in Figure 1 as *belief formation networks* or just belief networks. The process of belief propagation on such networks will be called *belief formation*, to emphasize the fact that the propagated belief ("Eva is guilty") could be different from the originating belief ("Eva is a murderer") and to differentiate this non-probabilistic approach from belief propagation on Bayesian networks [1, 2].

$$\mathsf{Alice{:}EvaMurdered} \wedge \mathsf{Bob{:}EvaMurdered} \to \mathsf{Jury{:}EvaIsGuilty}$$
$$\mathsf{Jury{:}EvaIsGuilty} \to \mathsf{Public{:}EvaIsGuilty}$$

Fig. 1. Belief Network $\mathcal{N}_1$

In the example above, the public forms its beliefs based only on the jury's beliefs, not directly on witnesses testimonies. The jury, on the other hand, bases its beliefs on the witnesses' testimonies, not on public opinion. These properties of the belief network are captured by belief dependency graph G_1 depicted in Figure 2.

Fault Tolerance. There are many different properties of belief networks and belief dependency graphs that one can consider. The focus of this paper is on *fault tolerance* – the ability of an agent in a belief network to resist forming false beliefs when some other agents exhibit faulty or malicious behavior. For example, assume, for the above discussed network $\mathcal{N}_1$, that neither Alice nor Bob have witnessed the murder. Then no beliefs will be formed in this network. If Bob misbehaves and lies to the jury, then, assuming Alice does not lie, neither the jury nor the public will form a false belief. We denote this by $Jury \parallel Bob$ and $Public \parallel Bob$. At the same time, if both Alice and Bob, misbehave, then this might result in the jury and the public forming a false belief: $\neg(Jury \parallel Alice, Bob)$ and $\neg(Public \parallel Alice, Bob)$.

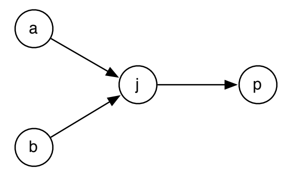

Fig. 2. Graph G_1 shows belief dependencies between agents *Alice* (a), *Bob* (b), *Jury* (j), and *Public* (p).

Different belief networks can have the same dependency graph. A fault tolerance claim which is true in one of these networks might be false in the other. For example, statements $Jury \parallel Bob$ and $Public \parallel Bob$ are not true in the belief network $\mathcal{N}_2$ depicted in Figure 3, although it has the same dependency graph G_1 as network $\mathcal{N}_1$.

Alice:EvaMurdered $\rightarrow$ Jury:EvaIsGuilty
Bob:EvaMurdered $\rightarrow$ Jury:EvaIsGuilty
Jury:EvaIsGuilty $\rightarrow$ Public:EvaIsGuilty

Fig. 3. Belief Network $\mathcal{N}_2$

1.1 Gateway

In this paper we study properties of fault tolerance that are common for all belief networks sharing the same dependency graph. An example of such a property for graph G_1 is $Public \parallel Jury \rightarrow Public \parallel Alice, Bob$. This statement is a special case of a more general *Gateway* principle. Let A, B, and W be three sets of vertices of a graph G. We say that set W is a gateway (see Figure 4) to set A from set B if every path connecting a vertex from set B with a vertex from A contains at least one vertex from set W. (For directed graphs, every *directed* path from set B to set A must contain a vertex from set W.)

Gateway Principle *If a set W is a gateway to a set A from a set B of a graph G, then the property $A \parallel W \rightarrow A \parallel B$ is true for any belief network with belief dependencies specified by the graph G.*

In the example above, $A = \{Public\}$, $W = \{Jury\}$, and $B = \{Alice, Bob\}$.

The gateway principle is an intuitively expected property of fault tolerance. Similar principles in other information flow settings were pro-

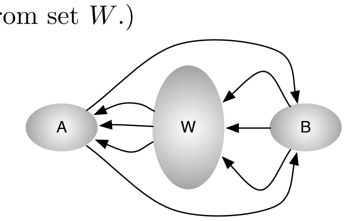

Fig. 4. Gateway

posed earlier for functional dependency relation on hypergraphs [3] as well as independence relation on graphs [4], directed acyclic graphs [5], and hypergraphs [6]. (Independence is also known in literature as nondeducibility [7].) In all these settings an appropriate version of the gateway principle was not only sound, but, together with several other relevant properties, gave a complete axiomatization of each of these relations over a fixed graph.

It turns out, however, that, unlike the above settings, the gateway principle is far from giving a complete description of all fault tolerance properties of an arbitrary graph. For example, the graph discussed above, G_1, has the following property: $Jury \parallel Alice, Bob \rightarrow Public \parallel Alice, Bob$. This property is a special case of a more general principle:

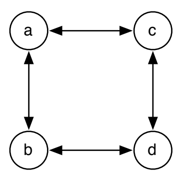

Fig. 5. Graph G_2

Second Gateway Principle *If a set W is a gateway to a set A from a set B of a graph G, then property $W \parallel B \rightarrow A \parallel B$ is true for any belief network with belief dependencies specified by the graph G.*

We will formally prove soundness of both gateway principles later. These two principles, though, still do not form a complete logical system for reasoning about fault tolerance on directed graphs. Indeed, consider dependency graph G_2 depicted in Figure 5.

Proposition 1. *$a \parallel c \rightarrow (b \parallel d \rightarrow a, b \parallel c, d)$, for any belief network with belief dependencies specified by graph G_2.*

Proof Idea. By "false" belief, we mean a belief that cannot be formed if all agents are behaving properly. Assume that a certain malfunctioning of agents c and d results in at least one of agents a and b forming a false belief. Let t_0 be the moment when such false belief is formed for the first time. Without loss of generality, assume that the belief is formed by the agent a. Note that it means that at any moment $t < t_0$ agent b did not hold any false beliefs. Thus, the only reason for agent a to form a false belief at moment t_0 is a malfunctioning of the agent c. Therefore, $\neg(a \parallel c)$, which is a contradiction with the assumption. This argument will be formalized later in the proof of Theorem 3.

It is easy to see that statement $a \parallel c \rightarrow (b \parallel d \rightarrow a, b \parallel c, d)$ cannot be shown using either of the two gateway principles above. It is an example of a more general principle that we call the Shield Wall principle.

1.2 Shield Wall

Ancient warriors (and modern police forces) used multiple shields to protect themselves. Each shield does not provide complete protection for its owner, but if a group of warriors arranges its members in a circle with all shields facing outward, then the whole group is completely protected. This idea is formally captured in our Shield Wall principle:

Shield Wall Principle *Let G be a graph. If B is a set of vertices, $\{A_i\}_{i \leq n}$ are pair-wise disjoint sets of vertices, and $\{S_i\}_{i \leq n}$ are sets of vertices such that*

1. sets A_i and S_i are disjoint for each $i \leq n$,
2. $S_i \cup (\bigcup_{j \neq i} A_j)$ is a gateway to A_i from B for each i,

then the property $\bigwedge_{i \leq n}(A_i \parallel S_i) \to \left(\bigcup_{i \leq n} A_i\right) \parallel B$ is true for any belief network with belief dependencies specified by the graph G.

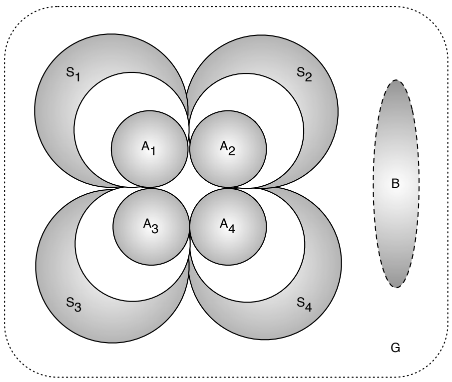

Fig. 6. Shield Wall

Informally, see Figure 6, sets $A_1, \ldots, A_n$ are "warriors" that are "protected" by "shields" $S_1, \ldots, S_n$. The fact that shield S_i protects warrior A_i is captured by the assumption $A_i \parallel S_i$. The fact that each warrior is blocked from the "enemy" B is expressed by the assumption that $S_i \cup (\bigcup_{j \neq i} A_j)$ is a gateway to set A_i from set B. The conclusion of the Shield Wall principle can be interpreted as a statement that the whole group $\bigcup_i A_i$ is protected against the enemy B.

For the graph on Figure 5, $n = 2$, $A_1 = \{a\}$, $A_2 = \{b\}$, $S_1 = \{c\}$, $S_2 = \{d\}$, and $B = \{c, d\}$. Thus, by the Shield Wall principle, $a \parallel c \to (b \parallel d \to a, b \parallel c, d)$. The Shield Wall principle also manifests itself in international visa and custom treaties, such as the European Schengen Agreement, under which each member ("warrior") enforces passport and custom control along external borders with non-members ("shields"). This results in all member-states being protected from undesired visitors and goods.

1.3 Logics of Fault Tolerance

Logical aspects of fault tolerance have been analyzed before. This work lead to the origination of a LICS workshop series on Logical Aspects of Fault Tolerance [8, 9]. Logical frameworks that have been used for such analysis are Temporal Logic (see, for example, [10]) and Deontic Logic [11]. Unlike the approach proposed in this paper, these frameworks do not integrate topological structure of the multi-agent system.

2 Graph Terminology

In this paper, all graphs are assumed to be directed. A directed path from vertex v_1 to vertex v_n in a graph $G = (V, E)$ is a sequence of vertices $v_1, \ldots, v_n$ such that $(v_i, v_{i+1}) \in E$ for each $i < n$. For example, a, j, p is a directed path in graph G_1, depicted in Figure 2.

Definition 1. *A set $W \subseteq V$ is a gateway to set $A \subseteq V$ from set $B \subseteq V$ of a graph $G = (V, E)$ if each path to a vertex in set A from a vertex in set B contains a vertex from set W.*

3 Semantics

In this section we give a formal definition of fault tolerance in belief networks that was discussed informally in the introduction.

Definition 2. *For any directed graph $G = (V, E)$, a belief formation network over G is a triple $\mathcal{N} = (\Omega, s, R)$, where*

1. *Ω is an arbitrary finite set (of "beliefs"),*
2. *the "initial state" s is an arbitrary mapping of vertices into sets of beliefs: $V \mapsto 2^{\Omega}$,*
3. *R is an arbitrary finite set of belief propagation rules*

$$(v_1 : \alpha_1) \wedge (v_2 : \alpha_2) \wedge \cdots \wedge (v_n : \alpha_n) \to w : \beta,$$

such that $v_1, \ldots, v_n, w \in V$, $\alpha_1, \ldots, \alpha_n, \beta \in \Omega$, and $(v_i, w) \in E$ for each $i \leq n$.

Definition 3. *For any belief network $\mathcal{N} = (\Omega, s, R)$ over $G = (V, E)$, state t of the network $\mathcal{N}$ is an arbitrary function from V to subsets of Ω.*

If a belief network $\mathcal{N}$ can go from state x to state y through a single application of a belief formation rule, then we write $x \rightarrowtriangle_{\mathcal{N}} y$. The formal definition of this relation is given below.

Definition 4. *Let x and y be two states of a belief network $\mathcal{N} = (\Omega, s, R)$ over $G = (V, E)$. We write $x \rightarrowtriangle_{\mathcal{N}} y$ if there is $v_0 \in V$ such that*

1. *$x(v) = y(v)$ for each vertex $v \neq v_0$,*
2. *there is a rule*

$$(v_1 : \alpha_1) \wedge (v_2 : \alpha_2) \wedge \cdots \wedge (v_n : \alpha_n) \to v_0 : \beta, \tag{1}$$

in set R such that (a) $\alpha_i \in x(v_i)$ for each $i \leq n$, (b) $y(v_0) = x(v_0) \cup \{\beta\}$.

Vertex v_0 will be called the "active" vertex of the rule (1). Let relation $x \rightarrowtriangle^*_{\mathcal{N}} y$ be the transitive and reflexive closure of the relation $x \rightarrowtriangle_{\mathcal{N}} y$.

Lemma 1. *Let $\mathcal{N} = (\Omega, s, R)$ be an arbitrary belief network over $G = (V, E)$. For any states t_1, t_2 such that $s \rightarrowtriangle^*_{\mathcal{N}} t_1$ and $s \rightarrowtriangle^*_{\mathcal{N}} t_2$, there is a state t such that $s \rightarrowtriangle^*_{\mathcal{N}} t$ and $t(v) = t_1(v) \cup t_2(v)$ for each $v \in V$.*

Proof. Since $s \rightarrowtriangle^*_{\mathcal{N}} t_1$, there is a sequence of belief formation rules, that, when applied, transform belief network $\mathcal{N}$ from state s into state t_1. By assumption $s \rightarrowtriangle^*_{\mathcal{N}} t_2$, there is a similar sequence for state t_2. Combine these sequences in, say, consecutive order. Let t be the resulting state. □

By $Mindset_{\mathcal{N}}(v)$ we mean all beliefs that can be potentially formed at vertex v.

Definition 5. *For any belief network $\mathcal{N} = (\Omega, s, R)$ over $G = (V, E)$ and any $v \in V$, $Mindset_{\mathcal{N}}(v) = \bigcup_{s \rightarrowtriangle^*_{\mathcal{N}} t} t(v)$.*

Note that $Mindest_{\mathcal{N}}$, being a function from vertices into subsets of Ω, can also be viewed as a state of the belief network.

Lemma 2. *If $\mathcal{N} = (\Omega, s, R)$ is an arbitrary belief network over $G = (V, E)$, then $s \rightarrow^*_{\mathcal{N}} Mindset_{\mathcal{N}}$.*

Proof. The statement follows from finiteness of the set Ω and Lemma 1. □

For any belief network $\mathcal{N}$ and any set of vertices B, by $\mathcal{N}_B$ we mean a modified network in which vertices in set B can misbehave or malfunction. We capture this formally in the definition below by adding new belief formation rules to $\mathcal{N}_B$ that allow vertices in set B to form any belief from set Ω without preconditions.

Definition 6. *For any belief network $\mathcal{N} = (\Omega, s, R)$ over a graph $G = (V, E)$ and any $B \subseteq V$, we define belief network $\mathcal{N}_B$ to be (Ω, s, R_B), where*

$$R_B = R \cup \{b : \alpha \mid b \in B, \alpha \in \Omega\}.$$

Definition 7. *For any graph $G = (V, E)$, by $\Phi(G)$ we mean the minimal set of formulas such that*

1. *$\bot \in \Phi(G)$,*
2. *$A \parallel B \in \Phi(G)$ for each $A, B \subseteq V$,*
3. *$\phi \rightarrow \psi \in \Phi(G)$ for each $\phi, \psi \in \Phi(G)$.*

The next definition is the key definition in this paper. It formally defines the fault tolerance relation $A \parallel B$ in belief networks as the inability of vertices in set A to form any additional beliefs if vertices in set B are malfunctioning.

Definition 8. *For any belief network $\mathcal{N} = (\Omega, s, R)$ over graph $G = (V, E)$ and any formula $\phi \in \Phi(G)$, we define truth relation $\mathcal{N} \vDash \phi$ by recursion on structural complexity of the formula ϕ:*

1. *$\mathcal{N} \nVdash \bot$*
2. *$\mathcal{N} \Vdash A \parallel B$ if and only if, for each $v \in A$, $Mindset_{\mathcal{N}_B}(v) \subseteq Mindset_{\mathcal{N}}(v)$*
3. *$\mathcal{N} \Vdash \phi_1 \rightarrow \phi_2$ if $\mathcal{N} \nVdash \phi_1$ or $\mathcal{N} \Vdash \phi_2$.*

4 Axioms

Our formal logical system for a graph G, in addition to Modus Ponens inference rules and propositional tautologies in the language $\Phi(G)$, contains the following three axioms:

1. **Empty Set:** $A \parallel \varnothing$,
2. **Monotonicity:** $A, B \parallel C \rightarrow A \parallel C$,
3. **Shield Wall:** $\bigwedge_{0<i\leq n}(A_i \parallel S_i) \rightarrow (\bigcup_{0<i\leq n} A_i) \parallel B$, where $\{A_i\}_i$ are pairwise disjoint sets and $S_i \cup (\bigcup_{j\neq i} A_j)$ is a gateway to A_i from B for each $0 < i \leq n$.

We write $\vdash_G \phi$ if $\phi \in \Phi(G)$ is provable from the axioms above and propositional tautologies in the language $\Phi(G)$ using the Modus Ponens inference rule. We write $X \vdash_G \phi$ if ϕ is provable using the additional set of axioms X. We often omit the parameter G when its value is clear from the context.

5 Examples

In this section, we give several examples of proofs in our formal system. We start by proving the two gateway principles mentioned in the introduction.

Proposition 2 (first gateway). *If W is a gateway (see Figure 4) to set of vertices A from set of vertices B of a graph G, then $\vdash_G A \parallel W \to A \parallel B$.*

Proof. Let $n = 1$, $A_1 = A$, $S_1 = W$, and $B = B$ in the Shield Wall axiom. □

Proposition 3 (second gateway). *If W is a gateway (see Figure 4) to set of vertices A from set of vertices B of a graph G, then $\vdash_G W \parallel B \to A \parallel B$.*

Proof. Let $n = 2$, $A_1 = A \setminus W$, $A_2 = W$, $S_1 = \varnothing$, $S_2 = B$ and $B = B$. By the Shield Wall axiom, $\vdash_G (A \setminus W) \parallel \varnothing \wedge W \parallel B \to A, W \parallel B$. By the Monotonicity axiom, $\vdash_G (A \setminus W) \parallel \varnothing \wedge W \parallel B \to A \parallel B$. Again by the Monotonicity axiom, $\vdash_G A \parallel \varnothing \wedge W \parallel B \to A \parallel B$. By the Empty Set axiom, $\vdash_G W \parallel B \to A \parallel B$. □

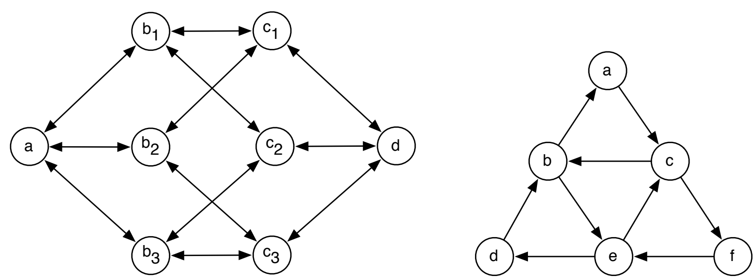

Fig. 7. Graph G_6 (left) and G_7 (right)

Proposition 4. *For G_6 depicted in Figure 7,*

$$\vdash_{G_6} (b_1 \parallel c_1, c_2) \wedge (b_2 \parallel c_1, c_3) \wedge (b_3 \parallel c_2, c_3) \to a \parallel d$$

Proof. Let $n = 4$, $A_1 = \{b_1\}$, $A_2 = \{b_2\}$, $A_3 = \{b_3\}$, $A_4 = \{a\}$, $S_1 = \{c_1, c_2\}$, $S_2 = \{c_1, c_3\}$, $S_3 = \{c_2, c_3\}$, $S_4 = \varnothing$, and $B = \{d\}$ in the Shield Wall axiom. Then, $\vdash_{G_6} (b_1 \parallel c_1, c_2) \wedge (b_2 \parallel c_1, c_3) \wedge (b_3 \parallel c_2, c_3) \wedge (a \parallel \varnothing) \to a, b_1, b_2, b_3 \parallel d$. By the Empty Set axiom,

$$\vdash_{G_6} (b_1 \parallel c_1, c_2) \wedge (b_2 \parallel c_1, c_3) \wedge (b_3 \parallel c_2, c_3) \to a, b_1, b_2, b_3 \parallel d.$$

By the Monotonicity axiom, $\vdash_{G_6} (b_1 \parallel c_1, c_2) \wedge (b_2 \parallel c_1, c_3) \wedge (b_3 \parallel c_2, c_3) \to a \parallel d$. □

Proposition 5. *For G_7 depicted in Figure 7, $\vdash_{G_7} (b \parallel c) \wedge (e \parallel f) \to d \parallel a$.*

Proof. Let $A_1 = \{d\}$, $A_2 = \{b\}$, $A_3 = \{e\}$, $S_1 = \varnothing$, $S_2 = \{c\}$, $S_3 = \{f\}$, and $B = \{a\}$. Thus, by the Shield Wall axiom, $\vdash_{G_7} (d \parallel \varnothing) \wedge (b \parallel c) \wedge (e \parallel f) \to b, d, e \parallel a$. By the Empty Set axiom, $\vdash_{G_7} (b \parallel c) \wedge (e \parallel f) \to b, d, e \parallel a$. By the Monotonicity axiom, $\vdash_{G_7} (b \parallel c) \wedge (e \parallel f) \to d \parallel a$. □

Proposition 6. *If G_8 is the graph depicted in Figure 8, then*

$$\vdash_{G_8} (x \parallel a, d) \wedge (y \parallel c, f) \wedge (z \parallel e, b) \rightarrow x, y, z \parallel a, c, e.$$

Proof. Let $n = 3$, $A_1 = \{x\}$, $A_2 = \{y\}$, $A_3 = \{z\}$, $S_1 = \{a, d\}$, $S_2 = \{c, f\}$, $S_3 = \{e, b\}$, and $B = \{a, c, e\}$. in the Shield Wall axiom. □

The next three examples are statements that are true for any graph G. Nevertheless, their proofs use the Shield Wall axiom.

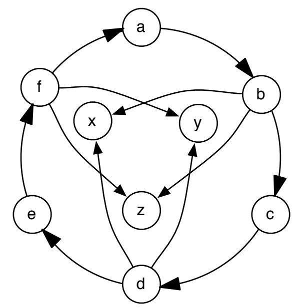

Fig. 8. Graph G_8

Proposition 7. $\vdash \varnothing \parallel A$*, for any set of vertices A of any graph G.*

Proof. Consider the Shield Wall axiom for $n = 0$. □

Proposition 8. $\vdash A \parallel B, C \rightarrow A \parallel B$*, for any sets of vertices A, B, and C of any graph G.*

Proof. Any path to a vertex in set A from a vertex in set B trivially contains a vertex from $B \cup C$. Hence, $B \cup C$ is a shield that separates set A from set B. The result, thus, follows from the Shield Wall axiom. □

Proposition 9. $\vdash (A \parallel B) \wedge (C \parallel B) \rightarrow A, C \parallel B$*, for any sets of vertices A, B, and C of any graph G.*

Proof. Let $A_1 = A$, $A_2 = C$, $S_1 = B$, and $S_2 = B$ in the Shield Wall axiom. □

6 Reverse Shield Wall

In the Shield Wall principle (see Figure 6), we assume that each of A_i can tolerate faulty behavior of the appropriate S_i and conclude that all A_i together can tolerate faulty behavior of B. What if the situation were reversed: each of S_i can tolerate faulty behavior of the appropriate A_i. Can we conclude in this case that B will tolerate the combined faulty behavior of all A_i? Intuitively, this seems to be true since "false" beliefs generated by the union of all A_i will be "locked" inside the wall formed by the shields S_i.

In other words, sets $A_1, \ldots, A_n$ can be thought of as groups of members of a secret society. Each group A_i is prohibited from revealing the society secrets to their non-member acquaintances that form the group S_i, but are free to discuss them with the other society members. As a result, the society secrets are not divulged to the outsiders in the group B.

Conjecture 1. $\bigwedge_{0<i\leq n}(S_i \parallel A_i) \rightarrow B \parallel \bigcup_{0<i\leq n} A_i$, where $\{A_i\}_i$ are pair-wise disjoint sets and $S_i \cup (\bigcup_{j\neq i} A_j)$ is a gateway **from** set A_i **to** set B for each $0 < i \leq n$.

It turns out, however, that this conjecture, as stated, is not true. Indeed, consider belief network $\mathcal{N}_1$ over graph G_1 depicted, respectively, in Figure 1 and Figure 2. Let $A_1 = \{Alice\}$, $A_2 = \{Bob\}$, $S_1 = S_2 = \{Jury\}$, $B = \{Public\}$. If Conjecture 1 is true, then the following implication is true for the belief network $\mathcal{N}_1$: $(Jury \parallel Alice) \wedge (Jury \parallel Bob) \to Public \parallel Alice, Bob$, which, as we have discussed in the introduction, is not true.

In spite of the example above, our "secret society" intuition is correct in the sense that assumptions on the topology of the graph in Conjecture 1 could be adjusted to make the statement true and provable in our axiomatic system:

Proposition 10 (reverse shield wall). $\vdash_G \bigwedge_{0<i\leq n}(S_i \parallel A_i) \to B \parallel \bigcup_{0<i\leq n} A_i$, *where* B,$S_1,\dots,S_n$ *are pair-wise disjoint sets and*

1. $\bigcup_i S_i$ *is a gateway to* B *from* $\bigcup_i A_i$,
2. $B \cup A_i \cup \bigcup_{j\neq i} S_j$ *is a gateway to* S_i *from* $\bigcup_j A_j$, *for each* $0 < i \leq n$.

Proof. From the Shield Wall axiom when the settings are such that $B, S_1, \dots, S_n$ are the "warriors" (formerly As), $\varnothing, A_1, \dots, A_n$ are the matching "shields" (formerly Ss) and $\bigcup_i A_i$ is the "enemy" (formerly B):

$$\vdash_G (B \parallel \varnothing) \wedge \bigwedge_{0<i\leq n} (S_i \parallel A_i) \to B, \bigcup_{0<i\leq n} S_i \parallel \bigcup_{0<i\leq n} A_i.$$

By the Monotonicity axiom, $\vdash_G (B \parallel \varnothing) \wedge \bigwedge_{0<i\leq n}(S_i \parallel A_i) \to B \parallel \bigcup_{0<i\leq n} A_i$. By the Empty Set axiom, $\vdash_G \bigwedge_{0<i\leq n}(S_i \parallel A_i) \to B \parallel \bigcup_{0<i\leq n} A_i$. □

7 Soundness

We prove soundness of our logical system by justifying separately each of its axioms.

Theorem 1 (Empty Set). $\mathcal{N} \vDash A \parallel \varnothing$, *for any belief network* $\mathcal{N} = (\Omega, s, R)$ *over a graph* $G = (E, V)$ *and any subset* $B \subseteq E$.

Proof. Note that, by Definition 6, $\mathcal{N}_\varnothing = \mathcal{N}$. Thus, $Mindset_{\mathcal{N}_\varnothing}(v) \subseteq Mindset_{\mathcal{N}}(v)$ for each $v \in A$. □

Theorem 2 (Monotonicity). *If* $\mathcal{N} \vDash A, B \parallel C$, *then* $\mathcal{N} \vDash A \parallel C$, *for any belief network* $\mathcal{N} = (\Omega, s, R)$ *over a graph* $G = (E, V)$ *and any subsets* $A, B, C \subseteq E$.

Proof. If $Mindset_{\mathcal{N}_B}(v) \subseteq Mindset_{\mathcal{N}}(v)$ for each $v \in A \cup C$, then it follows that $Mindset_{\mathcal{N}_B}(v) \subseteq Mindset_{\mathcal{N}}(v)$ for each $v \in A$. □

Before proving soundness of the Shield Wall axiom, we establish the following technical lemma.

Lemma 3 (Gateway Lemma). *Let set* W *be a gateway to set* A *from set* B *in graph* G *and let* $\mathcal{N} = (\Omega, s, R)$ *be an arbitrary belief network over* G. *For any state* t *such that* $s \twoheadrightarrow^*_{\mathcal{N}_B} t$ *and any state* s' *such that* $t(v) \subseteq s'(v)$ *for any* $v \in W$, *there is a state* t' *such that* $s' \twoheadrightarrow^*_{\mathcal{N}} t'$ *and* $t(v) \subseteq t'(v)$ *for each* $v \in A$.

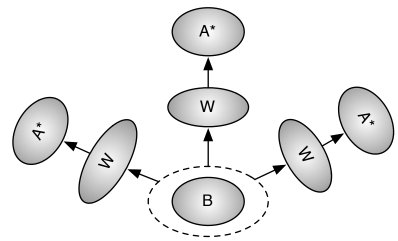

Fig. 9. Graph G_3

Proof. Let A^* be the set of all vertices $v \in V$ such that W is a gateway to set $\{v\}$ from set B (see Figure 9). By the assumption of the lemma, $A \subseteq A^*$.

Additionally, by the assumption $s \rightharpoondown^*_{\mathcal{N}_B} t$, there are states $s_1, \ldots, s_n$ such that $s = s_1$, $s_n = t$, and

$$s_1 \rightharpoondown_{\mathcal{N}_B} s_2 \rightharpoondown_{\mathcal{N}_B} \cdots \rightharpoondown_{\mathcal{N}_B} s_{n-1} \rightharpoondown_{\mathcal{N}_B} s_n \tag{2}$$

Starting from state s', reproduce all belief formation steps in path (2) in which the active vertex belongs to A^*. The result will be a path

$$s'_1 \rightharpoondown_{\mathcal{N}} s'_2 \rightharpoondown_{\mathcal{N}} \cdots \rightharpoondown_{\mathcal{N}} s'_{k-1} \rightharpoondown_{\mathcal{N}} s'_k,$$

where $s'(v) = s'_1(v)$ and $s'_k(v) = t(v)$ for each $v \in A^*$. Take s'_k to be t'. $\square$

Theorem 3 (Shield Wall). *Let $\mathcal{N} = (\Omega, s, R)$ be a belief network over a graph $G = (E, V)$, and $\{A_i\}_{i \le n}$, $\{S_i\}_{i \le n}$, and B be a sets of vertices such that*

1. *$\{A_i\}_{i \le n}$ are pair-wise disjoint,*
2. *A_i is disjoint with S_i for each $i \le n$,*
3. *B is disjoint with $\bigcup_i A_i$,*
4. *$S_i \cup (\bigcup_{j \ne i} A_j)$ is a gateway to A_i from B for each $i \le n$,*
5. *$\mathcal{N} \vDash A_i \parallel S_i$ for each $i \le n$.*

Then $\mathcal{N} \vDash \bigcup_i A_i \parallel B$.

Proof. We need to prove that $Mindset_{\mathcal{N}_B}(v) \subseteq Mindset_{\mathcal{N}}(v)$ for each $v \in \bigcup_i A_i$. Assume the opposite. Since beliefs are formed one at a time, there must be state t and $v_0 \in A_{i_0}$ such that

1. $s \rightharpoondown_{\mathcal{N}_B} t$,
2. $t(v_0) \nsubseteq Mindset_{\mathcal{N}}(v_0)$,
3. $t(v) \subseteq Mindset_{\mathcal{N}}(v)$ for each $v \in \bigcup_i A_i \setminus \{v_0\}$.

By Lemma 2, $s \rightharpoondown^*_{\mathcal{N}} Mindset_{\mathcal{N}}$. Thus, $s \rightharpoondown_{\mathcal{N}_{S_{i_0}}} s'$, where

$$s'(v) = \begin{cases} t(v) \cup Mindset_{\mathcal{N}}(v) & \text{if } v \in S_{i_0} \setminus \bigcup_i A_i, \\ Mindset_{\mathcal{N}}(v) & \text{otherwise.} \end{cases}$$

Note that $t(v) \subseteq Mindset_{\mathcal{N}}(v) = s'(v)$ for each $v \in \bigcup_{i \ne i_0} A_i$. Hence, $t(v) \subseteq s'(v)$ for each $v \in S_{i_0} \cup (\bigcup_{i \ne i_0} A_i)$, which, by an assumption of the theorem, is a gateway to A_{i_0} from B.

By Lemma 3, there is a state t' such that $s' \rightharpoondown^*_{\mathcal{N}} t'$ and $t(v) \subseteq t'(v)$ for each $v \in A_{i_0}$. Then, $s \rightharpoondown^*_{\mathcal{N}_{S_{i_0}}} t'$. At the same time, $t'(v_0) \nsubseteq Mindset_{\mathcal{N}}(v_0)$, because $t(v) \subseteq t'(v)$ and $t(v_0) \nsubseteq Mindset_{\mathcal{N}}(v_0)$. Since v_0 has been chosen from the set A_{i_0}, the above is a contradiction with the assumption $\mathcal{N} \vDash A_{i_0} \parallel S_{i_0}$. $\square$

8 Completeness

Let $G = (V, E)$ be a directed graph, X be a maximal consistent subset of $\Phi(G)$, and $\phi \in \Phi(G)$ such that $X \nvdash \phi$. In this section we construct a belief network $\mathcal{N} = (\Omega, s, R)$ over graph G such that $\mathcal{N} \nvDash \phi$.

In the physical world, each agent often leaves her "mark" on the beliefs that pass through her. In our construction, a belief is nothing but the collection of such "marks". Thus, we formally define a belief as a subset of vertices:

Definition 9. *Set of beliefs Ω is the powerset of V.*

Definition 10. *Initial state s is the function that maps each vertex into the empty set of beliefs.*

Definition 11. *Set $S \subseteq V$ is called a "shield" of vertex $v \in V$ if $X \vdash v \parallel S$.*

Definition 12. *The set of belief formation rules R of the network $\mathcal{N}$ is the minimal set such that for any $w \in V$, if $S_1, \dots S_n$ are all shields of vertex w, then R contains of all rules of the form $u : \beta \wedge \bigwedge_{i \le n} v_i : \alpha_i \to w : \beta \cup \{u\}$, where*

1. *$u, v_1, \dots, v_n, w \in V$,*
2. *$\beta, \alpha_1, \dots, \alpha_n \subseteq V$,*
3. *$(v_i, w) \in E$ for each $i \le n$,*
4. *$(u, w) \in E$,*
5. *$S_i \cap (\alpha_i \cup \{v_i\}) = \varnothing$ for each $i \le n$.*

The following lemma immediately follows from the above definition of the belief network $\mathcal{N}$:

Lemma 4. *For any $v \in V \setminus B$. If $x \in Mindset_{\mathcal{N}_B}(v)$, then set $x \cap B$ is not empty.* □

Lemma 5. *If $X \vdash A \parallel B$, then $\mathcal{N} \vDash A \parallel B$.*

Proof. We need to prove that $Mindset_{\mathcal{N}_B}(w) \subseteq Mindset_{\mathcal{N}}(w)$ for each $w \in A$. Indeed, the assumption $X \vdash A \parallel B$, by the Monotonicity axiom, implies that $X \vdash w \parallel B$. Thus, set B is a shield of vertex w. Thus, by Lemma 4, all beliefs in network $\mathcal{N}$ outside of set B are "marked" by at least one element of B. Hence, by Definition 12, vertex w will never form any beliefs. Therefore, set $Mindset_{\mathcal{N}_B}(w)$ is empty. □

Theorem 4. *If $\mathcal{N} \vDash A \parallel B$, then $X \vdash A \parallel B$.*

Proof. We divide all vertices of the graph G into two disjoint groups: red vertices and blue vertices. A vertex v is called blue if set $Mindset_{\mathcal{N}_B}(v)$ is empty. Otherwise, vertex v is called *red*.

Lemma 6. *$\Omega \subseteq Mindset_{\mathcal{N}_B}(v)$ for any $v \in B$.*

Proof. See Definition 6. □

Lemma 7 (Red Path). *If path $u_1, u_2, \ldots, u_k$ is a directed path of red vertices in graph G and $\gamma \in Mindset_{\mathcal{N}_B}(u_1)$, then $\gamma \cup \{u_1, \ldots, u_{k-1}\} \in Mindset_{\mathcal{N}_B}(u_k)$.*

Proof. Induction on k. If $k = 1$, then $\gamma \in Mindset_{\mathcal{N}_B}(u_1)$ by the assumption.

Let $k > 1$. If $u_k \in B$, then $\gamma \cup \{u_1, \ldots, u_{k-1}\} \in Mindset_{\mathcal{N}_B}(u_k)$ by Lemma 6. We will now assume that $u_k \notin B$. Thus, since vertex u_k is red, by Definition 12, there must exist vertices $v_1, \ldots, v_n$ and beliefs $\alpha_1, \ldots, \alpha_n$ such that

1. $(v_i, u_k) \in E$ for each $i \leq n$,
2. $\alpha_i \in Mindset_{\mathcal{N}_B}(v_i)$ for each $i \leq n$, and
3. $S_i \cap (\alpha_i \cup \{v_i\}) = \varnothing$ for each $i \leq n$.

Consider now the rule of our belief network

$$u_{k-1} : \gamma \cup \{u_1, \ldots, u_{k-2}\} \wedge \bigwedge_{i \leq n} v_i : \alpha_i \rightarrow u_k : \gamma \cup \{u_1, \ldots, u_{k-1}\},$$

By the Induction Hypothesis, $\gamma \cup \{v_1, \ldots, v_{k-2}\} \in Mindset_{\mathcal{N}_B}(u_{k-1})$. Therefore, $\gamma \cup \{v_1, \ldots, v_{k-1}\} \in Mindset_{\mathcal{N}_B}(u_k)$. □

Lemma 8. *If for every shield of a vertex u there is a red directed path (possibly except for vertex u) to vertex u from a vertex in B such that the path does not go through the shield, then vertex u itself is red.*

Proof. Let $S_1, \ldots, S_n$ be all the shields of the vertex u. By the Empty Set axiom, $X \vdash v \parallel \varnothing$. Thus, $n > 0$. Let $v_1, \ldots, v_n$ be vertices such that $(v_i, u) \in E$ and there is a directed red path $v_i^{(n_i)}, \ldots, v_i'', v_i', v_i$, from a vertex $v_i^{(n_i)} \in B$ to vertex v_i that does not go through S_i for each $i \leq n$. In other words, intersection $S_i \cap \left\{v_i^{(n_i)}, \ldots, v_i', v_i\right\}$ is empty.

By Lemma 7,

$$\left\{v_i^{(n_i)}, \ldots, v_i'\right\} \in Mindset_{\mathcal{N}_B}(v_i). \qquad (3)$$

Recall that $n > 0$, thus, by Definition 12, network $\mathcal{N}_B$ contains the rule

$$v_1 : \{v_1^{(n_1)}, \ldots, v_1'\} \wedge \bigwedge_{i \leq n} v_i : \{v_i^{(n_i)}, \ldots, v_i'\} \rightarrow u : \{v_1^{(n_1)}, \ldots, v_1', v_1\}.$$

Hence, due to (3), $\{v_1^{(n_1)}, \ldots, v_1', v_1\} \in Mindset_{\mathcal{N}_B}(u)$. Thus, set $Mindset_{\mathcal{N}_B}(u)$ is not empty, or, in other words, vertex u is red. □

Due to Lemma 8, the set of all blue vertices, $Blue$, satisfies the conditions of the Shield Wall axiom. Thus, $X \vdash Blue \parallel B$. By the assumption, $A \subseteq Blue$. Therefore, by the Monotonicity axiom, $X \vdash A \parallel B$. □

Theorem 5. *$X \vdash \phi$ if and only if $\mathcal{N} \vDash \phi$, for any formula $\phi \in \Phi(A)$.*

Proof. Induction on the structural complexity of formula ϕ. The base case follows from Lemma 5 and Theorem 4. The induction step follows in the standard way from maximality and consistency of the set X. □

Theorem 6 (Completeness). *For any graph G and for any $\phi \in \Phi(G)$. if $\nvdash \phi$, then there is a belief network $\mathcal{N}$ over graph G such that $\mathcal{N} \nvDash \phi$.*

Proof. Suppose that $\nvdash \phi$. Consider any maximal consistent set X containing $\neg\phi$. Let $\mathcal{N}$ be the canonical belief network defined in this section. By Theorem 5, $\mathcal{N} \nvDash \phi$. □

9 Conclusion

In this paper, we have studied fault tolerance properties of belief formation networks under an assumption that once a belief is formed by an agent, it is never forgotten by the agent. One can introduce "forgetting" belief networks by adding a second type of transition to Definition 4. During such a new transition $x \rightarrowtriangle_{\mathcal{N}} y$, the active vertex v_0 "forgets" some belief α. Thus, $y(v_0) = x(v_0) \setminus \{\alpha\}$. It is easy to show, however, that the soundness and completeness results in this paper can be generalized to the "forgetting" belief networks.

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