Strict Equilibria Interchangeability in Multi-Player Zero-Sum Games

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Abstract

The interchangeability property of Nash equilibria in two-player zero-sum games is well-known. This paper studies possible generalizations of this property to multi-party zero-sum games. A form of interchangeability property for strict Nash equilibria in such games is established. It is also shown, by proving a completeness theorem, that strict Nash equilibria do not satisfy any other non-trivial properties.

1 Introduction

Nash equilibria interchangeability [1] is a formal way to state that rational choices of players do not depend on each other. In case of a two-player game G, interchangeability can be defined as following: if (a_1, b_1) and (a_2, b_2) are two Nash equilibria of the game G, then (a_1, b_2) and (a_2, b_1) are also Nash equilibria of the same game. It is a well-known [3, p. 22] that the equilibria in any *zero-sum* two-player game is interchangeable.

One can similarly define interchangeability of strict Nash equilibria by replacing equilibria with strict equilibria in the above definition. It is easy to see, however, that two-player zero-sum game can have at most one strict Nash equilibrium[1]. This makes any two strict Nash equilibria in a zero-sum two-player game vacuously interchangeable.

In this paper we investigate strict equilibria interchangeability in multi-party zero-sum games. We define k-interchangeability property as follows: if $p_1, \dots, p_k$ are any k players in the game, and $e_1, \dots, e_k$ are any k strict Nash equilibria,

[1] If (a_1, b_2) and (a_2, b_1) are two strict Nash equilibria and u_a and u_b are the utility functions of the first and the second player respectively, then, by the definition of the strict equilibrium, $u_a(a_1, b_1) > u_a(a_2, b_1)$ and $u_b(a_2, b_1) < u_b(a_2, b_2)$. The second inequality implies $u_a(a_2, b_1) > u_a(a_2, b_2)$ due to the game being a zero-sum game. Thus, $u_a(a_1, b_1) > u_a(a_2, b_2)$. One can similarly show that $u_a(a_2, b_2) > u_a(a_1, b_1)$, which is a contradiction.

then there is a strict Nash equilibrium e such that strategy profiles e and e_i agree on strategy of player p_i for each $i \leq k$.

It is important to point out that there are multi-player zero-sum games with multiple strict Nash equilibria. We will give an example of such games later. The above trivial observation about uniqueness of *strict* equilibria of two-player zero-sum games can be generalized to multi-player games as statement that if strict Nash equilibria in n-player game are n-interchangeable, then the game has at most one strict equilibrium. A much less trivial observation, which is one of the main results of this paper is: if strict Nash equilibria in n-player zero-sum game are $(n-1)$-interchangeable, then the game has at most one strict equilibrium. To show that this result cannot be improved, we construct, for any $n \geq 3$, an example of n-player zero-sum game with multiple strict $(n-2)$-interchangeable Nash equilibria.

The above main result, generally speaking, is not true for regular (non-strict) Nash equilibria interchangeability. A counterexample will be given.

In the second part of this paper we investigate if there are any other non-trivial properties of strict equilibria interchangeability in multi-player zero-sum games that do not follow from our theorem. We give negative answer to this question even if "properties" are formulated in a more general language than the one considered so far.

This more general language uses not just k-interchangeability, but interchangeability of specific k players. Namely, for players $p_1, \dots, p_k$, we say that they are interchangeable if for any strict Nash equilibria $e_1, \dots, e_k$ there is a strict Nash equilibrium e such that strategy profiles e_i and e agree on strategies of player p_i for each $i \leq k$. We denote this relation by $[p_1, \dots, p_k]$. Using this more expressive language our interchangeability result can be captured by the following propositional formula:

$$\bigwedge_{|A|=n-1} [A] \to [P], \tag{1}$$

where P is the set of all players in the game and $n = |P|$. The other main result of this paper is a completeness theorem showing that the above principle, together with several other more trivial axioms, forms a logical system complete with respect to the strict interchangeability semantics.

The results in this paper apply to zero-sum games. Properties of interchangeability of an arbitrary multi-player strategic game have been studied previously by the first author [2] using a bit different, but related, notion of interchangeability. This previous work also contains a complete axiomatization, but it does not include anything similar to the axiom (1) above.

2 Interchangeability Theorem

Definition 1 *By a game we mean any triple $(P, \{S_p\}_{p\in P}, \{u_p\}_{p\in P})$, where P is an arbitrary finite set of "players", set S_p is a finite set of "strategies" for each player p, and $u_p : \prod_{q\in P} S_q \mapsto \mathbb{R}$ is the utility function of player p.*

For any player p and any strategy profile $e \in \prod_{p\in P} S_p$, by $pr_p(e)$ we mean the strategy of the player p in the profile e. All games in this paper are assumed to be zero-sum games:

$$\sum_{p\in P}(u_p(s)) = 0$$

for each $s \in \prod_{q\in P} S_q$. By $NE(G)$ we mean the set of *strict* Nash equilibria of the game G.

Definition 2 *Game G is k-interchangeable if for any players $p_1, \dots, p_k$ and any strict Nash equilibria $e_1, \dots, e_k$, there is a strict Nash equilibria e such that $pr_{p_i}(e) = pr_{p_i}(e_i)$ for each $i \le k$.*

The following theorem is one of the main results of this paper:

Theorem 1 (interchangeability) *For any game G with $n > 1$ players, if the game is $(n-1)$-interchangeable, then it has at most one strict Nash equilibrium.*

Proof. We start the proof of the theorem with a sequence of definitions and lemmas. For each player p we define subset $S_p^* \subseteq S_p$ of all strategies that are used by player p in at least one strict Nash equilibrium:

Definition 3 $S_p^* = \{pr_p(e) \mid e \in NE(G)\}$.

Lemma 1 *If game G with $n > 1$ players is $(n-1)$-interchangeable, then $|S_p^*| = |S_q^*|$ for all players p and q.*

Proof. If game G has no strict Nash equilibria, then $|S_p^*| = 0 = |S_q^*|$. Assume now that game G has at least one strict Nash equilibrium e. We will consider a function f from $S_p^* \times S_q^*$ into $\prod_{r\in P} S_r$. For any $x \in S_p^*$ and any $y \in S_q^*$, strategy profile $f(x, y)$ agrees with strategy profile e on all players except for player p and player q. On these two players, strategy profile $f(x, y)$ is equal to x and y respectively.

Note that due to $(n-1)$-interchangeability assumption, for each $x \in S_p^*$ there is at least one $y \in S_q^*$ such that $f(x, y) \in NE(G)$. Since we consider strict equilibria, for any y there can be no more than one x such that $f(x, y) \in NE(G)$. Thus, $|S_p^*| \le |S_q^*|$. One can similarly show that $|S_q^*| \le |S_p^*|$. Therefore, $|S_p^*| = |S_q^*|$. □

Definition 4 *For any $(n-1)$-interchangeable game G with n players,*

$$rank(G) = |S_p^*|.$$

The notion of the $rank(G)$ is well-defined due to Lemma 1.

Lemma 2 *For any $(n-1)$-interchangeable game G with $n > 1$ players,*

$$rank(G) \le 1.$$

Proof. Set $\prod_{r \in P} S_r^*$ contains $(rank(G))^n$ strategy profiles. Some of them belong to $NE(G)$, some do not. For any $s \in \prod_{r \in P} S_r^* \setminus NE(G)$ and any $e \in NE(G)$ we write $s \mathrel{-\!\!\triangleright}_p e$ if profiles s and e differ only on the player p. By the definition of the strict Nash equilibrium, if $s \mathrel{-\!\!\triangleright}_p e$, then

$$u_p(s) < u_p(e). \tag{2}$$

Assume that $rank(G) > 1$. By the definition of a strict Nash equilibrium, two strict equilibria profiles can not differ on just a single player. Thus, there is at least one triple s, e, p such that $s \mathrel{-\!\!\triangleright}_p e$. Let us add such inequalities (2) for each such triple s, e, p:

$$\sum_{s \mathrel{-\!\!\triangleright}_p e} u_p(s) < \sum_{s \mathrel{-\!\!\triangleright}_p e} u_p(e). \tag{3}$$

Due to the $(n-1)$-interchangeability of the game G, for each s and p there is at least one e such that $s \mathrel{-\!\!\triangleright}_p e$. Such e is unique because the equilibrium is strict. Thus,

$$\sum_{s \mathrel{-\!\!\triangleright}_p e} u_p(s) = \sum_s \sum_p u_p(s). \tag{4}$$

Since game G is a zero-sum game,

$$\sum_s \sum_p u_p(s) = \sum_s 0 = 0. \tag{5}$$

At the same time, for each e and each p there are $rank(G) - 1$ different s such that $s \mathrel{-\!\!\triangleright}_p e$. Hence,

$$\sum_{s \mathrel{-\!\!\triangleright}_p e} u_p(e) = (rank(G) - 1) \sum_e \sum_p u_p(e). \tag{6}$$

Since game G is a zero-sum game,

$$(rank(G) - 1) \sum_e \sum_p u_p(e) = (rank(G) - 1) \sum_e 0 = 0. \tag{7}$$

Combination of inequality (3) and equalities (4), (5), (6), and (7) implies that $0 < 0$, which is a contradiction. □

This concludes the proof of Theorem 1. □

Corollary 1 *For any game G with $n > 1$ players, if the game is $(n-1)$-interchangeable, then it is n-interchangeable.*

Proof. By Theorem 1, $rank(G) \leq 1$. If $rank(G) = 0$, then game G has no strict Nash equilibria; if $rank(G) = 1$, then game G has a unique strict Nash equilibrium. In either of these two cases, the game is vacuously n-interchangeable. □

k-interchangeability. The result given in Theorem 1 leads to the natural question whether k-interchangeability implies n-interchangeability for any value of k less than $n-1$. The negative answer to this question is given by the following game, which is based on the parity game previously described by the first author [2]. Unlike the previous work, however, the version of the parity game considered in this paper is a zero-sum game.

Definition 5 *For any set of players A and an additional player b ("banker"), parity game $PG(A,b)$ is defined as follows:*

1. *Each player in set A has two strategies: 0 and 1. Player b has a single strategy.*
2. *If the sum of all values chosen by the players in the set A is odd, then player b pays a fixed positive amount (say, one euro) to each player in set A. Otherwise, pay-off of each player is zero.*

Lemma 3 *If $|A| > 0$, then $NE(G(A,b))$ is the set of all strategy profiles in which sum of all strategies chosen by the players in the set A is odd.* □

Lemma 4 *If $|A| > 1$, then for each player $a \in A$ there is a strict Nash equilibrium of the game $G(A,b)$ in which player a is using strategy 0.*

Proof. Since $|A| > 1$, set A in addition to player a includes at least one more player a'. Consider strategy profile in which player a' chooses 1 and all other players choose 0. By Lemma 3, such strategy profile is a strict Nash equilibrium of the game $G(A,b)$. □

Lemma 5 *If $|A| = n > 1$, then game $G(A,b)$ is not n-interchangeable.*

Proof. By Lemma 4, each player $a \in A$ is using strategy 0 in at least one strict Nash equilibrium of the game $G(A,b)$. Yet, by Lemma 3, they can not use it in the same strict Nash equilibrium of this game. □

Lemma 6 *If $|A| = n$, then game $G(A,b)$ is k-interchangeable for each $k \leq n-2$.*

Proof. Let C be any subset of $A \cup \{b\}$ of size k. Since $k \leq n-2$, set $A \setminus C$ is not empty. Let $a_0 \in A \setminus C$. No matter what are the choices of the other players in set C, by Lemma 3, strategy of player a_0 can be adjusted to create a strict Nash equilibrium. □

Lemma 5 and Lemma 6 together show that the interchangeability result stated in Theorem 1 can not be improved by replacing $(n-1)$-interchangeability by k-interchangeability for some $k < n-1$.

Interchangeability of all Nash equilibria. The concept of interchangeability is not limited to strict Nash equilibria: one can generalize Definition 1 by simply replacing "strict Nash equilibria" with "Nash equilibria". The counterexample given below shows that under this new definition, Theorem 1 does not hold.

Definition 6 *Minority game is a three-player zero-sum game in which each of the three players has two strategies:* 0 *and* 1*. If all three players choose the same strategy, then pay-off of each player is zero. Otherwise, each of the two players in the majority pay a fixed positive amount, say one euro, to the player in the minority.*

Lemma 7 *Set of all Nash equilibria in the minority game is the set of all strategy profiles in which not all players chose the same strategy.* □

Lemma 8 *The set of equilibria in the minority game is 2-interchangeable, but is not 3-interchangeable.*

Proof. By Lemma 7, for any choice of strategies by any two players, there is a choice of strategy of the remaining player that creates a Nash equilibrium. Thus, the set of Nash equilibria of the minority game is 2-interchangeable.

Again by Lemma 7, each of the three players can choose strategy 0 in a Nash equilibria, but all three players can not choose this strategy in the same Nash equilibrium. Therefore, the set of Nash equilibria of the minority game is not 3-interchangeable. □

The minority game counterexample shows that Theorem 1 is not true if strict Nash equilibria is replaced with all Nash equilibria. One still can ask if Theorem 1 holds for all Nash equilibria in games with more than 3 players. To eliminate this possibility, we can consider the following generalization of the minority game:

Definition 7 *Minority game with judges is played between 3 regular players and any number of special players, called "judges". The three regular players play minority game as described above. Each of the judges has two strategies: "valid" and "void". Pay-offs of judges are always zero. If all judges choose "valid", then pay-off of regular players is determined by the rules of the minority games. If at least one judge chooses "void", then the game is considered to be nullified and all pay-offs are zero.*

Lemma 9 *The set of all Nash equilibria in the minority game with judges consists of all strategy profiles in which either the game is nullified by one of the judges, or three regular players choose strategies that are not all equal.* □

Lemma 10 *The set of all Nash equilibria in the minority game with j judges is $(j+2)$-interchangeable, but is not $(j+3)$-interchangeable.*

Proof. Follows from Lemma 9. □

In the conclusion we discuss that Theorem 1 is true for all (not just strict) Nash equilibria of a certain subclass of zero-sum games.

3 Axiomatization

In this section we will formally introduce logical system for the discussed in the introduction interchangeability predicate $[p_1, \dots, p_n]$ and prove soundness and completeness of this system.

3.1 Syntax and Semantics

Definition 8 *For any set finite of players P, by $\Phi(P)$ we mean the minimal set of formulas such that (i) $\bot \in \Phi(P)$, (ii) $[A] \in \Phi(P)$ for each $A \subseteq P$, (iii) if $\phi \in \Phi(P)$ and $\psi \in \Phi(P)$, then $\phi \to \psi \in \Phi(P)$.*

As usual, we will assume that the other propositional connectives are defined through constant false $\bot$ and implication $\to$.

Definition 9 *For any game G with the set of players P and any $\phi \in \Phi(P)$, we define truth relation $G \vDash \phi$ as follows:*

1. *$G \nvDash \bot$,*
2. *$G \vDash [p_1, \dots, p_n]$ if and only if for any $e_1, \dots, e_n \in NE(G)$ there is $e \in NE(G)$ such that $pr_{p_i}(e) = pr_{p_i}(e_i)$ for each $i \leq n$.*
3. *$G \vDash \phi \to \psi$ if and only if $G \nvDash \phi$ or $G \vDash \psi$.*

Lemma 11 *If game G has at least one Nash equilibrium, then $G \vDash [\varnothing]$.* □

3.2 Logical System

For any finite set of players P, our formal logical system contains propositional tautologies in the language $\Phi(P)$, the Modus Ponens inference rule, and the following five additional axioms:

1. No Players: $[\varnothing]$ if $P = \varnothing$,
2. Empty Set: $\neg[\varnothing] \to [A]$, where $|A| > 0$,
3. Singleton: $[A]$, where $|A| = 1$,
4. Monotonicity: $[A] \to [B]$, where $B \subseteq A$,
5. Proper Subset: $\bigwedge_{|A|=n-1}[A] \to [P]$, where $n = |P|$.

We write $\vdash_P \phi$ if formula $\phi \in \Phi(P)$ is provable in this logical system. We write $X \vdash_P \phi$ if it is provable in the same system extended by an additional set of axioms X. We sometimes do not write subscript P when doing so does not create ambiguity.

Theorem 2 (soundness) *If $\vdash_P \phi$, then $G \vDash \phi$ for any zero-sum game G with the set of players P.*

Proof. *No Players.* If game has no players, then empty tuple is the unique strict Nash equilibrium of the game. Thus, by Lemma 11, $G \vDash [\varnothing]$.
Empty Set. Suppose that $G \nvDash [\varnothing]$. Thus, by Lemma 11, the game has no strict Nash equilibria. Hence, $G \vDash [A]$ is vacuously true for each non-empty set A.

Soundness of the Singleton axiom and the Monotonicity axiom is obvious. Soundness of the Proper Subset axiom is established in Corollary 1. □

Theorem 3 (completeness) *For any set P and any $\phi \in \Phi(P)$, if $X \nvdash_P \phi$, then there is a zero-sum game G with set of players P such that $G \nvDash \phi$.*

To start the proof of this theorem, we fix a set of players P and a maximal consistent subset X of $\Phi(P)$ containing $\neg\phi$. We use set X to construct a "canonical" zero-sum game G with set of players P. The canonical game will be defined as a composition of several "atomic" zero-sum games paid in parallel.

Atomic Games. Atomic game $G(A, B)$ is the following slight modification of the parity game described in the introduction.

Definition 10 *For any partition $p = A \sqcup B$ of the set of all players such that set B (of "bankers") is not empty, game $G(A, B)$ is defined as following:*

1. *Each player in set A has two strategies: 0 and 1. Each player in set B has a single strategy.*
2. *If the sum of all values chosen by the players in the set A is odd, then each player in set B pays a fixed positive amount (say, one euro) to each player in set A. Otherwise, pay-off of each player is zero.*

Lemma 12 *If $|A| > 0$, then $NE(G(A, B))$ is the set of all strategy profiles in which sum of all strategies chosen by the players in the set A is odd.* □

Lemma 13 *If $|A| > 1$, then for each player $a \in A$ there is a strict Nash equilibrium of the game $G(A, B)$ in which player a is using strategy 0.*

Proof. Since $|A| > 1$, set A in addition to player a includes at least one more player a'. Consider strategy profile in which player a' chooses 1 and all other players choose 0. By Lemma 12, such strategy profile is a strict Nash equilibrium of the game $G(A, B)$. □

Lemma 14 *If $|A| > 1$, then $G(A,B) \nvDash [C]$ for each C such that $A \subseteq C$.*

Proof. By Lemma 13, each player $c \in C$ is using strategy 0 in at least one strict Nash equilibrium of the game $G(A,B)$. Yet, by Lemma 12, they can not all use this strategy in the same strict Nash equilibrium of the game. □

Lemma 15 *If $|A| > 1$, then $G(A,B) \vDash [C]$ for each C such that $A \nsubseteq C$.*

Proof. Let $a_0 \in A \setminus C$. No matter what are the choices of the other players, by Lemma 12, strategy of a_0 can be adjusted to create a strict Nash equilibrium. □

Game Composition. Informally, by a composition of several games with the same set of players we mean a game in which each of the composed games is played independently. Pay-off of any player is defined as the sum of the pay-offs in the individual games.

Definition 11 *Let $\{G^i\}_{i\in I} = \{(P, \{S^i_p\}_{p\in P}, \{u^i_p\}_{p\in P})\}_{i\in I}$ be a finite family of strategic between the same set of players P. By product game $\prod_i G_i$ we mean game $(P, \{S_p\}_{p\in P}, \{u_p\}_{p\in P})$ such that*

1. $S_p = \prod_i S^i_p$,
2. $u_p(s) = \sum_i u^i_p(pr_i(s))$ *for each strategy profile s of the game $\prod_i G^i$.*

Note that any strategy profile e of the game $\prod_i G^i$ can be thought off as a function $e(p,i)$ that maps player p and game number i into strategy $e(p,i) \in S^i_p$ used by the player p in the i-th game of the composition. We will use this view of e in the proofs of several lemmas below.

Lemma 16

$$NE\left(\prod_i G_i\right) = \prod_i NE(G_i).$$

Proof. First, assume that $e \in NE\left(\prod_i G_i\right)$. We will need to show that strategy profile $e^i = \langle e(p,i)\rangle_{p\in P}$ is a strict Nash equilibrium of each individual game G_i for each $i \in I$. Indeed, suppose that for some $k \in I$, some $q \in P$, and some $s_q \in S_q$ we have

$$u^k_q(e^k_{-q}, s_q) \geq u^k_q(e^k). \tag{8}$$

Define strategy profile $\hat{e}(p,i)$ of the game $\prod_i G^i$ as follows:

$$\hat{e}(p,i) \equiv \begin{cases} s_q & \text{if } i = k \text{ and } p = q, \\ e(p,i) & \text{otherwise.} \end{cases}$$

Let $\hat{e}^i = \langle \hat{e}(p,i)\rangle_{p\in P}$. Note that, taking into account inequality (8),

$$\begin{aligned} u_q(\hat{e}) &= \sum_{i\in I} u^i_q(\hat{e}^i) = u^k_q(\hat{e}^k) + \sum_{i\neq k} u^i_q(\hat{e}^i) = u^k_q(e^k_{-q}, s_q) + \sum_{i\neq k} u^i_q(\hat{e}^i) \geq \\ &\geq u^k_q(e^k) + \sum_{i\neq k} u^i_q(e^i) = u_q(e), \end{aligned}$$

which is a contradiction with the assumption that e is a strict Nash equilibrium of the game $\prod_i G^i$.

Next, assume that $\{e^i\}_{i\in I}$ is such a set that for any $i \in I$,

$$e^i \in NE(G^i) \tag{9}$$

Let $e(p,i) = pr_p(e^i)$. We need to prove that $e \in NE\left(\prod_i G^i\right)$. Indeed, consider any q and any $s_q = \langle s^i_q\rangle_{i\in I} \in \prod_{i\in I} S^i_q$. By assumption (9) and the definition of a strict equilibrium, $u^i_q(e^i_{-q}, s^i_q) < u^i_q(e^i)$ for any $i \in I$. Thus,

$$u_q(e_{-q}, s_q) = \sum_{i\in I} u^i_q(e^i_{-q}, s^i_q) < \sum_{i\in I} u^i_q(e^i) = u_q(e).$$

Therefore, $e \in NE\left(\prod_i G^i\right)$. □

Lemma 17 *For any subset $C = \{c_1, \ldots, c_m\}$ of the set P, if each of the games $\{G^i\}_{i\in I}$ has at least one strict Nash equilibrium and $\prod_{i\in I} G^i \vDash [C]$, then $G^i \vDash [C]$ for each $i \in I$.*

Proof. Consider any i and any $e^i_1, \ldots, e^i_m \in NE(G^i)$. We will show that there is $e^i \in NE(G^i)$ such that $pr_{c_k}(e^i) = pr_{c_k}(e^i_k)$ for each $k \leq m$. By the assumption of the lemma, each of the games $\{G^i\}_{i\in I}$ has at least one equilibrium. We denote it by $\bar{e}^i$. Consider strategy profiles $e_1, \ldots, e_m$ for the game $\prod_i G^i$ such that

$$e_m(q,j) = \begin{cases} pr_q(e^i) & \text{if } j = i, \\ pr_q(\bar{e}^i) & \text{otherwise.} \end{cases}$$

By Lemma 16, each of the profiles $e_1, \ldots, e_m$ is a strict Nash equilibrium of the game $\prod_i G^i$. By the assumption $\prod_i G^i \vDash [C]$, there is $e \in NE(\prod_i G^i)$ such that $e(c_k, j) = e_k(c_k, j)$ for each j. Let us define action profile e^i of the game G^i to be equal to $\langle e(p,i)\rangle_{p\in P}$. By Lemma 16, $e^i \in NE(G^i)$. At the same time, $pr_{c_k}(e^i) = e(c_k, i) = pr_{c_k}(e^i)$. □

Lemma 18 *For any subset $C = \{c_1, \ldots, c_m\}$ of the set P, if $G^i \vDash [C]$ for each $i \in I$, then $\prod_{i\in I} G^i \vDash [C]$.*

Proof. Consider any $e_1, \ldots, e_m \in NE(\prod_i G^i)$. We will show that there is $e \in NE(\prod_i G^i)$ such that $pr_{c_k}(e) = pr_{c_k}(e_k)$ for each $k \leq m$. Indeed, by Lemma 16, strategy profile $e^i_k = \langle e_k(p,i)\rangle_{p\in P}$ is a strict Nash equilibrium of the game G^i

for each $i \in I$ and each $k \leq m$. By the assumption of the lemma, there is $e^i \in NE(G^i)$ such that $pr_{c_k}(e^i) = pr_{c_k}(e^i_k)$ for each $k \leq m$. Let $e(p,i) = pr_p(e^i)$ for each $p \in P$ and each $i \in I$. By Lemma 16, $e \in NE(\prod_i G^i)$. At the same time, $pr_{c_k}(e) = \langle pr_{c_k}(e^i)\rangle_{i\in I} = \langle pr_{c_k}(e^i_k)\rangle_{i\in I} = \langle e_k(c_k,i)\rangle_{i\in I} = pr_{c_k}(e_k)$. □

Final Steps. To prove the completeness theorem, we will first consider a composition of atomic games $G(A, P \setminus A)$, where A is any nonempty proper subset of P such that $X \nvdash_P A$.

Definition 12 $G_* = \prod\{G(A, P \setminus A) \mid \varnothing \subsetneq A \subsetneq P$ *and* $X \nvdash_P A\}$.

Lemma 19 *If* $G_* \vDash [C]$*, then* $G(A, P \setminus A) \vDash [C]$ *for each set* A *such that* $\varnothing \subsetneq A \subsetneq P$ *and* $X \nvdash_P A$.

Proof. By Lemma 12, game $G(A, P \setminus A)$ has a Nash equilibrium for each A such that $\varnothing \subsetneq A \subsetneq P$. Hence, the required follows from Lemma 17. □

Lemma 20 *If* $[C] \in X$*, then* $G_* \vDash [C]$*, for each* $C \subseteq P$.

Proof. Suppose that $G_* \nvDash [C]$. Thus, by Lemma 18, there must be $\varnothing \subsetneq A \subsetneq P$ such that $X \nvdash_P [A]$ and $G(A, P \setminus A) \nvDash [C]$. Hence, $|A| > 1$ by the Singleton axiom. Then, by Lemma 15, $A \subseteq C$. By the the Monotonicity axiom, $X \nvdash_P [C]$. Therefore, $[C] \notin X$. □

Lemma 21 *If* $G_* \vDash [C]$*, then* $[C] \in X$*, for each* C *such that* $\varnothing \subsetneq C \subseteq P$.

Proof. *Case I*: $C \subsetneq P$. Suppose $[C] \notin X$. Hence, $X \nvdash_P [C]$ by maximality of set X. Thus, $|C| > 1$ due to the Singleton axiom. By Lemma 14, $G(C, P \setminus C) \nvDash [C]$. Hence, by Lemma 19, $G_* \nvDash [C']$. Therefore, $G_* \nvDash [C]$.

Case II: $C = P$. Suppose that $[P] \notin X$. Hence, $X \nvdash_P [P]$ by maximality of set X. By Proper Subset axiom, there must be $A \subsetneq P$ such that $|P \setminus A| = 1$ and $X \nvdash_P [A]$. Thus, $|A| > 1$ due to the Singleton axiom. By Lemma 14, $G(A, P \setminus A) \nvDash [A]$. Hence, by Lemma 19, $G_* \nvDash [A]$. Therefore, $G_* \nvDash [P]$ due to the soundness of the Monotonicity axiom. □

Lemma 22 *If* $[\varnothing] \in X$*, then* $G_* \vDash \psi$ *if and only if* $\psi \in X$*, for each formula* ψ *in* $\Phi(P)$.

Proof. Induction on the structural complexity of ψ.
Base Case: $(\Rightarrow)$ Assume that $G_* \vDash [C]$. If C is not-empty, then $[C] \in X$ by Lemma 21. If C is empty, then $[C] \in X$ by the assumption of the lemma. $(\Leftarrow)$ Suppose that $[C] \in X$, then $G_* \vDash [C]$ by Lemma 20. *Induction Step:* follows in the usual way from the maximality and consistency of set X. □

Let G_0 be any zero-sum game between the players in the set P that does not have any strict Nash equilibria assuming that set P has at least one player. For example, game G_0 could be the game in which all players choose between two strategies and pay-off of each player is always zero.

Lemma 23 *If $[\varnothing] \notin X$ and $P \neq \varnothing$, then $G_0 \vDash$ if and only if $\psi \in X$, for each formula $\psi \in \Phi(P)$.*

Proof. Induction on the structural complexity of ψ.
Base Case: ($\Rightarrow$) If $C = \varnothing$, then $G_0 \nvDash [C]$ because game G_0 has no strict Nash equilibrium. Suppose now that $C \neq \varnothing$. Let $[C] \notin X$. Thus, $\neg[C] \in X$, due to maximality of set X. Hence, $X \vdash_P [\varnothing]$, by the contrapositive of the Empty Set axiom. Thus, $[\varnothing] \in X$, due to maximality of the set X, which is a contradiction with the assumption of the lemma. ($\Leftarrow$) Assume that $[C] \in X$. Thus, $C \neq \varnothing$ due to the assumption of the lemma. Hence, $G_0 \vDash [C]$ is vacuously true because game G_0 has no strict Nash equilibria. *Induction Step:* follows in the usual way from the maximality and consistency of the set X. □

Finally, in case if $P = \varnothing$, then let G_1 be the game between zero players. Technically, it is a zero-sum game with empty tuple being the only strategy profile of the game, and, thus, its unique strict Nash equilibrium.

Lemma 24 *If $P = \varnothing$, then $G_1 \vDash \psi$ if and only if $\psi \in X$, for each formula $\psi \in \Phi(\varnothing)$.*

Proof. Induction on the structural complexity of ψ.
Base Case: $G_1 \vDash [\varnothing]$ is true because game G_1 has a Nash equilibrium. $[\varnothing] \in X$ due to No Players axiom and maximality of the set X. *Induction Step:* follows in the usual way from the maximality and consistency of the set X. □

To finish the proof of Theorem 3, recall that $\neg\phi \in X$. Thus, $\phi \notin X$ due to consistency of X. If $P = \varnothing$, then let G be the game G_1. By Lemma 24, $G \nvDash \phi$. Assume now that $P \neq \varnothing$. If $[\varnothing] \in X$, then let G be the game G_*. By Lemma 22, $G \nvDash \phi$. If $[\varnothing] \notin X$, then let G be the game G_0. By Lemma 23, $G \nvDash \phi$. □

4 Conclusion

We have proved the interchangeability theorem for strict Nash equilibria and have shown that a similar result is not true for all (not just strict) Nash equilibria. It could be observed by analyzing our proof of this theorem, however, that the same result is true for all Nash equilibria in "sparse" games, where by sparse game we mean any game which hamming distance between any two Nash equilibria is at least two. In other words, game is sparse if any two of Nash equilibria of the game differ at more than one player.

Finally, in Definition 1 we have assumed that each player has finitely many strategies. This assumption is significant for our proof since otherwise $rank(G)$ is not a well-defined notion. Furthermore, we can construct an example of a game with infinitely many strategies for which interchangeability theorem for strict equilibria does not hold.

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