**Archive for Mathematical Logic manuscript No.**
(will be inserted by the editor)

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# Upper bounds on complexity of Frege proofs with limited use of certain schemata

the date of receipt and acceptance should be inserted later

**Abstract** The paper considers a commonly used axiomatization of the classical propositional logic and studies how different axiom schemata in this system contribute to proof complexity of the logic. The existence of a polynomial bound on proof complexity of every statement provable in this logic is a well-known open question.

The axiomatization consists of three schemata. We show that any statement provable using unrestricted number of axioms from the first of the three schemata and polynomially-bounded in size set of axioms from the other schemata, has a polynomially-bounded proof complexity. In addition, it is also established, that any statement, provable using unrestricted number of axioms from the remaining two schemata and polynomially-bounded in size set of axioms from the first scheme, also has a polynomially-bounded proof complexity.

## 1 Introduction

The question whether the minimal proof size, as a function of the theorem size, has a polynomial bound, goes back to Gödel (7). The existence of such polynomial bound for the propositional logic would imply equality of the computational complexity classes NP and co-NP.

Most existing results in this area fall into one of the following two categories: lower exponential bounds for some deductively weak versions of propositional calculus and upper polynomial bounds for certain classes of propositional tautologies. In the first group are Haken (8) exponential lower bound for the resolution proof system, the exponential lower bound for bounded depth propositional proofs, independently obtained by Pitassi, Beame, and Impagliazzo (10) and by Krajíček, Pudlák, and Woods (9), as well as the results of Pudlák (11) and Bonet, Pitassi, and Raz (1) on lower bounds for the cutting planes proof system. The

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same group also contains Buss and Pudlák (3) lower bound for the intuitionistic propositional calculus, which is conditional upon a conjuncture about class NP. The second group is less densely populated. It includes Buss (2) polynomial upper bound on the proof size of the pigeonhole principle in propositional calculus and Cook, Coullard, and Turán (5) polynomial upper bound on proof size of the same propositional pigeonhole principle in the cutting planes proof system.

This paper also falls into the second category, but unlike previous works, it focuses on a particular axiomatization of the classical propositional logic and investigates how different schemata of this axiomatization contribute to proof complexity. The axiomatization of the propositional logic that we study goes back to Frege (6), except that he deals with the substitution rule instead of axiom schemata. Frege's original axioms for propositional logic, written in modern notations, are:

$$a \to (b \to a), \tag{1}$$

$$(a \to (b \to c)) \to ((a \to b) \to (a \to c)), \tag{2}$$

$$\neg\neg a \to a. \tag{3}$$

In addition, Frege considers axiom $(a \to b) \to (\neg b \to \neg a)$, but it could be derived from axioms (1), (2), and (3) if $\neg\phi$ is defined as $\phi \to \perp$. As a result, Church (4), along with many others, uses axioms (1), (2), and (3) as the axiomatization of the propositional logic. Frege calls formula (3) *Duplex negatio affirmat*, and does not give any names to the other axioms. Church calls axioms (1), (2), and (3) the *law of affirmation of consequent*, the *self-distributive law of (material) implication*, and the *law of double negation* correspondingly. In this paper, we also will treat negation $\neg\phi$ as an abbreviation for $\phi \to \perp$ and we will use shorter names for schemata defined by formulas (1), (2), and (3) – *weakening* schema, *distributivity* schema, and *double negation* schema, accordingly.

We establish upper polynomial bounds on the size of minimal proofs in two fragments of propositional logic. The first fragment, called *distributivity logic*, is the set of all sequences $\Gamma \vdash \phi$ such that propositional formula $\phi$ is derivable from the finite set of hypothesis $\Gamma$ using Modus Ponens inference rule and instances of distributivity schema only. Note that in the absence of the weakening schema the deduction theorem is not valid, so sequent $\Gamma \vdash \phi$ is not, generally speaking, equivalent to implication $\bigwedge\Gamma \to \phi$. In Theorem 5, we give an upper bound on the size of minimal proof of sequent $\Gamma \vdash \phi$ in the distributivity logic. The bound is a polynomial function of the combined size of $\Gamma$ and $\phi$. This upper bound is established by proving a variation of the subformula property for distributivity logic. Namely, we show that if formula $\phi$ is provable from set $\Gamma$ in the distributivity logic, then there is a derivation of $\phi$ from $\Gamma$ which is only using formulas $\psi_1 \to (\psi_2 \to \psi_3)$, where $\psi_1$, $\psi_2$, and $\psi_3$ are subformulas of $\Gamma \cup \{\phi\}$.

Note that this result could be interpreted as a statement about proof complexity in the (full) propositional logic. Namely, any proof in propositional logic of formula $\phi$ could be considered as a proof of sequent $\Gamma \vdash \phi$ in the distributivity logic, where set $\Gamma$ consists of all instances of weakening and double negation schemata used in the proof. Thus, if a formula $\phi$ is provable in the propositional logic, then the size of a minimal proof of formula $\phi$ has an upper bound, that is a polynomial function of size of formula $\phi$ and the total size of all instances of weakening and double negation schemata required to derive $\phi$.

The second fragment of the propositional logic, that we consider, is *weakening and double negation logic*. It is defined as the set of all sequences $\Gamma \vdash \phi$ such that propositional formula $\phi$ is derivable from the finite set of hypothesis $\Gamma$ using Modus Ponens inference rules and instances of weakening and double negation schemata only. If set $\Gamma$ is consistent in the weakening and double negation logic, then we can give a polynomial upper bound on the size of the minimal proof of $\Gamma \vdash \phi$ (Theorem 7). Thus, we can say that if a formula $\phi$ is provable in (full) propositional logic, then the size of a minimal proof of formula $\phi$ has an upper bound, which is a polynomial function of size of $\phi$ and the total size of all instances of distributivity schema required to derive $\phi$. The upper bound for the weakening and double negation logic is also established using an appropriate version of the subformula property.

Although the question remains open whether there are propositional proofs that can not be reduced to simpler proofs that will have polynomial bound on their size, the results, presented in this paper, show that such proofs would have to use a mix of distributivity axioms on one hand and weakening and double negation axioms on the other. In fact, total sizes of required axioms from each of these two groups would have to have no polynomial upper bounds.

## 2 Syntax

### 2.1 Formula

By a propositional formula we mean a well-formed expression built from the propositional variables, the constant $\bot$, and the binary connective $\rightarrow$. By $\sqsubseteq$ we mean *subformula* relation on formulas. This relation defines partial order without a least element on the set of all formulas. For technical reasons, we add the least element to this partial order. It will be called *the empty formula* and denoted by symbol $\varepsilon$. Thus, formula $\varepsilon$ is a subformula of any other formula. Note that $\varepsilon$ is the only "non-standard" formula added to the syntax. For instance, expression $\varepsilon \rightarrow \varepsilon$ is not considered to be a valid formula. The Greek letters, other than $\varepsilon$, will be used to denote only nonempty formulas.

We say that formula $x$ is a *proper subformula* of formula $y$, denoted by $x \sqsubset y$, if $x \sqsubseteq y$ and $x \neq y$. For an arbitrary set of formulas $Y$, we write $x \sqsubseteq Y$ if $x \sqsubseteq y$ for at least one $y \in Y$. Similarly, we write $x \sqsubset Y$ if $x \sqsubset y$ for at least one $y \in Y$.

**Definition 1** The size $|x|$ of a propositional formula $x$ is the number of connectives $\rightarrow$ in formula $x$.

### 2.2 Proof

**Definition 2** A proof of formula $\phi$ from a finite set of propositional formulas $\Gamma$ is a finite sequence of non-empty formulas $\phi_1, \ldots, \phi_k$ such that $\phi_k = \phi$ and each formula in the sequence is either an element of set $\Gamma$, or is an axiom, or is obtained by an inference rule from one or several formulas with lower indices.

We write $\Gamma \vdash_\pi \phi$ if $\pi$ is a proof of formula $\phi$ from set $\Gamma$.

By tree proof we mean a proof in which every formula, except for the last one, is used as an argument in exactly one inference rule application. Tree proofs also can be viewed as tree graphs, in which each node has an associated formula. A formula associated with a leaf node, is either an element of $\Gamma$ or an axiom. The formula associated with a non-leaf node is obtained by inferences rules from formulas associated with children nodes.

**Definition 3** The size $|\pi|$ of a proof $\pi = \phi_1, \dots, \phi_k$ is $|\phi_1| + \cdots + |\phi_k|$.

## 3 Distributivity Logic

The axioms of *distributivity logic*, $L_d$, are instances of distributivity schema: $(\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi))$, the inference rule of this logic is Modus Ponens $(MP)$.

**Theorem 1** *The following two distributivity inference rules:*

$$\frac{\phi \to (\psi \to \chi)}{(\phi \to \psi) \to (\phi \to \chi)}\ (D_1) \qquad \frac{\phi \to \psi, \quad \phi \to (\psi \to \chi)}{\phi \to \chi}\ (D_2)$$

*are derivable in the distributivity logic.*

*Proof* Rule $(D_1)$ is a combination of distributivity axiom and a single instance of the Modus Ponens rule. The rule $(D_2)$ is a combination of $(D_1)$ and Modus Ponens rules.

For the rest of the paper, the system $L_d$ is augmented to include $(D_1)$ and $(D_2)$ as primitive rules of inference. This can be done without loss of generality by Theorem 1 and since addition of these rules can decrease minimal proof length and size by only a constant factor.

**Lemma 1** *For any propositional formulas $\phi$, $\psi$, $\chi$, and $\sigma$, if* $\dfrac{\chi \to \phi, \quad \chi \to \psi}{\chi \to \sigma}$ *is a $(D_2)$ inference, then* $\dfrac{\phi, \quad \psi}{\sigma}$ *is an $(MP)$ inference.* □

### 3.1 Projections

**Definition 4** For any propositional formula $x$, we define hypothesis $h(x)$ and conclusion $c(x)$ of formula $x$ as follows:

1. $h(y \to z) = y$, and $c(y \to z) = z$, for any propositional formulas $y$ and $z$.
2. $h(x) = \varepsilon$, and $c(x) = x$ if formula $x$ is the empty formula $\varepsilon$, the constant $\perp$, or a propositional variable.

**Lemma 2** *If $x$ is a propositional formula, then $h(x) \sqsubseteq x$ and $c(x) \sqsubseteq x$. If $x$ is a nonempty formula, then $h(x) \sqsubset x$ and $\varepsilon \sqsubset c(x)$.* □

If formula $\phi$ has form $\phi_1 \to (\phi_2 \to \phi_3)$, then formulas $\phi_1$, $\phi_2$, $\phi_3$ will be called *the first*, *the second*, and *the third projections* of formula $\phi$. In general, the three projections are defined as follows:

**Definition 5** For an arbitrary formula x, let $pr_1(x) = h(x)$, $pr_2(x) = h(c(x))$, and $pr_3(x) = c(c(x))$.

**Lemma 3** *If $\phi$ is a nonempty propositional formula, then $pr_1(\phi) \sqsubset \phi$, $pr_2(\phi) \sqsubset \phi$, and $\varepsilon \sqsubset pr_3(\phi) \sqsubseteq \phi$.*

*Proof* By Lemma 2, $pr_1(\phi) = h(h(\phi)) \sqsubseteq h(\phi) \sqsubset \phi$. In addition, by the same lemma, $pr_2(\phi) = h(c(\phi)) \sqsubset c(\phi) \sqsubseteq \phi$ and $\varepsilon \sqsubset c(c(\phi)) = pr_3(\phi) = c(c(\phi)) \sqsubseteq c(\phi) \sqsubseteq \phi$.

**Definition 6** For any proof $\pi$, by $Pr_1(\phi)$ we mean the set of the hypotheses of all formulas in proof $\pi$. That is: $Pr_1(\pi) = \{h(\phi) \mid \phi \in \pi\}$.

**Definition 7** For any set of propositional formulas $\Phi$,

$$C(\Phi) = \{\phi_1 \to \phi_3 \mid \phi_1 \to (\phi_2 \to \phi_3) \in \Phi\}.$$

**Theorem 2** *If $\Gamma \vdash_\pi \phi$, then $pr_3(\phi) \sqsubseteq \Gamma \cup C(Pr_1(\pi))$.*

*Proof* Induction on the size of the derivation. If $\phi \in \Gamma$, then, by Lemma 3, we have $pr_3(\phi) \sqsubseteq \phi \in \Gamma$. Hence, $pr_3(\phi) \sqsubseteq \Gamma$. If formula $\phi$ is a distributivity axiom $(\phi_1 \to (\phi_2 \to \phi_3)) \to ((\phi_1 \to \phi_2) \to (\phi_1 \to \phi_3))$, then $pr_3(\phi) = \phi_1 \to \phi_3 \in C(Pr_1(\pi))$, because $\phi_1 \to (\phi_2 \to \phi_3) = pr_1(\phi) \in Pr_1(\pi)$.

Assume that $\phi$ is obtained by Modus Ponens from formulas $\psi$ and $\psi \to \phi$. By Lemma 2, $pr_3(\phi) = c(c(\phi)) = c(c(c(\psi \to \phi))) = c(pr_3(\psi \to \phi)) \sqsubseteq pr_3(\psi \to \phi)$. Thus, by the Induction Hypothesis, $pr_3(\phi) \sqsubseteq pr_3(\psi \to \phi) \sqsubseteq \Gamma \cup C(Pr_1(\pi))$. Therefore, $pr_3(\phi) \sqsubseteq \Gamma \cup C(Pr_1(\pi))$.

Let formula $\phi = (\phi_1 \to \phi_2) \to (\phi_1 \to \phi_3)$ be obtained by rule $(D_1)$ from formula $\phi_1 \to (\phi_2 \to \phi_3)$. Thus, by the Induction Hypothesis, $pr_3(\phi) = \phi_3 = pr_3(\phi_1 \to (\phi_2 \to \phi_3)) \sqsubseteq \Gamma \cup C(Pr_1(\pi))$. Finally, suppose that $\phi = \phi_1 \to \phi_3$ is derived from $\phi_1 \to \phi_2$ and $\phi_1 \to (\phi_2 \to \phi_3)$ using rule $(D_2)$. Then, by Lemma 2, $pr_3(\phi) = c(c(\phi_1 \to \phi_3)) = c(\phi_3) \sqsubseteq \phi_3$. Therefore, by the Induction Hypothesis, $pr_3(\phi) \sqsubseteq \phi_3 = pr_3(\phi_1 \to (\phi_2 \to \phi_3)) \sqsubseteq \Gamma \cup C(Pr_1(\pi))$.

**Lemma 4** *If $\phi \sqsubset C(\Phi)$, then $\phi \sqsubset \Phi$.*

*Proof* Assume that $\phi \sqsubset C(\Phi)$. Thus, $\phi \sqsubset \phi_1 \to \phi_3$ for some $\phi_1 \to (\phi_2 \to \phi_3) \in \Phi$. Hence, $\phi \sqsubseteq \phi_1$ or $\phi \sqsubseteq \phi_3$. Therefore, $\phi \sqsubset \Phi$.

**Theorem 3** *If $\Gamma \vdash_\pi \phi$ and $\phi$ is not an axiom, then $pr_2(\phi) \sqsubset \Gamma \cup Pr_1(\pi)$.*

*Proof* Suppose $\phi \in \Gamma$. Thus, by Lemma 3, $pr_2(\phi) \sqsubset \phi \in \Gamma$. Hence, $pr_2(\phi) \sqsubset \Gamma$.

Assume that $\phi$ is obtained by Modus Ponens from formulas $\psi$ and $\psi \to \phi$. By Lemma 2, $pr_2(\phi) = h(c(\phi)) \sqsubset c(\phi) = c(c(\psi \to \phi)) = pr_3(\psi \to \phi)$. By Theorem 2, $pr_3(\psi \to \phi) \sqsubseteq \Gamma \cup C(Pr_1(\pi))$. Hence, $pr_2(\phi) \sqsubset \Gamma \cup C(Pr_1(\pi))$. Therefore, by Lemma 4, $pr_2(\phi) \sqsubset \Gamma \cup Pr_1(\pi)$.

Let formula $\phi = (\phi_1 \to \phi_2) \to (\phi_1 \to \phi_3)$ be obtained by rule $(D_1)$ from formula $\phi_1 \to (\phi_2 \to \phi_3)$. Thus, $pr_2(\phi) = \phi_1 \sqsubset \phi_1 \to \phi_2 = pr_1(\phi) \in Pr_1(\pi)$.

Finally, suppose that formula $\phi = \phi_1 \to \phi_3$ is derived from formula $\phi_1 \to \phi_2$ and formula $\phi_1 \to (\phi_2 \to \phi_3)$ using rule $(D_2)$. By Lemma 2, $pr_2(\phi) = h(c(\phi)) = h(c(\phi_1 \to \phi_3)) = h(\phi_3) \sqsubset \phi_3 = pr_3(\phi_1 \to (\phi_2 \to \phi_3))$. By Theorem 2, $pr_3(\phi_1 \to (\phi_2 \to \phi_3)) \sqsubseteq \Gamma \cup C(Pr_1(\pi))$. Thus, $pr_2(\phi) \sqsubset \Gamma \cup C(Pr_1(\pi))$. Hence, by Lemma 4, $pr_2(\phi) \sqsubset \Gamma \cup Pr_1(\pi)$.

### 3.2 Proof tree rank

Let $\Gamma$ be a set of propositional formulas and $\phi$ be a propositional formula provable from $\Gamma$. We will define rank of any proof tree that derives formula $\phi$ from set $\Gamma$. Later, we will prove a theorem by induction on the proof tree rank.

**Definition 8** Consider an arbitrary proof of $\phi$ from $\Gamma$. A formula $\psi$ in this proof is called excessive if $pr_1(\psi) \not\sqsubseteq \Gamma \cup \{\phi\}$. An excessive formula is called extreme if its first projection has the maximal size among first projections of all excessive formulas in the proof.

**Definition 9** The rank of a proof tree is a pair of natural numbers $\langle s, n\rangle$, where $s$ is the size of the first projection of extreme formulas in the proof tree and $n$ is the number of the extreme formulas in the proof tree. If the proof tree has no excessive formulas, then its rank is $\langle 0, 0\rangle$.

We will assume the lexicographical order on the proof tree ranks. The set of all ranks is linearly ordered by this relation.

### 3.3 $\alpha$-reductions and canonical proofs

**Definition 10** An $\alpha_1$-reduction is a transformation of a proof tree, that replaces an instance of Modus Ponens rule, whose second argument is a distributivity axiom:

$$\frac{\dfrac{\mathscr{T}}{\phi \to (\psi \to \chi)}, \qquad (\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi))}{(\phi \to \psi) \to (\phi \to \chi)}\ (MP),$$

with a single instance of $(D_1)$ rule:

$$\frac{\dfrac{\mathscr{T}}{\phi \to (\psi \to \chi)}}{(\phi \to \psi) \to (\phi \to \chi)}\ (D_1).$$

**Definition 11** An $\alpha_2$-reduction is a transformation of a proof tree, that replaces an instance of Modus Ponens rule, whose second argument is obtained by $(D_1)$ rule:

$$\frac{\dfrac{\mathscr{T}_1}{\phi \to \psi} \qquad \dfrac{\dfrac{\mathscr{T}_2}{\phi \to (\psi \to \chi)}}{(\phi \to \psi) \to (\phi \to \chi)}\ (D_1)}{\phi \to \chi}\ (MP),$$

with a single instance of $(D_2)$ rule:

$$\cfrac{\cfrac{\mathscr{T}_1}{\phi \to \psi} \qquad \cfrac{\mathscr{T}_2}{\phi \to (\psi \to \chi)}}{\phi \to \chi}\ (D_2).$$

By an $\alpha$-reduction we mean an $\alpha_1$- or an $\alpha_2$-reduction.

**Lemma 5** *An $\alpha$-reduction does not increase the rank of a proof tree.*

*Proof* Both $\alpha$-reductions do not add any new formulas to the proof tree.

**Definition 12** A proof tree is in an $\alpha$-canonical form if it does not contain instances of Modus Ponens rule that can be eliminated using an $\alpha$-reduction.

**Lemma 6** *Any proof tree can be transformed to an $\alpha$-canonical form by a finite number of $\alpha$-reductions.*

*Proof* Each $\alpha$-reduction reduces the number of nodes in the proof tree. Thus, they can not be applied more times than the number of nodes in the original proof tree.

### 3.4 $D_2$-trees

**Definition 13** A node of a proof tree is called irregular if it is obtained by Modus Ponens inference rule.

**Definition 14** For any irregular node $n$ of a proof tree, we define left $D_2$-tree and right $D_2$-tree grounded at node $n$ as the minimal sets of nodes that satisfy the following conditions:

1. The left $D_2$-tree contains the node representing the first argument of the Modus Ponens rule used to derive $n$ and the right $D_2$-tree contains the node representing the second argument of the same rule.
2. If one of the $D_2$-trees contains a node derived by $(D_2)$ rule, then the same $D_2$-tree contains two nodes representing the arguments of this rule.

**Lemma 7** *All formulas in any $D_2$-tree have the same first projection.*

*Proof* The conclusion and both arguments of any instance of $(D_2)$ rule have the same first projection.

**Definition 15** A $D_2$-tree in a proof tree is called extreme if at least one formula in this $D_2$-tree is extreme.

**Lemma 8** *Any formula in an extreme $D_2$-tree is an extreme formula.*

*Proof* See Lemma 7.

**Lemma 9** *Any extreme $D_2$-tree is a right $D_2$-tree.*

*Proof* We will show that the first projection of formulas in the left $D_2$-tree of any node is smaller than the first projection of formulas in the right $D_2$-tree of the same node. Indeed, let irregular node be obtained by $(MP)$ rule from nodes labeled with formulas $\phi$ and $\phi \to \psi$. By Lemma 7, all formulas in the left $D_2$-tree have first projection $pr_1(\phi)$ and all formulas in the right $D_2$-tree have first projection $\phi$. We are left to notice that, by Lemma 3, $pr_1(\phi) \sqsubset \phi$. Therefore, $|pr_1(\phi)| < |\phi|$.

**Lemma 10** *If the formula, associated with a node $n_0$ of a proof tree, is extreme, then node $n_0$ belongs to a $D_2$-tree.*

*Proof* Consider the path $n_0, n_1, n_2, \dots$ in the proof tree that starts at node $n_0$ and leads towards the root of the proof tree. Let $\psi_1, \psi_2, \dots$ be formulas associated with these nodes. Assume that $n_k$ is the first node along this path that is *not* an argument of an instance of the $(D_2)$ rule. Note that propositional formulas $\psi_0, \psi_1, \dots, \psi_k$ have the same first projection. Since node $n_k$ is not an argument of an $(D_2)$ rule instance, one of the following cases takes place:
*Case 1*: Node $n_k$ is the root of the proof tree. Thus, $\psi_k = \phi$. Hence, $pr_1(\psi_0) = pr_1(\psi_k) \sqsubseteq \phi$. Therefore, formula $\psi_0$ is not excessive. Contradiction.
*Case 2*: Node $n_k$ is an argument of an instance of $(D_1)$ rule. Hence, $\psi_k = a \to (b \to c)$ and $\psi_{k+1} = (a \to b) \to (a \to c)$ for some propositional formulas $a$, $b$, and $c$. Thus,

$$|pr_1(\psi_{k+1})| = |a \to b| > |a| = |pr_1(\psi_k)| = |pr_1(\psi_0)|.$$

Since $\psi_0$ is an extreme formula, the above inequality implies that formula $\psi_{k+1}$ can *not* be excessive. Thus, $pr_1(\psi_{k+1}) \sqsubseteq \Gamma \cup \{\phi\}$. Hence

$$pr_1(\psi_0) = pr_1(\psi_k) = a \sqsubset a \to b = pr_1(\psi_{k+1}) \sqsubseteq \Gamma \cup \{\phi\}.$$

Therefore, formula $\psi_0$ is not excessive either. Contradiction.
*Case 3*: Node $n_k$ is an argument of an instance of $(MP)$ rule. Therefore, node $n_0$ belongs to a $D_2$-tree grounded at node $n_{k+1}$.

**Lemma 11** *If the proof tree is in an $\alpha$-canonical form, then any of its extreme $D_2$-trees contains more than one node.*

*Proof* Let $n$ be an irregular node of the proof tree $\pi$ derived from nodes labeled with formulas $\chi$ and $\chi \to \psi$. Assume that one of $D_2$-trees, grounded at node $n$, is extreme. By Lemma 9, this must be the right $D_2$-tree. We will show that it contains more than one node.

If $\chi \to \psi \in \Gamma$, then $pr_1(\chi \to \psi) = \chi \sqsubseteq \Gamma$. Thus, $\chi \to \psi$ is not an excessive formula. Therefore, by Lemma 8, the right $D_2$-tree at node $n$ is not extreme. If $\chi \to \psi$ is a distributivity axiom or is obtained by $(D_1)$ rule, then either an $\alpha_1-$ or an $\alpha_2-$ reduction could be applied at node $n$. Hence, the proof tree is not in an $\alpha$-canonical form.

Assume that formula $\chi \to \psi$ is derived by another instance of Modus Ponens rule from formulas $\sigma$ and $\sigma \to (\chi \to \psi)$. Note that $\sigma \to (\chi \to \psi)$ could not be an axiom, because the tree is in an $\alpha$-canonical form. Thus, by Theorem 3, $\chi = pr_2(\sigma \to (\chi \to \psi)) \sqsubset \Gamma \cup Pr_1(\pi)$. Note that $\chi \not\sqsubseteq \Gamma$, because $\chi \to \psi$ is excessive. Hence, $\chi \sqsubset Pr_1(\pi)$. Thus, $\chi \sqsubset pr_1(\tau)$ for some formula $\tau$ in the proof $\pi$. Yet, we know that $\chi \to \psi$ is an extreme formula. Thus, formula $\tau$ is not excessive. Hence

$$\cfrac{\cfrac{\mathscr{T}_0}{\chi} \qquad \cfrac{\cfrac{\cfrac{\mathscr{T}_1}{\chi \to \psi_1} \quad \dots \quad \cfrac{\mathscr{T}_n}{\chi \to \psi_n}}{\dots \ (D_2)\text{-inferences} \ \dots}}{\chi \to \phi}\,(D_2)}{\phi}\,(MP)$$

**Fig. 1** Before $\beta$-reduction

$pr_1(\tau) \sqsubseteq \Gamma \cup \{\phi\}$. Then, $pr_1(\chi \to \psi) = \chi \sqsubset pr_1(\tau) \sqsubseteq \Gamma \cup \{\phi\}$. Consequently, formula $\chi \to \psi$ is not excessive either. Contradiction.

Therefore, formula $\chi \to \psi$ must be derived by $(D_2)$ rule, which means that the right $D_2$-tree grounded at node $n$ contains more than one node.

**Definition 16** Let $T_1$ and $T_2$ be two right $D_2$-trees grounded at nodes $n_1$ and $n_2$ correspondingly. We say that $T_1 \prec T_2$ if node $n_1$ lays on the path that connects $n_2$ with the root of the proof tree.

**Lemma 12** *Relation $\prec$ induces a partial order on the set of all extreme $D_2$-trees of any proof tree.*

*Proof* By Lemma 9, only right $D_2$-trees can be extreme. But any two different right $D_2$-trees can not be grounded at the same node.

**Definition 17** An extreme $D_2$-tree is called critical if it is a maximal, with respect to partial order $\prec$, $D_2$-tree among all extreme $D_2$-trees in the proof tree.

### 3.5 $\beta$-reduction

Suppose (see Figure 1) that formula $\phi$ at an irregular node $n$ of a proof tree is obtained by applying Modus Ponens to formulas $\chi$ and $\chi \to \psi$. Let $\mathscr{T}_0$ be the derivation of formula $\chi$, formulas $\chi \to \psi_1, \dots, \chi \to \psi_n$ be associated with the leaves of the right $D_2$-tree grounded at node $n$, and $\mathscr{T}_1, \dots, \mathscr{T}_n$ be the derivations of these formulas. Note that by Lemma 7, the first projection of any formula in this right $D_2$-tree is $\chi$.

By $\beta$-reduction at node $n$ (see Figure 2) we mean the transformation of the proof that removes hypothesis $\chi$ from all formulas in the $D_2$-tree, replacing each inference of the $(D_2)$ rule in the $D_2$-tree by an instance of a Modus Ponens rule, and moves application of the Modus Ponens with the first argument $\chi$ from node $n$ to all leaves of the former $D_2$-tree. The node $n$ is removed from the proof tree. This reduction could be applied at any, not necessary root, irregular node $n$ of a proof tree.

**Lemma 13** *The result of any $\beta$-reduction is a valid proof tree.*

*Proof* See Lemma 1.

**Lemma 14** *If the right $D_2$-tree grounded at node $n$ is critical, then $\beta$-reduction at node $n$ decreases the rank of the proof tree.*

$$\cfrac{\cfrac{\cfrac{\mathscr{T}_0}{\chi} \quad \cfrac{\mathscr{T}_1}{\chi \to \psi_1}}{\psi_1}\,(MP) \quad \dots \quad \cfrac{\cfrac{\mathscr{T}_0}{\chi} \quad \cfrac{\mathscr{T}_n}{\chi \to \psi_n}}{\psi_n}\,(MP)}{\begin{array}{c} \dots\ \dots\ \dots\ \dots\ \dots\ \dots \\ \dots\ (MP)\text{-inferences}\ \dots \\ \dots\ \dots\ \dots \\ \hline \phi \end{array}}\,(MP)$$

**Fig. 2** After $\beta$-reduction

*Proof* First of all, note that although $\beta$-reduction removes some formulas with the first projection $\chi$, at least one of them is left in the proof tree.

Second, although $\beta$-reduction can introduce a new formula $\psi$ into the proof tree, such formula can not be extreme. Indeed, this formula will have to be obtained by removing hypothesis $\chi$ from formula $\chi \to \psi$ in the $D_2$-tree. If $\chi \to \psi$ is an axiom, then $\chi = \phi_1 \to (\phi_2 \to \phi_3)$ and $\psi = (\phi_1 \to \phi_2) \to (\phi_1 \to \phi_3)$ for some propositional formulas $\phi_1$, $\phi_2$, and $\phi_3$. Hence, $|pr_1(\psi)| = |\phi_1 \to \phi_2| < |\phi_1 \to (\phi_2 \to \phi_3)| = |\chi| = |pr_1(\chi \to \psi)|$. Thus, formula $\psi$ can not be extreme because $\chi \to \psi$ is a larger excessive formula. If $\chi \to \psi$ is not an axiom, then, by Lemma 3, $pr_1(\psi) = pr_2(\chi \to \psi) \sqsubset \Gamma \cup Pr_1(\pi)$. Thus, formula $\psi$ is not excessive.

Third, although $\beta$-reduction can increase number of occurrences of the same formula in the proof tree, it does not increase number of occurrences of extreme formulas. Indeed, the only formulas whose number of occurrences is increased are those that are in proof subtree $\mathscr{T}_0$. But this subtree can not include any extreme formulas because the right $D_2$-tree at node $n$ is critical.

Finally, by Lemma 11, the eliminated $D_2$-tree had more than one node. Thus, it had at least one node which is not a leaf of the $D_2$-tree. This node is modified. Hence, the total number of extreme formulas in the proof tree is decreased.

### 3.6 Upper Bounds

**Theorem 4** *If $\Gamma \vdash_\pi \phi$, then there is a proof $\hat{\pi}$ such that $\Gamma \vdash_{\hat{\pi}} \phi$ and $pr_1(\psi) \sqsubseteq \{\phi\} \cup \Gamma$ for any $\psi \in \hat{\pi}$.*

*Proof* Induction on the rank of proof tree $\pi$. If rank is $\langle 0, 0\rangle$ then the proof has no extreme formulas. Hence, it has no excessive formulas. Thus, $pr_1(\psi) \sqsubseteq \{\phi\} \cup \Gamma$ for any any formula $\psi$ in proof $\pi$.

Assume that the rank of proof tree $\pi$ is not $\langle 0, 0\rangle$. By Lemma 6, the proof tree can be transformed into a canonical proof $\pi'$ without increase of the rank. If proof tree $\pi'$ has no excessive formulas, then $pr_1(\psi) \sqsubseteq \{\phi\} \cup \Gamma$ for any formula $\psi$ in proof $\pi'$. If proof tree $\pi'$ does have some excessive formulas, then it has at least one extreme formula. By Lemma 10, this formula is a part of an extreme $D_2$-tree. Thus, the proof tree contains at least one extreme $D_2$-tree. Consider a critical extreme $D_2$-tree. Let it be grounded at node $n$. By Lemma 9, this critical tree is a right $D_2$-tree grounded at node $n$. Apply $\beta$-reduction at node $n$. By Lemma 14, the resulting proof tree has smaller rank. Therefore, by the Induction Hypothesis, it can be transformed into proof $\hat{\pi}$ of formula $\phi$ from set $\Gamma$, such that $pr_1(\psi) \sqsubseteq \{\phi\} \cup \Gamma$ for any $\psi \in \hat{\pi}$.

**Theorem 5** *If $\Gamma \vdash \phi$, then there is a proof $\pi$ such that $\Gamma \vdash_\pi \phi$, $length(\pi) \in O((|\Gamma|+|\phi|)^3)$, and $|\pi| \in O((|\Gamma|+|\phi|)^4)$.*

*Proof* Let $\pi$ be the proof of $\phi$ form $\Gamma$ such that $pr_1(\psi) \sqsubseteq \{\phi\} \cup \Gamma$ for any $\psi \in \pi$. This proof exists by Theorem 4. We will assume that repeating occurrences of the same formula are eliminated from proof $\pi$. Each formula in the proof could be completely described by specifying its first, second, and third projection. The first and the second projections could be empty formulas.

By the choice of proof $\pi$, first projections are limited to subformulas of $\Gamma \cup \{\phi\}$. Thus, the number of such projections and the size of each of them belong to $O(|\Gamma|+|\phi|)$.

By Theorem 3, each second projection is either a subformula of a formula from $\Gamma \cup Pr_1(\pi)$ or the second projection of an axiom. In other words, each second projection is a subformula of a formula from set

$$\Gamma \cup Pr_1(\pi) \cup \{\phi_1 \to \phi_2 \mid \phi_1 \to (\phi_2 \to \phi_3) \in Pr_1(\pi)\}.$$

By the choice of $\pi$, the number of such subformulas and the individual size of each of them belong to $O(|\Gamma|+|\phi|)$. Finally, by Theorem 2 and the choice of $\pi$, the number of the third projections and the individual size of each of them are also in $O(|\Gamma|+|\phi|)$.

Thus, the total number of formulas in the proof $\pi$ is $O((|\Gamma|+|\phi|)^3)$ and the size of each formula in $\pi$ is $O(|\Gamma|+|\phi|)$. Therefore, $|\pi| \in O((|\Gamma|+|\phi|)^4)$.

**Corollary 1** *If formula $\phi$ is provable in the classical propositional logic using only instances $W_1,\ldots,W_n$ of weakening schema and only instances $N_1,\ldots,N_k$ of double negation schema, then there is a proof $\pi$ in the classical propositional logic of $\phi$ such that*

$$length(\pi) \in O((\sum_{i=1}^{n}|W_i| + \sum_{i=1}^{k}|N_i| + |\phi|)^3)$$

*and*

$$|\pi| \in O((\sum_{i=1}^{n}|W_i| + \sum_{i=1}^{k}|N_i| + |\phi|)^4).$$

## 4 Weakening and Double Negation Logic

### 4.1 Axioms

The axioms of the *logic of weakening*, $L_w$, are instances of weakening schema $\phi \to (\psi \to \phi)$. The inference rule of $L_w$ is Modus Ponens. The following weakening rule

$$\frac{\phi}{\psi \to \phi}\ (W)$$

is derivable in $L_w$. We will treat it as a primitive rule of the system. Addition of this rule can decrease the minimal proof size and length only by a constant factor.

The *logic of weakening and double negation*, $L_{wn}$, in addition to weakening axiom and Modus Ponens inference rule, also includes the double negation schema $((\phi \to \bot) \to \bot) \to \phi$. The weakening rule $(W)$ and following double negation rule

$$\frac{(\phi \to \bot) \to \bot}{\phi}\ (N)$$

are derivable in $L_{wn}$. We will treat them both as primitive rules of the system. Addition of these rules can decrease the minimal proof size and length only by a constant factor. We say that a finite set of propositional formulas $\Gamma$ is $L_{wn}$-consistent if sequent $\Gamma \vdash \phi$ is not derivable in this logic.

In this section, unless stated otherwise, by proof we will mean a proof in either logic $L_w$ or logic $L_{wn}$.

## 4.2 $\gamma$-reductions

**Definition 18** A $\gamma_1$-reduction is a transformation of a proof tree, that replaces an instance of Modus Ponens rule, whose second argument is a weakening axiom:

$$\frac{\frac{\mathscr{T}}{\phi}, \qquad \phi \to (\psi \to \phi)}{\psi \to \phi}\ (MP),$$

with a single instance of $(W)$ rule:

$$\frac{\frac{\mathscr{T}}{\phi}}{\psi \to \phi}\ (W).$$

**Definition 19** A $\gamma_2$-reduction is a transformation of a proof tree, that replaces an instance of Modus Ponens rule, whose second argument is a double negation axiom:

$$\frac{\frac{\mathscr{T}}{(\phi \to \bot) \to \bot}, \qquad ((\phi \to \bot) \to \bot) \to \phi}{\phi}\ (MP),$$

with a single instance of $(N)$ rule:

$$\frac{\frac{\mathscr{T}}{(\phi \to \bot) \to \bot}}{\phi}\ (N).$$

**Definition 20** A $\gamma_3$-reduction is a transformation of a proof tree, that eliminates an instance of Modus Ponens rule, whose second argument is obtained by $(W)$ rule:

$$\frac{\frac{\mathscr{T}_1}{\psi} \qquad \frac{\frac{\mathscr{T}_2}{\phi}}{\psi \to \phi}\ (W)}{\phi}\ (MP),$$

by replacing it with the argument of the $(W)$ rule:

$$\frac{\mathscr{T}_2}{\phi}.$$

By a $\gamma$-reduction we mean a $\gamma_1$-, a $\gamma_2$-, or a $\gamma_3$-reduction.

**Definition 21** A proof tree is in $\gamma$-canonical form if it does not contain instances of Modus Ponens rule that can be eliminated using a $\gamma$-reduction.

**Lemma 15** *Any proof tree can be transformed to a $\gamma$-canonical form by a finite number of $\gamma$-reductions.*

*Proof* Each $\gamma$-reduction reduces the number of nodes in the proof tree. Thus, they can not be applied more times than the number of nodes in the original proof tree.

## 4.3 Pivot Point

**Definition 22** The right-most path in a proof tree is the sequence of formulas $\phi_1, \ldots, \phi_n$, associated with the nodes of the path of the proof tree that starts at the right-most leaf and goes down to the root of the tree.

Thus, if the proof tree is a proof of formula $\phi$ from set $\Gamma$, then $\phi_1$ is either an element of $\Gamma$ or an axiom, each $\phi_i$, for $i > 0$, is obtained from $\phi_{i-1}$ by an inference rule, and $\phi_n = \phi$.

**Lemma 16** *Let $\phi_1, \ldots, \phi_n$ be the right-most path in a $\gamma$-canonical proof. If formula $\phi_i$, for $0 < i < n$, is derived by $(W)$ rule, then formula $\phi_{i+1}$ can not be derived by Modus Ponens.*

*Proof* Assume that $\phi_{i+1}$ is obtained by an instance of Modus Ponens rule whose second argument is $\phi_i$ and formula $\phi_i$, in turn, is obtained by $(W)$ inference rule. It means that $\gamma_3$-reduction can be applied at the node labeled with formula $\phi_{i+1}$. Thus, the proof is not in a $\gamma$-canonical form. Contradiction.

**Lemma 17** *Let $\Gamma$ be a $L_{wn}$-consistent set of formulas and $\phi_1, \ldots, \phi_n$ be the right-most path in a proof tree from $\Gamma$. If formula $\phi_i$, for $0 < i < n$, is derived by $(W)$ rule, then formula $\phi_{i+1}$ can not be derived by $(N)$ rule.*

*Proof* Assume that formula $\phi_{i+1}$ is derived by $(N)$ rule. Thus, formula $\phi_i$ is equal to $(\phi_{i+1} \to \bot) \to \bot$. If formula $\phi_i$ is derived by rule $(W)$, then it will have to be derived from $\bot$. Therefore, set $\Gamma$ is not $L_{wn}$-consistent.

**Definition 23** The pivot point of the right-most path $\phi_1, \ldots, \phi_n$ in a proof tree is an index $0 \le p \le n$ such that for $0 < i \le p$, formula $\phi_i$ is derived by Modus Ponens or $(N)$ inference rule and for $p < i \le n$, formula $\phi_i$ is derived by $(W)$ inference rule.

**Lemma 18** *(a) The right-most path in a $\gamma$-canonical proof in $L_w$ has a pivot point. (b) If set $\Gamma$ is $L_{wn}$-consistent, then the right-most path in any $\gamma$-canonical proof in $L_{wn}$ has a pivot point.*

*Proof* See Lemma 16 and Lemma 17.

**Lemma 19** *If $p$ is a pivot point of a right-most path $\phi_1,\dots,\phi_n$, then*

$$\phi_1 \sqsupset \phi_2 \sqsupset \cdots \sqsupset \phi_{p-1} \sqsupset \phi_p \sqsubset \phi_{p+1} \sqsubset \cdots \sqsubset \phi_n.$$

*Proof* If $\phi_{i+1}$ is obtained from $\phi_i$ by Modus Ponens or rule $(N)$, then $\phi_i \sqsupset \phi_{i+1}$. If $\phi_{i+1}$ is obtained from $\phi_i$ by rule $(W)$, then $\phi_i \sqsubset \phi_{i+1}$.

**Lemma 20** *Let $\phi_1,\dots,\phi_n$ be the right-most path in a $\gamma$-canonical proof tree of formula $\phi_n$ from set of formulas $\Gamma$. If this right-most chain contains a pivot point, then $\phi_i \sqsubseteq \Gamma \cup \{\phi_n\}$ for any $0 \le i < n$.*

*Proof* If $\phi_1 \in \Gamma$ or $n = 1$, then the statement of the lemma follows from Lemma 19. Assume that $\phi_1$ is an axiom and $n \ge 2$.
*Case 1:* Formula $\phi_2$ is derived from axiom $\phi_1$ using Modus Ponens rule. Thus, either $\gamma_1$-reduction (if $\phi_1$ is a weakening axiom) or $\gamma_2$-reduction (if $\phi_1$ is a double negation axiom) could be applied at the node of the proof tree labeled with formula $\phi_2$. Thus, the proof tree is not $\gamma$-canonical.
*Case 2:* Formula $\phi_2$ is derived from axiom $\phi_1$ using $(N)$ inference rule. Thus, formula $\phi_1$ is equal to $((\phi_2 \to \bot) \to \bot$. Note that such formula could be obtained neither from weakening schema nor from double negation schema. Hence, $\phi_1$ is not an axiom. Contradiction.
*Case 3:* Formula $\phi_2$ is derived from axiom $\phi_1$ using $(W)$ inference rule. Hence, pivot point of the right-most path is 1. Therefore, by Lemma 19, $\phi_i \sqsubset \phi_n$, for any $1 \le i < n$.

**Lemma 21** *For any set $\Delta$ of propositional formulas, any finite subset $\Gamma \subseteq \Delta$, and any formula $\phi \sqsubseteq \Delta$, let $\pi$ be a $\gamma$-canonical proof tree of formula $\phi$ from set $\Gamma$, such that either*

1. *$\pi$ is a proof tree in weakening logic $L_w$, or*
2. *$\pi$ is a proof tree in logic $L_{wn}$ and set $\Gamma$ is $L_{wn}$-consistent.*

*Then every formula in proof tree $\pi$ is a subformula of set $\Delta$.*

*Proof* Induction on the size of the derivation $\pi$. Note that by the assumption, $\phi \sqsubseteq \Delta$. If $\phi$ is the only formula in the proof tree then the required is established.

Suppose that formula $\phi$ is derived using one of the inference rules. Note that, by Lemma 18, the right-most path in the proof tree has a pivot point. Thus, by Lemma 20, any formula on the right-most path is a subformula of $\Gamma \cup \{\phi\}$. Hence, any formula on the right-most path is a subformula of set $\Delta$.
*Case 1:* Formula $\phi$ is derived by either inference rule $(W)$ or inference rule $(N)$ from formula $\psi$. Since formula $\psi$ belongs to the right-most path in the proof tree, we have $\psi \sqsubseteq \Delta$. Thus, by the induction hypothesis, every formula in the derivation of $\psi$ is a subformula of set $\Delta$. Therefore, every formula in $\pi$ is a subformula of set $\Delta$.
*Case 2:* Formula $\phi$ is derived by Modus Ponens from formulas $\psi$ and $\psi \to \phi$. Since formula $\psi \to \phi$ belongs to the right-most path in the proof tree, $\psi \to \phi \sqsubseteq \Delta$. Hence, $\psi \sqsubseteq \Delta$. By the induction hypothesis, every formula in the derivations of $\psi$ and $\psi \to \phi$ is a subformula of set $\Delta$. Therefore, every formula in $\pi$ is a subformula of set $\Delta$.

**Theorem 6** *If $\Gamma \vdash \phi$ in $L_w$, then there is a proof $\pi$ of $\phi$ from $\Gamma$ in $L_w$ such that $length(\pi) \in O(|\Gamma| + |\phi|)$ and $|\pi| \in O((|\Gamma| + |\phi|)^2)$.*

*Proof* Assume that $\Gamma \vdash \phi$. By Lemma 15, there is a $\gamma$-canonical proof tree of $\phi$ from $\Gamma$. By Lemma 21, each formula in this proof tree is a subformula of $\Delta = \Gamma \cup \{\phi\}$. By eliminating all repeating occurrences of formulas in the tree proof, the proof can be converted to a proof $\pi$ of length no more than the number of subformulas in $\Gamma \cup \{\phi\}$. In other words, $length(\pi) \in O(|\Gamma| + |\phi|)$. Since every formula in the proof $\pi$ is a subformula of $\Gamma \cup \{\phi\}$, the total size of the proof $\pi$ is $O((|\Gamma| + |\phi|)^2)$.

**Theorem 7** *If $\Gamma \vdash \phi$ in $L_{wn}$ and set $\Gamma$ is $L_{wn}$-consistent, then there is a proof $\pi$ of $\phi$ from $\Gamma$ in $L_{wn}$ such that $length(\pi) \in O(|\Gamma| + |\phi|)$ and $|\pi| \in O((|\Gamma| + |\phi|)^2)$.*

*Proof* Identical to the proof of Theorem 6.

**Corollary 2** *If formula $\phi$ is provable in the classical propositional logic using only instances $D_1, \ldots, D_n$ of distributivity schema, then there is a proof $\pi$ in the classical propositional logic of $\phi$ such that $length(\pi) \in O(\sum_{i=1}^{n} |D_i| + |\phi|)$ and $|\pi| \in O((\sum_{i=1}^{n} |D_i| + |\phi|)^2)$.*

*Proof* Consistency of the propositional logic implies that set $\{D_1, \ldots, D_n\}$ is $L_{wn}$-consistent.

## 5 Conclusions

We have established upper polynomial bounds on the proof complexity of two fragments of the classical propositional logic, defined by limiting the use of certain axiom schemata. Both results are obtained by finding valid versions of the subformula property for these fragments.

These results could be viewed as a step towards establishing a hypothetical polynomial upper bound on proof complexity in the full propositional logic. The next logical step in this direction is finding upper polynomial bounds for richer fragments of the logic using a variation of subformula property or some other technique.

## References


1. Bonet, M., Pitassi, T., Raz, R.: Lower bounds for cutting planes proofs with small coefficients. J. Symbolic Logic **62**(3), 708–728 (1997)
2. Buss, S.R.: Polynomial size proofs of the propositional pigeonhole principle. J. Symbolic Logic **52**(4), 916–927 (1987)
3. Buss, S.R., Pudlák, P.: On the computational content of intuitionistic propositional proofs. Ann. Pure Appl. Logic **109**(1-2), 49–64 (2001). Dedicated to Petr Vopěnka
4. Church, A.: Introduction to mathematical logic. Vol. I. Princeton University Press, Princeton, N. J. (1956)

5. Cook, W., Coullard, C.R., Turán, G.: On the complexity of cutting-plane proofs. Discrete Appl. Math. **18**(1), 25–38 (1987)
6. Frege, G.: Begriffsschrift, a formal language, modeled upon that of arithmetic, for pure thought. In: J. van Heijenoort (ed.) From Frege to Gödel, pp. 1–82. Harvard University Press, Cambridge, MA (2002). A source book in mathematical logic, 1879–1931, Reprint of the third printing of the 1967 original
7. Gödel, K.: Letter to John von Neumann, March 20th, 1956. In: Kurt Gödel: collected works. Vol. V, pp. 372–377. The Clarendon Press Oxford University Press, Oxford (2003). Correspondence H–Z, Edited by Solomon Feferman, John W. Dawson, Jr., Warren Goldfarb, Charles Parsons, and Wilfried Sieg
8. Haken, A.: The intractability of resolution. Theoret. Comput. Sci. **39**(2-3), 297–308 (1985)
9. Krajíček, J., Pudlák, P., Woods, A.: An exponential lower bound to the size of bounded depth Frege proofs of the pigeonhole principle. Random Structures Algorithms **7**(1), 15–39 (1995)
10. Pitassi, T., Beame, P., Impagliazzo, R.: Exponential lower bounds for the pigeonhole principle. Comput. Complexity **3**(2), 97–140 (1993)
11. Pudlák, P.: Lower bounds for resolution and cutting plane proofs and monotone computations. J. Symbolic Logic **62**(3), 981–998 (1997)