Equilibria Interchangeability in Cellular Games

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Abstract

The notion of interchangeability has been introduced by John Nash in one of his original papers on equilibria. This paper studies properties of Nash equilibria interchangeability in cellular games that model behavior of infinite chain of homogeneous economic agents. The paper shows that there are games in which a strategy of any given player is interchangeable with strategies of players in an arbitrary large neighborhood of the given player, but is not interchangeable with the strategy of a remote player outside of the neighborhood. The main technical result is a sound and complete logical system describing universal properties of interchangeability common to all cellular games.

1 Introduction

Cellular Games. An one-dimensional cellular automaton is an infinite row of cells that transition from one state to another under certain rules. The rules are assumed to be identical for all cells. Usually, rules are chosen in such a way that the next state of each cell is determined by the current states of the cell itself and its two neighboring cells.

Harjes and Naumov [2] introduced an object similar to cellular automaton that they called *cellular game*. They proposed to view each cell as a player, whose pay-off function depends on the strategy of the cell itself and the strategies of its two neighbors. The cellular games are *homogeneous* in the sense that all players of a given game have the same pay-off function. Such games can model rational behavior of linearly-spaced homogeneous agents. Linearly-spaced economies have been studied by economists before [11, 10].

Consider an example of a cellular game that we call G_1. In this game each player has only three strategies. We identify these strategies with congruence classes [0], [1], and [2] of $\mathbb{Z}_3$. Each player is rewarded for either matching the strategy of her left neighbor or choosing strategy which is one more (in $\mathbb{Z}_3$) than the strategy of the left neighbor. An example of a Nash equilibrium in this game is shown on Figure 1.

Interchangeability. The notion of interchangeability goes back to one of Nash's original papers [6] on equilibria in strategic games. Interchangeability is easiest to define in a two-player game: players in such a game are interchangeable if for any two

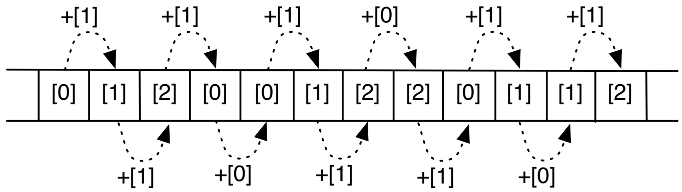

Figure 1: A Nash equilibrium of game G_1. Arrows explain dependencies between the strategy of a player and the strategy of her right neighbor.

equilibria $\langle a_1, b_1\rangle$ and $\langle a_2, b_2\rangle$, strategy profiles $\langle a_1, b_2\rangle$ and $\langle a_2, b_1\rangle$ are also equilibria. Players in any two-player *zero-sum* game are interchangeable [6].

Consider now a multiplayer game with set of players P. We say that players $p \in P$ and $q \in P$ are interchangeable if for any two equilibria $\langle e'_i\rangle_{i\in P}$ and $\langle e''_i\rangle_{i\in P}$ of the game, there is an equilibrium $\langle e_i\rangle_{i\in P}$ of the same game such that $e_p = e'_p$ and $e_q = e''_q$. The strategies of the other players in the strategy profile $\langle e_i\rangle_{i\in P}$ are not restricted. We denote this by $p \parallel q$. For example, it is easy to see that for the game described in the previous section, players p and q are interchangeable if they are not adjacent. In other words, $p \parallel q$ if and only if $|p - q| > 1$ and $p \neq q$. This is the relation whose properties in cellular games we study in this paper.

We now consider another game, that we call G_2. Each player in game G_2 can either pick a strategy from $\mathbb{Z}_3$ or switch to playing matching pennies game with both of her neighbors. In the latter case, the strategy is a pair (y_1, y_2), where $y_1, y_2 \in \{head, tail\}$. Value y_1 is the strategy in the matching pennies game against the left neighbor and value y_2 is the strategy against the right neighbor. The formal definition of this game is given in Section 5.3.

If the left and the right neighbors of a player choose, respectively, elements x and z from set $\mathbb{Z}_3$ such that $z - x \in \{[0], [1]\}$, then the player is not paid no matter what her strategy is. Otherwise, player is rewarded to start matching pennies games with both neighbors. If two adjacent players both play matching pennies game, then player on the right is rewarded to *match* the penny of the player on the left and player on the left is rewarded to *mismatch* the penny of the player on the right. An example of a Nash equilibria in such game is shown on Figure 2. The set of all Nash equilibria of this game

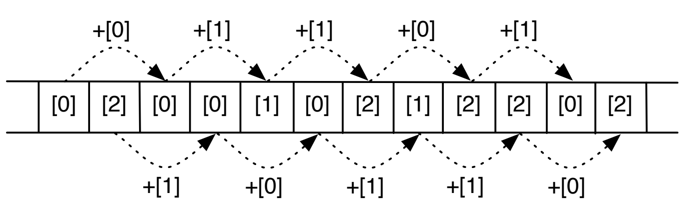

Figure 2: A Nash equilibrium of game G_2.

consists of all strategy profiles in which each player chooses an element of $\mathbb{Z}_3$ in such a way that for each player p player $p+2$ never chooses strategy that is two-more (in $\mathbb{Z}_3$)

than the strategy of player p. An interesting property of this game (see Theorem 3) is that $p \parallel q$ if and only if $|p-q| \neq 2$. Thus, any two adjacent players are interchangeable, but players that are two-apart are not interchangeable. Note that we have achieved this by using each player to synchronize strategies of her two neighbors. The ability of a player to do this significantly relies on the fact that the pay-off function of each player is computed based on the choice of strategies by the player herself and her two adjacent neighbors. A player can not synchronize in the same way strategies of the players that are 2-players away to the left and 2-players away to the right. Thus, it would be natural to assume that it is impossible to construct a cellular game in which players, say 1000-players away, are non-interchangeable, but players that are closer are interchangeable. If this hypothesis is true, then the following property is true for all cellular games:

$$(p \parallel (p+1) \wedge p \parallel (p+2) \wedge \cdots \wedge p \parallel (p+999)) \rightarrow p \parallel (p+1000). \tag{1}$$

The main surprising result of this paper is that such a game does exist. Namely, we prove that for any $n \geq 1$ there is a cellular game G_n in which any two players p and q are interchangeable if and only if $|p-q| = n$. The construction of such game for $n > 2$ is non-trivial. Strategies of players in our game are special $(n-1) \times 2$ matrices of elements from $\mathbb{Z}_{n+1}$.

Another way to state our result is to say that statement (1) is not a universal property of cellular games. Naturally, one can ask what statements are universal properties of all cellular games. We answer this question by giving a sound and complete axiomatization of such properties consisting of just the following three axioms:

1. Reflexivity: $a \parallel a \rightarrow a \parallel b$,
2. Homogeneity: $a \parallel b \rightarrow (a+c) \parallel (b+c)$,
3. Symmetry: $a \parallel b \rightarrow b \parallel a$.

The proof of completeness takes multiple instances of the discussed above cellular game G_n and combines them into a single cellular game needed to finish the proof.

The interchangeability relation between players of multi-player game could be further generalized to a relation between two sets of players. Properties of this relation are completely axiomatizable [7] by Geiger, Paz, and Pearl axioms originally proposed to describe properties of independence in probability theory [1]. The same axioms also describe properties of Sutherland's [12] nondeducibility relation in information flow theory [4] and of a non-interference relation in concurrency theory [5]. Naumov and Simonelli [9] described interchangeability properties between two sets of players in zero-sum games.

Functional Dependence. Our work is closely related to paper by Harjes and Naumov [2] on functional dependence in cellular games. Strategy of player p functionally determines strategy of player q in a cellular game if any two Nash equilibria of the game that agree on the strategy of player p also agree on the strategy of player q. We denote this by $p \rhd q$. The functional dependence relation between players can not be expressed through interchangeability and vice versa. Harjes and Naumov gave complete axiomatization of functional dependence relation for cellular games with finite set of strategies:

1. Reflexivity: $a \rhd a$,

2. Transitivity: $a \rhd b \to (b \rhd c \to a \rhd c)$,

3. Homogeneity: $a \rhd b \to (a + c) \rhd (b + c)$,

4. Symmetry: $a \rhd b \to b \rhd a$.

In spite of certain similarity between these axioms and our axioms for interchangeability, the proofs of completeness are very different. The completeness proof techniques used by Harjes and Naumov are based on properties of Fibonacci numbers and, to the best of our knowledge, can not be adopted to our setting. Similarly, the $(n-1) \times 2$-matrix based game G_n that we use in the current paper can not be used to prove the results obtained in [2].

The paper is structured as following. In Section 2, we give the formal definition of a cellular game and introduce formal syntax and semantics of our theory. In Section 3, we list the axioms of our logical systems and review some related notations. In Section 4, we prove soundness of this logical system. In Section 5 we define games G_n and sketch the proof of completeness. Detailed proof of completeness is available in the full version of this paper [8]. Section 6 concludes.

2 Syntax and Semantics

In this section we formally define cellular games, Nash equilibrium, and introduce the formal syntax and the formal semantics of our logical system. The definition of interchangeability predicate $a \parallel b$ is a part of the formal semantics specification in Definition 4 below.

Definition 1 *Let Φ be the minimal set of formulas that satisfies the following conditions:*

1. $\bot \in \Phi$,

2. $a \parallel b \in \Phi$ for each integer $a, b \in \mathbb{Z}$,

3. if $\varphi \in \Phi$ and $\psi \in \Phi$, then $\varphi \to \psi \in \Phi$.

Definition 2 *Cellular game is a pair (S, u), where*

1. S is any set of "strategies",

2. u is a "pay-off" function from S^3 to the set of real numbers $\mathbb{R}$.

The domain of the function u in the above definition is S^3 because the pay-off of each player is determined by her own strategy and the strategies of her two neighbors. By a strategy profile of a cellular game (S, u) we mean any tuple $\langle s_i \rangle_{i \in \mathbb{Z}}$ such that $s_i \in S$ for each $i \in \mathbb{Z}$.

Definition 3 *A Nash equilibrium of a game (S, u) is a strategy profile $\langle e_i \rangle_{i \in \mathbb{Z}}$ such that $u(e_{i-1}, s, e_{i+1}) \leq u(e_{i-1}, e_i, e_{i+1})$, for each $i \in \mathbb{Z}$ and each $s \in S$.*

By $NE(G)$ we denote the set of all Nash equilibria of a cellular game G.

Lemma 1 *For each $k \in \mathbb{Z}$, if $\langle e_i \rangle_{i\in\mathbb{Z}}$ is a Nash equilibrium of a cellular game, then $\langle e_{i+k} \rangle_{i\in\mathbb{Z}}$ is a Nash equilibrium of the same game.* ⊠

Definition 4 *For any formula $\varphi \in \Phi$ and any cellular game G, relation $G \vDash \varphi$ is defined recursively as follows:*

1. $G \nvDash \bot$,
2. $G \vDash a \parallel b$ *if and only if for each* $\langle e'_i \rangle_{i\in\mathbb{Z}} \in NE(G)$ *and each* $\langle e''_i \rangle_{i\in\mathbb{Z}} \in NE(G)$, *there is* $\langle e_i \rangle_{i\in\mathbb{Z}} \in NE(G)$ *such that* $e_a = e'_a$ *and* $e_b = e''_b$,
3. $G \vDash \psi \to \chi$ *if and only if* $G \nvDash \psi$ *or* $G \vDash \chi$.

3 Axioms

Our logical system, in addition to propositional tautologies in the language Φ and the Modus Ponens inference rule, contains the following axioms:

1. Reflexivity: $a \parallel a \to a \parallel b$,
2. Homogeneity: $a \parallel b \to (a+c) \parallel (b+c)$,
3. Symmetry: $a \parallel b \to b \parallel a$.

We write $\vdash \varphi$ if formula φ is provable in our logical system. The next lemma gives an example of a proof in our logical system. This lemma will later be used in the proof of the completeness theorem.

Lemma 2 *If $|a-b| = |c-d|$, then $\vdash a \parallel b \to c \parallel d$.*

Proof. Due to Symmetry axiom, without loss of generality we can assume that $a > b$ and $c > d$. Thus, assumption $|a-b| = |c-d|$ implies that $a-b = c-d$. Hence, $c-a = d-b$. Then, by Homogeneity axiom,

$$\vdash a \parallel b \to (a + (c-a)) \parallel (b + (d-b)).$$

In other words, $\vdash a \parallel b \to c \parallel d$. ⊠

4 Soundness

Soundness of propositional tautologies and Modus Ponens inference rules is straightforward. We prove soundness of each of the remaining axioms of our logical system as a separate lemma.

Lemma 3 (reflexivity) *If $G \vDash a \parallel a$, then $G \vDash a \parallel b$ for each $a, b \in \mathbb{Z}$.*

Proof. Let $\langle e'_i \rangle_{i\in\mathbb{Z}} \in NE(G)$ and $\langle e''_i \rangle_{i\in\mathbb{Z}} \in NE(G)$, we need to show that there exists $\langle e_i \rangle_{i\in\mathbb{Z}} \in NE(G)$ such that $e_a = e'_a$ and $e_b = e''_b$. Indeed, by assumption $G \vDash a \parallel a$, there exists $\langle e'''_i \rangle_{i\in\mathbb{Z}} \in NE(G)$ such that $e'''_a = e'_a$ and $e'''_a = e''_a$. Thus, $e'_a = e''_a$. Take $\langle e_i \rangle_{i\in\mathbb{Z}} \in NE(G)$ to be $\langle e''_i \rangle_{i\in\mathbb{Z}} \in NE(G)$. Then $e_a = e''_a = e'_a$ and $e_b = e''_b$. ⊠

Lemma 4 (homogeneity) *If $G \vDash a \parallel b$, then $G \vDash (a+c) \parallel (b+c)$, for each $a, b, c \in \mathbb{Z}$.*

Proof. Let $\langle e'_i \rangle_{i\in\mathbb{Z}} \in NE(G)$ and $\langle e''_i \rangle_{i\in\mathbb{Z}} \in NE(G)$, we need to show that there exists $\langle e_i \rangle_{i\in\mathbb{Z}} \in NE(G)$ such that $e_{a+c} = e'_{a+c}$ and $e_{b+c} = e''_{b+c}$. By Lemma 1, $\langle e'_{i+c} \rangle_{i\in\mathbb{Z}} \in NE(G)$ and $\langle e''_{i+c} \rangle_{i\in\mathbb{Z}} \in NE(G)$. Then, by assumption $G \vDash a \parallel b$, there exists $\langle e'''_i \rangle_{i\in\mathbb{Z}} \in NE(G)$ such that $e'''_a = e'_{a+c}$ and $e'''_b = e''_{b+c}$. Lemma 1 implies that $\langle e'''_{i-c} \rangle_{i\in\mathbb{Z}} \in NE(G)$. Take $\langle e_i \rangle_{i\in\mathbb{Z}} \in NE(G)$ to be $\langle e'''_{i-c} \rangle_{i\in\mathbb{Z}}$. Then, $e_{a+c} = e'''_{(a+c)-c} = e'''_a = e'_{a+c}$ and $e_{b+c} = e'''_{(b+c)-c} = e'''_b = e''_{b+c}$. ⊠

Lemma 5 (symmetry) *If $G \vDash a \parallel b$, then $G \vDash b \parallel a$ for each $a, b \in \mathbb{Z}$.*

Proof. Let $\langle e'_i \rangle_{i\in\mathbb{Z}} \in NE(G)$ and $\langle e''_i \rangle_{i\in\mathbb{Z}} \in NE(G)$. We need to show that there is $\langle e_i \rangle_{i\in\mathbb{Z}} \in NE(G)$ such that $e_b = e'_b$ and $e_a = e''_a$. Indeed, by assumption $G \vDash a \parallel b$, there exists $\langle e_i \rangle_{i\in\mathbb{Z}} \in NE(G)$ such that $e_a = e''_a$ and $e_b = e'_b$. ⊠

5 Completeness

In this section we sketch completeness of our logical system by arguing that for each formula φ such that $\nvdash \varphi$ there exists a cellular game G such that $G \nvDash \varphi$. The detailed proof of completeness is available in the full version of this paper [8]. The game G will be constructed as a composition of multiple cellular "mini" games G_n. Throughout this paper, by $[k]_n$ we mean the equivalence class of k modulo n. In other words, $[k]_n \in \mathbb{Z}_n$. We sometimes omit subscript n in the expression $[k]_n$ if the value of the subscript is clear from the context.

5.1 Game G_0

We start with a very simple game G_0.

Definition 5 *Let G_0 be pair $(\mathbb{Z}_2, 0)$, where pay-off function is constant 0.*

Theorem 1 *$G_0 \vDash a \parallel b$ if and only if $a \neq b$.*

5.2 Game G_1

Let us now recall from the introduction the definition of game G_n for $n = 1$ and prove its important property. Each player in this game has only three strategies. We identify these strategies with congruence classes in $\mathbb{Z}_3$. Each player is rewarded if she either matches the strategy of her left neighbor or chooses the strategy one more (in $\mathbb{Z}_3$) than the strategy of the left neighbor. This is formally specified by the definition below.

Definition 6 *Let game G_1 be pair $(\mathbb{Z}_3, u)$, where*

$$u(x, y, z) = \begin{cases} 1 & \text{if } y \neq x + [2]_3, \\ 0 & \text{otherwise.} \end{cases}$$

An example of a Nash equilibrium of game G_1 is depicted in Figure 1 in the introduction.

Theorem 2 *$G_1 \vDash a \parallel b$ if and only if $|a - b| > 1$.*

5.3 Game G_2

We now recall definition of game G_n for $n = 2$ from the introduction. Each player in this game can either pick a strategy from $\mathbb{Z}_3$ or switch to playing matching pennies game with both of her neighbors. In the latter case, the strategy is a pair (y_1, y_2), where $y_1, y_2 \in \{head, tail\}$. Value y_1 is the strategy in the matching pennies game against the left neighbor and value y_2 is the strategy against the right neighbor.

If the left and the right neighbors of a player choose, respectively, elements x and z from set $\mathbb{Z}_3$ such that $z - x \in \{[0]_3, [1]_3\}$, then the player is not paid no matter what her strategy is. Otherwise, player is rewarded to start matching pennies games with both neighbors. If two adjacent players both play matching pennies game, then player on the right is rewarded to *match* the penny of the player on the left and player on the left is rewarded to *mismatch* the penny of the player on the right. We formally capture the above description of the game G_2 in the following definition.

Definition 7 *Let game G_2 be pair (S, u), where*

1. $S = \mathbb{Z}_3 \cup \{head, tail\}^2$. *In other words, strategy of each player in this game could be either a congruence class from $\mathbb{Z}_3$ or a pair (y_1, y_2) such that each of u and v is either "head" or "tail".*

2. *pay-off function $u(x, y, z) = u_1(x, y, z) + u_2(x, y) + u_3(y, z)$ is the sum of three separate pay-offs specified below:*

   (a) *if either at least one of x and z is not in $\mathbb{Z}_3$ or if they are both in $\mathbb{Z}_3$ and $x + [2]_3 = z$, then pay-off $u_1(x, y, z)$ rewards player y not to be an element of $\mathbb{Z}_3$:*

   $$u_1(x, y, z) = \begin{cases} 1 & \text{if } y \in \{head, tail\}^2, \\ 0 & \text{otherwise.} \end{cases}$$

   in all other cases $u_1(x, y, z)$ is equal to zero.

   (b) *if both $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in $\{head, tail\}^2$, then pay-off $u_2(x, y)$ rewards player y if $x_2 = y_1$:*

   $$u_2((x_1, x_2), (y_1, y_2)) = \begin{cases} 1 & \text{if } x_2 = y_1, \\ 0 & \text{otherwise.} \end{cases}$$

   in all other cases $u_2(x, y)$ is equal to zero.

   (c) *if both $y = (y_1, y_2)$ and $z = (z_1, z_2)$ are in $\{head, tail\}^2$, then pay-off $u_3(y, z)$ rewards player y if $y_2 \neq z_1$:*

   $$u_2((y_1, y_2), (z_1, z_2)) = \begin{cases} 1 & \text{if } y_2 \neq z_1, \\ 0 & \text{otherwise.} \end{cases}$$

   in all other cases $u_3(y, z)$ is equal to zero.

An example of a Nash equilibrium of the game G_2 has been given in the introduction in Figure 2.

Theorem 3 $G_2 \vDash a \parallel b$ *if and only if either $|a - b| = 1$ or $|a - b| > 2$.*

5.4 Game G_n: general case

In this section we define game G_n for $n \geq 3$. The set of strategies S^n of the game G_n is $(\mathbb{Z}_{n+1} \times \mathbb{Z}_{n+1})^{n-1}$. We visually represent elements of S^n as $(n-1) \times 2$ matrices whose elements belong to $\mathbb{Z}_{n+1}$.

Definition 8 *Pay-off function*

$$u\left(\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \\ \vdots & \vdots \\ x_{n-1,1} & x_{n-1,2} \end{pmatrix}, \begin{pmatrix} y_{1,1} & y_{1,2} \\ y_{2,1} & y_{2,2} \\ \vdots & \vdots \\ y_{n-1,1} & y_{n-1,2} \end{pmatrix}, \begin{pmatrix} z_{1,1} & z_{1,2} \\ z_{2,1} & z_{2,2} \\ \vdots & \vdots \\ z_{n-1,1} & z_{n-1,2} \end{pmatrix}\right) \tag{2}$$

is equal to 1 if the following conditions are satisfied:

1. $y_{1,1} = [0]_{n+1}$,
2. $y_{k+1,2} + z_{k+1,1} - x_{k,2} - y_{k,1} \in \{[0]_{n+1}, [1]_{n+1}\}$, *for every* $1 \leq k < n-1$,
3. $z_{1,2} - x_{n-1,2} - y_{n-1,1} \in \{[0]_{n+1}, [1]_{n+1}\}$.

if at least one of the above conditions is not satisfied, then pay-off function (2) is equal to 0.

Theorem 4 $G_n \vDash a \parallel b$ *if and only if* $|a-b| \neq n$ *and* $a \neq b$, *where* $n \geq 3$.

5.5 Product of games

In this section we define product operation on cellular games. Informally, product is a composition of several cellular games played concurrently and independently. The pay-off of a player in the product of the games is the sum of the pay-offs in the individual games.

Definition 9 *Product* $\prod_{i=1}^{n} G^i$ *of any finite family of cellular games* $\{G^i\}_{i=1}^{n} = \{(S^i, u^i)\}_{i=1}^{n}$ *is the cellular game* $G = (S, u)$, *where*

1. S *is Cartesian product* $\prod_{i=1}^{n} S^i$,
2. $u(\langle x^i \rangle_{i \leq n}, \langle y^i \rangle_{i \leq n}, \langle z^i \rangle_{i \leq n}) = \sum_{i=1}^{n} u^i(x^i, y^i, z^i)$.

Theorem 5 *If* $a, b \in \mathbb{Z}$ *and each of the cellular games in family* $\{G^i\}_{i=1}^{n} = \{(S^i, u^i)\}_{i=1}^{n}$ *has at least one Nash equilibrium, then* $\prod_{i=1}^{n} G^i \vDash a \parallel b$ *if and only if* $G^i \vDash a \parallel b$ *for each* $i \leq n$.

5.6 Game G_∞

Definition 10 *Game* G_∞ *is pair* $(\{0\}, 0)$, *whose first element is the single-element set containing number* 0 *and whose second component is the constant function equal to* 0.

Lemma 6 $G_\infty \vDash a \parallel b$ *for each* $a, b \in \mathbb{Z}$.

5.7 Completeness: final steps

We are now ready to state and prove the completeness theorem for our logical system.

Theorem 6 (completeness) *For each formula $\varphi \in \Phi$, if $G \vDash \varphi$ for each cellular game G, then $\vdash \varphi$.*

Proof. Suppose that $\nvdash \varphi$. Let M be any maximal consistent subset of Φ such that $\neg\varphi \in M$. Thus, $\varphi \notin M$ due to assumption of the consistency of M. There are two cases that we consider separately.
Case I: $0 \parallel 0 \in M$.

Lemma 7 *$\psi \in M$ if and only if $G_\infty \vDash \psi$ for each $\psi \in \Phi$.*

Proof. Induction on structural complexity of formula ψ. If φ is $\bot$, then $\psi \notin M$ due to consistency of set M. At the same time, $G_\infty \nvDash \psi$ by Definition 4.

Suppose now that ψ is formula $a \parallel b$. By Reflexivity axiom, $0 \parallel 0 \to a \parallel b$. Thus, $a \parallel b \in M$, by the assumption $0 \parallel 0 \in M$ and due to maximality of set M. At the same time, $G_\infty \vDash a \parallel b$ by Lemma 6.

Finally, case when ψ is an implication $\sigma \to \tau$ follows in the standard way from maximality and consistency of set M and Definition 4. ⊠
To finish the proof of the theorem, note that $\varphi \notin M$ implies, by the above lemma, that $G_\infty \nvDash \varphi$.
Case II: $0 \parallel 0 \notin M$. Let $Sub(\varphi)$ be the finite set of all $(p, q) \in \mathbb{Z}^2$ such that $p \parallel q$ is a subformula of φ. Define game G to be

$$\prod\{G_{|p-q|} \mid (p,q) \in Sub(\varphi) \text{ and } p \parallel q \notin M\}.$$

Lemma 8 *For each subformula ψ of formula φ,*

$$\psi \in M \quad \textit{if and only if} \quad G \vDash \psi.$$

To finish the proof of the theorem, note that $\varphi \notin M$ implies, by Lemma 8, that $G \nvDash \varphi$. ⊠

6 Conclusion

In this paper we have shown the existence of cellular games in which Nash equilibria are interchangeable for near by players, but not interchangeable for far-away players. We also gave complete axiomatization of all propositional properties of interchangeability of cellular games. Possible next step in this work could be the axiomatization of properties of Nash equilibria for two-dimensional or even circular cellular games. Circular economies has been studies in the economics literature before [3].

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