Pavel Naumov

On Modal Logics of Partial Recursive Functions

Abstract. The classical propositional logic is known to be sound and complete with respect to the set semantics that interprets connectives as set operations. The paper extends propositional language by a new binary modality that corresponds to partial recursive function type constructor under the above interpretation. The cases of deterministic and non-deterministic functions are considered and for both of them semantically complete modal logics are described and decidability of these logics is established.

Keywords: modal logic, recursive function, Curry-Howard isomorphism

Introduction

We are interested in the use of logical connectives to describe properties of the set and type operations. Historically, there have been two major ways to interpret logical connectives as such operations: Curry-Howard isomorphism and set semantics.

Under Curry-Howard isomorphism ([3, 4, 5, 8]), propositional formulas are interpreted as types and connectives $\wedge, \vee$, and $\rightarrow$ are interpreted as Cartesian product, disjoint union, and constructive function type constructors. It can be shown that a formula is provable in intuitionistic propositional logic if and only if it is always evaluated to an inhabited type. Thus, intuitionistic logic could be viewed as a calculus that describes properties of Cartesian product, disjoint union, and function type constructors.

Since the list of possible type constructors is not limited to just the trio of product, disjoint union, and function, one can raise a question about logical principles describing behavior of other type constructors. For example, list, partial object [16] and squash [1] types can be viewed as modalities while inductive and co-inductive constructors ([11] and [2]) may be considered as quasi-quantifiers. In fact, [9] established that modal logic of squash operator is Lax Logic [6].

According to the set semantics, every propositional formula is evaluated to a subset of a given universe U and propositional connectives conjunction $\wedge$, disjunction $\vee$, and negation $\neg$ are identified with set operations intersection $\cap$, union $\cup$, and complement $\complement_U$, correspondingly. It is easy to see that a formula is provable in the classical propositional logic if and only if it is evaluated to the entire universe U under any interpretation of propositional variables.

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Several possible extensions of the classical logic by modal operators corresponding, under the above set semantics, to additional set operations have been considered. [10] established that if the universe U is a topological space, then modal logic S4 describes properties of the interior operator. If the universe U is the set of all words in some alphabet, then properties of the logical connectives corresponding to product and star operations are axiomatized by Interval Temporal Logic [12]. In [13, 14], the author describes an extension of the classical propositional logic by binary modalities, corresponding to the operations disjoint union and Cartesian product.

This paper considers an extension of the classical propositional logic by a binary modality $\rhd$, corresponding to computable function type constructor. Namely, if U is the universe of all words in some alphabet, then $(\phi \rhd \psi)^*$ is the set of all Turing machine descriptions of partial recursive functions from ϕ^* into ψ^*. We consider cases of deterministic and nondeterministic Turing machines. For both of them complete Hilbert-style axiomatizations of the appropriate modal logics is given. It turns out that modal logic of deterministic functions $\Re_d$ is an extension of the modal logic of nondeterministic functions $\Re$ by just one additional axiom.

The modality $\phi \rhd \psi$ of the logics of partial recursive functions is, essentially, a form of Hoare triple $\phi\{\alpha\}\psi$ with a fixed program variable α. Thus, there is some similarity between modal logics of recursive functions and the dynamic logic [7]. For example, introduced below axiom of logic $\Re$: $\phi \rhd \psi \to (\chi \rhd \psi \to (\phi \vee \chi) \rhd \psi)$ could be related to dynamic logic theorem $\phi\{\alpha\}\psi \to (\chi\{\alpha\}\psi \to (\phi \vee \chi)\{\alpha\}\psi)$. This similarity, however, ends once iterative applications of the modality are considered. For example, formula $(\top\{\alpha\}\phi)\{\alpha\}\phi$ is also a theorem of the dynamic logic but modal formula $(\top \rhd \phi) \rhd \phi$ is valid neither in $\Re$ nor in $\Re_d$.

This paper focuses on soundness and completeness of logics $\Re$ and $\Re_d$ with respect to the class of partial recursive functions. As one can expect, the results can be easily relativized by an oracle. It is worth mentioning, though, that soundness and completeness proofs, presented in this paper, could also be adopted for some subclasses of the class of partial recursive functions such as, for example, polynomial functions and finite-domain functions. Hence, both of these logics capture very general properties of "complete", in some informal sense, classes of enumerable functions. The downside of this, of course, is that more specific properties of recursive functions are not reflected in these logics. For example, many of the properties of recursive functions captured by the intuitionistic logic under Curry-Howard isomorphism, such as closure under composition, could not be expressed in logics $\Re$ and $\Re_d$. One should think of these logics more as an attempt to reason about functions in (a modal extension of) the classical propositional logic rather than a modal axiomatization of recursiveness. Similarly defined logics of *total* recursive functions, as will be mentioned in the conclusion, would provide a significantly more expressive language. Our investigation of logics $\Re$ and $\Re_d$ could be viewed as a first step towards study of such more expressive logics.

The results for logics $\Re$ and $\Re_d$ will be presented together. In the next section we discuss the definition of the recursive functions and the Kleene recursion theorem on which our completeness results are based. In Section 3, a formal semantics of the modal logics of recursive functions is given. Section 4 lists axioms and inference rules for both logics and verifies their soundness. The rest of the paper is dedicated to the completeness proof. In Section 5, Kripke-style models for $\Re$ and $\Re_d$ are introduced and completeness of these logics with respect to appropriate classes of the Kripke models is proven. In Section 6, in order to finish the proof of the completeness theorem, we show how Kripke models could be converted into sets of partial recursive functions. Decidability of the logics follows from finiteness of the corresponding Kripke models. Section 7 concludes with the discussion of an alternative definition of the logic of nondeterministic partial recursive functions and the logics of total recursive functions.

Recursive functions

We study modal logic descriptions of partial recursive functions. The two classes of recursive functions – deterministic and nondeterministic – will be considered. Nondeterministic partial recursive functions could be described, for example, as nondeterministic Turing machines. Value $f(x)$ of a nondeterministic function f on an argument x is defined as the set of all values that a nondeterministic machine representing f can return on input x. Deterministic partial recursive function is a special case of nondeterministic function whose value is a set that has no more than one element.

We consider an enumeration $\{\xi_u\}_{u\in U}$ of partial recursive functions from a universe U into U by the elements of the same universe U. The two major cases that will be considered are: a) $\{\xi_u\}_{u\in U}$ is an enumeration of all nondeterministic partial recursive functions and b) $\{\xi_u\}_{u\in U}$ is an enumeration of all deterministic partial recursive functions. The exact choice of the universe, such as Turing machines or λ-terms, and the enumeration will not be important as long as the following version of Kleene recursion theorem is satisfied:

THEOREM 1. *For any finite set $f_1, \dots, f_n$ of total recursive functions from U^n to U there are elements $u_1, \dots, u_n \in U$ such that $\xi_{u_i} \equiv \xi_{f_i(u_1,\dots,u_n)}$ for any $0 \le i \le n$.*

Note that reproduced below standard (see, for example, [15]) proof of the recursion theorem for enumeration $\{\xi_u\}_{u\in U}$ of deterministic partial recursive functions is also valid for enumerations of nondeterministic partial recursive functions.

PROOF. Let $\{\delta^n_x\}_{x\in U}$ be an enumeration of deterministic partial recursive functions of arity n by elements of the universe U. Consider recursive functions $g_i : U^n \mapsto U$ such that for any $x_1, \dots, x_n \in U$,

$$\xi_{g_i(x_1,\dots,x_n)}(y) = \begin{cases} \xi_{\delta^n_{x_i}(x_1,\dots,x_n)}(y) & \text{if } \delta^n_{x_i}(x_1,\dots,x_n) \text{ convergent} \\ \text{divergent} & \text{otherwise} \end{cases}$$

Note that $h_i(x_1, \dots, x_n) = f_i(g_1(x_1, \dots, x_n), \dots, g_n(x_1, \dots, x_n))$ is a total recursive function $U^n \mapsto U$ for any i. Let w_i be such that $\delta^n_{w_i} \equiv h_i$. Thus,

$$\xi_{g_i(w_1,\dots,w_n)} \equiv \xi_{\delta^n_{w_i}(w_1,\dots,w_n)} \equiv \xi_{h_i(w_1,\dots,w_n)} \equiv \xi_{f_i(g_1(w_1,\dots,w_n),\dots,g_n(w_1,\dots,w_n))}.$$

Take u_i to be $g_i(w_1, \dots, w_n)$. ■

Modal Tautologies

DEFINITION 1. *The formulas of the modal language $\mathcal{L}$ are built from propositional variables $p, q, r \dots$ and false constant $\bot$ using implication $\to$ and binary modality $\rhd$.*

As usual, boolean connectives conjunction $\wedge$, disjunction $\vee$, negation $\neg$, and constant true $\top$ are assumed to be defined through implication and false. Let $\bigwedge\!\!\!\bigwedge\Gamma$ be the conjunction of all formulas from a finite set Γ. By definition, $\bigwedge\!\!\!\bigwedge\varnothing$ is $\top$.

DEFINITION 2. *Let $\{\xi_u\}_{u\in U}$ be an arbitrary enumeration of (deterministic or nondeterministic) recursive functions. Valuation $*$ is an arbitrary mapping of propositional variables into subsets of the universe. We define mapping $(\cdot)^*$ that extends $*$ to a mapping from modal formulas into subsets of U:*

1. $\bot^* = \varnothing$,
2. $(\phi \to \psi)^* = \complement_U(\phi^*) \cup \psi^*$,

3. $(\phi \rhd \psi)^* = \{w \in U \mid \forall u \in \phi^* \ (\xi_w(u) \neq \varnothing \to \xi_w(u) \cap \psi^* \neq \varnothing)\}$.

If $\phi^* = U$ *for any valuation* $*$, *then we say that propositional modal formula* ϕ *is a tautology of enumeration* $\{\xi_u\}_{u\in U}$. *Notation:* $\{\xi_u\}_{u\in U} \vDash \phi$.

In this paper we will provide complete axiomatizations of all tautologies for an arbitrary enumeration of deterministic or nondeterministic functions. They will be called modal logic of deterministic partial recursive functions and modal logic of nondeterministic partial recursive functions.

Note that part three of the above definition stipulates that a nondeterministic function belongs to $(\phi \rhd \psi)^*$ if for any argument from ϕ^*, on which this function is defined, at least one of its values belongs to ψ^*. An alternative definition, when all such values are required to belong to ψ^*, is discussed in the conclusion.

Axiomatizations

Definition 3. *The modal logic* $\Re$ *of nondeterministic partial recursive functions is an extension of the classical propositional logic, formulated in the language* $\mathcal{L}$, *by the following axioms*

A1. $\phi \rhd \psi \to (\chi \rhd \psi \to (\phi \vee \chi) \rhd \psi)$,

A2. $\bot \rhd \phi$,

A3. $\phi \rhd \top$,

and, in addition to Modus Ponens, the following monotonicity inference rule:

$$M. \ \frac{\phi_1 \to \phi_2, \qquad \psi_1 \to \psi_2}{\phi_2 \rhd \psi_1 \to \phi_1 \rhd \psi_2}$$

Definition 4. *The modal logic* $\Re_d$ *of deterministic partial recursive functions, in addition to the axioms and the inference rules of* $\Re$, *contains the following additional axiom:*

A4. $\phi \rhd \psi \to (\phi \rhd \chi \to \phi \rhd (\psi \wedge \chi))$.

A formula is a theorem in either of these logics if it can be proven from the appropriate axioms using Modus Ponens and Monotonicity inference rules. Let $\Delta \vdash_L \phi$ denotes that formula ϕ is provable from a set of formulas Δ and the *theorems* (not just axioms) of modal logic L using *only* Modus Ponens inference rule. In particular, with Δ being empty, $\vdash_L \phi$ expresses that ϕ is a *theorem* of logic L.

LEMMA 1.
$$(a \wedge c) \rhd b, (a \wedge \neg c) \rhd b \vdash_{\Re} a \rhd b$$

PROOF. Assume $(a \wedge c) \rhd b$ and $(a \wedge \neg c) \rhd b$. By axiom A1,

$$((a \wedge c) \vee (a \wedge \neg c)) \rhd b. \tag{1}$$

On the other hand, since $a \to (a \wedge c) \vee (a \wedge \neg c)$ and $b \to b$ are propositional tautologies, by rule M,

$$\vdash_{\Re} ((a \wedge c) \vee (a \wedge \neg c)) \rhd b \to a \rhd b.$$

This, in combination with formula (1), implies $a \rhd b$. ∎

LEMMA 2.
$$a \rhd \neg(b \wedge c), a \rhd \neg(b \wedge \neg c) \vdash_{\Re_d} a \rhd \neg b$$

PROOF. Assume $a \rhd \neg(b \wedge c)$ and $a \rhd \neg(b \wedge \neg c)$. By axiom A4,

$$a \rhd (\neg(b \wedge c) \wedge \neg(b \wedge \neg c)) \tag{2}$$

On the other hand, since $a \to a$ and $\neg(b \wedge c) \wedge \neg(b \wedge \neg c) \to \neg b$ are propositional tautologies, by rule M,

$$\vdash_{\Re_d} a \rhd (\neg(b \wedge c) \wedge \neg(b \wedge \neg c)) \to a \rhd \neg b.$$

This, in combination with formula (2), implies $a \rhd \neg b$. ∎

THEOREM 2. *For any propositional modal formula ϕ,*

1. *If $\vdash_{\Re} \phi$, then $\{\xi_u\}_{u \in U} \vDash \phi$ for any enumeration $\{\xi_u\}_{u \in U}$ of nondeterministic recursive functions,*
2. *If $\vdash_{\Re_d} \phi$, then $\{\xi_u\}_{u \in U} \vDash \phi$ for any enumeration $\{\xi_u\}_{u \in U}$ of deterministic recursive functions.*

PROOF. Both parts of the theorem will be proven simultaneously by the induction on the size of the derivation of formula ϕ. Cases of classical logic axioms and Modus Ponens inference rule are trivial. Let us consider axioms A1-A4 and the monotonicity rule M:

A1. Suppose that $w \in (\phi \rhd \psi)^*$ and $w \in (\chi \rhd \psi)^*$. We will show that $w \in ((\phi \vee \chi) \rhd \psi)^*$. Indeed, assume that there is $u \in (\phi \vee \chi)^*$ such that $\xi_w(u) \neq \varnothing$. Note that $(\phi \vee \chi)^* = \phi^* \cup \chi^*$. Thus, $u \in \phi^*$ or $u \in \chi^*$. In the first case, because $w \in (\phi \rhd \psi)^*$, we can conclude that $\xi_w(u) \cap \psi^* \neq \varnothing$. Therefore, $w \in ((\phi \vee \chi) \rhd \psi)^*$. The second case is similar.

A2. For any $w \in U$ and any valuation $*$, statement

$$\forall u \in \bot^* \ (\xi_w(u) \neq \varnothing \rightarrow \xi_w(u) \cap \psi^* \neq \varnothing)$$

is true because $\bot^* = \varnothing$.

A3. For any $w \in U$ and any valuation $*$, statement

$$\forall u \in \phi^* \ (\xi_w(u) \neq \varnothing \rightarrow \xi_w(u) \cap \top^* \neq \varnothing)$$

is true because $\top^* = U$.

A4. Applicable only to the second part of the theorem. Suppose that $w \in (\phi \rhd \psi)^*$ and $w \in (\psi \rhd \chi)^*$. We will show that $w \in (\phi \rhd (\psi \wedge \chi))^*$. Indeed, assume that there is $u \in \phi^*$ such that $\xi_w(u) \neq \varnothing$. Note that $w \in (\phi \rhd \psi)^*$ and $w \in (\psi \rhd \chi)^*$ imply that $\xi_w(u) \cap \psi^* \neq \varnothing$ and $\xi_w(u) \cap \chi^* \neq \varnothing$. Since $\xi_w(u)$ cannot contain more than one element, $\xi_w(u) \cap (\psi^* \cap \chi^*) \neq \varnothing$. Therefore, $w \in (\phi \rhd (\psi \wedge \chi))^*$.

M. If $\phi_1^* \subseteq \phi_2^*$ and $\psi_1^* \subseteq \psi_2^*$, then any function from ϕ_2^* into ψ_1^* is also a function from ϕ_1^* into ψ_2^*.

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Kripke Models

The intended semantics of logics $\Re$ and $\Re_d$ is the semantics of partial recursive functions given in Definition 2. The soundness of these logics with respect to this semantics is established in Theorem 2. The rest of the paper is dedicated to the completeness proof. As a first step, we introduce auxiliary Kripke-style semantics for both logics. Unlike the universe of partial functions, Kripke semantics provides finite models for both logics. The completeness results for Kripke semantics, which are also presented below, are obtained using standard modal logic techniques. In the next section, Kripke models will be converted into valuations over the universe of partial functions using Kleene recursion theorem. This will conclude the completeness proof for semantics of partial recursive functions.

Definition 5. *A Kripke model is a triple $\langle W, \rightarrow, \Vdash \rangle$, where W is a finite set of "worlds", $\rightarrow$ is a ternary "computability" relation on worlds, and $\Vdash$ is a binary "forcing" relation between worlds and propositional formulas.*

Informally, worlds could be viewed as nondeterministic program codes and $u \to_w v$ as a statement that program w on input u might terminate with output v.

Definition 6. *A Kripke model is called deterministic if for any worlds* w_1, w_2*, and* u *there could not be two different worlds* v_1 *and* v_2 *such that* $u \to_{w_1} v_1$ *and* $u \to_{w_2} v_2$.

Under the above informal interpretation, deterministic Kripke model is such that for every input there is no more than one program that terminates on this input.

Definition 7. *For any Kripke model the forcing relation is extended to relations* $\Vdash$ *between worlds and modal formulas as follows:*

1. $w \nVdash \bot$,
2. $w \Vdash \phi \to \psi$ *if and only if either* $w \nVdash \phi$ *or* $w \Vdash \psi$,
3. $w \Vdash \phi \rhd \psi$ *iff for any worlds* u *and* v *such that* $u \to_w v$ *and* $u \Vdash \phi$ *there is world* v' *such that* $u \to_w v'$ *and* $v' \Vdash \psi$.

Note that the last part of the above definition does not claim any forcing relation at world v. Informally, $w \Vdash \phi \rhd \psi$ iff for any ϕ-forcing input u if nondeterministic program w can terminate on input u then it can terminate with a ψ-forcing output. Of course, in the case of a deterministic Kripke model, worlds v and v' in the above definition will have to be the same.

Theorem 3. *For any propositional modal formula* ϕ_0,

1. *If* $\nvdash_{\mathfrak{R}} \phi_0$*, then there is a world* w *of a Kripke model* $\langle W, \to, \Vdash \rangle$ *such that* $w \nVdash \phi_0$.
2. *If* $\nvdash_{\mathfrak{R}_d} \phi_0$*, then there is a world* w *of a deterministic Kripke model* $\langle W, \to, \Vdash \rangle$ *such that* $w \nVdash \phi_0$.

Proof. Justifications of the two parts of this theorem are similar. We will present them in one proof. Let symbol $\vdash$ below stand for $\vdash_{\mathfrak{R}}$ or $\vdash_{\mathfrak{R}_d}$, depending on whether we prove the first or the second part of the theorem.

Definition 8. *Let us define operation* $\sim$ *on modal propositional formulas as follows:* $\sim(\neg\phi)$ *is* ϕ *for any propositional modal formula* ϕ *and* $\sim\phi$ *is* $\neg\phi$ *if* ϕ *is not, syntactically, a negation of some formula.*

One can easily see that $\sim \phi$ is equivalent to $\neg\phi$ in the classical propositional logic. Since logics $\Re$ and $\Re_d$ are extensions of the classical logic, the same equality holds there too.

DEFINITION 9. *Let Φ_0 be a finite extension of $\{\phi_0\}$ closed with respect to subformulas and operation $\sim$.*

DEFINITION 10. *For any subsets u, v, and w of Φ_0, pair (u, v) is w-consistent if $w \nvdash \bigwedge u \rhd \neg\bigwedge v$.*

LEMMA 3. *If pair (u, v) is w-consistent, then sets u and v are consistent.*

PROOF. Assume that u is not consistent: $\vdash \bigwedge u \to \bot$. Thus, by rule M, we have $\vdash \bot \rhd \neg\bigwedge v \to \bigwedge u \rhd \neg\bigwedge v$. Hence, by axiom A2, $\vdash \bigwedge u \rhd \neg\bigwedge v$. This contradicts to w-consistency of pair (u, v).

Next, suppose that v is inconsistent: $\vdash \top \to \neg\bigwedge v$. Thus, by rule M, one can conclude that $\vdash \bigwedge u \rhd \top \to \bigwedge u \rhd \neg\bigwedge v$. Taking into account axiom A3, $\vdash \bigwedge u \rhd \neg\bigwedge v$. Again contradiction with w-consistency of pair (u, v). ■

LEMMA 4. *For any w-consistent pair (u, v) of subsets of Φ_0, subset u can be extended to a complete consistent subset u' of Φ_0 such that pair (u', v) is still w-consistent.*

PROOF. We only need to prove that for any formula ϕ either ϕ or $\sim \phi$ could be added to u to keep pair (u, v) consistent. Assume that $w \vdash (\bigwedge u \wedge \phi) \rhd \neg\bigwedge v$ and $w \vdash (\bigwedge u \wedge \sim \phi) \rhd \neg\bigwedge v$. By rule M, $\vdash ((\bigwedge u \wedge \sim \phi) \rhd \neg\bigwedge v) \to ((\bigwedge u \wedge \neg\phi) \rhd \neg\bigwedge v)$. Thus, $w \vdash (\bigwedge u \wedge \neg\phi) \rhd \neg\bigwedge v$. By Lemma 1, $w \vdash \bigwedge u \rhd \neg\bigwedge v$. Therefore, (u, v) is not w-consistent. Contradiction. ■

LEMMA 5. *For any w-consistent in logic $\Re_d$ pair (u, v) of subsets of Φ_0, subset v can be extended to a complete and consistent in $\Re_d$ subset v' of Φ_0 such that pair (u, v') is still w-consistent in logic $\Re_d$.*

PROOF. Similarly to the proof of Lemma 4, assume that $w \vdash_{\Re_d} \bigwedge u \rhd \neg(\bigwedge v \wedge \phi)$ and $w \vdash_{\Re_d} \bigwedge u \rhd \neg(\bigwedge v \wedge \sim \phi)$. By rule M, $\vdash_{\Re_d} (\bigwedge u \rhd \neg(\bigwedge v \wedge \sim \phi)) \to (\bigwedge u \rhd \neg(\bigwedge v \wedge \neg\phi))$. Thus, $w \vdash_{\Re_d} \bigwedge u \rhd \neg(\bigwedge v \wedge \neg\phi)$. By Lemma 2, $w \vdash_{\Re_d} \bigwedge u \rhd \neg\bigwedge v$. Therefore, (u, v) is not w-consistent. ■

DEFINITION 11. *Let Kripke model $K = \langle W, \to, \Vdash\rangle$ be defined as follows: W is the set of all pairs (w, ϕ) where w is a maximal consistent in $\Re$ subset of Φ_0 and ϕ is a formula from Φ_0, $(u, \psi) \to_{(w,\phi)} (v, \chi)$ is true iff $(u, \{\psi\})$ is a w-consistent in $\Re$ pair and $\psi \in v$, and $(w, \phi) \Vdash p$ is true iff $p \in w$.*

LEMMA 6. *For any formula $\phi \in \Phi_0$ and any world (w, σ) of model K,*

$$\phi \in w \iff (w, \sigma) \Vdash \phi.$$

PROOF. Induction on the complexity of formula ϕ. The only non-trivial case is when $\phi \equiv \phi_1 \rhd \phi_2$.

- $\Rightarrow$ Suppose that $\phi_1 \rhd \phi_2 \in w$. Consider any world (u, ψ) such that $(u, \psi) \Vdash \phi_1$. Case 1: (u, ψ) is not w-consistent. Thus, by Definition 11, there is no (v, χ) such that $(u, \psi) \to_{(w,\sigma)} (v, \chi)$. Therefore, $(w, \sigma) \Vdash \phi_1 \rhd \phi_2$. Case 2: (u, ψ) is w-consistent. By the induction hypothesis, $\phi_1 \in u$. Thus, $\vdash_{\Re} \bigwedge\!\!\!\bigwedge u \to \phi_1$. We will show that set $\{\phi_2, \psi\}$ is consistent. Indeed, if $\phi_2 \vdash_{\Re} \neg\psi$, then, by rule M, we have $\vdash_{\Re} \phi_1 \rhd \phi_2 \to \bigwedge\!\!\!\bigwedge u \rhd \neg\psi$. Hence, $w \vdash_{\Re} \bigwedge\!\!\!\bigwedge u \rhd \neg\psi$. This means that the pair $(u, \{\psi\})$ is not w-consistent. Contradiction. Thus, $\{\phi_2, \psi\}$ is a consistent set. Let v be any maximal consistent extension of this set and χ be any formula from Φ_0. By the induction hypothesis, $(v, \chi) \Vdash \phi_2$. By Definition 11, $(u, \psi) \to_{(w,\sigma)} (u, \chi)$.

- $\Leftarrow$ Suppose that $\phi_1 \rhd \phi_2 \notin w$. By the maximality of set w, we have $w \nvdash_{\Re} \phi_1 \rhd \phi_2$. By rule M, $w \nvdash_{\Re} \phi_1 \rhd \neg \sim \phi_2$. Thus, pair $(\{\phi_1\}, \{\sim \phi_2\})$ is w-consistent. By Lemma 4, there is a complete consistent extension u of $\{\phi_1\}$ such that $(u, \{\sim \phi_2\})$ is w-consistent. By the induction hypothesis, $(u, \sim \phi_2) \Vdash \phi_1$. By Lemma 3, set $\{\sim \phi_2\}$ is consistent. Consider an arbitrary complete and consistent extension v of this set and an arbitrary formula χ of Φ_0. Trivially, $(u, \sim \phi_2) \to_{(w,\sigma)} (u, \chi)$. At the same time, for any (v', χ') such that $(u, \sim \phi_2) \to_{(w,\sigma)} (v', \chi')$ we will have $\sim \phi_2 \in v'$. Consistency of v' implies that $\phi_2 \notin v'$. Thus, by the induction hypothesis, $(v', \chi') \nVdash \phi_2$. Therefore, $(w, \sigma) \nVdash \phi$.

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DEFINITION 12. *Let deterministic Kripke model $K_d = \langle W, \to, \Vdash \rangle$ be defined as follows: W is the set of all pairs of maximal consistent in $\Re_d$ subset of Φ_0, $(u_1, u_2) \to_{(w_1,w_2)} (v_1, v_2)$ is true iff (u_1, v_1) is a w_1-consistent in $\Re_d$ pair and $u_2 = v_1 = v_2$, and $(w_1, w_2) \Vdash p$ is true iff $p \in w_1$.*

LEMMA 7. *For any formula $\phi \in \Phi_0$ and any world (w_1, w_2) of model K_d,*

$$\phi \in w_1 \iff (w_1, w_2) \Vdash \phi.$$

PROOF. Induction on complexity of formula ϕ. The only non-trivial case is when ϕ is $\phi_1 \rhd \phi_2$ for some modal formulas ϕ_1 and ϕ_2.

⇒ Assume that $\phi_1 \rhd \phi_2 \in w_1$. Consider an arbitrary w_1-consistent pair (u, v) of maximal consistent subsets of Φ_0. It will be sufficient to show that if $(u, v) \Vdash \phi_1$, then $(v, v) \Vdash \phi_2$. Indeed, assume that $(u, v) \Vdash \phi_1$ and $(v, v) \nVdash \phi_2$. By the induction hypothesis, $\phi_1 \in u$ and $\phi_2 \notin v$. Thus, by maximality of v, we have $\sim \phi_2 \in v$. Hence formulas $\bigwedge\!\!\!\bigwedge u \rightarrow \phi_1$ and $\phi_2 \rightarrow \neg \bigwedge\!\!\!\bigwedge v$ are provable in the classical propositional logic. By rule M, $\vdash_{\Re_d} \phi_1 \rhd \phi_2 \rightarrow \bigwedge\!\!\!\bigwedge u \rhd \neg \bigwedge\!\!\!\bigwedge v$. Given that $\phi_1 \rhd \phi_2 \in w_1$, we can conclude that $w_1 \vdash_{\Re_d} \bigwedge\!\!\!\bigwedge u \rhd \neg \bigwedge\!\!\!\bigwedge v$. Therefore, (u, v) is not a w_1-consistent pair. Contradiction.

⇐ Suppose $\phi_1 \rhd \phi_2 \notin w_1$. By the maximality of w_1, we have $w_1 \nvdash_{\Re_d} \phi_1 \rhd \phi_2$. By rule M, $w_1 \nvdash_{\Re_d} \phi_1 \rhd \neg \sim \phi_2$. Thus, $(\{\phi_1\}, \{\sim \phi_2\})$ is a w_1-consistent pair of sets. By Lemma 4 and Lemma 5, it can be extended to a pair (u, v) of maximal consistent sets which is also w_1-consistent. By Definition 12, $(u, v) \rightarrow_{(w_1, w_2)} (v, v)$. By the induction hypothesis, $(u, v) \Vdash \phi_1$ and $(v, v) \nVdash \phi_2$. Therefore, $(w_1, w_2) \nVdash \phi_1 \rhd \phi_2$.

■

Let us finish the proof of the completeness theorem. If $\nvdash_{\Re} \phi_0$, then consistent subset $\{\sim \phi_0\}$ of Φ_0 could be extended to a maximal consistent subset w of Φ_0. By Lemma 6, $(w, \phi_0) \nVdash \phi_0$. Similarly, if $\nvdash_{\Re_d} \phi_0$, then $\{\sim \phi_0\}$ is consistent subset of Φ_0. It can be extended to a maximal consistent subset w of Φ. By Lemma 7, $(w, w) \nVdash \phi_0$. ■

Computational Completeness

THEOREM 4. *For any propositional modal formula ϕ_0,*

1. *If $w \nVdash \phi_0$ for some world w of a Kripke model K, then $\{\xi_u\}_{u \in U} \nvDash \phi_0$ for any enumeration $\{\xi_u\}_{u \in U}$ of nondeterministic partial recursive functions.*

2. *If $w \nVdash \phi_0$ for some world w of a deterministic Kripke model K, then $\{\xi_u\}_{u \in U} \nvDash \phi_0$ for any enumeration $\{\xi_u\}_{u \in U}$ of deterministic partial recursive functions.*

PROOF. The two parts of this theorem will be proven simultaneously. The main idea is to define an embedding of the Kripke model into the universe of partial recursive functions and a valuation of propositional formulas in such a way that a propositional formula is forced in a world of the Kripke model if and only if the function, corresponding to the world of the model, belongs to the valuation of the formula.

Suppose $w_1 \nVdash \phi_0$ for some world w_1 of the Kripke model K. Let set $\{w_1, \ldots, w_n\}$ be the set of all worlds of this Kripke model. Consider functions $f_i(x_1, \ldots, x_n)$ such that

$$\xi_{f_i(x_1,\ldots,x_n)}(u) = \{x_k \mid \exists j \ (u = x_j \wedge w_j \rightarrow_{w_i} w_k)\}.$$

Note that if Kripke model K is deterministic, then w_k, mentioned in the above definition, is unique. Thus, partial recursive function $\xi_{f_i(x_1,\ldots,x_n)}$ is deterministic. No matter if the model K is deterministic or nondeterministic, let us consider fixed points $u_1, \ldots, u_n$ of functions $f_1, \ldots, f_n$ whose existence follows from Theorem 1. Also, let valuation $*$ be defined on propositional variables as follows: $p^* = \{u_i \mid w_i \Vdash p\}$.

Lemma 8. *For any propositional modal formula ϕ and any $1 \le i \le n$,*

$$u_i \in \phi^* \quad \Longleftrightarrow \quad w_i \Vdash \phi.$$

Proof. Induction on the complexity of formula ϕ. By the definition of $*$, the lemma is true for propositional variables. We will consider the only non-trivial inductive case: $\phi = \phi_1 \rhd \phi_2$.

- $\Rightarrow$ Suppose $w_i \nVdash \phi_1 \rhd \phi_2$. Thus, by Definition 7, there are j and k such that $w_j \Vdash \phi_1$, $w_j \rightarrow_{w_i} w_k$, and for any k' such that $w_j \rightarrow_{w_i} w_{k'}$, we have $w_{k'} \nVdash \phi_2$. Thus, $u_k \in \xi_{f_i(u_1,\ldots,u_n)}(u_j)$ and, at the same time, $w_{k'} \nVdash \phi_2$ for any k' such that $u_{k'} \in \xi_{f_i(u_1,\ldots,u_n)}(u_j)$. Hence $\xi_{f_i(u_1,\ldots,u_n)}(u_j)$ is not empty and, by the induction hypothesis,

  $$\xi_{f_i(u_1,\ldots,u_n)}(u_j) \cap \phi_2^* = \varnothing$$

  By the choice of elements $u_1, \ldots, u_n$, they are fixed points of functions $f_1, \ldots, f_n$. Hence, $\xi_{u_i}(u_j)$ is not empty and $\xi_{u_i}(u_j) \cap \phi_2^* = \varnothing$. At the same time, by the induction hypothesis, $u_j \in \phi_1^*$. Thus,

  $$\neg \forall u \in \phi_1^* (\xi_{u_i}(u) \neq \varnothing \rightarrow \xi_{u_i}(u) \cap \phi_2^* \neq \varnothing).$$

  Therefore, by Definition 2, $u_i \notin (\phi_1 \rhd \phi_2)^*$.

- $\Leftarrow$ Assume that $u_i \notin (\phi_1 \rhd \phi_2)^*$. Thus, by Definition 2, there is an element $y \in U$ such that $y \in \phi_1^*$, $\xi_{u_i}(y) \neq \varnothing$, and $\xi_{u_i}(y) \cap \phi_2^* = \varnothing$. Note that since $\xi_{u_i} \equiv \xi_{f_i(u_1,\ldots,u_n)}$, we can conclude that $\xi_{f_i(u_1,\ldots,u_n)}(y)$ is also non-empty. This, by the definition of f_i can happen only if

$y = u_j$ for some $0 \leq j \leq n$. In this case, by the same definition, $\xi_{u_i}(u_j) = \xi_{f_i(u_1,\dots,u_n)}(u_j) = \{u_k \mid w_j \rightarrow_{w_i} w_k\}$. Given that $y \in \phi_1^*$ and $\xi_{u_i}(y) \cap \phi_2^* = \varnothing$, we can conclude, by the induction hypothesis, that $w_j \Vdash \phi_1$ and $w_k \nVdash \phi_2$ for any k such that $w_j \rightarrow_{w_i} w_k$. Therefore, by Definition 7, $w_i \nVdash \phi_1 \rhd \phi_2$.

■

To finish the proof of Theorem 4, note that $w_1 \nVdash \phi_0$ implies, by Lemma 8, that $u_1 \notin \phi_0^*$. Therefore, $\{\xi_u\}_{u\in U} \nvDash \phi_0$. ■

THEOREM 5. *For any propositional modal formula ϕ and any enumeration $\{\xi_u\}_{u\in U}$ of nondeterministic partial recursive functions, the following statements are equivalent:*

1. $\{\xi_u\}_{u\in U} \vDash \phi$,
2. $w \Vdash \phi$ *for every world w of any Kripke model,*
3. $\vdash_{\Re} \phi$.

PROOF. Statement 1 implies statement 2 by Theorem 4. Statement 2 implies statement 3 by Theorem 3. Statement 3 implies statement 1 by Theorem 2. ■

COROLLARY 1. *Modal logic $\Re$ is decidable.*

THEOREM 6. *For any propositional modal formula ϕ and any enumeration $\{\xi_u\}_{u\in U}$ of deterministic partial recursive functions, the following statements are equivalent:*

1. $\{\xi_u\}_{u\in U} \vDash \phi$,
2. $w \Vdash \phi$ *for every world w of any deterministic Kripke model,*
3. $\vdash_{\Re_d} \phi$.

PROOF. The same as the proof of Theorem 5. ■

COROLLARY 2. *Modal logic $\Re_d$ is decidable.*

Conclusions

In this paper we have introduced two modal logics of partial recursive functions, gave their complete axiomatizations, and proved decidability of both logics. These results, of course, depend on the exact interpretation of connective $\rhd$ as given in Definition 2. Let us consider two natural alternatives to this interpretation.

First of all, there are at least two different ways to define partial nondeterministic functions from set A to set B. One approach is to require that all computational paths that start with an element in A either do not terminate or terminate in B. The second approach is to say that if the terminating paths exist, then at least one of them ends in B. The second approach is normally used to define computation of a nondeterministic finite automaton and it is the approach adopted in Definition 2 of this paper. It is also possible to consider the logic of nondeterministic partial computable functions under the first approach. One can easily see that not only are all axioms of logic $\Re$ valid in this situation, but the axiom A4 of logic $\Re_d$ is valid too. Simple review of the given above completeness proof for logic $\Re_d$ shows that the same proof establishes completeness of $\Re_d$ as a logic of nondeterministic partial functions under the first approach.

Secondly, one can define $(\phi \rhd \psi)^*$ to be the set of all *total* recursive functions from ϕ^* to ψ^*. This definition seems to be especially appropriate given that under Curry-Howard isomorphism implication in the intuitionistic logic corresponds to the type of total recursive functions. In the case of modal logics of recursive functions, transition from partial to total functions is not trivial. Indeed, if $(\phi \rhd \psi)^*$ is interpreted as the set of all total (deterministic or nondeterministic) recursive functions from ϕ^* into ψ^*, then let's consider unary modality $\Diamond\phi \equiv \neg(\phi \rhd \bot)$. Note that a function from ϕ^* to $\varnothing$ exists only if ϕ^* is empty. Thus, set $(\Diamond\phi)^*$ is equal to the entire universe U if set ϕ^* contains at least one element and set $(\Diamond\phi)^*$ is empty if ϕ^* is empty. The ability to define $\Diamond$ in the logics of total recursive functions makes it possible to express many properties that can not be expressed in the logics of partial functions. For example, formula $\Diamond(\phi \rhd \psi) \wedge \Diamond(\psi \rhd \chi) \rightarrow \Diamond(\phi \rhd \psi)$ states, essentially, that the set of total functions is closed with respect to composition. A complete description of logics of total functions remains an open question.

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Pavel Naumov
Department of Mathematics
and Computer Science
McDaniel College
Westminster, Maryland, USA
pnaumov@mcdaniel.edu