On Interchangeability of Nash Equilibria in Multi-Player Strategic Games

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Abstract. The article studies properties of interchangeability of pure, mixed, strict, and strict mixed Nash equilibria. The main result is a sound and complete axiomatic system that describes properties of interchangeability in all four settings. It has been previously shown that the same axiomatic system also describes properties of independence in probability theory, nondeducibility in information flow, and non-interference in concurrency theory.

1. Introduction

1.1. Interchangeability

The interchangeability property (Nash, 1951) of equilibria in strategic games is a formal way to state that rational choices of two groups of players do not depend on each other.

1.1.1. *Two-player games*

In a two-player game, interchangeability of the set of pure Nash equilibria is especially easy to define: if a_1 and a_2 are any two strategies of the first player, b_1 and b_2 are any two strategies of the second player such that $\langle a_1, b_1\rangle$ and $\langle a_2, b_2\rangle$ are two Nash equilibria, then $\langle a_1, b_2\rangle$ and $\langle a_2, b_1\rangle$ are also Nash equilibria. Some two-party strategic games satisfy this property, others do not.

Table I. Game G_1

	b_1	b_2	b_3
a_1	1,1	0,0	1,1
a_2	0,0	1,1	0,0
a_3	1,1	0,0	1,1

For example, the set of Nash equilibria of game G_1, with pay-off matrix given by Table I, is not interchangeable because strategy profiles

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$\langle a_1, b_1\rangle$ and $\langle a_2, b_2\rangle$ are Nash equilibria, but strategy profile $\langle a_1, b_2\rangle$ is not.

Table II. Game II

	b_1	b_2	b_3
a_1	2,2	1,1	2,2
a_2	1,1	0,0	1,1
a_3	2,2	1,1	2,2

On the other hand, the set of equilibria of game G_2, whose pay-off matrix is given by Table II, is the following interchangeable set

$$\{\langle a_1, b_1\rangle, \langle a_1, b_3\rangle, \langle a_3, b_1\rangle, \langle a_3, b_3\rangle\}.$$

Interchangeability has a clear epistemic meaning in terms of information available to an external observer of the game. If by knowing the strategy of one of the players in a Nash equilibrium, the external observer can, at least sometimes, deduce extra information about the strategy of the other player in the same equilibrium, then the set of equilibria is *not* interchangeable. If, however, external observer can never deduce any additional information about the strategy of the second player, then the set of equilibria is interchangeable.

For example, in game G_1, played between players Alice and Bob, an external observer a priori only knows that, in a Nash equilibrium, Bob is using one of strategy b_1, b_2, or b_3. If, however, the observer learns that Alice is using strategy a_1, then it can deduce that Bob must be using either strategy b_1 or strategy b_3, but not strategy b_2. In other words, by learning Alice's strategy, the observer also learns some additional information about Bob's strategy. The same is not true, however, for the game G_2 in which equilibria are interchangeable.

Another meaning of the interchangeability of Nash equilibria is in terms of choices available to the players in the game. If the game has multiple Nash equilibria, than each player is facing with dilemma which of the several strategies corresponding to Nash equilibria she should choose. If the equilibria are interchangeable, than, in some sense, it does not matter: the outcome is guaranteed to be a Nash equilibrium no matter which of these strategies she decides to choose.

It is a well-known (Nash, 1951) that the set of all equilibria in any *zero-sum* two-player game is interchangeable. We discuss applicability of our main results to zero-sum games in the conclusion.

1.1.2. *Multi-player games*

Epistemic interpretation of interchangeability gives intuition on how one can generalize this property to multi-player games. We will start with interchangeability on two players in a multi-player games. The set of Nash equilibria is interchangeable on two players if by learning the strategy of one of the players in an equilibrium, an external observer can not deduce any extra information about the strategy of the other player in the same equilibrium. For example, consider game G_3 defined below:

DEFINITION 1. *In game G_3, players Alice, Bob, and Cathy vote "yes" or "no". If votes split, then everyone in the majority pays one dollar to the voter in the minority.*

This game has six Nash equilibria: all strategy profile except for unanimous votes $\langle yes, yes, yes\rangle$ and $\langle no, no, no\rangle$. In some situations, knowing strategies of Alice and Bob in a Nash equilibrium, one can predict Cathy's strategy in the same equilibrium. For example, if Alice and Bob both voted "yes", then, in a Nash equilibrium, Cathy's vote is "no". At the same time, knowing only Alice's vote, an external observer can not say anything about Bob's vote. Thus, the set of equilibria of this game is interchangeable on players Alice and Bob. We denote this by $Game_3 \vDash Alice \parallel Bob$. Of course, statements $Game_3 \vDash Alice \parallel Cathy$ and $Game_3 \vDash Cathy \parallel Bob$ are also true.

Next, let us consider another three-player game:

DEFINITION 2. *In game G_4, players Alice, Bob, and Cathy vote "yes" or "no". If votes split, then the voter in the minority pays one dollar to each voter in the majority.*

Game G_4 has only two Nash equilibria: $\langle yes, yes, yes\rangle$ and $\langle no, no, no\rangle$. Thus, knowing any of the three votes in a Nash equilibrium reveals the other two votes. Therefore, $Game_4 \nvDash Alice \parallel Bob$; $Game_4 \nvDash Alice \parallel Cathy$; and $Game_4 \nvDash Cathy \parallel Bob$.

1.1.3. *Interchangeability on sets of players*

Finally, if A and B are two *sets* of players, then we say that Nash equilibria is interchangeable on sets A and B if knowing strategies of all players in set A in a Nash equilibrium reveals no additional information about the strategies of the players in set B in the same equilibrium. We denote this relation by $A \parallel B$. Formally, it means that for any Nash equilibria e_1 and e_2 there is a Nash equilibrium e such that e agrees with e_1 on all players in A and with e_2 on all players in B. For example, $Game_3 \nvDash Alice, Bob \parallel Cathy$, because $\langle yes, yes, no\rangle$ and $\langle no, yes, yes\rangle$ are Nash equilibria and $\langle yes, yes, yes\rangle$ is not.

In this article we do not study interchangeability properties of particular games. Instead, we consider properties of the interchangeability that are common to all strategic games. The following are two non-trivial examples of such properties that are true for any four players a, b, c, and d in an arbitrary strategic game with at least four players:

$$a \parallel b \rightarrow (c \parallel d \rightarrow (a, b \parallel c, d \rightarrow a, c \parallel b, d)), \tag{1}$$

$$a \parallel b \rightarrow (a, b \parallel c \rightarrow (a, b, c \parallel d \rightarrow a \parallel b, c, d)). \tag{2}$$

We prove sounds of these principles in Section 4. In this paper we describe all properties of interchangeability in the positional language. Namely, we show that all such properties can be derived in the following axiomatic system:

1. Empty Set: $A \parallel \varnothing$,
2. Symmetry: $A \parallel B \rightarrow B \parallel A$,
3. Monotonicity: $A \parallel B, C \rightarrow A \parallel B$,
4. Exchange: $A, B \parallel C \rightarrow (A \parallel B \rightarrow A \parallel B, C)$,

where here and everywhere below A, B means the union of sets A and B. Of course, soundness of the first three axioms is obvious. Soundness of the Exchange axiom will be shown in Theorem 1.

1.1.4. *Strict equilibria interchangeability*

One can define interchangeability of the set of *strict* Nash equilibria on two sets of players by replacing words "Nash equilibria" with "strict Nash equilibria" in the above definition of interchangeability. Note that interchangeability of strict equilibria is not equivalent to interchangeability of all equilibria. For example, all five equilibria in two-player game G_5 depicted in Table III are not interchangeable, but four strict equilibria are interchangeable.

Table III. Game G_5

	b_1	b_2	b_3
a_1	1,1	0,0	1,1
a_2	0,0	0,0	0,0
a_3	1,1	0,0	1,1

In spite of this, in Theorem 2, we will show that the same logical system defined by the axioms 1.-4. is sound and complete with respect to strict equilibria interchangeability just like it is with respect to all equilibria.

1.1.5. *Mixed equilibria interchangeability*
Interchangeability of (all or strict only) *mixed* Nash equilibria can be defined similarly. Interchangeability of pure and mixed equilibria are two non-equivalent properties. For example, two-player game G_6, whose pay-off matrix is given in Table IV, has four *pure* Nash equilibria: $\langle a_1, b_1\rangle$, $\langle a_1, b_4\rangle$, $\langle a_4, b_1\rangle$, and $\langle a_4, b_4\rangle$. This set of equilibria is interchangeable.

Table IV. Game G_6

	b_1	b_2	b_3	b_4
a_1	1,1	0,0	0,0	1,1
a_2	0,0	2,1	1,2	0,0
a_3	0,0	1,2	2,1	0,0
a_4	1,1	0,0	0,0	1,1

At the same time, the set of mixed equilibria of the same game G_6 is not interchangeable because mixed strategy profile

$$\langle(0, 0.5, 0.5, 0), (0, 0.5, 0.5, 0)\rangle,$$

in which both players pick between second and third strategies with equal probabilities, and mixed strategy profile

$$\langle(1, 0, 0, 0), (1, 0, 0, 0)\rangle,$$

in which both players always pick their first strategies, are Nash equilibria, but mixed strategy profile $\langle(0, 0.5, 0.5, 0), (1, 0, 0, 0)\rangle$ is not an equilibrium.

Nevertheless, in Theorem 2, we will show that the same logical system defined by the axioms 1.-4. is sound and complete with respect to mixed equilibria interchangeability and mixed strict equilibria interchangeability.

1.1.6. *Independence in Probability Theory*
Surprisingly, given the above axioms 1.-4. axiomatize not only properties of the Nash equilibria interchangeability, but several others seemingly unrelated relations. This system of axioms[1] first was introduced by Geiger, Paz, and Pearl (1991) to describe properties of independence in probability theory.

[1] The axiom *names* used above are ours.

Two events are called independent if the probability of their intersection is equal to the product of their probabilities. It is believed (Encyclopædia Britannica, 1998) that this notion was first introduced by de Moivre (de Moivre, 1711; de Moivre, 1718). If $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_m\}$ are two disjoint sets of random variables with finite ranges of values, then these two sets of variables are called independent if for any values $v_1, \ldots, v_n$ and any values $w_1, \ldots, w_m$, events $\bigwedge_{i \leq n}(a_i = v_i)$ and $\bigwedge_{i \leq m}(b_i = w_i)$ are independent. We denote this relation by $A \parallel B$. This definition can be generalized to independence of sets of variables with infinite ranges through the independence of appropriate σ-algebras.

The axioms 1.-4. above give complete axiomatization of propositional properties of the independence relation between two sets of random variables (Geiger et al., 1991). In a related work, Studený (1990) showed that *conditional* probabilistic independence does not have a complete finite axiomatization.

1.1.7. *Independence in Information Flow*

Sutherland (1986) introduced a relation between two pieces of information, which are sometimes called “secrets”, that later became known as the “nondeducibility” relation. Two secrets are in this relation if any possible value of the first secret is consistent with any possible value of the second secret. More and Naumov (2010b) generalized this relation to a relation $A \parallel B$ between two sets of secrets and called it independence: sets of secrets A and B are independent if each possible combination of the values of secrets in A is consistent with each possible combination of the values of secrets in B. More and Naumov (2010b) have shown that the same system of axioms 1.-4. is sound and complete with respect to defined this way semantics of secrets[2].

Cohen (1977) presented a related notion called *strong dependence*. More recently, Halpern and O’Neill (2008) introduced f-secrecy to reason about multiparty protocols. In our notation, f-secrecy is a version of the nondeducibility predicate whose left or right side contains a certain function of the secret rather than the secret itself. More, Naumov, and Donders also axiomatized a variation of the independence relation between secrets over graphs (More and Naumov, 2011; Donders et al., 2011) and hypergraphs (Miner More and Naumov, 2010a). *Conditional independence* relation is a generalization of independence. It is also known in the database theory as embedded multi-valued dependency. Parker and Parsaye-Ghomi (1980) have shown that conditional inde-

[2] As long as the same secret can not appear simultaneously on the left and right hand side of the independence symbol. Otherwise, one more axiom should be added to achieve completeness.

pendence can not be described by a finite system of inference rules. Herrmann (1995, 2006) proved the undecidability of the propositional theory of this relation. Lang, Liberatore, and Marquis (2002) studied complexity of conditional independence between sets of propositional variables. Naumov and Nicholls (2013) gave complete recursively enumerable axiomatiztion of the conditional independence relation between sets of secrets.

1.1.8. *Independence in Concurrency Theory.*
The third semantics for the same axioms 1.-4. was proposed by More, Naumov, and Sapp (More et al., 2011). Under this semantics, independence is interpreted as "non-interference" between two sets of concurrent processes. A set of processes A interferes with a set of processes B if these two sets can reach a deadlocked state where either set A or set B is not internally deadlocked. For example, if p_1, p_2, p_3, p_4, p_5 are five philosophers seating at a table with five forks in the classical Dijkstra's (1971) dining philosopher problem, then neither set $\{p_1, p_2, p_3\}$ nor set $\{p_4, p_5\}$ can deadlock by itself (if the other philosophers leave the table). However, the complete set $\{p_1, p_2, p_3, p_4, p_5\}$ can deadlock. Thus, using our notations we can say that statement $p_1, p_2, p_3 \parallel p_4, p_5$ is false.

More, Naumov, and Sapp (2011) have shown that the same system of axioms 1.-4. is sound and complete with respect to the concurrency semantics.

1.1.9. *Article Outline*
In this article we will show that the same axioms 1.-4. give a sound and complete axiomatization of properties of Nash equilibria interchangeability for all equilibria, strict equilibria, mixed equilibria, and strict mixed equilibria. It is easy to see that any strategic game could be viewed as an information flow protocol. Thus, soundness of these axioms in the game setting trivially follows from their soundness in the information flow setting (Miner More and Naumov, 2010b). The main technical contribution of this work is the proof of completeness. More and Naumov (2010b) have shown that if a formula is not provable from axioms 1.-4., then there is an information flow protocol for which this formula is false. In this article, we show that such protocol can be described in terms of a strategic game.

The significant implication of these results is that *the same non-trivial set of axioms captures the properties of independence-like relations in four different settings: probability, information flow, concurrency, and game theory.*

The preliminary version of this work appeared as (Naumov and Nicholls, 2011). This article adds proofs of soundness and completeness with respect to mixed equilibria. To achieve this, the second parity game is added in Section 6.2.

In the conclusion we discuss what appears to be a more general interchangeability relation $A_1 \parallel A_2 \parallel \ldots \parallel A_n$ on several sets of players. We will show, however, that this relation can be expressed through interchangeability on just two sets of players.

2. Semantics

In this section, we give formal definitions of the interchangeability and related notions that have been informally discussed in the previous section.

DEFINITION 3. *A game is a triple* $(P, \{S_p\}_{p\in P}, \{u_p\}_{p\in P})$, *where*

1. *P is a non-empty finite set of "players".*
2. *S_p is a non-empty finite set of "strategies" of a player $p \in P$. Elements of the product $\prod_{p\in P} S_p$ are called "strategy profiles".*
3. *u_p is a "pay-off" function from strategy profiles into the real numbers.*

For any tuple $a = \langle a_i \rangle_{i\in I}$, any $i_0 \in I$ and any value b, by $\langle a_i \rangle_{i\in I}[i_0 \mapsto b]$ we mean the tuple a in which i_0-th component is changed from a_{i_0} to b. In the game theory literature the same modified tuple is sometimes denoted by (a_{-i_0}, b).

DEFINITION 4. *For any game* $(P, \{S_p\}_{p\in P}, \{u_p\}_{p\in P})$, *a (pure) Nash equilibrium of this game is a strategy profile* $\langle s_p \rangle_{p\in P}$ *such that*

$$u_p(\langle s_p \rangle_{p\in P}[p_0 \mapsto s_0]) \leq u_p(\langle s_p \rangle_{p\in P}) \tag{3}$$

for any $p_0 \in P$ *and any* $s_0 \in S_{p_0}$.

Alternatively, one can define *strict* Nash equilibrium by replacing relation $\leq$ in inequality (3) with strict inequality sign $<$. The set of all Nash equilibria of a game G will be denoted by $NE(G)$ and the set of all strict equilibria by $sNE(G)$.

DEFINITION 5. *For any game* $(P, \{S_p\}_{p\in P}, \{u_p\}_{p\in P})$, *a mixed strategy of a player q is a random variable with the values in S_q. Mixed strategy profile is any tuple $\langle s^*_p \rangle_{p\in P}$ such that s^*_p is a mixed strategy of the player p.*

DEFINITION 6. *For any game $(P, \{S_p\}_{p\in P}, \{u_p\}_{p\in P})$, a mixed Nash equilibrium of this game is any mixed strategy profile $\langle s_p^*\rangle_{p\in P}$ such that*

$$E[u_p(\langle s_p^*\rangle_{p\in P}[p_0 \mapsto s_0])] \leq E[u_p(\langle s_p^*\rangle_{p\in P})] \tag{4}$$

for any $p_0 \in P$ and any mixed strategy s_0^* of the player p_0. Expected value $E[\cdot]$ in the above formula is computed assuming independence of distributions of mixed strategies of different players.

One can define *strict* mixed Nash equilibrium by replacing relation $\leq$ in inequality (4) with strict inequality sign $<$. The set of all mixed Nash equilibria of a game G will be denoted by $NE^*(G)$ and the set of all strict mixed equilibria by $sNE^*(G)$.

Next, we formally define the set of all formulas that we consider.

DEFINITION 7. *For any finite set of players P, the set of formulas $\Phi(P)$ is defined recursively: (i) $\perp \in \Phi(P)$, (ii) $(A \parallel B) \in \Phi(P)$, where A and B are two disjoint subsets of P, (iii) $\phi \to \psi \in \Phi(P)$, where $\phi, \psi \in \Phi(P)$.*

If $x = \langle x_i\rangle_{i\in I}$ and $y = \langle y_i\rangle_{i\in I}$ are two tuples such that $x_a = y_a$ for any $a \in A$, then we write $x \equiv_A y$. We use this notation to define truth relation $G \vDash \phi$ between a game G and a formula ϕ:

DEFINITION 8. *For any game $G = (P, \{S_p\}_{p\in P}, \{u_p\}_{p\in P})$ and any formula $\phi \in \Phi(P)$, binary relation $G \vDash \phi$ is defined as follows:*

1. *$G \nvDash \perp$,*
2. *$G \vDash \phi \to \psi$ if and only if $G \nvDash \phi$ or $G \vDash \psi$,*
3. *$G \vDash A \parallel B$ if and only if for any $e_1, e_2 \in NE(G)$ there is $e \in NE(G)$ such that $e_1 \equiv_A e \equiv_B e_2$.*

The third part of the above definition is the key definition of this article. It formally specifies interchangeability of Nash equilibria on two sets of players in a strategic game. One can similarly define relation $\vDash$ for strict, mixed, and strict mixed Nash equilibria by replacing $NE(G)$ in this definition by $sNE(G)$, $NE^*(G)$, and $sNE^*(G)$ respectively.

3. Axioms

DEFINITION 9. *The logic of interchangeability, in addition to propositional tautologies and the Modus Ponens inference rule, consists of the following axioms:*

1. *Empty Set:* $A \parallel \varnothing$,
2. *Symmetry:* $A \parallel B \to B \parallel A$,
3. *Monotonicity:* $A \parallel B, C \to A \parallel B$,
4. *Exchange:* $A, B \parallel C \to (A \parallel B \to A \parallel B, C)$.

Recall from the introduction that these axioms first appeared in a work on independence of random variables in probability theory (Geiger et al., 1991). The same axioms also describe properties of nondeducibility in information flow (Miner More and Naumov, 2010b) and properties of non-interference in concurrency theory (More et al., 2011).

4. Examples

The soundness and completeness of the logic of interchangeability will be shown in the next two sections. The main point of this work is that the same logical principles describe properties of very different relations in probability, information flow, concurrency, and game theory. Additionally, this logical system can be used to prove non-trivial properties of these relations. Here we will consider two examples of such properties. Our first example generalizes principle (1) discussed in the introduction:

$$A \parallel B \to (C \parallel D \to (A, B \parallel C, D \to A, C \parallel B, D))$$

Note that this principle at first appears to be a result of applying the Exchange axiom on both sides of the assumption $A, B \parallel C, D$ together with the Symmetry axiom. This, however, is *not* true. Indeed, the first application of the Exchange axiom results in $A \parallel B, C, D$. By the Symmetry axiom we can conclude $B, C, D \parallel A$. In order to apply the Exchange axiom again, however, we need assumption $C \parallel B, D$, but we only are given $C \parallel D$. To prove the required, as we will see below, a more sophisticated argument is needed:

PROPOSITION 1. *For any disjoint sets of players A, B, C, and D,*

$$\vdash A \parallel B \to (C \parallel D \to (A, B \parallel C, D \to A, C \parallel B, D)).$$

Proof. Assume $A, B \parallel C, D$ and $A \parallel B$, as well as $C \parallel D$. We will prove that $A, C \parallel B, D$. First, by the Monotonicity Axiom, assumption $A, B \parallel C, D$ implies $B \parallel C, D$. By the Symmetry Axiom, $C, D \parallel B$. From assumption $C \parallel D$ and the Exchange axiom,

$$C \parallel D, B. \tag{5}$$

Next, let us return to assumption $A, B \parallel C, D$. Taking into account assumption $A \parallel B$, by Exchange axiom we have $A \parallel B, C, D$. By Symmetry, $B, C, D \parallel A$. Again by the Exchange axiom and using (5), we have $B, D \parallel A, C$. Finally, by Symmetry, $A, C \parallel B, D$. □

Next, we will prove a generalized version of the principle (2) also discussed in the introduction.

PROPOSITION 2. *For any $n > 0$ and any disjoint sets of players $A_1, \dots, A_n$ of the game,*

$$\vdash \bigwedge_{2 \le k \le n} (A_1, \dots, A_{k-1} \parallel A_k) \rightarrow (A_1 \parallel A_2, \dots, A_n).$$

Proof. Meta induction on n. If $n = 1$, then the conclusion $\varnothing \parallel A_1$ is an instance of the Empty Set axiom. Suppose now that

$$\vdash \bigwedge_{2 \le k \le n-1} (A_1, \dots, A_{k-1} \parallel A_k) \rightarrow (A_1 \parallel A_2, \dots, A_{n-1}). \tag{6}$$

We will show that

$$\vdash \bigwedge_{2 \le k \le n} (A_1, \dots, A_{k-1} \parallel A_k) \rightarrow (A_1 \parallel A_2, \dots, A_n).$$

Indeed, suppose that $\bigwedge_{2 \le k \le n} (A_1, \dots, A_{k-1} \parallel A_k)$. Thus,

$$\left(\bigwedge_{2 \le k \le n-1} (A_1, \dots, A_{k-1} \parallel A_k) \right) \wedge (A_1, \dots, A_{n-1} \parallel A_n).$$

Taking into account the induction hypothesis 6,

$$(A_1 \parallel A_2, \dots, A_{n-1}) \wedge (A_1, \dots, A_{n-1} \parallel A_n).$$

Therefore, by the Exchange axiom, $A_1 \parallel A_2, \dots, A_{n-1}, A_n$ □

5. Soundness

In this section we prove soundness of axioms 1.-4. for pure Nash equilibrium semantics. The proof of soundness for the other three semantics is similar.

THEOREM 1. *For any finite set of parties P and any $\phi \in \Phi(P)$, if $\vdash \phi$, then $G \vDash \phi$ for each game $G = (P, \{S_p\}_{p \in P}, \{u_p\}_{p \in P})$.*

Proof. It will be sufficient to verify that $G \vDash \phi$ for each axiom ϕ of the logic of interchangeability. Soundness of the Modus Ponens rule is trivial.

Empty Set Axiom. Consider any two Nash equilibria $e_1, e_2 \in NE(G)$. Let $e = e_2$. Then, $e \equiv_\varnothing e_1$ and $e \equiv_A e_2$.

Monotonicity Axiom. Consider any two Nash equilibria $e_1, e_2 \in NE(G)$. If $e \equiv_{A,B} e_1$ and $e \equiv_C e_2$, then $e \equiv_A e_1$ and $e \equiv_C e_2$.

Exchange Axiom. Consider any two Nash equilibria $e_1, e_2 \in NE(G)$. By the assumption that $A \parallel B$, there is a Nash equilibrium $e_3 \in NE(G)$ such that $e_3 \equiv_A e_1$ and $e_3 \equiv_B e_2$. Since $D \parallel C$, there is a Nash equilibrium $e_4 \in NE(G)$ such that $e_4 \equiv_D e_2$ and $e_4 \equiv_C e_1$. Finally, by the assumption the $A, B \parallel C, D$, there is a Nash equilibrium $e \in NE(G)$ such that $e \equiv_{A,B} e_3$ and $e \equiv_{C,D} e_4$. Thus, $e \equiv_A e_3 \equiv_A e_1$, $e \equiv_C e_4 \equiv_C e_1$, $e \equiv_B e_3 \equiv_B e_2$, and $e \equiv_D e_4 \equiv_D e_2$. Therefore, $e \equiv_{A,C} e_1$ and $e \equiv_{B,D} e_2$. □

6. Completeness

In this section we will prove the completeness of the axioms 1.-4. with respect to all four game semantics. This result is stated in Theorem 2. We start, however, with a sequence of lemmas in which we assume a fixed finite set of players P and a fixed maximal consistent set of formulas $X \subseteq \Phi(P)$.

6.1. Critical Sets

The key to understanding axioms 1.-4. is the notions of critical pair and critical set. Below is their combined definition and their basic properties. Later we will define a separate strategic game for each critical subset of P.

DEFINITION 10. *A set $C \subseteq P$ is called critical if there is a disjoint partition $C_1 \sqcup C_2$ of C, called a "critical partition", such that*

1. $X \nvdash C_1 \parallel C_2$,
2. $X \vdash C_1 \cap D \parallel C_2 \cap D$, *for any* $D \subsetneq C$.

LEMMA 1. *Any critical partition is a non-trivial partition.*

Proof. It will be sufficient to prove that for any set A, we have $X \vdash A \parallel \varnothing$ and $X \vdash \varnothing \parallel A$. The first statement is an instance of the Empty Set axiom, the second statement follows from the Empty Set and the Symmetry axioms. □

LEMMA 2. *$X \nvdash A \parallel B$, for any non-trivial (but not necessarily critical) partition $A \sqcup B$ of a critical set C.*

Proof. Suppose $X \vdash A \parallel B$ and let $C_1 \sqcup C_2$ be a critical partition of C. By the Monotonicity and Symmetry axioms, $X \vdash A \cap C \parallel B \cap C$. Thus,

$$X \vdash A \cap C_1, A \cap C_2 \parallel B \cap C_1, B \cap C_2. \tag{7}$$

Since $A \sqcup B$ is a non-trivial partition of C, sets A and B are both non-empty. Thus, $A \subsetneq C$ and $B \subsetneq C$. Hence, by the definition of a critical set, $X \vdash A \cap C_1 \parallel A \cap C_2$ and $X \vdash B \cap C_1 \parallel B \cap C_2$.

Note that $A \cap C$ is not empty since $A \sqcup B$ is a non-trivial partition of C. Thus, either $A \cap C_1$ or $A \cap C_2$ is not empty. Without loss of generality, assume that $A \cap C_1 \neq \varnothing$. From (7) and our earlier observation that $X \vdash A \cap C_1 \parallel A \cap C_2$, the Exchange axiom yields

$$X \vdash A \cap C_1 \parallel A \cap C_2, B \cap C_1, B \cap C_2.$$

By the Symmetry axiom,

$$X \vdash A \cap C_2, B \cap C_1, B \cap C_2 \parallel A \cap C_1. \tag{8}$$

The assumption $A \cap C_1 \neq \varnothing$ implies that $(A \cap C_2) \cup (B \cap C_1) \cup (B \cap C_2) \subsetneq C$. Hence, by the definition of a critical set,

$$X \vdash B \cap C_1 \parallel A \cap C_2, B \cap C_2.$$

By Symmetry axiom,

$$X \vdash A \cap C_2, B \cap C_2 \parallel B \cap C_1.$$

From (8) and the above statement, using the Exchange axiom,

$$X \vdash A \cap C_2, B \cap C_2 \parallel A \cap C_1, B \cap C_1.$$

Since $A \sqcup B$ is a partition of C, we can conclude that $X \vdash C_2 \parallel C_1$. By the Symmetry axiom, $X \vdash C_1 \parallel C_2$, which contradicts the assumption that $C_1 \sqcup C_2$ is a critical partition. □

LEMMA 3. *For any two disjoint subsets $A, B \subseteq P$, if $X \nvdash A \parallel B$, then there is a critical partition $C_1 \sqcup C_2$, such that $C_1 \subseteq A$ and $C_2 \subseteq B$.*

Proof. Consider the partial order $\preceq$ on set $2^A \times 2^B$ such that $(E_1, E_2) \preceq (F_1, F_2)$ if and only if $E_1 \subseteq F_1$ and $E_2 \subseteq F_2$. Define

$$\mathcal{E} = \{(E_1, E_2) \in 2^A \times 2^B \mid X \nvdash E_1 \parallel E_2\}.$$

$X \nvdash A \parallel B$ implies that $(A, B) \in \mathcal{E}$. Thus, $\mathcal{E}$ is a non-empty finite set. Take (C_1, C_2) to be a minimal element of set $\mathcal{E}$ with respect to partial order $\preceq$. □

6.2. Parity Games

In this section our proof of completeness is using slightly different arguments for the different types of semantics we consider.

For any subset $Q \subseteq P$, we define two "parity" games $PG_1(Q)$ and $PG_2(Q)$. Later we will consider such games only for Q which are critical subsets of P. For now, however, Q is just an arbitrary subset of P.

We start with an informal description of both games. Players in set Q will be referred to as "active" players, since they will be able to influence outcomes of the games. Players in the set $P \setminus Q$ are "passive": they get a pay-off, but can not influence its amount.

In the first parity game, $PG_1(Q)$, each active player picks an integer number. If the sum of all picked numbers is odd, then *each player who picked an odd number* pays a one-dollar penalty.

In the second parity game, $PG_2(Q)$, each active player also picks an integer number. If the sum of all picked numbers is odd, then *each player in the game* pays a one-dollar penalty.

In the formalization of these two games below, we assume that players only pick numbers from the set $\{0, 1\}$ and that passive players always pick number 0.

DEFINITION 11. *For any set of players $Q \subseteq P$, by parity game $PG_1(Q)$ we mean game $(P, \{S_p\}_{p \in P}, \{u_p\}_{p \in P})$ such that*

1. *set of strategies of player $p \in P$ is*

$$S_p = \begin{cases} \{0,1\} & \textit{if } p \in Q, \\ \{0\} & \textit{otherwise.} \end{cases}$$

2. *for any $p \in P$, pay-off function $u_p(\langle s_r \rangle_{r \in P}) \in \{-1, 0\}$ such that*[3]

$$u_p(\langle s_r \rangle_{r \in P}) \equiv 1 + s_p \cdot \sum_{r \in P} s_r \pmod 2.$$

LEMMA 4. *For any set of players $Q \subseteq P$,*

$$NE(PG_1(Q)) = \left\{ \langle s_p \rangle_{p \in P} \in \prod_{p \in P} S_p \;\middle|\; \sum_{p \in P} s_p \equiv 0 \pmod 2 \right\}.$$

[3] It is important to keep in mind that $-1 \equiv 1 \pmod 2$.

Proof. Follows from Definition 4 and Definition 11. □

Next, we will show that all mixed Nash equilibria of the game $PG_1(Q)$ are pure. Of course, formally speaking a pure strategy profile is a tuple of strategies and a mixed strategy profile is a tuple of tuples of real numbers, representing probabilities. For example, $\langle a_1, b_2\rangle$ is a (pure) strategy profile and

$$\langle(1.0, 0.0, 0.0), (0.0, 1.0, 0.0)\rangle$$

is a corresponding mixed strategy profile in a two-player game in which each player has three strategies. However, in the discussion below we will allow certain informality by not distinguishing these two formally different, but intuitively identical objects. This will allow us to consider pure equilibria of a game to be a subset of mixed equilibria of the same game.

LEMMA 5. *For any set of players $Q \subseteq P$,*

$$NE^*(PG_1(Q)) = \left\{ \langle s_p\rangle_{p\in P} \in \prod_{p\in P} S_p \;\middle|\; \sum_{p\in P} s_p \equiv 0 \pmod 2 \right\}.$$

Proof. It is easy to see that each strategy profile $\langle s_p\rangle_{p\in P}$ such that $\sum_{p\in P} s_p \equiv 0 \pmod 2$ is a mixed Nash equilibrium. To show the converse, suppose that $\langle s_p\rangle_{p\in P}$ is mixed Nash equilibrium of the game $PG_1(Q)$. If at least one of players p_0 has a mixed strategy in this equilibrium, then there is a non-zero probability that the sum $\sum_{p\in P} s_p$ is odd. Thus, strongly dominant strategy for all other players is to pick 0 with probability 1. If each of the other players picks 0 with probability 1, then strongly dominant strategy for player p_0 is also to pick 0 with probability 1. Thus, all strategies in the equilibrium $\{s_p\}_{p\in P}$ are pure. □

We will use game $PG_1(Q)$ to prove completeness with respect to semantics of interchangeability of pure equilibria and semantics of interchangeability of mixed equilibria. Note, however, that game $PG_1(Q)$ has only one *strict* Nash equilibrium – action profile consisting of all 0s. This forces us to use the other parity game, $PG_2(Q)$, to handle the strict equilibria cases.

DEFINITION 12. *For any set of players $Q \subseteq P$, by parity game $PG_2(Q)$ we mean the game $(P, \{S_p\}_{p\in P}, \{u_p\}_{p\in P})$ such that*

1. *set of strategies of party $p \in P$ is*

$$S_p = \begin{cases} \{0,1\} & \text{if } p \in Q, \\ \{0\} & \text{otherwise.} \end{cases}$$

2. *pay off function* $u_p(\langle s_p\rangle_{p\in P}) \in \{-1, 0\}$ *is the same for all players* $p \in P$. *We denote it simply by* u *and require that*

$$u(\langle s_p\rangle_{p\in P}) \equiv 1 + \sum_{p\in P} s_p \pmod 2.$$

LEMMA 6. *For any set of players* $Q \subseteq P$,

$$sNE(PG_2(Q)) = \left\{ \langle s_p\rangle_{p\in P} \in \prod_{p\in P} S_p \;\middle|\; \sum_{p\in P} s_p \equiv 0 \pmod 2 \right\}.$$

Proof. Follows from Definition 4 and Definition 12. □

LEMMA 7. *For any set of players* $Q \subseteq P$,

$$sNE^*(PG_2(Q)) = \left\{ \langle s_p\rangle_{p\in P} \in \prod_{p\in P} S_p \;\middle|\; \sum_{p\in P} s_p \equiv 0 \pmod 2 \right\}.$$

Proof. It is easy to see that each strategy profile $\langle s_p\rangle_{p\in P}$ such that $\sum_{p\in P} s_p \equiv 0 \pmod 2$ is a strict mixed Nash equilibrium. To show the converse, suppose that $\langle s_p\rangle_{p\in P}$ is strict mixed Nash equilibrium of the game $PG_1(Q)$. Assume that at least one of players p_0 is using a mixed strategy $(\rho, 1-\rho)$ in this equilibrium, where $0 < \rho < 1$. That is, the mixed strategy of player p_0 is to pick 0 with probability ρ and 1 with probability $1-\rho$. In addition, assume that in the equilibrium $\langle s_p\rangle_{p\in P}$, probability that the sum of all picks by the other players (rather than p_0) being odd is τ and being even is $1-\tau$. Then, expected value of the penalty of the player p_0 is

$$E = \rho\tau + (1-\rho)(1-\tau).$$

Case I: If $\tau < 1/2$, then

$$\begin{aligned} E &= \rho\tau + (1-\rho)(1-\tau) = \rho\tau + (1-\rho)(1-\tau) - \tau + \tau \\ &= (\rho-1)\tau + (1-\rho)(1-\tau) + \tau = (1-\rho)(1-2\tau) + \tau > \tau \\ &= 1\cdot\tau + 0\cdot(1-\tau). \end{aligned}$$

Thus, expected value of penalty of player p_0 will decrease if she switches from mixed strategy $(\rho, 1-\rho)$ to pure strategy $(1, 0)$. Therefore, $\langle s_p\rangle_{p\in P}$ is not a mixed Nash equilibrium, which is a contradiction.
Case II: If $\tau = 1/2$, then

$$E = \frac{1}{2}\rho + \left(1 - \frac{1}{2}\right)(1-\rho) = \frac{1}{2}\rho + \frac{1}{2}(1-\rho) = 1.$$

Thus, expected value of penalty of p_0 does not depend on the choice of mixed strategy $(\rho, 1-\rho)$. Therefore, although $\langle s_p\rangle_{p\in P}$ could be a mixed Nash equilibrium, it cannot be a strict mixed Nash equilibrium, which is a contradiction.
Case III: If $\tau > 1/2$, then

$$\begin{aligned} E &= \rho\tau + (1-\rho)(1-\tau) = \rho\tau + (1-\rho)(1-\tau) - (1-\tau) + (1-\tau) \\ &= \rho\tau - \rho(1-\tau) + (1-\tau) = \rho(2\tau - 1) + (1-\tau) > 1-\tau \\ &= 0\cdot\tau + 1\cdot(1-\tau). \end{aligned}$$

Thus, expected value of penalty of player p_0 will decrease if it switches from mixed strategy $(\rho, 1-\rho)$ to pure strategy $(0,1)$. Therefore, $\langle s_p\rangle_{p\in P}$ is not a mixed Nash equilibrium, which again is a contradiction. □

6.3. Parity Games of Critical Sets

The rest of all four proofs of completeness is identical. When below we talk about "Nash equilibria of the parity game $PG(Q)$", one should interpret any such statement as a set of four different statements: (i) about pure Nash equilibria of game $PG_1(Q)$, (ii) about mixed Nash equilibria of game $PG_1(Q)$, (iii) about pure strict Nash equilibria of game $PG_2(Q)$, and (iv) about mixed strict Nash equilibria of game $PG_2(Q)$. Thus, for example, we can combine Lemma 4, Lemma 5, Lemma 6, and Lemma 7 into the following single statement:

LEMMA 8. *For any set of players $Q \subseteq P$,*

$$NE(PG(Q)) = \left\{ \langle s_p\rangle_{p\in P} \in \prod_{p\in P} S_p \;\middle|\; \sum_{p\in P} s_p \equiv 0 \pmod 2 \right\}.$$

□

LEMMA 9. *For any set of players $Q \subseteq P$, set $NE(PG(Q))$ is not empty.*

Proof. Follows from Lemma 8.

LEMMA 10. *If A and B are two disjoint subsets of Q, then $PG(Q) \nvDash A \parallel B$ if and only if $A \sqcup B$ is a non-trivial partition of the set Q.*

Proof. $(\Rightarrow)$: Suppose that $A \sqcup B$ is *not* a non-trivial partition of Q. There are three possible cases to consider:
Case I: A is empty. Thus, $PG(P) \vDash A \parallel B$ due to soundness of the Empty Set and Symmetry axioms (See Theorem 1).

Case II: B is empty. Thus, $PG(P) \vDash A \parallel B$ due to soundness of the Empty Set axiom.
Case III: there is $q_0 \in Q \setminus (A \cup B)$. Let e', e'' be any two Nash equilibria of the game $PG(Q)$. We will show that there is $e \in NE(PG(Q))$ such that $e' \equiv_A e \equiv_B e''$. Indeed, consider strategy profile $\langle e_p \rangle_{p \in P}$ such that

$$e_p \equiv \begin{cases} e'_p & \text{if } p \in A, \\ e''_p & \text{if } p \in B, \\ \sum_{a \in A} e'_a + \sum_{b \in B} e''_b & \text{if } p = q_0, \\ 0 & \text{otherwise.} \end{cases} \pmod 2$$

Note that

$$\begin{aligned} \sum_{p \in P} e_p &= e_{q_0} + \sum_{a \in A} e_a + \sum_{b \in B} e_b \equiv \\ &\equiv \sum_{a \in A} e'_a + \sum_{b \in B} e''_b + \sum_{a \in A} e'_a + \sum_{b \in B} e''_b \equiv 0 \pmod 2. \end{aligned}$$

Therefore, by Lemma 8, $e \in NE(PG(Q))$.
$(\Leftarrow)$: Suppose that $PG(P) \vDash A \parallel B$ and $A \sqcup B$ is a non-trivial partition of Q. Let $a_0 \in A$ and $b_0 \in B$. Consider strategy profiles $e^A = \langle e^A_p \rangle_{p \in P}$ and $e^B = \langle e^B_p \rangle_{p \in P}$ such that $e^A_p = 0$ for each $p \in P$ and

$$e^B_p \equiv \begin{cases} 1 & \text{if } p = a_0, \\ 1 & \text{if } p = b_0, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 8, $e^A, e^B \in NE(PG(Q))$. By assumption $PG(Q) \vDash A \parallel B$ there must be $e \in NE(PG(Q))$ such that $e^A \equiv_A e \equiv_B e^B$. Since $A \sqcup B$ is a partition of Q, we have

$$\sum_{p \in P} e_p = \sum_{a \in A} e_a + \sum_{b \in B} e_b = \sum_{a \in A} e^A_a + \sum_{b \in B} e^B_b = e^A_{a_0} + e^B_{b_0} = 0 + 1 = 1.$$

Contradiction with Lemma 8. □

6.4. Game Composition

Informally, by a composition of several games we mean a game in which each of the composed games is played independently. Pay-off of any player is defined as the sum of the pay-offs in the individual games.

DEFINITION 13. *Let $\{G^i\}_{i \in I} = \{(P, \{S^i_p\}_{p \in P}, \{u^i_p\}_{p \in P})\}_{i \in I}$ be a finite family of strategic games between the same set of players P. By product game $\prod_i G^i$ we mean game $(P, \{S_p\}_{p \in P}, \{u_p\}_{p \in P})$ such that*

1. $S_p = \prod_i S_p^i$,
2. $u_p(\langle\langle s_p^i\rangle_{i\in I}\rangle_{p\in P}) = \sum_i u_p^i(\langle s_p^i\rangle_{p\in P})$.

Of course, any mixed strategy of the product game could be viewed as product of mixed strategies in the original games, where distributions of mixed strategies in the original game are assumed to be stochastically independent. The following theorem is true for Nash equilibria, strict Nash equilibria, mixed Nash equilibria, and strict mixed Nash equilibria. The proof that we give is written for standard Nash equilibria, but similar proofs can be given in the three other cases.

LEMMA 11.
$$NE\left(\prod_i G^i\right) = \prod_i NE(G^i).$$

Proof. First, assume that $\langle e_p\rangle_{p\in P} = \langle\langle e_p^i\rangle_{i\in I}\rangle_{p\in P} \in NE\left(\prod_i G^i\right)$. We will need to show that $\langle e_p^i\rangle_{p\in P} \in NE(G^i)$ for any $i \in I$. Indeed, suppose that for some $i_0 \in I$, some $p_0 \in P$, and some $s_0 \in S_{p_0}$ we have

$$u_{p_0}^{i_0}(\langle e_p^{i_0}\rangle_{p\in P}[p_0 \mapsto s_0]) > u_{p_0}^{i_0}(\langle e_p^{i_0}\rangle_{p\in P}). \tag{9}$$

Define strategy profile $\langle \hat{e}_p\rangle_{p\in P} = \langle\langle \hat{e}_p^i\rangle_{i\in I}\rangle_{p\in P}$ of the game $\prod_i G^i$ as follows:
$$\hat{e}_p^i \equiv \begin{cases} s_0 & \text{if } i = i_0 \text{ and } p = p_0, \\ e_p^i & \text{otherwise.} \end{cases}$$

Note that, taking into account inequality (9),

$$\begin{aligned} u_{p_0}(\langle \hat{e}_p\rangle_{p\in P}) &= \sum_{i\in I} u_{p_0}^i(\langle \hat{e}_p^i\rangle_{p\in P}) = u_{p_0}^{i_0}(\langle \hat{e}_p^{i_0}\rangle_{p\in P}) + \sum_{i\neq i_0} u_{p_0}^i(\langle \hat{e}_p^i\rangle_{p\in P}) = \\ &= u_{p_0}^{i_0}(\langle e_p^{i_0}\rangle_{p\in P}[p_0 \mapsto s_0]) + \sum_{i\neq i_0} u_{p_0}^i(\langle e_p^i\rangle_{p\in P}) > \\ &> u_{p_0}^{i_0}(\langle e_p^{i_0}\rangle_{p\in P}) + \sum_{i\neq i_0} u_{p_0}^i(\langle e_p^i\rangle_{p\in P}) = \\ &= \sum_i u_{p_0}^i(\langle e_p^i\rangle_{p\in P}) = u_{p_0}(\langle e_p\rangle_{p\in P}), \end{aligned}$$

which is a contradiction with the assumption that $\langle e_p\rangle_{p\in P}$ is a Nash equilibrium of the game $\prod_i G^i$.

Next, assume that $\{\langle e_p^i\rangle_{p\in P}\}_{i\in I}$ is such a set that for any $i \in I$,

$$\langle e_p^i\rangle_{p\in P} \in NE(G^i) \tag{10}$$

We will prove that $\langle\langle e_p^i\rangle_{i\in I}\rangle_{p\in P} \in NE\left(\prod_i G^i\right)$. Indeed, consider any p_0 and any $\langle s_0^i\rangle_{i\in I} \in \prod_{i\in I} S_{p_0}^i$. By assumption (10) and Definition 4, for any $i \in I$

$$u_{p_0}^i(\langle e_p^i\rangle_{p\in P}[p_0 \mapsto s_0^i]) \le u_{p_0}^i(\langle e_p^i\rangle_{p\in P}).$$

Thus,

$$\begin{aligned} u_{p_0}(\langle\langle e_p^i\rangle_{i\in I}\rangle_{p\in P}[p_0 \mapsto \langle s_0^i\rangle_{i\in I}]) &= \sum_{i\in I} u_{p_0}^i(\langle e_p^i\rangle_{p\in P}[p_0 \mapsto s_0^i]) \le \\ &\le \sum_{i\in I} u_{p_0}^i(\langle e_p^i\rangle_{p\in P}) = u_{p_0}(\langle\langle e_p^i\rangle_{i\in I}\rangle_{p\in P}). \end{aligned}$$

Therefore, $\langle\langle e_p^i\rangle_{i\in I}\rangle_{p\in P} \in NE\left(\prod_i G^i\right)$. □

LEMMA 12. *For any disjoint subsets A and B of the set P, if each of the games $\{G_i\}_{i\in I}$ has at least one Nash equilibrium, then*

$$\prod_i G^i \vDash A \parallel B \qquad \textit{iff} \qquad \forall i\,(G^i \vDash A \parallel B).$$

Proof. $(\Rightarrow)$: By the assumption of the theorem, for any $i \in I$ there is at least one Nash equilibrium $\langle e_p^i\rangle_{p\in P}$ of the game G^i. Suppose that $\prod_i G^i \vDash A \parallel B$ and consider any $i_0 \in I$. We will prove that $G^{i_0} \vDash A \parallel B$. Indeed, let $f = \langle f_p\rangle_{p\in P} \in NE(G^{i_0})$ and $g = \langle g_p\rangle_{p\in P} \in NE(G^{i_0})$. We will construct $h = \langle h_p\rangle_{p\in P} \in NE(G^{i_0})$ such that $f \equiv_A h \equiv_B g$. To construct such equilibrium, consider strategy profiles $\hat{f} = \langle\langle \hat{f}_p^i\rangle_{i\in I}\rangle_{p\in P}$ and $\hat{g} = \langle\langle \hat{g}_p^i\rangle_{i\in I}\rangle_{p\in P}$ for the game $\prod_i G^i$ such that

$$\hat{f}_p^i = \begin{cases} f_p & \text{if } i = i_0 \\ e_p^i & \text{otherwise} \end{cases} \tag{11}$$

and

$$\hat{g}_p^i = \begin{cases} g_p & \text{if } i = i_0 \\ e_p^i & \text{otherwise} \end{cases} \tag{12}$$

By Lemma 11, $\hat{f}, \hat{g} \in NE(\prod_i G^i)$. Thus, by assumption $\prod_i G^i \vDash A \parallel B$, there must be $\hat{h} \in NE(\prod_i G^i)$ such that

$$\hat{f} \equiv_A \hat{h} \equiv_B \hat{g} \tag{13}$$

Define strategy profile h for the game G^{i_0} to be $\langle h_p^{i_0}\rangle_{p\in P}$. By Lemma 11, $h \in NE(G^{i_0})$. From statements (13), (11), and (12), it follows that $f \equiv_A h \equiv_B g$.

$(\Leftarrow)$: Assume that $\forall i\,(G^i \vDash A \parallel B)$. Let $f = \langle\langle f_p^i\rangle_{i\in I}\rangle_{p\in P} \in NE(\prod_i G^i)$ and $g = \langle\langle g_p^i\rangle_{i\in I}\rangle_{p\in P} \in NE(\prod_i G^i)$. We will show that there is $e \in$

$NE(\prod_i G^i)$ such that $f \equiv_A e \equiv_B g$. Indeed, by Lemma 11, $\langle f^i_p \rangle_{p\in P} \in NE(G^i)$ and $\langle g^i_p \rangle_{p\in P} \in NE(G^i)$ for any $i \in I$. Thus, by the assumption, for any $i \in I$ there is $\langle e^i_p \rangle_{p\in P} \in NE(G^i)$ such that $\langle g^i_p \rangle_{p\in P} \equiv_A \langle e^i_p \rangle_{p\in P} \equiv_B \langle g^i_p \rangle_{p\in P}$. Thus,

$$\langle\langle f^i_p \rangle_{i\in I}\rangle_{p\in P} \equiv_A \langle\langle e^i_p \rangle_{i\in I}\rangle_{p\in P} \equiv_B \langle\langle g^i_p \rangle_{i\in I}\rangle_{p\in P}.$$

Pick strategy profile e to be $\langle\langle e^i_p \rangle_{i\in I}\rangle_{p\in P}$ and notice that, by Lemma 11, $e \in NE(\prod_i G^i)$. □

6.5. Completeness: the final steps

We are now ready to prove the completeness theorem, which is stated below.

THEOREM 2. *For any set of players P and any $\phi \in \Phi(P)$, if $\nvdash \phi$, then*

1. *there is a game G with set of players P such that $G \nvDash \phi$ with respect to Nash equilibrium semantics,*
2. *there is a game G with set of players P such that $G \nvDash \phi$ with respect to strict Nash equilibrium semantics,*
3. *there is a game G with set of players P such that $G \nvDash \phi$ with respect to mixed Nash equilibrium semantics,*
4. *there is a game G with set of players P such that $G \nvDash \phi$ with respect to mixed strict Nash equilibrium semantics.*

Proof. All four parts will be proven together. Suppose that $\nvdash \phi$ and let X be a maximal consistent set of formulas containing $\neg\phi$. Let $\{C_i\}_{i\in I}$ be the finite set of all critical subsets of P. Let $PG(C_i)$ be the parity game between set of players P. Pick game G to be $\prod_{i\in I} PG(C_i)$.

LEMMA 13. *For any disjoint subsets A and B of the set P,*

$$G \vDash A \parallel B \quad \textit{iff} \quad A \parallel B \in X.$$

Proof. $(\Rightarrow)$: Assume that $A \parallel B \notin X$. Thus, $X \nvdash A \parallel B$ due to the maximality of set X. Hence, by Lemma 3, there is a critical set $C \subseteq P$ such that $(A\cap C) \sqcup (B\cap C)$ is a critical partition of C. Thus, by Lemma 1, $(A\cap C) \sqcup (B\cap C)$ is a non-trivial partition of the set C. Hence, by Lemma 10, $PG(C) \nvDash A\cap C \parallel B \cap C$. Thus, due to soundness of the Monotonicity and Symmetry axioms (Theorem 1), $PG(C) \nvDash A \parallel B$.

Hence, by Lemma 9 and Lemma 12, $\prod_{i \in I} G^i \nvDash A \parallel B$. In other words, $G \nvDash A \parallel B$.
$(\Leftarrow)$: Suppose that $A \parallel B \in X$. Due to Lemma 9 and Lemma 12, it will be sufficient to show that $PG(C_i) \vDash A \parallel B$ for any $i \in I$. Assume that $PG(C_{i_0}) \nvDash A \parallel B$ for some $i_0 \in I$. Thus, due to soundness of Symmetry and Monotonicity axioms, $PG(C_{i_0}) \nvDash A \cap C_{i_0} \parallel B \cap C_{i_0}$. Then, by Lemma 10, $A \cap C_{i_0} \sqcup B \cap C_{i_0}$ is a non-trivial partition of C_{i_0}. Hence, by Lemma 2, $X \nvdash A \cap C_{i_0} \parallel B \cap C_{i_0}$. Therefore, by Monotonicity and Symmetry axioms, $X \nvdash A \parallel B$. □

LEMMA 14. *For any formula ψ in $\Phi(P)$,*

$$G \vDash \psi \qquad \textit{iff} \qquad \psi \in X$$

Proof. Induction on the structural complexity of ψ. Base case is proven in Lemma 13. The induction step follows from the maximality and the consistency of the set X. □
To finish the proof of the completeness theorem, note that $\neg\phi \in X$. Thus, $\phi \notin X$ due to consistency of X. Therefore, by Lemma 14, $G \nvDash \phi$. □

7. Conclusion

7.1. An n-ary Interchangeability Relation

In this article, we have considered the interchangeability relation $A \parallel B$ between two sets of players. This binary relation can be naturally generalized to the n-ary relation

$$A_1 \parallel A_2 \parallel \ldots \parallel A_n$$

between n sets of players by changing part 3 of Definition 8 to

3. $G \vDash A_1 \parallel A_2 \parallel \ldots \parallel A_n$ if and only if for any $e_1, e_2, \ldots, e_n \in NE(G)$ there is $e \in NE(G)$ such that $e \equiv_{A_i} e_i$ for each $i \leq n$.

It turns out, however, that the n-ary interchangeability relation can be expressed through the binary interchangeability relation studied in this article. For example, in the case $n = 3$, the following result holds:

THEOREM 3. *For any game $G = (P, \{S_p\}_{p \in P}, \{u_p\}_{p \in P})$ and any disjoint subsets A, B, and C of set the P,*

$$G \vDash (A \parallel B \parallel C) \Longleftrightarrow (A \parallel B, C) \wedge (B \parallel C).$$

Proof.
($\Rightarrow$) : Assume $G \vDash A \parallel B \parallel C$. To prove $G \vDash A \parallel B, C$, consider any two equilibria $e_1, e_2 \in NE(G)$. We will show that there is equilibrium $e \in NE(G)$ such that $e_1 \equiv_A e \equiv_{B,C} e_2$. Indeed, by the assumption, there must be equilibrium $e \in NE(G)$ such that $e \equiv_A e_1$, $e \equiv_B e_2$, and $e \equiv_C e_2$.

To prove $G \vDash B \parallel C$, consider any two equilibria $e_1, e_2 \in NE(G)$. We will show that there is equilibrium $e \in NE(G)$ such that $e_1 \equiv_B e \equiv_C e_2$. Indeed, by the assumption, there must be equilibrium $e \in NE(G)$ such that $e \equiv_A e_1$, $e \equiv_B e_1$, and $e \equiv_C e_2$.
($\Leftarrow$) : Assume $G \vDash A \parallel B, C$ and $G \vDash B \parallel C$. To prove $G \vDash A \parallel B \parallel C$, consider any three equilibria $e_1, e_2, e_3 \in NE(G)$. We will show that there is equilibrium $e \in NE(G)$ such that $e \equiv_A e_1$, $e \equiv_B e_2$, and $e \equiv_C e_3$. Indeed, by the assumption $G \vDash B \parallel C$, there must be equilibrium $e_4 \in NE(G)$ such that $e_2 \equiv_B e_4 \equiv_C e_3$. By the assumption $G \vDash A \parallel B, C$, there must be equilibrium $e \in NE(G)$ such that $e_1 \equiv_A e \equiv_{B,C} e_4$. Therefore, $e \equiv_A e_1$, $e \equiv_B e_4 \equiv_B e_2$, and $e \equiv_C e_4 \equiv_C e_3$. □

7.2. Interchangeability in Zero-sum Games

Axioms 1.-4. are sound for all games, including zero-sum games. However, these axioms are not complete with respect to zero-sum games. For example, if $P = \{a, b\}$, then $G \vDash a \parallel b$ for any zero-sum game G with the set of players P. This follows from interchangeability property of Nash equilibria in zero-sum two-player games (Nash, 1951). At the same time, Game G_3, that has been discussed in introduction to this article, shows that three-player zero-sum games might have a non-trivial set of Nash equilibria. Thus, it is not obvious how and if the interchangeability theorem could be generalized to multi-player games. The complete axiomatization of properties of interchangeability in multi-player zero-sum games remains an open question.

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