Rationally Functional Dependence

Pavel Naumov and Brittany Nicholls

Department of Mathematics and Computer Science
McDaniel College, Westminster, Maryland 21157, USA

{pnaumov,brn002}@mcdaniel.edu

Abstract. Two different types of functional dependencies are compared: dependencies that are functional due to the laws of nature and dependencies that are functional if all involved agents behave rationally. The first type of dependencies was axiomatized by Armstrong. This paper gives a formal definition of the second type of functional dependencies in terms of strategic games and describes a sound and complete axiomatization of their properties. The axiomatization is significantly different from the Armstrong's axioms.

1 Introduction

Let A and B be two sets of arbitrary variables. We say that the set of variables A functionally determines the set of values B if knowing the values of variables in set A one can predict the values of variables in set B. We denote this relation by $A \rhd B$.

Functional dependency manifests itself in two different forms. In the physical world, dependency is a result of the laws of nature: knowing the mass of a particle and the forces applied to it, one can determine the acceleration of the particle. It is generally assumed that there is no way to disobey these laws. We will refer to this type of dependence as *necessity* functional.

On the other hand, an agent with free will can make her own choices. The choice of a rational agent, however, might be pre-determined by the information available to the agent: although an army general is free to choose to move the army in any direction, this choice might be pre-determined once the general learns specifics of the enemy's attack plan. We call this type of dependence *rationally* functional, to emphasize that it comes from our assumption of an agent behaving rationally. Unlike necessity functional dependence, rationally functional dependence can be formally defined only in a contest of a game.

In this paper we compare the two types of functional dependence and show that they have different mathematical properties.

1.1 Armstrong's Axioms

The functional dependency that we call necessity functional has been previously studied by several authors. Armstrong [1] presented the following sound and complete axiomatization of this relation:

1. *Reflexivity*: $A \rhd B$, if $A \supseteq B$,
2. *Augmentation*: $A \rhd B \to A, C \rhd B, C$,
3. *Transitivity*: $A \rhd B \to (B \rhd C \to A \rhd C)$,

where here and everywhere below A, B denotes the union of sets A and B. The above axioms are known in database literature as Armstrong's axioms [3, p. 81]. Beeri, Fagin, and Howard [2] suggested a variation of Armstrong's axioms that describe properties of multi-valued dependence. An extension of the Armstromg's logical system that captures properties of the functional dependencies over a given "hypergraph of secrets" was described by More and Naumov [5].

1.2 Rationally Functional Dependence

In this paper we suggest a possible precise definition of rationally functional dependence and give a complete axiomatization of this relation, which is different from the Armstrong axioms. We define rationally functional dependence $A \rhd B$ as a relation between two sets of players, A and B, in a strategic game. Informally, $A \rhd B$ means that if all players in set A publicly announce their choice of strategies, then each of the players in set B will be left with only one "rational" choice of strategy. For example, if a and b are two players in the the rock-paper-scissors game and player a announces her move in advance, then player b is left with only one rational choice, thus $a \rhd b$. Note that here and everywhere below we write $a \rhd b$ instead of $\{a\} \rhd \{b\}$.

Formally, once players in set A commit to their strategies $\hat{A}$ in a game G, the game is reduced to a new game $G[A \mapsto \hat{A}]$ in which each player in set A now has only a single strategy to choose from. If players in set B have the same strategies in any Nash equilibrium of the game $G[A \mapsto \hat{A}]$, then we say that strategies of the players in B are rationally determined by the choice $\hat{A}$ of players A and write $A \rhd B$. Alternatively, strongly dominant strategy elimination can be used instead of Nash equilibrium to capture rationality. We will discuss this alternative in the conclusion.

It is easy to see that Armstrong's Augmentation and Transitivity axioms are not sound for the rationally functional dependence. Indeed, to show that the Augmentation axiom is not true even in the case when all there sets are single-element sets:

$$a \rhd b \to (a, c \rhd b, c),$$

consider a game G_1 in which players a, b, and c each have a choice of two strategies: 0 and 1. Player a is paid the same no matter what is the outcome. Player c is paid if players a and c use the same strategy. Player b is paid if all three players use the same strategy. If player a publicly announces her strategy, then player c will have only one rational strategy: to match a. Knowing this, player b also will be rationally predetermined to match them. Thus $a \rhd b$. On the other hand, if players a and c publicly commit to two *different* strategies, then the strategy of player b will not be rationally predetermined. Therefore, $a, c \rhd b, c$ is false.

The above example actually demonstrates even more. It shows that rationally functional dependence, unlike its necessity functional counterpart, does not satisfy even the seemingly obvious monotonicity property:

$$a \rhd b \to a, c \rhd b.$$

To show that the Transitivity axiom is not true even in the case when all three sets are single-element sets:

$$a \rhd b \to (b \rhd c \to a \rhd c),$$

consider a game G_2 in which players a, b, and c each have a choice of two strategies: 0 and 1. Player a is paid if she chooses strategy 1. Player b is paid if she matches the choice of player a. Player c is paid if players a and c choose strategy 1. Note that $a \rhd b$ because player b is paid if she matches the choice of player a. To show that $b \rhd c$, recall that no matter what the choice of player b is, player a is rationally predetermined to choose 1. Player c knows this and, thus, she is also rationally predetermined to choose 1. Finally, if player a choses strategy 0, then player c is not paid no matter what her strategy is. Therefore, $a \rhd c$ is false.

The notion of rationally functional dependence is a special case of dependence [4, 6] in strategic games.

1.3 Axioms

In this paper we prove that the following four axioms give a sound and complete description of the properties of the rationally functional dependence:

1. *Reflexivity*: $A \rhd A$,
2. *Monotonicity*: $A \rhd B, C \to A \rhd B$,
3. *Union*: $A \rhd B \to (A \rhd C \to A \rhd B, C)$,
4. *Weak Transitivity*: $A \rhd B \to (A, B \rhd C \to A \rhd C)$.

All four of these axioms can be derived from Armstrong's axioms. Thus, the proposed logical system for the rationally functional dependence is logically weaker than logical system for the necessity functional dependence.

In the rest of the paper we formally define game semantics for the rationally functional dependence and prove soundness and completeness of this axiomatic system with respect to the game semantics.

2 Semantics

Definition 1. *A strategic game is a triple* $G = (P, \{S_p\}_{p \in P}, \{u_p\}_{p \in P})$, *where*

1. P *is a non-empty finite set of "players".*
2. S_p *is a non-empty set of "strategies" of a player* $p \in P$. *Elements of the cartesian product* $\prod_{p \in P} S_p$ *are called "strategy profiles".*

3. u_p *is a "pay-off" function from strategy profiles into the set of real numbers.*

As is common in the game theory literature, for any tuple $a = \langle a_i \rangle_{i \in I}$, any $i_0 \in I$, and any value b, by (a_{-i_0}, b) we mean the tuple a in which i_0-th component is changed from a_{i_0} to b.

Definition 2. *Nash equilibrium of a strategic game* $G = (P, \{S_p\}_{p \in P}, \{u_p\}_{p \in P})$, *is a strategy profile* s *such that*

$$u_p(s_{-p_0}, s_0) \le u_p(s) \tag{1}$$

for each $p_0 \in P$ *and each* $s_0 \in S_{p_0}$.

The set of all Nash equilibria of a game G is denoted by $NE(G)$. Alternatively, one can define *strict* Nash equilibrium by replacing the relation $\le$ in inequality (1) with strict inequality sign $<$. The soundness and completeness theorems in this paper are true for both types of equilibria.

Definition 3. *Let* $G = (P, \{S_p\}_{p \in P}, \{u_p\}_{p \in P})$ *be any strategic game,* $A \subseteq P$ *be any subset of players, and* $\hat{A}$ *be any tuple* $\langle \hat{s}_a \rangle_{a \in A}$ *such that* $\hat{s}_a \in S_a$ *for each* $a \in A$*. By restricted game* $G[A \mapsto \hat{A}]$ *we mean game* $(P, \{S'_p\}_{p \in P}, \{u_p\}_{p \in P})$, *where*

$$S'_p = \begin{cases} \{\hat{s}_p\} & \text{if } p \in A, \\ S_p & \text{otherwise.} \end{cases}$$

If $x = \langle x_i \rangle_{i \in I}$ and $y = \langle y_i \rangle_{i \in I}$ are two tuples such that $x_a = y_a$ for any $a \in A$, then we write $x \equiv_A y$.

Lemma 1. *Let* $G = (P, \{S_p\}_{p \in P}, \{u_p\}_{p \in P})$ *be a strategic game,* $e = \langle e_p \rangle_{p \in P}$ *be a Nash equilibrium of game* G*, and* A *be an arbitrary subset of* P*. If* $\hat{A} = \langle \hat{e}_a \rangle_{a \in A}$, *then*

$$\{e \in NE(G) \mid e \equiv_A \hat{e}\} \subseteq NE(G[A \mapsto \hat{A}]).$$

□

Lemma 2. *Let* $G = (P, \{S_p\}_{p \in P}, \{u_p\}_{p \in P})$ *be a strategic game,* A *and* B *be two arbitrary subsets of* P*, and* $\hat{A}$ *and* $\hat{B}$ *be arbitrary tuples from sets* $\prod_{a \in A} S_a$ *and* $\prod_{b \in B} S_b$ *respectively. If sets* A *and* B *are disjoint, then* $G[A \mapsto \hat{A}][B \mapsto \hat{B}] = G[A, B \mapsto \hat{A}, \hat{B}]$. □

Definition 4. *For any finite set of players* P*, the set of formulas* $\Phi(P)$ *is the minimal set of formulas such that: (i)* $\bot \in \Phi(P)$*, (ii)* $A \rhd B \in \Phi(P)$*, where* A *and* B *are two subsets of* P*, (iii)* $\phi \to \psi \in \Phi(P)$*, if* $\phi, \psi \in \Phi(P)$.

Next, we define the truth relation $G \vDash \phi$ between a game G and a formula ϕ:

Definition 5. *For any game* $G = (P, \{S_p\}_{p \in P}, \{u_p\}_{p \in P})$ *and any formula* $\phi \in \Phi(P)$*, binary relation* $G \vDash \phi$ *is defined as follows:*

1. $G \nvDash \bot$,

2. $G \vDash \phi \to \psi$ *if and only if* $G \nvDash \phi$ *or* $G \vDash \psi$*,*
3. $G \vDash A \rhd B$ *if and only if* $e \equiv_B e'$ *for each tuple* $\hat{A} \in \prod_{a \in A} S_a$ *and each* $e, e' \in NE(G[A \mapsto \hat{A}])$*.*

The third part of the above definition is the key definition of this paper. It formally specifies rationally functional dependence of two sets of players in a strategic game.

3 Logical Systems

For any fixed set of players P, our formal logical system contains propositional tautologies in the language $\Phi(P)$, the Modus Ponens inference rule, and the following four additional axioms:

1. *Reflexivity*: $A \rhd A$,
2. *Monotonicity*: $A \rhd B, C \to A \rhd B$,
3. *Union*: $A \rhd B \to (A \rhd C \to A \rhd B, C)$,
4. *Weak Transitivity*: $A \rhd B \to (A, B \rhd C \to A \rhd C)$.

We write $\vdash \phi$ if formula $\phi \in \Phi(P)$ is provable in this logical system. We write $X \vdash \phi$ if it is provable in the same system extended by an additional set of axioms X.

4 Soundness

Theorem 1. *For any set of players* P*, any formula* $\phi \in \Phi(P)$*, and any game* G *with the set of players* P*, if* $\vdash \phi$*, then* $G \vDash \phi$*.*

Proof. It will be sufficient to prove soundness of axioms 1.-4. We present each of these results as a separate lemma.

Lemma 3 (reflexivity). $G \vDash A \rhd A$*, for each game* $G = (P, \{S_p\}_{p \in P}, \{u_p\}_{p \in P})$ *and each* $A \subseteq P$*.*

Proof. Let $\hat{A} = \langle \hat{s}_a \rangle_{a \in A} \in \prod_{a \in A} S_a$. Suppose that $e = \langle e_i \rangle_{i \in I} \in NE(G[A \mapsto \hat{A}])$ and $e' = \langle e'_i \rangle_{i \in I} \in NE(G[A \mapsto \hat{A}])$. Then, $e_a = \hat{s}_a = e'_a$ for all $a \in A$. Therefore, $e \equiv_A e'$. □

Lemma 4 (monotonicity). *If* $G \vDash A \rhd B, C$*, then* $G \vDash A \rhd B$ *for each game* $G = (P, \{S_p\}_{p \in P}, \{u_p\}_{p \in P})$ *and each* $A, B, C \subseteq P$*.*

Proof. Let $\hat{A} \in \prod_{a \in A} S_a$ and $e, e' \in NE(G[A \mapsto \hat{A}])$. Then, by the assumption of the lemma, $e \equiv_{B,C} e'$. Therefore, $e \equiv_B e'$. □

Lemma 5 (union). *If* $G \vDash A \rhd B$ *and* $G \vDash A \rhd C$*, then* $G \vDash A \rhd B, C$ *for each game* $G = (P, \{S_p\}_{p \in P}, \{u_p\}_{p \in P})$ *and each* $A, B, C \subseteq P$*.*

Proof. Let $\hat{A} \in \prod_{a\in A} S_a$ and $e, e' \in NE(G[A \mapsto \hat{A}])$. Then, by the assumptions of the theorem, $e \equiv_B e'$ and $e \equiv_C e'$. Therefore, $e \equiv_{B,C} e'$. □

Lemma 6 (weak transitivity). *If $G \vDash A \rhd B$ and $G \vDash A, B \rhd C$, then $G \vDash A \rhd C$ for each game $G = (P, \{S_p\}_{p\in P}, \{u_p\}_{p\in P})$ and each $A, B, C \subseteq P$.*

Proof. Let $\hat{A} = \langle \hat{s}_a \rangle_{a\in A} \in \prod_{a\in A} S_a$. Suppose that $e = \langle e_i \rangle_{i\in I} \in NE(G[A \mapsto \hat{A}])$ and $e' = \langle e'_i \rangle_{i\in I} \in NE(G[A \mapsto \hat{A}])$. Then, by the first assumption of the theorem, $e \equiv_B e'$. Define $\hat{B} \in \prod_{b\in B\setminus A} S_b$ to be $\langle e_b \rangle_{b\in B\setminus A}$. By Lemma 1,

$$e \in NE(G[A \mapsto \hat{A}][B \setminus A \mapsto \hat{B}])$$

$$e' \in NE(G[A \mapsto \hat{A}][B \setminus A \mapsto \hat{B}])$$

By Lemma 2,

$$e \in NE(G[A, B \mapsto \hat{A}, \hat{B}])$$

$$e' \in NE(G[A, B \mapsto \hat{A}, \hat{B}])$$

Therefore, by the second assumption of the theorem, $e \equiv_C e'$. □

This concludes the proof of the soundness theorem. □

5 Completeness

In this section we will prove completeness theorem for our logical system. This result is stated as Theorem 2 in the end of the section. The proof of the completeness theorem consists in constructing a counterexample game for each formula not provable in our logical system. The counterexample game will be a composition of several mini games played in parallel. Each of the mini games will be a modification of the Circle of Blame game, which is described below.

5.1 Circle of Blame

Informally, Circle of Blame game can be thought of as a game between n players positioned in a circle. Each player has two strategy: 0 and 1. Each player in the game is paid a fixed amount, say one euro, if the choice of the player matches the choice of the her left neighbor. We call this setting a "circle of blame" because each player is not penalized as long as she can blame the left neighbor for her choice of strategy. Formally, the game is defined as followings:

Definition 6. *For any ordered set of parties $W = \{w_1, \ldots, w_n\}$, circle of blame game $CB(W)$ is a triple $(W, \{S_w\}_{w\in W}, \{u_w\}_{w\in W})$, where*

1. *$S_w = \{0, 1\}$ for each $w \in W$,*

2. for any $\langle s_w \rangle_{w \in W} \in \{0,1\}^n$ *and any* $i \leq n$,

$$u_{w_i}(\langle s_w \rangle_{w \in W}) = \begin{cases} 1 & \text{if } s_{w_i} = s_{w_{i+1}}, \\ 0 & \text{otherwise,} \end{cases}$$

where by w_{n+1} *we mean* w_1.

Lemma 7. *If* $|W| > 0$, *then game* $CB(W)$ *has exactly two Nash equilibria.*

Proof. These equilibria are the tuple of all 0s and the tuple of all 1s. Indeed, if tuple $\langle s_w \rangle_{w \in W} \in \{0,1\}^n$ contains at least one 0 and at least one 1, then there is such i that $s_{w_i} \neq s_{w_{i+1}}$. In this case, player w_i can switch strategy from s_{w_i} to $s_{w_{i+1}}$ in order to increase her pay-off. Therefore, tuple $\langle s_w \rangle_{w \in W}$ is not a Nash equilibrium of the game. □

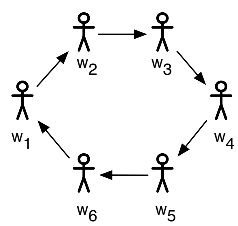

Fig. 1. Circle of Blame

Assume now that players w_1 and w_3 in the Circle of Blame in Figure 1 publicly commit to strategies 1 and 0 respectively. Once this happens, the game has only one Nash equilibrium: the one in which players w_4, w_5, w_6 use strategy 1 and player w_2 uses strategy 0. This observation can be generalized as follows:

Lemma 8. *For any set* W, *any subset* $W_0 \subseteq W$ *of size* $n \geq 1$, *and any* $\hat{W}_0 \in \{0,1\}^n$, *game* $CBG(W)[W_0 \mapsto \hat{W}_0]$ *has exactly one Nash equilibrium.* □

5.2 Game $G_A(P)$

Assume now that P is a fixed finite set of players and X is a fixed maximal consistent subset of $\Phi(P)$.

Definition 7. *For any set of players* A, *let* A^* *be set* $\{a \in P \mid X \vdash A \rhd a\}$.

Lemma 9. $A \subseteq A^*$ *for each* $A \subseteq P$.

Proof. Assume that $a \in A$. We will show that $a \in A^*$. Indeed, by the Reflexivity axiom, $\vdash A \rhd A$. Thus, by the Monotonicity axiom, $\vdash A \rhd a$. Therefore, $a \in A^*$.

Lemma 10. $X \vdash A \rhd A^*$, *for each* $A \subseteq P$.

Proof. Let $A^* = \{a_1, \dots, a_n\}$. By the definition of A^*, we have $X \vdash A \rhd a_i$, for all $i \leq n$. We prove, by induction on k, that $X \vdash A \rhd a_1, \dots, a_k$ for any $0 \leq k \leq n$. *Base Case*: $X \vdash A \rhd A$ by the Reflexivity axiom. Hence, by the Monotonicity axiom, $X \vdash A \rhd \varnothing$. *Induction Step*: Assume that $X \vdash A \rhd a_1, \dots, a_k$. By our assumption, $X \vdash A \rhd a_{k+1}$. Thus, by the Union axiom, $X \vdash A \rhd a_1, \dots, a_k, a_{k+1}$. □

Next, we will describe a certain generalization of the Circle of Blame game between players in set P. In this new game, there will be three groups of players: group A ("masters"), group $A^* \setminus A$ ("slaves"), and group $P \setminus A^*$ ("blamers"). Each of the players in each of the groups has two strategies. Masters can either sleep (strategy 0) or be awake (strategy 1). They are paid a fixed positive amount, say one euro, to be awake. Slaves have the same two strategies as masters, but they are paid to be awake if at least one of the masters is awake and they are paid to sleep if all of the masters are asleep. Finally, blamers are paid to choose 1 if at least one of the masters is awake, otherwise, they play between themselves the standard Circle of Blame game described above. Formally, this game can be described as follows:

Definition 8. *Assume $A \subseteq P$. Let game $G_A(P)$ be triple $(P, \{S_p\}_{p\in P}, \{u_p\}_{p\in P})$ such that*

1. $S_p = \{0, 1\}$ *for any* $p \in P$,
2. *if* $a \in A$, *then* $u_a(\langle s_p \rangle_{p\in P}) = s_a$,
3. *if* $z \in A^* \setminus A$, *then*

$$u_z(\langle s_p \rangle_{p\in P}) = \begin{cases} 1 & \text{if } s_z = 1 \text{ and } \exists a \in A\,(s_a = 1), \\ 1 & \text{if } s_z = 0 \text{ and } \forall a \in A\,(s_a = 0), \\ 0 & \text{otherwise,} \end{cases}$$

4. *if* $w_i \in \{w_1, \dots, w_n\} = P \setminus A^*$, *then*

$$u_{w_i}(\langle s_p \rangle_{p\in P}) = \begin{cases} 1 & \text{if } s_{w_i} = 1 \text{ and } \exists a \in A\,(s_a = 1), \\ 1 & \text{if } s_{w_i} = s_{w_{i+1}} \text{ and } \forall a \in A\,(s_a = 0), \\ 0 & \text{otherwise,} \end{cases}$$

where by w_{n+1} we mean w_1.

Lemma 11. *If $A \nsubseteq C$, then $G_A(P) \vDash C \rhd d$,*

Proof. If $d \in C$, then $G_A(P) \vDash C \rhd d$ because player d has only one strategy in the game $G_A(P)[C \mapsto \hat{C}]$. Assume now that $d \notin C$. Let n be the size of the set C. Consider any $\hat{C} \in \{0,1\}^n$. Let $a_0 \in A \setminus C$. By Definition 8, $s_{a_0} = 1$ in each Nash equilibrium of the game $G_A(P)[C \mapsto \hat{C}]$. Thus, by Definition 8, $s_d = 1$ in each Nash equilibrium of the game $G_A(P)[C \mapsto \hat{C}]$ no matter if $d \in A$, $d \in A^* \setminus A$, or $d \in P \setminus A^*$. □

Lemma 12. $G_A(P) \vDash C \rhd d$ *if* $A \subseteq C$ *and* $d \in A^*$.

Proof. The statement of the lemma follows from Definition 8. $\square$

Lemma 13. *If $A \subseteq C$ and $C \nsubseteq A^*$ and $d \in P \setminus A^*$, then $G_A(P) \vDash C \rhd d$.*

Proof. Let n be the size of the set C. Consider any $\hat{C} = \langle \hat{s}_c \rangle_{c \in C} \in \{0,1\}^n$.
Case I: $\hat{s}_{a_0} = 1$ for at least one $a_0 \in A \subseteq C$. Then, by Definition 8, $s_d = 1$ in each Nash equilibrium of the game $G_A(P)[C \mapsto \hat{C}]$.
Case II: $\hat{s}_a = 0$ for all $a \in A$. Thus, by Definition 8, players in $W = P \setminus A^*$ are playing Circle of Blame game (see Definition 6) between themselves. Note that $W \cap C$ is not empty because $C \nsubseteq A^*$. Thus, by Lemma 8, game $G_A(P)[C \mapsto \hat{C}]$ has a unique Nash equilibrium. $\square$

Lemma 14. *If $X \vdash C \rhd D$, then $G_A(P) \vDash C \rhd D$, for any subset $A \subseteq P$.*

Proof. Suppose that $G_A(P) \nvDash C \rhd D$. Hence, by Definition 5, $G_A(P) \nvDash C \rhd d$ for some $d \in D$. Thus, by Lemma 11, $A \subseteq C$. Hence, by Lemma 12, $d \notin A^*$. In other words, $d \in P \setminus A^*$. Then, by Lemma 13, $C \subseteq A^*$. Thus, $A \subseteq C \subseteq A^*$. Then, $C = A \cup (C \cap A^*)$. Hence, by the assumption of the lemma, $X \vdash A, (C \cap A^*) \rhd D$. By Monotonicity Axiom,

$$X \vdash A, (C \cap A^*) \rhd d. \tag{2}$$

At the same time, by Lemma 9, $X \vdash A \rhd A^*$. By the Monotonicity axiom, $X \vdash A \rhd (C \cap A^*)$. By the Weak Transitivity axiom and (2), $X \vdash A \rhd d$. Therefore, $d \in A^*$, which is a contradiction. $\square$

Lemma 15. *If $G_A(P) \vDash A \rhd B$, then $X \vdash A \rhd B$.*

Proof. Assume that $X \nvdash A \rhd B$. Thus, by Lemma 9 and the Monotonicity axiom, $B \nsubseteq A^*$. Let $b_0 \in B \setminus A^*$. Consider $\hat{A} \in \prod_{a \in A} S_a$ such that all components of the tuple $\hat{A}$ are equal to 0. By Definition 8, in game $G_A(P)[A \mapsto \hat{A}]$, all players in $P \setminus A^*$ are playing circle of blame game between themselves. Thus, by Lemma 7, game $G_A(P)[A \mapsto \hat{A}]$ has two Nash equilibria with different strategies of player b. Therefore, by Definition 5, $G_A(P) \nvDash A \rhd B$. $\square$

5.3 Game Composition

Informally, by a composition of several games with the same set of players we mean a game in which each of the composed games is played independently. Pay-off of each player is defined as the sum of the pay-offs in the individual games. In the definition below by $pr_p(s)$ we mean the strategy of the player p in a strategy profile s.

Definition 9. *Let $\{G^i\}_{i \in I} = \{(P, \{S^i_p\}_{p \in P}, \{u^i_p\}_{p \in P})\}_{i \in I}$ be a finite family of strategic games between the same set of players P. By product game $\prod_i G^i$ we mean such game $(P, \{S_p\}_{p \in P}, \{u_p\}_{p \in P})$ that*

1. $S_p = \prod_i S^i_p$,

2. $u_p(s) = \sum_i u_p^i(pr_p(s))$ *for each strategy profile* s *of the game* $\prod_i G^i$.

Note that any strategy profile e of the game $\prod_i G^i$ can be thought of as a function $e(p,i)$ that maps player p and game number i into strategy $e(p,i) \in S_p^i$ used by the player p in the i-th game of the composition. We will use this view of e in the proofs of several lemmas below.

Lemma 16. *If* $\hat{A}^j \in \prod_{a\in A} S_a^j$ *for each* $j \in I$, *then*

$$(\prod_i G^i)[A \mapsto \prod_j \hat{A}^j] = \prod_i (G^i[A \mapsto \hat{A}^i]).$$

□

Lemma 17.

$$NE\left(\prod_i G^i\right) = \prod_i NE(G^i).$$

Proof. First, assume that $e \in NE\left(\prod_i G^i\right)$. We will need to show that strategy profile $e^i = \langle e(p,i)\rangle_{p\in P}$ is a Nash equilibrium of each individual game G^i for each $i \in I$. Indeed, suppose that for some $k \in I$, some $q \in P$, and some $s_q \in S_q$ we have

$$u_q^k(e_{-q}^k, s_q) > u_q^k(e^k). \tag{3}$$

Define strategy profile $\hat{e}(p,i)$ of the game $\prod_i G^i$ as follows:

$$\hat{e}(p,i) \equiv \begin{cases} s_q & \text{if } i = k \text{ and } p = q, \\ e(p,i) & \text{otherwise.} \end{cases}$$

Let $\hat{e}^i = \langle \hat{e}(p,i)\rangle_{p\in P}$. Note that, taking into account inequality (3),

$$u_q(\hat{e}) = \sum_{i\in I} u_q^i(\hat{e}^i) = u_q^k(\hat{e}^k) + \sum_{i\neq k} u_q^i(\hat{e}^i) = u_q^k(e_{-q}^k, s_q) + \sum_{i\neq k} u_q^i(\hat{e}^i) >$$
$$> u_q^k(e^k) + \sum_{i\neq k} u_q^i(e^i) = u_q(e),$$

which is a contradiction with the assumption that e is a Nash equilibrium of the game $\prod_i G^i$.

Next, assume that $\{e^i\}_{i\in I}$ is such a set that for any $i \in I$,

$$e^i \in NE(G^i) \tag{4}$$

Let $e(p,i) = pr_p(e^i)$. We need to prove that $e \in NE\left(\prod_i G^i\right)$. Indeed, consider any q and any $s_q = \langle s_q^i\rangle_{i\in I} \in \prod_{i\in I} S_q^i$. By assumption (4) and the definition of a equilibrium, $u_q^i(e_{-q}^i, s_q^i) \leq u_q^i(e^i)$ for any $i \in I$. Thus,

$$u_q(e_{-q}, s_q) = \sum_{i\in I} u_q^i(e_{-q}^i, s_q^i) \leq \sum_{i\in I} u_q^i(e^i) = u_q(e).$$

Therefore, $e \in NE\left(\prod_i G^i\right)$. □

Lemma 18. *For any subset A and B of the set P, if each of the games $\{G^i\}_{i\in I}$ has at least one Nash equilibrium, then*

$$\prod_i G^i \vDash A \rhd B \qquad \textit{iff} \qquad \forall i\,(G^i \vDash A \rhd B).$$

Proof. ($\Rightarrow$) Consider any i_0, any $\hat{A}^{i_0} \in \prod_{a\in A} S_a^{i_0}$ and any $f^{i_0}, g^{i_0} \in NE(G^{i_0}[A \mapsto \hat{A}^{i_0}])$. We will show that $f^{i_0} \equiv_B g^{i_0}$. Indeed, by the assumption of the lemma, each of the games $\{G^i\}_{i\in I}$ has at least one equilibrium. We denote it by $\bar{e}^i$. Let $\hat{A}^i = \{\langle \bar{e}^i(a)\rangle_{a\in A}\}$ for each $i \neq i_0$.

By Lemma 1, $\bar{e}^i \in NE(G^i[A \mapsto \hat{A}^i])$ for each $i \in I$. Consider strategy profiles F and G in the game $\prod_i (G^i[A \mapsto \hat{A}^i])$ such that

$$F(p,i) = \begin{cases} f^i(p) & \text{if } i = i_0, \\ \bar{e}^i(p) & \text{otherwise.} \end{cases}$$

$$G(p,i) = \begin{cases} g^i(p) & \text{if } i = i_0, \\ \bar{e}^i(p) & \text{otherwise.} \end{cases}$$

By Lemma 17, profiles F and G are Nash equilibria of the game $\prod_i (G^i[A \mapsto \hat{A}^i])$. By Lemma 16, F and G are Nash equilibria of the game $(\prod_i G^i)[A \mapsto \prod_i \hat{A}^i]$. Thus, by the assumption $\prod_i G^i \vDash A \rhd B$, we have $F \equiv_B G$. Therefore, by the definition of F and G, we can conclude that $f^{i_0} \equiv_B g^{i_0}$.
($\Leftarrow$) Consider any $e_1, e_2 \in NE((\prod_i G^i)[A \mapsto \prod_j \hat{A}^j])$. Thus, Lemma 16,

$$e_1, e_2 \in NE(\prod_i (G^i[A \mapsto \hat{A}^i])).$$

We will show that $e_1 \equiv_B e_2$. Indeed, for any $i \in I$ we can consider strategy profiles $e_1^i = \lambda p.(e_1(p,i))$ and $e_2^i = \lambda p.(e_2(p,i))$ for the game $G^i[A \mapsto \hat{A}^i]$. By Lemma 17, $e_1^i, e_2^i \in NE(G^i[A \mapsto \hat{A}^i])$. Hence, by the assumption of the lemma, $e_1^i \equiv_B e_2^i$. Thus, $e_1(p,i) = e_1^i(p) = e_2^i(p) = e_2(p,i)$ for each $p \in B$. Recall, that we have assumed that i is an arbitrary element of I. Therefore, $e_1 \equiv_B e_2$. □

5.4 Completeness: the Final Steps

Theorem 2 (completeness). *For any set of player P and any $\phi \in \Phi(P)$, if $\nvdash \phi$, then there is a strategic game G, with the set of players P, such that $G \nvDash \phi$.*

Proof. Let X be a maximal consistent subset of $\Phi(P)$ containing $\neg\phi$. Consider game $G = \prod_{A\subseteq P} G_A(P)$.

Lemma 19. *$G \vDash \psi$ if and only if $X \vdash \psi$, for each formula $\psi \in \Phi(P)$.*

Proof. Induction on the structural complexity of formula ψ. For the base case, assume that $\psi \equiv A \rhd B$. Note that game $G_A(P)$ has at least one Nash equilibrium for each $A \subseteq P$ – the strategy profile in which all players pick strategy 0.

(⇒) If $G \vDash A \rhd B$, then, by Lemma 18, $G_A(P) \vDash A \rhd B$. Hence, $X \vdash A \rhd B$, by Lemma 15.
(⇐) If $X \vdash A \rhd B$, then, by Lemma 14, $G_C(P) \vDash A \rhd B$ for each $C \subseteq P$. Hence, by Lemma 18, $\prod_{A \subseteq P} G_A(P) \vDash A \rhd B$. Therefore, $G \vDash A \rhd B$. Induction step case follows from the maximality and consistency of the set X. □

To finish the proof of the completeness theorem, recall that $\neg\phi \in X$. Thus, $\phi \notin X$, due to consistency of the set X. Therefore, by Lemma 19, $G \nvDash \phi$. □

6 Conclusion

Dominated strategy elimination. In this paper rationality is captured be the assumption that players' strategies are in a Nash equilibrium. This, of cause is not the only possible definition of rationality. For example, we can alternatively define $A \rhd B$ as statement *if players in set A publicly announced their choices of strategies, then strongly dominated strategy elimination procedure yields a unique choice of strategies for all players in set B*. Not only axioms 1.-4. are sound under this alternative semantics, but our logical system, defined by axioms 1.-4. is also complete with respect to this semantics. In fact, the completeness proof given in this paper could be easily modified for this case. This is because strongly dominated strategy elimination procedure applied to the Circle of Blame game, after some of the players publicly announce their strategies, yields exactly the unique Nash equilibrium.
Zero-sum games. If one restricts consideration only to zero-sum games, then the results of this paper will also not change. The key point here is that the Circle of Blame game (with at least three players) can be modified to be a zero-sum game. In the modified game each player is paid to match her *left* neighbor and the payment is done by the *right* neighbor. It is easy to see that this modification does not change the set of Nash equilibria of the Circle of Blame game.

Possible extensions of this work could consider settings when not all agents are assumed to be rational or when commitments are not made public and are only known to some players in the game.

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