Cellular Games, Nash Equilibria, and Fibonacci Numbers

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Abstract. The paper introduces a notion of cellular game that is intended to represent rationally behaving cells of a cellular automaton. The focus is made on studying properties of functional dependence between strategies of different cells in a Nash equilibrium of such games. The main result is a sound and complete axiomatization of these properties. The construction in the proof of completeness is based on the Fibonacci numbers.

1 Introduction

Cellular Games. A (one-dimensional) cellular automaton is a two-way-infinite row of cells that transition from one state to another under certain rules. The rules are assumed to be identical for all cells. Usually, rules are chosen in such a way that the next state of each cell is determined by the current states of the cell itself and its two neighboring cells.

In this paper we consider an object similar to cellular automaton that we call *cellular game*. Each cell of the row is now viewed as a player, whose pay-off function only depends on the strategy of the cell itself and the strategies of its two neighbors. The cellular game is *homogeneous* in the sense that all players have the same pay-off function. Such games can model linearly-spaced homogeneous economic agents who only interact with their neighbors.

As an example, consider a cellular game G in which each player has only two strategies: 0 and 1. Let $\{s_i\}_{i\in\mathbb{Z}}$ be any strategy profile of this game. Let the pay-off of player i be defined as follows: the pay-off is positive if $s_{i-1}+s_i+s_{i+1}\equiv 0 \pmod 2$ and is zero otherwise. A Nash equilibrium of this game will be any strategy profile for which condition $s_{i-1}+s_i+s_{i+1}\equiv 0 \pmod 2$ is satisfied for each $i\in\mathbb{Z}$. Hence, Nash equilibria in this game have one of the following two forms:

$$\ldots 000000000000000000 \ldots$$

$$\ldots 110110110110110110 \ldots$$

Functional Dependence. The main focus of this work is on properties of Nash equilibria of cellular games. Specifically, we study if knowing the strategy of one of the players in a Nash equilibria one can predict the strategy of the other player. If knowing the strategy of a player $a \in \mathbb{Z}$ one can predict the strategy of player $b \in \mathbb{Z}$, then we say that the strategy of b is functionally dependent on the strategy of player a and denote it by $a \rhd b$. For example, in any Nash equilibrium of the game G above, strategies of two players that are three-cells apart are always the same. Thus, $G \vDash 1 \rhd 4$, $G \vDash 2 \rhd 5$, etc. In more general terms, $G \vDash a \rhd a + 3$ for each $a \in \mathbb{Z}$. Note that the property $a \rhd a + 3$ is true for cellular game G, but is not true for many other cellular games. For example, it is not true for the cellular game where each player has a constant pay-off. In this paper we are interested in the universal properties of functional dependence that are true for all cellular games. The trivial examples of such properties are Reflexivity and Transitivity:

$$a \rhd a, \tag{1}$$

$$a \rhd b \to (b \rhd c \to a \rhd c). \tag{2}$$

Since all players have the same pay-off function, we also have Homogeneity:

$$a \rhd b \to (a + c) \rhd (b + c). \tag{3}$$

Just like it is common to assume that cellular automata have a finite number of states, we assume that in cellular games each player has only finitely many strategies. As we will show in Lemma 4, this, perhaps unexpectedly, implies Symmetry:

$$a \rhd b \to b \rhd a. \tag{4}$$

In this paper we will answer the question whether there are any other universal properties of functional dependence in cellular games in addition to properties (1), (2), (3), and (4). To state a hypothetical example of such property, let us get back to the previously discussed game G. As we have seen, $G \vDash a \rhd a + 3$ for each $a \in \mathbb{Z}$. At the same time, by simply analyzing the listed above Nash equilibria of this game one can observe that $G \nvDash a \rhd a + 1$ and $G \nvDash a \rhd a + 2$. In other words,

$$a \rhd (a + 3) \to (a \rhd (a + 1) \vee a \rhd (a + 2))$$

is *not* a universal property of the functional dependence, because game G provides a counterexample. Note that this counterexample appears to heavily rely on the fact that the pay-off function of each player takes into account the strategies of exactly three players: the player herself and her two neighbors. Thus, one might expect that there will be no game G' for which the following property is false:

$$a \rhd (a + 100) \to (a \rhd (a + 1) \vee a \rhd (a + 2) \vee \cdots \vee a \rhd (a + 99)). \tag{5}$$

This would make formula (5) a valid universal principle of functional dependence in cellular games.

Surprisingly, however, such game G' does exist. Indeed, if F_{100} is the 100th Fibonacci number, then one can consider a "Fibonacci" cellular game in which the set of strategies of each player is $\mathbb{Z}_{F_{100}}$. Let the pay-off of player i be a fixed positive number if $s_{i-1} + s_i = s_{i+1}$ in $\mathbb{Z}_{F_{100}}$ and be zero otherwise. As we will see later, for this game the assumption of formula (5) is true and each disjunct in the conclusion is false.

Furthermore, the main result of this paper is that there are no other universal properties of the cellular games except for Reflexivity (1), Transitivity (2), Homogeneity (3), and Symmetry (4). In other words, the set of these axioms is complete with respect to the cellular game semantics.

Related Work. Assumptions on linear structure of the game could be relaxed to an arbitrary "dependency graph" setting where a pay-off function of each player is determined only by its own strategy and the strategies of its neighbors. Various properties of such games have been studied before [1–4]. In particular, in our work [5], we axiomatized properties of functional dependency between *sets* of players universal to all games that share the same dependence graph. The key difference of this work is the "homogeneity" assumption that all players have the same pay-off function. This assumption leads to Symmetry and Homogeneity axioms that were not present in any form in [5]. What is possibly even more interesting, this assumption leads to the use of Fibonacci numbers in the proof of completeness, which was not needed in [5].

One might argue that this work is not really about cellular games, but rather about general information flow properties in cellular-automaton-like structures. We would agree, except that the information flow setting that we study seems to be the most natural when described in terms of Nash equilibria. Properties of functional dependence relation in another network flow setting has been studied by More and Naumov [6]. Information flow properties of linear communication chains expressible in modal epistemic language were axiomatized by Kane and Naumov [7].

2 Syntax and Semantics

Definition 1. *Let Φ be the minimal set of formulas that satisfies the following conditions:*

1. $\bot \in \Phi$,
2. $a \rhd b \in \Phi$ *for each* $a, b \in \mathbb{Z}$,
3. *if* $\varphi \in \Phi$ *and* $\psi \in \Phi$, *then* $\varphi \to \psi \in \Phi$.

Definition 2. *Cellular game is a pair* (S, u), *where*

1. S *is a finite set of "strategies",*
2. u *is a "pay-off" function from* S^3 *to* $\mathbb{R}$.

In the above definition domain of the function u is S^3 because the pay-off of each player is determined by her own strategy and the strategies of her two neighbors. By a strategy profile of a cellular game (S, u) we mean any set $\{s_i\}_{i\in\mathbb{Z}}$ such that $s_i \in S$ for each $i \in \mathbb{Z}$.

Definition 3. *A Nash Equilibrium $\{e_i\}_{i\in\mathbb{Z}}$ of a game (S, u) is any strategy profile such that*

$$u(e_{i-1}, s, e_{i+1}) \leq u(e_{i-1}, e_i, e_{i+1}),$$

for each $i \in \mathbb{Z}$ and each $s \in S$.

Lemma 1. *For each $k \in \mathbb{Z}$, if $\{e_i\}_{i\in\mathbb{Z}}$ is a Nash equilibrium of a cellular game, then $\{e_{i+k}\}_{i\in\mathbb{Z}}$ is a Nash equilibrium of the same cellular game.* □

The set of all Nash equilibria of a cellular game $G = (S, u)$ will be denoted by $NE(G)$. The next definition is one of the key definitions of this paper. In part 2 we formally specify the semantics of the predicate $\rhd$.

Definition 4. *For any formula $\varphi \in \Phi$ and any cellular game G, relation $G \vDash \varphi$ is defined recursively as follows:*

1. *$G \nvDash \bot$,*
2. *$G \vDash a \rhd b$ iff for every $\{e'_i\}_{i\in\mathbb{Z}} \in NE(G)$ and $\{e''_i\}_{i\in\mathbb{Z}} \in NE(G)$, if $e'_a = e''_a$, then $e'_b = e''_b$,*
3. *$G \vDash \psi \to \chi$ iff $G \nvDash \psi$ or $G \vDash \chi$.*

3 Axioms

Our logical system, in addition to propositional tautologies in the language Φ and the Modus Ponens inference rule, contains the following axioms:

1. Reflexivity: $a \rhd a$,
2. Transitivity: $a \rhd b \to (b \rhd c \to a \rhd c)$,
3. Homogeneity: $a \rhd b \to (a + c) \rhd (b + c)$,
4. Symmetry: $a \rhd b \to b \rhd a$.

We write $X \vdash \varphi$ if formula ϕ is provable in our system extended by the set of additional axioms X. We write $\vdash \varphi$ instead of $\varnothing \vdash \varphi$.

4 Example

Soundness of our logical system will be shown in the next section. Here we give an example of a non-trivial property provable in our logical system. We will later use this result in the proof of the completeness theorem.

Lemma 2. *For any $a, b \in \mathbb{Z}$,*

$$\vdash 0 \rhd a \to 0 \rhd a \cdot b.$$

Proof. If $b = 0$, then $0 \rhd a \cdot b$ is an instance of Reflexivity axiom.

Suppose next that $b > 0$. From the assumption $0 \rhd a$ by Homogeneity axiom,

$$0 \rhd a \qquad a \rhd 2a \qquad 2a \rhd 3a \qquad 3a \rhd 4a \qquad \dots \qquad (b-1)a \rhd ab$$

Thus, by multiple applications of Transitivity axiom, $0 \rhd a \cdot b$.

Finally, assume that $b < 0$. As we have just shown above, assumption $0 \rhd a$ implies $0 \rhd a \cdot |b|$. Hence, by Homogeneity axiom, $-a \cdot |b| \rhd 0$. In other words, $a \cdot b \rhd 0$. Therefore, $0 \rhd a \cdot b$ by Symmetry axiom. □

5 Soundness

Theorem 1 (soundness). *If* $\vdash \phi$, *then* $G \vDash \phi$ *for each cellular game* G.

We prove soundness of each of the axioms as a separate lemma using Definition 4.

Lemma 3 (reflexivity). $G \vDash a \rhd a$.

Proof. For every $\{e'_i\}_{i\in\mathbb{Z}}, \{e''_i\}_{i\in\mathbb{Z}} \in NE(G)$, if $e'_a = e''_a$, then $e'_a = e''_a$. □

Lemma 4 (symmetry). *If* $G \vDash a \rhd b$, *then* $G \vDash b \rhd a$.

Proof. Let $V(a) = \{e_a \mid \{e_i\}_{i\in\mathbb{Z}} \in NE(G)\}$ for each $a \in \mathbb{Z}$. Assume now that $G \vDash a \rhd b$. Thus, there is a function $f : V(a) \to V(b)$ such that $e_b = f(e_a)$ for each $\{e_i\}_{i\in\mathbb{Z}} \in NE(G)$. We will show that function f is a surjection from $V(a)$ onto $V(b)$. Let $y \in V(b)$. Hence, $y = e_b$ for some $\{e_i\}_{i\in\mathbb{Z}} \in NE(G)$ due to the definition of $V(b)$. Then, $f(e_a) = e_b = y$. Therefore, f is a surjection of $V(a)$ onto $V(b)$.

By Lemma 1, $V(a) = V(b)$. Thus, f is a surjection of a finite set into a finite set of the same size. It is well-known in set theory that any such function is a bijection. Hence, $e_a = f^{-1}(e_b)$ for each $e \in NE(G)$. Therefore, $G \vDash b \rhd a$. □

Lemma 5 (transitivity). *If* $G \vDash a \rhd b$ *and* $G \vDash b \rhd c$, *then* $G \vDash a \rhd c$.

Proof. Consider any $\{e'_i\}_{i\in\mathbb{Z}}, \{e''_i\}_{i\in\mathbb{Z}} \in NE(G)$. Suppose that $e'_a = e''_a$. We will show that $e'_c = e''_c$. Indeed, by the first assumption of the lemma, $e'_b = e''_b$. Therefore, by the second assumption of the lemma, $e'_c = e''_c$. □

Lemma 6 (homogeneity). *If* $G \vDash a \rhd b$, *then* $G \vDash (a + c) \rhd (b + c)$.

Proof. Consider any $\{e'_i\}_{i\in\mathbb{Z}}, \{e''_i\}_{i\in\mathbb{Z}} \in NE(G)$. Suppose that $e'_{a+c} = e''_{a+c}$. We will need to show that $e'_{b+c} = e''_{b+c}$. Indeed, by Lemma 1, $\{e'_{i+c}\}_{i\in\mathbb{Z}}, \{e''_{i+c}\}_{i\in\mathbb{Z}} \in NE(G)$. Hence, by the assumption of the lemma, $e'_{b+c} = e''_{b+c}$. □

6 Completeness

In Theorem 2, given in the end of this section, we prove the completeness theorem for our logical system. We start, however, with several technical definitions and lemmas.

6.1 Rank

For any nonempty set of integers A, by $gcd(A)$ we mean the greatest common divisor of all integers in the set A. If set A contains only zeros, then $gcd(A)$ is assumed to be equal to the smallest infinite ordinal ω.

The following lemma is well-known result in elementary number theory which is commonly referred to as Bézout identity or Bézout lemma. It is usually proven through an analysis of Euclidian algorithm [8, p. 7].

Lemma 7. *For any integers a and b there are integers u and v such that $gcd(a, b) = ua + vb$.* □

In this paper, in addition to Bézout identity, we also refer to a lesser-known more general result whose proof is reproduced below.

Lemma 8. *For any non-empty set of integers $\{a_1, a_2, \ldots, a_n\}$, there are integers $c_1, \ldots, c_n$ such that $gcd(\{a_1, a_2, \ldots, a_n\}) = c_1a_1 + \cdots + c_na_n$.*

Proof. Induction on n. If $n = 1$, then let c_1 be 1. Thus, $gcd(\{a_1\}) = a_1 = c_1a_1$. If $n > 1$, then by Lemma 7, there must exist u and v such that

$$gcd(\{a_1, \ldots, a_n\}) = gcd(\{gcd(\{a_1, \ldots, a_{n-1}\}, a_n\}) = u{\cdot}gcd(\{a_1, \ldots, a_{n-1}\}) + va_n.$$

At the same time, by the Induction Hypothesis, there are integers $c_1, \ldots, c_{n-1}$ such that $gcd(\{a_1, a_2, \ldots, a_{n-1}\}) = c_1a_1 + \cdots + c_{n-1}a_{n-1}$. Therefore,

$$gcd(\{a_1, \ldots, a_n\}) = u{\cdot}gcd(\{a_1, \ldots, a_{n-1}\}) + va_n = uc_1a_1 + \cdots + uc_{n-1}a_{n-1} + va_n.$$

□

Lemma 9. *For any set of integer numbers $A = \{a_1, a_2, \ldots\}$, if $gcd(A)$ is finite, then there are an integer $k > 0$ and integers $c_1, \ldots, c_k$ such that*

$$c_1a_1 + \cdots + c_ka_k = gcd(A).$$

Proof. Consider monotonic sequence

$$gcd\{a_1\} \geq gcd\{a_1, a_2\} \geq gcd\{a_1, a_2, a_3\} \geq \ldots$$

Due to the well-ordering principle, this sequence must have a smallest element, which, thus, is equal to $gcd(A)$. In other words, there is k such that $gcd(A) = gcd(\{a_1, a_2, a_3, \ldots, a_k\})$. Finally, by Lemma 8, there are integers $c_1, \ldots, c_k$ such that

$$c_1a_1 + \cdots + c_ka_k = gcd\{a_1, a_2, a_3, \ldots, a_k\} = gcd(A).$$

□

Definition 5. *For any set of statements X in language Φ, let*

$$rank(X) = gcd\{\ d \mid X \vdash 0 \rhd d\ \}.$$

Since we have only defined $gcd(A)$ for nonempty set A, for the above definition to be valid we need to show that set $\{ d \mid X \vdash 0 \rhd d \}$ is not empty, which is true since $X \vdash 0 \rhd 0$ by Reflexivity axiom.

Lemma 10. *If $rank(X)$ is finite, then $X \vdash 0 \rhd rank(X)$.*

Proof. By Lemma 9, there are $k \geq 1$ and $c_1, \ldots, c_k, d_1, \ldots, d_k \in \mathbb{Z}$ such that

$$c_1 d_1 + \cdots + c_k d_k = gcd\{d \mid X \vdash 0 \rhd d\} \tag{6}$$

and

$$X \vdash 0 \rhd d_1, \qquad X \vdash 0 \rhd d_2, \qquad \ldots \qquad X \vdash 0 \rhd d_k.$$

By Lemma 2,

$$X \vdash 0 \rhd c_1 d_1, \qquad X \vdash 0 \rhd c_2 d_2, \qquad \ldots \qquad X \vdash 0 \rhd c_k d_k.$$

By the Homogeneity axiom,

$$\begin{aligned}
&X \vdash 0 \rhd c_1 d_1,\\
&X \vdash c_1 d_1 \rhd c_1 d_1 + c_2 d_2,\\
&\ldots\\
&X \vdash c_1 d_1 + \cdots + c_{k-1} d_{k-1} \rhd c_1 d_1 + \cdots + c_k d_k.
\end{aligned}$$

By the Transitivity axiom, applied $k-1$ times,

$$X \vdash 0 \rhd c_1 d_1 + \cdots + c_k d_k.$$

Therefore, $X \vdash 0 \rhd rank(X)$ due to equation (6). □

Lemma 11. *If $rank(X)$ is finite, then $X \vdash a \rhd b$ if and only if $rank(X) \mid (b-a)$.*

Proof. (⇒). Suppose that $X \vdash a \rhd b$. Thus, $X \vdash 0 \rhd (b-a)$ by the Homogeneity axiom. Therefore, $rank(X) \mid (b-a)$ by Definition 5. (⇐). Suppose that $rank(X) \mid (b-a)$. Thus, $X \vdash 0 \rhd (b-a)$ due to Lemma 2 and Lemma 10. Therefore, $X \vdash a \rhd b$ by the Homogeneity axiom. □

Lemma 12. *If $rank(X) = \omega$, then $X \vdash a \rhd b$ if and only if $a = b$.*

Proof. (⇒). If $X \vdash a \rhd b$, then, by the Homogeneity axiom, $X \vdash 0 \rhd (b-a)$. Hence, $b - a = 0$ due to the assumption $rank(X) = \omega$. Therefore, $a = b$. (⇐). $X \vdash a \rhd a$ by Reflexivity axiom. □

6.2 Game G_d for $2 < d < \omega$

Definition 6. *For any integer $d > 2$, let G_d be the game $(\mathbb{Z}_{F_d}, u)$, where*

$$u(x, y, z) = \begin{cases} 1 & \text{if } x + y = z \text{ in } \mathbb{Z}_{F_d},\\ 0 & \text{otherwise.} \end{cases}$$

Lemma 13. $e_{n-1}+e_n=e_{n+1}$ *for each* $\{e_i\}_{i\in\mathbb{Z}}\in NE(G_d)$ *and each* $n\in\mathbb{Z}$. $\square$

By $F_0,F_1,F_2,F_3,F_4,F_5,\ldots$ we mean Fibonacci numbers $0,1,1,2,3,5,\ldots$ and by $[F_0],[F_1],[F_2],\ldots$ their congruence classes in $\mathbb{Z}_{F_d}$.

Lemma 14. *For each* $n,k\in\mathbb{Z}$ *and each* $\{e_i\}_{i\in\mathbb{Z}}\in NE(G_d)$*, if* $k>0$*, then*

$$e_{n+k}=[F_{k-1}]e_n+[F_k]e_{n+1}.$$

Proof. Induction on k. If $k=1$, then

$$e_{n+1}=[0]\cdot e_n+[1]\cdot e_{n+1}=[F_0]\cdot e_n+[F_1]\cdot e_{n+1}.$$

If $k=2$, then, by Lemma 13,

$$e_{n+2}=e_n+e_{n+1}=[1]\cdot e_n+[1]\cdot e_{n+1}=[F_1]\cdot e_n+[F_2]\cdot e_{n+1}.$$

If $k>2$, then by Lemma 13, the Induction Hypothesis, and the recurrence relation for Fibonacci numbers,

$$\begin{aligned}
e_{n+k}&=e_{n+k-2}+e_{n+k-1}\\
&=([F_{k-3}]\cdot e_n+[F_{k-2}]\cdot e_{n+1})+([F_{k-2}]\cdot e_n+[F_{k-1}]\cdot e_{n+1})\\
&=([F_{k-3}]+[F_{k-2}])\cdot e_n+([F_{k-2}]+[F_{k-1}])\cdot e_{n+1}\\
&=[F_{k-1}]\cdot e_n+[F_k]\cdot e_{n+1}.
\end{aligned}$$

$\square$

Lemma 15. *For each* $n,q,d\in\mathbb{Z}$*, if* $q\geq 0$*, then*

$$e_{n+qd}=[F_{d-1}^q]\cdot e_n,$$

where F_{d-1}^q *is* F_{d-1} *raised to power* q.

Proof. Induction on q. If $q=0$, then

$$e_{n+qd}=e_n=[1]\cdot e_n=[F_{d-1}^0]\cdot e_n.$$

Note that in the above we rely on the fact that $F_{d-1}^0=1$, which is true because $d>2$ and, thus, $F_{d-1}>0$.

Assume now that $q>0$. Thus, by Lemma 14, the Induction Hypothesis, and due to the fact that $F_d=[0]$ in $\mathbb{Z}_{F_d}$,

$$\begin{aligned}
e_{n+qd}&=e_{n+(q-1)d+d}\\
&=[F_{d-1}]\cdot e_{n+(q-1)d}+[F_d]\cdot e_{n+(q-1)d+1}\\
&=[F_{d-1}]\cdot e_{n+(q-1)d}+[0]\cdot e_{n+(q-1)d+1}\\
&=[F_{d-1}]\cdot e_{n+(q-1)d}=[F_{d-1}]\cdot[F_{d-1}^{q-1}]e_n=[F_{d-1}^q]\cdot e_n.
\end{aligned}$$

$\square$

Lemma 16. *If $d \mid (b-a)$, then $G_d \vDash a \rhd b$.*

Proof. Due to Lemma 4, we can assume that $a < b$. Suppose that $b = a + qd$ and that $e'_a = e''_a$ for some $\{e'_i\}_{i\in\mathbb{Z}}, \{e''_i\}_{i\in\mathbb{Z}} \in NE(G_d)$. We will show that $e'_b = e''_b$. Indeed, by Lemma 15,

$$e'_b = e'_{a+qd} = [F^q_{d-1}] \cdot e'_a.$$

Similarly, $e''_b = [F^q_{d-1}] \cdot e''_a$. Thus,

$$e'_b = [F^q_{d-1}] \cdot e'_a = [F^q_{d-1}] \cdot e''_a = e''_b.$$

□

By **0** we mean the constant function from $\mathbb{Z}$ to $\mathbb{Z}_{F_d}$ equal to $[0]$ on all integer numbers.

Lemma 17. $\mathbf{0} \in NE(G_d)$.

Proof. See Definition 6. □

Note that the Fibonacci sequence

$$F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, \dots$$

can be expanded to negative subscripts in a way that preserves the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for the Fibonacci numbers:

$$F_{-1} = 1, F_{-2} = 0, F_{-3} = 1, F_{-4} = -1, F_{-5} = 2, \dots$$

By **F** we mean the function from $\mathbb{Z}$ to $\mathbb{Z}_{F_d}$ such that $\mathbf{F}(z) = [F_z]$.

Lemma 18. $\mathbf{F} \in NE(G_d)$.

Proof. See Definition 6. □

Lemma 19. *Element $[F_{d-1}]$ is invertible in $\mathbb{Z}_{F_d}$ for each $d > 2$.*

Proof. Due to the identity $F_n = F_{n-1} + F_{n-2}$, the Euclidian algorithm, when applied to any two consecutive Fibonacci numbers, generates the complete sequence of Fibonacci numbers in reverse order:

$$F_n, F_{n-1}, F_{n-2}, F_{n-3}, \dots, 13, 8, 5, 3, 2, 1, 1, 0.$$

Hence, $gcd(F_n, F_{n-1}) = 1$ for each $n \geq 1$. In particular, $gcd(F_d, F_{d-1}) = 1$. Thus, due to Lemma **??**, there are integers a and b such that

$$a \cdot F_d + b \cdot F_{d-1} = 1.$$

Therefore, $b \cdot F_{d-1} \equiv 1 \pmod{F_d}$. In other words, $[b] \cdot [F_{d-1}] = [1]$ in $\mathbb{Z}_{F_d}$. □

Lemma 20. *If $d > 2$ and $G_d \vDash a \rhd b$, then $d \mid (b-a)$.*

Proof. Assume that $G_d \vDash a \rhd b$. Due to Lemma 4, we can also assume that $a < b$. Suppose that $b - a = qd + r$ for $0 \leq r < d$. We will show that $r = 0$.

Consider strategy profiles $\{e'_i\}_{i\in\mathbb{Z}}$ and $\{e''_i\}_{i\in\mathbb{Z}}$ such that

$$e'_i = \mathbf{0}(i - a)$$

and

$$e''_i = \mathbf{F}(i - a)$$

for each $i \in \mathbb{Z}$. By Lemma 1, $\{e'_i\}_{i\in\mathbb{Z}}, \{e''_i\}_{i\in\mathbb{Z}} \in NE(G)$. Note that

$$e'_a = \mathbf{0}(a - a) = [0] = \mathbf{F}(a - a) = e''_a \pmod{F_d}.$$

Thus, $e'_b = e''_b$ due to the assumption $G_d \vDash a \rhd b$. Hence, $\mathbf{0}(b - a) = \mathbf{F}(b - a)$. In other words, $\mathbf{F}(b - a) = [0]$. Hence, $\mathbf{F}(r + qd) = [0]$. At the same time, by Lemma 15,

$$\mathbf{F}(r + qd) = [F^q_{d-1}]\mathbf{F}(r) = [F^q_{d-1} \cdot F_r] = [F_{d-1}]^q \cdot [F_r].$$

Then, $[F_{d-1}]^q \cdot [F_r] = [0]$. By Lemma 19, element $[F_{d-1}]$ is invertible in $\mathbb{Z}_{F_d}$. Thus,

$$[F_{d-1}]^{-q} \cdot [F_{d-1}]^q \cdot [F_r] = [F_{d-1}]^{-q} \cdot [0].$$

Hence, $[F_r] = [0]$. In other words, $F_r \equiv 0 \pmod{F_d}$. Recall that $0 \leq r < d$ by the choice of r. Thus, $0 \leq F_r < F_d$. Taking into account $F_r \equiv 0 \pmod{F_d}$, we can conclude that $F_r = 0$. Recall now that $r \geq 0$. Therefore, $r = 0$. □

6.3 Special cases: game G_ω, G_1, and G_2

Definition 7. *Let G_ω be the game $(\{0, 1\}, u)$, where $u(x, y, z) = 0$ for all $x, y, z \in \mathbb{Z}$.*

Lemma 21. *$G_\omega \vDash a \rhd b$ if and only if $a = b$.*

Proof. Any strategy profile of game G_ω is a Nash equilibrium. □

Definition 8. *let G_1 be the game $(\{0, 1\}, u)$, where*

$$u(x, y, z) = \begin{cases} 1 & \text{if } y = z \\ 0 & \text{otherwise.} \end{cases}$$

In other words, in game G_1 each player is paid to be equal to her right neighbor.

Lemma 22. *$G_1 \vDash a \rhd b$ for any $a, b \in \mathbb{Z}$.*

Proof. Game G_1 has only two Nash equilibria: $\ldots 000 \ldots$ and $\ldots 111 \ldots$. □

Definition 9. *let G_2 be the game $(\{0, 1, panic\}, u)$, where $u(x, y, z)$ is defined as follows: if $x = z \neq panic$, then y is not paid no matter what its value is; otherwise y is paid a positive amount if it is equal to panic.*

Lemma 23. *Game G_2 has five Nash equilibria:*

1. *all players are in the state of panic,*
2. *all players are in the state 0,*
3. *all players are in the state 1,*
4. *even-indexed players are in state 0, odd-indexed players are in state 1,*
5. *even-indexed players are in state 1, odd-indexed players are in state 0.*

□

Lemma 24. *$G_2 \vDash a \rhd b$ if and only if $2 \,|\, (a-b)$.*

Proof. Follows from Lemma 23. □

7 Completeness: final steps

Lemma 25. *For any set of formulas X in the language Φ, there is a game G such that, $X \vdash a \rhd b$ if and only if $G \vDash a \rhd b$, for each $a, b \in \mathbb{Z}$.*

Proof. Case I: $rank(X) = 1$. Consider game G_1. $(\Rightarrow)$: Note that $G_1 \vDash a \rhd b$ by Lemma 22. $(\Leftarrow)$: By the assumption of this case, $rank(X) = 1$. Thus, $X \vdash 0 \rhd 1$ by Lemma 10. Hence, $X \vdash 0 \rhd (b-a)$, by Lemma 2. Therefore, $X \vdash a \rhd b$, by the Homogeneity axiom.

Case II: $rank(X) = 2$. Consider game G_2. $(\Rightarrow)$: Let $G_2 \nvDash a \rhd b$. Thus, by Lemma 24, $b - a = 2q + 1$ for some $q \in \mathbb{Z}$. We need to show that $X \nvdash a \rhd b$. Suppose the opposite, $X \vdash a \rhd b$. Hence, $X \vdash 0 \rhd (b-a)$ by the Homogeneity axiom. Thus, $X \vdash 0 \rhd 2q+1$. This is a contradiction with the assumption $rank(X) = 2$, because $2q+1$ is not divisible by 2. $(\Leftarrow)$: Assume that $G_2 \vDash a \rhd b$. Thus, $b - a = 2q$, by Lemma 24. Recall the assumption $rank(X) = 2$. Thus, by Lemma 10, $X \vdash 0 \rhd 2$. Hence, $X \vdash 0 \rhd 2q$ by Lemma 2. In other words, $X \vdash 0 \rhd (b-a)$. Thus, by the Homogeneity axiom, $X \vdash a \rhd b$.

Case III: $2 < d = rank(X) < \omega$. Consider game G_d. $(\Rightarrow)$: Let $X \vdash a \rhd b$. Hence, by the Homogeneity axiom, $X \vdash 0 \rhd (b-a)$. Thus, $d \mid (b-a)$ due to Definition 5. Thus, $G_d \vDash a \rhd b$, by Lemma 16. $(\Leftarrow)$: Assume that $G_d \vDash a \rhd b$. Hence, $d \mid (b-a)$ by Lemma 20. Thus, $b - a = qd$ for some $q \in \mathbb{Z}$. Recall that $d = rank(X) < \omega$. Thus, $X \vdash 0 \rhd d$ by Lemma 10. Hence, $X \vdash 0 \rhd qd$, by Lemma 2. In other words, $X \vdash 0 \rhd (b-a)$. Therefore, $X \vdash a \rhd b$, by the Homogeneity axiom.

Case IV: $rank(X) = \omega$. Consider game G_ω. $(\Rightarrow)$: Let $X \vdash a \rhd b$. Thus, by the Homogeneity axiom, $X \vdash 0 \rhd (b-a)$. Hence, $b - a = 0$, due to the fact that $rank(X) = \omega$. Then $G_\omega \vDash a \rhd b$ by Lemma 21. $(\Leftarrow)$: Assume that $G_\omega \vDash a \rhd b$. Thus, $a = b$, by Lemma 21. Therefore, $X \vdash a \rhd b$ by the Reflexivity axiom. □

Lemma 26. *For any maximal consistent set of formulas $X \subset \Phi$, there is a game G such that $X \vdash \psi$ if and only if $G \vDash \psi$ for each $\psi \in \Phi$.*

Proof. Consider any maximal consistent set $X \subset \Phi$. Let G be the game that exists by Lemma 25. We will prove that $X \vdash \psi$ if and only if $G \vDash \psi$ by induction on the structural complexity of formula ψ. The case when ψ is an atomic proposition follows from Lemma 25. If ψ is constant $\bot$, then $G \nvDash \psi$ by Definition 4 and $X \nvdash \psi$ due to consistency of the set X. The case when ψ is an implication follows from the assumption of maximality and consistency of X in the standard way. □

Theorem 2 (completeness). *If $G \vDash \phi$ for each cellular game G, then $\vdash \phi$.*

Proof. Suppose that $\nvdash \phi$. Let X be a maximal consistent subset of Φ containing formula $\neg\phi$. By Lemma 26, there is a cellular game G such that $G \vDash \neg\phi$. □

8 Conclusion

In this paper we gave a complete axiomatization of functional dependence in linear cellular games with finite number of strategies. The two natural extensions of this work are games with infinite number of strategies and cellular games on a plane.

If players are allowed to have infinite number of strategies, then the Symmetry axiom is no longer sound. Indeed, consider a game in which strategies are infinite boolean sequences. Let $a_1, a_2, a_3, \ldots$ be the strategies of a player u and $b_1, b_2, b_3, \ldots$ be the strategies of her right neighbor $u+1$. The player u gets a fixed positive pay-off if and only if

$$a_i = b_{i-1} \qquad \text{for all } i \geq 2. \tag{7}$$

Note that equation (7) puts no restriction on the value of a_1. It is easy to see that the Nash equilibria of this game are all strategy profiles in which equation (7) is satisfied for all adjacent players u and $u+1$. Thus, for this game $u \rhd u+1$ is true. At the same time, equation (7) puts no restrictions on a_1 and, thus, formula $u+1 \rhd u$ is false. The complete axiomatization of functional dependence in linear cellular games with infinite number of strategies remains an open problem.

By a cellular game on the plane we mean a game on a square grid where the pay-off of each player is determined by her own strategy and the strategies of her eight neighbors. Our attempts, made together with Jeffrey Kane, to generalize results of this paper to such games were unsuccessful due to the fact that we were not able to find the right "two-dimensional" version of of Fibonacci numbers. We even do not know, for example, if there is a game (see Figure 1) in which formula $a \rhd c$ is true, but formulas $b_k \rhd c$ are false for each $k \in \{1, 2, \ldots, 10\}$.

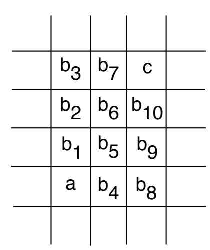

Fig. 1. Plane Game.

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