

Equilibria Interchangeability in Cellular Games

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Abstract

The notion of interchangeability has been introduced by John Nash in one of his original papers on equilibria. This article studies properties of Nash equilibria interchangeability in cellular games that model behavior of infinite chain of homogeneous economic agents. The article shows that there are games in the which strategy of any given player is interchangeable with strategies of players in an arbitrary large neighborhood of the given player, but is not interchangeable with the strategy of a remote player outside of the neighborhood. The main technical result is a sound and complete logical system describing universal properties of interchangeability common to all cellular games.

1 Introduction

Cellular Games. A one-dimensional cellular automaton is an infinite row of cells that transition from one state to another under certain rules. The rules are assumed to be identical for all cells. Usually, rules are chosen in such a way that the next state of each cell is determined by the current states of the cell itself and its two neighboring cells.

Harjes and Naumov [5] introduced an object similar to cellular automaton that they called *cellular game*. They proposed to view each cell as a player, whose pay-off function depends on the strategy of the cell itself and the strategies of its two neighbors. The cellular games are *homogeneous* in the sense that all players of a given game have the same pay-off function. Such games can model rational behavior of linearly-spaced homogeneous agents. Linearly-spaced economies have been studied by economists before [15, 14]. Cellular game is special case of a graphical game [7], where pay-off function of any player depends only on the strategies of her neighbors in an adjacency graph. Existence of Nash equilibria [2] in such games, algorithmic complexity of finding equilibria [4], efficient algorithms for finding equilibria in special cases [1], and information flow in graphical games [6] have been studied before.

Consider an example of a cellular game that we call G_1 . In this game each player has only three strategies. We identify these strategies with congruence

classes $[0]$, $[1]$, and $[2]$ of \mathbb{Z}_3 . Each player is rewarded for either matching the strategy of her left neighbor or choosing strategy which is one more (in \mathbb{Z}_3) than the strategy of the left neighbor. An example of a Nash equilibrium in this game is shown in Figure 1.

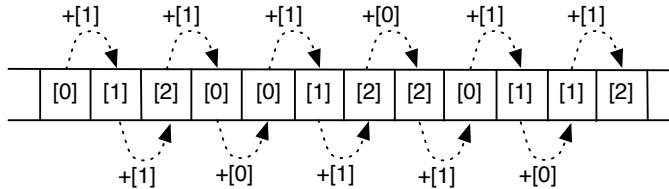


Figure 1: A Nash equilibrium of game G_1 . Arrows explain dependencies between the strategy of a player and the strategy of her right neighbor.

Interchangeability. The notion of interchangeability goes back to one of Nash’s original papers [11] on equilibria in strategic games. Interchangeability is easiest to define in a two-player game: players in such a game are interchangeable if for any two equilibria $\langle a_1, b_1 \rangle$ and $\langle a_2, b_2 \rangle$, strategy profiles $\langle a_1, b_2 \rangle$ and $\langle a_2, b_1 \rangle$ are also equilibria. Players in any two-player *zero-sum* game are interchangeable [11].

Consider now a multiplayer game with set of players P . We say that players $p \in P$ and $q \in P$ are interchangeable if for any two equilibria $\langle e'_i \rangle_{i \in P}$ and $\langle e''_i \rangle_{i \in P}$ of the game, there is an equilibrium $\langle e_i \rangle_{i \in P}$ of the same game such that $e_p = e'_p$ and $e_q = e''_q$. We denote this by $p \parallel q$. The strategies of the other players in the strategy profile $\langle e_i \rangle_{i \in P}$ are not restricted. For example, it is easy to see that for the game described in the previous section, players p and q are interchangeable if they are not adjacent. In other words, $p \parallel q$ if and only if $|p - q| > 1$. This is the relation whose properties in cellular games we study in this article.

We now consider another game, that we call G_2 . Each player in game G_2 can either pick a strategy from \mathbb{Z}_3 or switch to playing the matching pennies game with both of her neighbors. In the latter case, the strategy is a pair (y_1, y_2) , where $y_1, y_2 \in \{head, tail\}$. Value y_1 is the strategy in the matching pennies game against the left neighbor and value y_2 is the strategy against the right neighbor. The formal definition of this game is given in Section 5.3.

If the left and the right neighbors of a player choose, respectively, elements x and z from set \mathbb{Z}_3 such that $z - x \in \{[0], [1]\}$, then the player is not paid no matter what her strategy is. Otherwise, the player is rewarded to start matching pennies games with both neighbors. If two adjacent players both play matching pennies game, then the player on the right is rewarded to *match* the penny of the player on the left and player on the left is rewarded to *mismatch* the penny of the player on the right. An example of a Nash equilibria in such game is shown on Figure 2. The set of all Nash equilibria of this game consists of all strategy

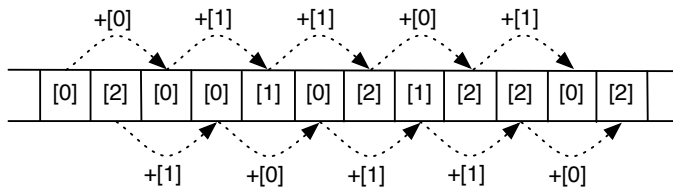


Figure 2: A Nash equilibrium of game G_2 .

profiles in which each player chooses an element of \mathbb{Z}_3 in such a way that for each player p player $p + 2$ never chooses strategy that is two-more (in \mathbb{Z}_3) than the strategy of player p . An interesting property of this game (see Theorem 3) is that $p \parallel q$ if and only if $|p - q| \neq 2$ and $p \neq q$. Thus, any two adjacent players are interchangeable, but players that are two-apart are not interchangeable. Note that we have achieved this by using each player to synchronize strategies of her two neighbors. The ability of a player to do this significantly relies on the fact that the pay-off function of each player is computed based on the choice of strategies by the player herself and her two adjacent neighbors. A player can not synchronize in the same way strategies of the players that are 2-players away to the left and 2-players away to the right. Thus, it would be natural to assume that it is impossible to construct a cellular game in which players, say 1000-players away, are non-interchangeable, but players that are closer are interchangeable. If this hypothesis is true, then the following property is true for all cellular games:

$$(p \parallel (p + 1) \wedge p \parallel (p + 2) \wedge \dots \wedge p \parallel (p + 999)) \rightarrow p \parallel (p + 1000). \quad (1)$$

The main surprising result of this article is that such a game does exist. Namely, we prove that for any $n \geq 1$ there is a cellular game G_n in which any two players p and q are interchangeable if and only if $|p - q| = n$. The construction of such game for $n > 2$ is non-trivial. Strategies of players in our game are special $(n - 1) \times 2$ matrices of elements from \mathbb{Z}_{n+1} .

Another way to state our result is to say that statement (1) is not a universal property of cellular games. Naturally, one can ask what statements are universal properties of all cellular games. We answer this question by giving a sound and complete axiomatization of such properties consisting of just the following three axioms:

1. Reflexivity: $a \parallel a \rightarrow a \parallel b$,
2. Homogeneity: $a \parallel b \rightarrow (a + c) \parallel (b + c)$,
3. Symmetry: $a \parallel b \rightarrow b \parallel a$.

The proof of completeness takes multiple instances of the discussed above cellular game G_n and combines them into a single cellular game needed to finish the proof.

The interchangeability relation between players of a multi-player game could be further generalized to a relation between two sets of players. Properties of this relation are completely axiomatizable [12] by Geiger, Paz, and Pearl axioms originally proposed to describe properties of independence in probability theory [3]. The same axioms also describe properties of Sutherland’s [16] nondeducibility relation in information flow theory [9] and of a non-interference relation in concurrency theory [10]. Naumov and Simonelli [13] described interchangeability properties between two sets of players in zero-sum games.

Functional Dependence. Our work is closely related to a paper by Harjes and Naumov [5] on functional dependence in cellular games. Strategy of player p functionally determines strategy of player q in a cellular game if any two Nash equilibria of the game that agree on the strategy of player p also agree on the strategy of player q . We denote this by $p \triangleright q$. The functional dependence relation between players can not be expressed through interchangeability and vice versa. Harjes and Naumov gave complete axiomatization of functional dependence relation for cellular games with finite set of strategies:

1. Reflexivity: $a \triangleright a$,
2. Transitivity: $a \triangleright b \rightarrow (b \triangleright c \rightarrow a \triangleright c)$,
3. Homogeneity: $a \triangleright b \rightarrow (a + c) \triangleright (b + c)$,
4. Symmetry: $a \triangleright b \rightarrow b \triangleright a$.

In spite of certain similarity between these axioms and our axioms for interchangeability, the proofs of completeness are very different. The completeness proof techniques used by Harjes and Naumov are based on properties of Fibonacci numbers and, to the best of our knowledge, can not be adapted to our setting. Similarly, the $(n - 1) \times 2$ -matrix based game G_n that we use in the current article can not be used to prove the results obtained in [5].

The article is structured as follows. In Section 2, we give the formal definition of a cellular game and introduce formal syntax and semantics of our theory. In Section 3, we list the axioms of our logical systems and review some related notations. In Section 4, we prove soundness of this logical system. The rest of the article is dedicated to the proof of completeness. In Section 5.1, Section 5.2, and Section 5.3, we define special cases of the game G_n for $n = 0, 1, 2$ and prove their key properties. Games G_1 and G_2 have already been informally discussed above. In Section 8, we give general definition of G_n for $n \geq 3$ and prove its properties. We combine results about games G_n for all $n \geq 0$ in Section 5.5. In Section 5.7, we introduce a very simple game G_∞ and prove its properties. In Section 5.6, we define a product operation on cellular games that can be used to combine several cellular games into one. In Section 5.8, we use the product of multiple games G_n to finish the proof of completeness. Section 6 concludes.

2 Syntax and Semantics

In this section we formally define cellular games, Nash equilibrium, and introduce the formal syntax and the formal semantics of our logical system. The definition of interchangeability predicate $a \parallel b$ is a part of the formal semantics specification in Definition 4 below.

Definition 1 *Let Φ be the minimal set of formulas that satisfies the following conditions:*

1. $\perp \in \Phi$,
2. $a \parallel b \in \Phi$ for each integer $a, b \in \mathbb{Z}$,
3. if $\varphi \in \Phi$ and $\psi \in \Phi$, then $\varphi \rightarrow \psi \in \Phi$.

Definition 2 *A cellular game is a pair (S, u) , where*

1. S is any set of “strategies”,
2. u is a “pay-off” function from S^3 to the set of real numbers \mathbb{R} .

The domain of the function u in the above definition is S^3 because the pay-off of each player is determined by her own strategy and the strategies of her two neighbors. By a strategy profile of a cellular game (S, u) we mean any tuple $\langle s_i \rangle_{i \in \mathbb{Z}}$ such that $s_i \in S$ for each $i \in \mathbb{Z}$.

Definition 3 *A Nash equilibrium of a game (S, u) is a strategy profile $\langle e_i \rangle_{i \in \mathbb{Z}}$ such that $u(e_{i-1}, s, e_{i+1}) \leq u(e_{i-1}, e_i, e_{i+1})$, for each $i \in \mathbb{Z}$ and each $s \in S$.*

Note that in this article we only consider pure Nash equilibria. By $NE(G)$ we denote the set of all Nash equilibria of a cellular game G . Below by $\langle e_{i+k} \rangle_{i \in \mathbb{Z}}$ we mean strategy profile in which player i uses strategy of player $i+k$ from profile $\langle e_i \rangle_{i \in \mathbb{Z}}$.

Lemma 1 *For each $k \in \mathbb{Z}$, if $\langle e_i \rangle_{i \in \mathbb{Z}}$ is a Nash equilibrium of a cellular game, then $\langle e_{i+k} \rangle_{i \in \mathbb{Z}}$ is a Nash equilibrium of the same game. \square*

Definition 4 *For any formula $\varphi \in \Phi$ and any cellular game G , relation $G \models \varphi$ is defined recursively as follows:*

1. $G \not\models \perp$,
2. $G \models a \parallel b$ if and only if for each $\langle e'_i \rangle_{i \in \mathbb{Z}} \in NE(G)$ and each $\langle e''_i \rangle_{i \in \mathbb{Z}} \in NE(G)$, there is $\langle e_i \rangle_{i \in \mathbb{Z}} \in NE(G)$ such that $e_a = e'_a$ and $e_b = e''_b$,
3. $G \models \psi \rightarrow \chi$ if and only if $G \not\models \psi$ or $G \models \chi$.

3 Axioms

Our logical system, in addition to propositional tautologies in the language Φ and the Modus Ponens inference rule, contains the following axioms:

1. Reflexivity: $a \parallel a \rightarrow a \parallel b$,
2. Homogeneity: $a \parallel b \rightarrow (a + c) \parallel (b + c)$,
3. Symmetry: $a \parallel b \rightarrow b \parallel a$.

We write $\vdash \varphi$ if formula φ is provable in our logical system. The next lemma gives an example of a proof in our logical system. This lemma will later be used in the proof of the completeness theorem.

Lemma 2 *If $|a - b| = |c - d|$, then $\vdash a \parallel b \rightarrow c \parallel d$.*

Proof. Due to Symmetry axiom, without loss of generality we can assume that $a > b$ and $c > d$. Thus, assumption $|a - b| = |c - d|$ implies that $a - b = c - d$. Hence, $c - a = d - b$. Then, by Homogeneity axiom,

$$\vdash a \parallel b \rightarrow (a + (c - a)) \parallel (b + (d - b)).$$

In other words, $\vdash a \parallel b \rightarrow c \parallel d$. \(\square\)

4 Soundness

Soundness of propositional tautologies and Modus Ponens inference rules is straightforward. We prove soundness of each of the remaining axioms of our logical system as a separate lemma.

Lemma 3 (reflexivity) *If $G \models a \parallel a$, then $G \models a \parallel b$ for each $a, b \in \mathbb{Z}$.*

Proof. Let $\langle e'_i \rangle_{i \in \mathbb{Z}} \in NE(G)$ and $\langle e''_i \rangle_{i \in \mathbb{Z}} \in NE(G)$, we need to show that there exists $\langle e_i \rangle_{i \in \mathbb{Z}} \in NE(G)$ such that $e_a = e'_a$ and $e_b = e''_b$. Indeed, by assumption $G \models a \parallel a$, there exists $\langle e'''_i \rangle_{i \in \mathbb{Z}} \in NE(G)$ such that $e'''_a = e'_a$ and $e'''_a = e''_a$. Thus, $e'_a = e''_a$. Take $\langle e_i \rangle_{i \in \mathbb{Z}} \in NE(G)$ to be $\langle e'''_i \rangle_{i \in \mathbb{Z}} \in NE(G)$. Then $e_a = e''_a = e'_a$ and $e_b = e''_b$. \(\square\)

Lemma 4 (homogeneity) *If $G \models a \parallel b$, then $G \models (a + c) \parallel (b + c)$, for each $a, b, c \in \mathbb{Z}$.*

Proof. Let $\langle e'_i \rangle_{i \in \mathbb{Z}} \in NE(G)$ and $\langle e''_i \rangle_{i \in \mathbb{Z}} \in NE(G)$, we need to show that there exists $\langle e_i \rangle_{i \in \mathbb{Z}} \in NE(G)$ such that $e_{a+c} = e'_{a+c}$ and $e_{b+c} = e''_{b+c}$. By Lemma 1, $\langle e'_{i+c} \rangle_{i \in \mathbb{Z}} \in NE(G)$ and $\langle e''_{i+c} \rangle_{i \in \mathbb{Z}} \in NE(G)$. Then, by assumption $G \models a \parallel b$, there exists $\langle e'''_i \rangle_{i \in \mathbb{Z}} \in NE(G)$ such that $e'''_a = e'_{a+c}$ and $e'''_b = e''_{b+c}$. Lemma 1 implies that $\langle e'''_{i-c} \rangle_{i \in \mathbb{Z}} \in NE(G)$. Take $\langle e_i \rangle_{i \in \mathbb{Z}} \in NE(G)$ to be $\langle e'''_{i-c} \rangle_{i \in \mathbb{Z}}$. Then, $e_{a+c} = e'''_{(a+c)-c} = e'''_a = e'_{a+c}$ and $e_{b+c} = e'''_{(b+c)-c} = e'''_b = e''_{b+c}$. \(\square\)

Lemma 5 (symmetry) *If $G \models a \parallel b$, then $G \models b \parallel a$ for each $a, b \in \mathbb{Z}$.*

Proof. Let $\langle e'_i \rangle_{i \in \mathbb{Z}} \in NE(G)$ and $\langle e''_i \rangle_{i \in \mathbb{Z}} \in NE(G)$. We need to show that there is $\langle e_i \rangle_{i \in \mathbb{Z}} \in NE(G)$ such that $e_b = e'_b$ and $e_a = e''_a$. Indeed, by assumption $G \models a \parallel b$, there exists $\langle e_i \rangle_{i \in \mathbb{Z}} \in NE(G)$ such that $e_a = e''_a$ and $e_b = e'_b$. \square

5 Completeness

In this section we prove completeness of our logical system by showing that for each formula φ such that $\not\models \varphi$ there exists a cellular game G such that $G \not\models \varphi$. The game G will be constructed as a composition of multiple cellular “mini” games G_n . The section is structured as following: first we define game G_n in special cases of $n = 0$ and also recall games G_1 and G_2 already defined in the introduction. Next, we define a significantly more complicated game G_n for $n > 2$. For each value of n , game G_n is such that two players i and j in this are interchangeable if and only if $|i - j| = n$. In the end of this section, we define game G_∞ and combine some of the games G_n into a single game that provides “canonical model” for the proof of completeness.

Throughout this article, by $[k]_n$ we mean the equivalence class of k modulo n . In other words, $[k]_n \in \mathbb{Z}_n$. We sometimes omit subscript n in the expression $[k]_n$ if the value of the subscript is clear from the context. While proving properties of the game G_n , we will find useful the following technical lemma:

Lemma 6 *For any $n \geq 1$, any $u, v \in \mathbb{Z}_n$, and any $k \geq n$, there is a sequence of classes $z_1, \dots, z_k \in \mathbb{Z}_n$ such that*

1. $z_1 = u$,
2. $z_k = v$,
3. $z_{i+1} - z_i \in \{[0]_n, [1]_n\}$ for each $i < k$.

\square

5.1 Game G_0

We start with a very simple game G_0 .

Definition 5 *Let G_0 be pair $(\mathbb{Z}_2, 0)$, where pay-off function is constant 0.*

Lemma 7 *The set of all Nash equilibria of the game G_0 is set of all possible strategy profiles of this game.* \square

Theorem 1 $G_0 \models a \parallel b$ if and only if $a \neq b$.

Proof. (\Rightarrow) : Suppose that $G_0 \models a \parallel b$ and $a = b$. Consider strategy profiles $\langle e'_k \rangle_{k \in \mathbb{Z}}$ and $\langle e''_k \rangle_{k \in \mathbb{Z}}$ such that $e'_k = [0]$ and $e''_k = [1]$ for each $k \in \mathbb{Z}$. By Lemma 7, strategy profiles $\langle e'_k \rangle_{k \in \mathbb{Z}}$ and $\langle e''_k \rangle_{k \in \mathbb{Z}}$ are Nash equilibria of the game G_0 . Thus, by the assumption $G_0 \models a \parallel b$, there must exist Nash equilibrium $\langle e_k \rangle_{k \in \mathbb{Z}}$ such that $e_a = e'_a$ and $e_b = e'_b$. Recall that $a = b$. Thus, $[0]_2 = e'_a = e_a = e_b = e'_b = [1]_2$, which is a contradiction.

(\Leftarrow) : Assume that $a \neq b$ and consider any two Nash equilibria $\langle e'_k \rangle_{k \in \mathbb{Z}}$ and $\langle e''_k \rangle_{k \in \mathbb{Z}}$ of the game G_0 . We need to show that there is Nash equilibrium $\langle e_k \rangle_{k \in \mathbb{Z}}$ such that $e_a = e'_a$ and $e_b = e'_b$. Indeed, consider strategy profile $\langle e_k \rangle_{k \in \mathbb{Z}}$ such that

$$e_k = \begin{cases} e'_a & \text{if } k = a, \\ e'_b & \text{if } k = b, \\ [0]_2 & \text{otherwise.} \end{cases}$$

By Lemma 7, strategy profile $\langle e_k \rangle_{k \in \mathbb{Z}}$ is a Nash equilibrium of the game G_0 . \square

5.2 Game G_1

Let us now recall from the introduction the definition of game G_n for $n = 1$ and prove its important property. Each player in this game has only three strategies. We identify these strategies with congruence classes in \mathbb{Z}_3 . Each player is rewarded if she either matches the strategy of her left neighbor or chooses the strategy one more (in \mathbb{Z}_3) than the strategy of the left neighbor. This is formally specified by the definition below.

Definition 6 Let game G_1 be pair (\mathbb{Z}_3, u) , where

$$u(x, y, z) = \begin{cases} 1 & \text{if } y \neq x + [2]_3, \\ 0 & \text{otherwise.} \end{cases}$$

An example of a Nash equilibrium of game G_1 is depicted in Figure 1 in the introduction.

Lemma 8 Strategy profile $\langle e_k \rangle_{k \in \mathbb{Z}}$ is a Nash equilibrium of the game G_1 if and only if $e_k - e_{k-1} \in \{[0]_3, [1]_3\}$ for each $k \in \mathbb{Z}$. \square

Theorem 2 $G_1 \models a \parallel b$ if and only if $|a - b| > 1$.

Proof. Without loss of generality, we can assume that $a \leq b$.

(\Rightarrow) First, suppose that $a = b$. Consider strategy profiles $e' = \langle e'_k \rangle_{k \in \mathbb{Z}}$ and $e'' = \langle e''_k \rangle_{k \in \mathbb{Z}}$ of the game G_1 such that $e'_k = [0]_3$ and $e''_k = [1]_3$ for each $k \in \mathbb{Z}$. By Lemma 8, $e', e'' \in NE(G_1)$. Assume that $G_1 \models a \parallel b$, then there must exist $e = \langle e_k \rangle_{k \in \mathbb{Z}} \in NE(G_1)$ such that $e_a = e'_a$ and $e_b = e'_b$. Thus, $[0]_3 = e'_a = e_a = e_b = e'_b = [1]_3$, due to the assumption $a = b$. Therefore, $[0]_3 = [1]_3$, which is a contradiction.

Next, assume that $b = a + 1$. Consider strategy profile $e' = \langle e'_k \rangle_{k \in \mathbb{Z}}$ such that $e'_k = [0]_3$ for each $k \in \mathbb{Z}$ and strategy profile $e'' = \langle e''_k \rangle_{k \in \mathbb{Z}}$ such that $e''_k = [2]_3$ for each $k \in \mathbb{Z}$. By Lemma 8, $e', e'' \in NE(G_1)$. At the same time, due to the same Lemma 8, there can not be $e = \langle e_k \rangle_{k \in \mathbb{Z}} \in NE(G_1)$ such that $e_a = [0]_3$ and $e_b = e_{a+1} = [2]_3$. Therefore, $G_1 \not\equiv a \parallel b$.

(\Leftarrow) Assume that $|a - b| > 1$. Thus, $a + 1 < b$ due to the assumption $a \leq b$. Consider any two equilibria $e' = \langle e'_k \rangle_{k \in \mathbb{Z}}$ and $e'' = \langle e''_k \rangle_{k \in \mathbb{Z}}$ of game G_1 . We will show that there is $e = \langle e_k \rangle_{k \in \mathbb{Z}} \in NE(G_1)$ such that $e_a = e'_a$ and $e_b = e''_b$. Indeed, since $a + 1 < b$, by Lemma 6, there must exist sequence of congruence classes $x_a, x_{a+1}, x_{a+2}, \dots, x_b$ in \mathbb{Z}_3 such that $x_k - x_{k-1} \in \{[0]_3, [1]_3\}$ for each $a < k \leq b$. Define strategy profile $e = \langle e_k \rangle_{k \in \mathbb{Z}}$ as

$$e_k = \begin{cases} x_a & \text{if } k < a, \\ x_k & \text{if } a \leq k \leq b, \\ x_b & \text{if } b < k. \end{cases}$$

By Lemma 8, $e \in NE(G_1)$. \(\square\)

5.3 Game G_2

We now recall definition of game G_n for $n = 2$ from the introduction. Each player in this game can either pick a strategy from \mathbb{Z}_3 or switch to playing matching pennies game with both of her neighbors. In the latter case, the strategy is a pair (y_1, y_2) , where $y_1, y_2 \in \{head, tail\}$. Value y_1 is the strategy in the matching pennies game against the left neighbor and value y_2 is the strategy against the right neighbor.

If the left and the right neighbors of a player choose, respectively, elements x and z from set \mathbb{Z}_3 such that $z - x \in \{[0]_3, [1]_3\}$, then the player is not paid no matter what her strategy is. Otherwise, player is rewarded to start matching pennies games with both neighbors. If two adjacent players both play matching pennies game, then player on the right is rewarded to *match* the penny of the player on the left and player on the left is rewarded to *mismatch* the penny of the player on the right. We formally capture the above description of the game G_2 in the following definition.

Definition 7 Let game G_2 be pair (S, u) , where

1. $S = \mathbb{Z}_3 \cup \{head, tail\}^2$. In other words, strategy of each player in this game could be either a congruence class from \mathbb{Z}_3 or a pair (y_1, y_2) such that each of y_1 and y_2 is either “head” or “tail”.
2. pay-off function (informally, of player “y”) is $u(x, y, z) = u_1(x, y, z) + u_2(x, y) + u_3(y, z)$ which is the sum of three separate pay-offs specified below:

- (a) if either at least one of x and z is not in \mathbb{Z}_3 or if they are both in \mathbb{Z}_3 and $x + [2]_3 = z$, then pay-off $u_1(x, y, z)$ rewards player y not to be an element of \mathbb{Z}_3 :

$$u_1(x, y, z) = \begin{cases} 1 & \text{if } y \in \{\text{head}, \text{tail}\}^2, \\ 0 & \text{otherwise.} \end{cases}$$

in all other cases $u_1(x, y, z)$ is equal to zero.

- (b) if both $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in $\{\text{head}, \text{tail}\}^2$, then pay-off $u_2(x, y)$ rewards player y if $x_2 = y_1$:

$$u_2((x_1, x_2), (y_1, y_2)) = \begin{cases} 1 & \text{if } x_2 = y_1, \\ 0 & \text{otherwise.} \end{cases}$$

in all other cases $u_2(x, y)$ is equal to zero.

- (c) if both $y = (y_1, y_2)$ and $z = (z_1, z_2)$ are in $\{\text{head}, \text{tail}\}^2$, then pay-off $u_3(y, z)$ rewards player y if $y_2 \neq z_1$:

$$u_3((y_1, y_2), (z_1, z_2)) = \begin{cases} 1 & \text{if } y_2 \neq z_1, \\ 0 & \text{otherwise.} \end{cases}$$

in all other cases $u_3(y, z)$ is equal to zero.

An example of a Nash equilibrium of the game G_2 has been given in the introduction in Figure 2.

Lemma 9 Strategy profile $\langle e_k \rangle_{k \in \mathbb{Z}}$ is a Nash equilibrium of game G_2 if and only if the following two conditions are satisfied:

1. $e_k \in \mathbb{Z}_3$ for each $k \in \mathbb{Z}$,
2. $e_{k+2} - e_k \in \{[0]_3, [1]_3\}$ for each $k \in \mathbb{Z}$.

Proof. The lemma immediately follows from Definition 7. Indeed, If the conditions are not satisfied, then players start matching pennies game which has no Nash equilibria. \square

Theorem 3 $G_2 \models a \parallel b$ if and only if either $|a - b| = 1$ or $|a - b| > 2$.

Proof. Without loss of generality, suppose that $a \leq b$.

(\Rightarrow) First, assume that $G_2 \models a \parallel b$ and $a = b$. Consider strategy profiles $e' = \langle e'_k \rangle_{k \in \mathbb{Z}}$ and $e'' = \langle e''_k \rangle_{k \in \mathbb{Z}}$ such that $e'_k = [0]_3$ and $e''_k = [1]_3$ for each $k \in \mathbb{Z}$. Note that $e', e'' \in NE(G_2)$ by Lemma 9. Thus, by the assumption $G_2 \models a \parallel b$, there must exist $e = \langle e_k \rangle_{k \in \mathbb{Z}}$ such that $e_a = e'_a = [0]_3$ and $e_b = e''_b = [1]_3$. Hence, because $a = b$, we have $[0]_3 = e_a = e_b = [1]_3$, which is a contradiction.

Next, suppose that $G_2 \models a \parallel b$ and $b = a + 2$. Consider strategy profiles $e' = \langle e'_k \rangle_{k \in \mathbb{Z}}$ and $e'' = \langle e''_k \rangle_{k \in \mathbb{Z}}$ such that $e'_k = [0]_3$ and $e''_k = [2]_3$ for each $k \in \mathbb{Z}$. Note that $e', e'' \in NE(G_2)$ by Lemma 9. Thus, by the assumption $G_2 \models a \parallel b$, there must exist $e = \langle e_k \rangle_{k \in \mathbb{Z}}$ such that $e_a = e'_a = [0]_3$ and $e_{a+2} = e_b = e''_b = [2]_3$, which is a contradiction to Lemma 9.

(\Leftarrow) Assume now that $f = \langle f_k \rangle_{k \in \mathbb{Z}}$ and $g = \langle g_k \rangle_{k \in \mathbb{Z}}$ are two Nash equilibria of the game G_2 . We need to show that there is an equilibrium $e = \langle e_k \rangle_{k \in \mathbb{Z}}$ of the same game G_2 such that $e_a = f_a$ and $e_b = g_b$. Note that by Lemma 9, $f_k, g_k \in \mathbb{Z}_3$ for each $k \in \mathbb{Z}$ and

$$f_{k+2} - f_k \in \{[0]_3, [1]_3\}, \quad (2)$$

$$g_{k+2} - g_k \in \{[0]_3, [1]_3\}, \quad (3)$$

for each $k \in \mathbb{Z}$. We will consider two separate cases: $b = a + 1$ and $b > a + 2$.

f_{a-4}	g_{a-3}	f_{a-2}	g_{a-1}	f_a	g_{a+1}	f_{a+2}	g_{a+3}	f_{a+4}	g_{a+5}	f_{a+6}	g_{a+7}
-----------	-----------	-----------	-----------	-------	-----------	-----------	-----------	-----------	-----------	-----------	-----------

Figure 3: Nash equilibrium $e = \langle e_k \rangle_{k \in \mathbb{Z}}$.

Case I. Suppose that $b = a + 1$. Consider (see Figure 3) strategy profile $e = \langle e_k \rangle_{k \in \mathbb{Z}}$ such that

$$e_k = \begin{cases} f_k & \text{if } k \equiv a \pmod{2}, \\ g_k & \text{if } k \equiv a + 1 \pmod{2}. \end{cases}$$

Note that $e_{k+2} - e_k \in \{[0]_3, [1]_3\}$ for each $k \in \mathbb{Z}$ due statements (2) and (3). Thus, by Lemma 9, strategy profile e is a Nash equilibrium of game G_2 . Note that $e_a = f_a$ and $e_b = e_{a+1} = g_{a+1} = g_b$.

Case II. Assume now that $b > a + 2$. Thus, $b - a > 2$. Hence, by Lemma 6, there must exist $z_a, z_{a+1}, \dots, z_b \in \mathbb{Z}_3$ such that $z_a = f_a$, $z_b = g_b$, and $z_{k+2} - z_k \in \{[0]_3, [1]_3\}$ for each k such that $a \leq k \leq b - 2$. Consider strategy profile $e = \langle e_k \rangle_{k \in \mathbb{Z}}$ such that

$$e_k = \begin{cases} z_a & \text{if } k < a, \\ z_k & \text{if } a \leq k \leq b, \\ z_b & \text{if } b < k. \end{cases}$$

By Lemma 9, strategy profile e is a Nash equilibrium of the game G_2 . Note that $e_a = z_a = f_a$ and $e_b = z_b = g_b$, by the choice of the sequence z_a, z_{a+1}, \dots, z_b . \square

5.4 Game G_n : general case

In this section we define game G_n for $n \geq 3$. Recall that the main desired property of game G_n is that any two players i and j are interchangeable if and

only if $|i - j| = n$. Since pay-off function of any player can only depend on strategies of the player and her two adjacent neighbors, construction of such a game for large values of n becomes a very non-trivial task.

The set of strategies S^n of the game G_n is $(\mathbb{Z}_{n+1} \times \mathbb{Z}_{n+1})^{n-1}$. We visually represent elements of S^n as $(n-1) \times 2$ matrices whose elements belong to \mathbb{Z}_{n+1} .

Definition 8 *Pay-off function*

$$u \left(\left(\begin{array}{cc} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \\ \vdots & \vdots \\ x_{n-1,1} & x_{n-1,2} \end{array} \right), \left(\begin{array}{cc} y_{1,1} & y_{1,2} \\ y_{2,1} & y_{2,2} \\ \vdots & \vdots \\ y_{n-1,1} & y_{n-1,2} \end{array} \right), \left(\begin{array}{cc} z_{1,1} & z_{1,2} \\ z_{2,1} & z_{2,2} \\ \vdots & \vdots \\ z_{n-1,1} & z_{n-1,2} \end{array} \right) \right) \quad (4)$$

is equal to 1 if the following conditions are satisfied:

1. $y_{1,1} = [0]_{n+1}$,
2. $y_{k+1,2} + z_{k+1,1} - x_{k,2} - y_{k,1} \in \{[0]_{n+1}, [1]_{n+1}\}$, for every $1 \leq k < n-1$,
3. $z_{1,2} - x_{n-1,2} - y_{n-1,1} \in \{[0]_{n+1}, [1]_{n+1}\}$.

if at least one of the above conditions is not satisfied, then pay-off function (4) is equal to 0.

5.4.1 Perfect strategy profiles

While describing properties of game G_n , it will be convenient to use terms “perfect strategy profile” and “semi-perfect strategy profile” at a particular player. We introduce the notion of a perfect profile in this section and the notion of a semi-perfect profile in the next section.

Definition 9 *Strategy profile*

$$\left\langle \left(\begin{array}{cc} x_{1,1}^i & x_{1,2}^i \\ x_{2,1}^i & x_{2,2}^i \\ \vdots & \vdots \\ x_{n-1,1}^i & x_{n-1,2}^i \end{array} \right) \right\rangle_{i \in \mathbb{Z}}$$

of the game G_n , where $n \geq 3$, is perfect at player i if

1. $x_{1,1}^i = [0]_{n+1}$,
2. $x_{k,2}^{i-1} + x_{k,1}^i = x_{k+1,2}^i + x_{k+1,1}^{i+1}$ in \mathbb{Z}_{n+1} for every $1 \leq k < n-1$,
3. $x_{n-1,2}^{i-1} + x_{n-1,1}^i = x_{1,2}^{i+1}$ in \mathbb{Z}_{n+1} .

By a sum of two strategies in the game G_n , where $n \geq 3$, we mean element-wise sum of the two matrices.

Lemma 10 For any two strategy profiles $\langle s_i \rangle_{i \in \mathbb{Z}}$ and $\langle s'_i \rangle_{i \in \mathbb{Z}}$, of the game G_n perfect at a player $i \in \mathbb{Z}$, strategy profile $\langle s_i + s'_i \rangle_{i \in \mathbb{Z}}$ is also perfect at player i . \square

Proof. See Definition 9. \square

Lemma 11 For any

$$M = \begin{pmatrix} [0] & x_{1,2} \\ x_{2,1} & x_{2,2} \\ x_{3,1} & x_{3,2} \\ \vdots & \vdots \\ x_{n-2,1} & x_{n-2,2} \\ x_{n-1,1} & x_{n-1,2} \end{pmatrix} \in (\mathbb{Z}_{n+1} \times \mathbb{Z}_{n+1})^{n-1}$$

and any $a, b \in \mathbb{Z}$, if $0 < |a - b| < n$, then there is a strategy profile $s = \langle s_i \rangle_{i \in \mathbb{Z}}$ of the game G_n such that

1. $s_a = M$,

2. $s_b = \begin{pmatrix} [0] & [0] \\ [0] & [0] \\ [0] & [0] \\ \vdots & \vdots \\ [0] & [0] \\ [0] & [0] \end{pmatrix}$,

3. strategy profile s is perfect at each player i such that $a < i < b$.

Proof. We assume that $a < b$. The case $b < a$ could be shown in a similar way. *Case $b = a + 1$:* Consider strategy profile $s = \langle s_i \rangle_{i \in \mathbb{Z}}$ such that $s_a = M$ and all other strategies are equal to

$$\begin{pmatrix} [0] & [0] \\ [0] & [0] \\ [0] & [0] \\ \vdots & \vdots \\ [0] & [0] \\ [0] & [0] \end{pmatrix}.$$

The third condition of the lemma is satisfied vacuously.

Case $a+1 < b < a+n$: Consider strategy profile $s = \langle s_i \rangle_{i \in \mathbb{Z}}$ such that strategies s_a, s_{a+1}, \dots, s_b are equal to

$$\begin{pmatrix} [0] & x_{1,2} \\ x_{2,1} & x_{2,2} \\ x_{3,1} & x_{3,2} \\ x_{4,1} & x_{4,2} \\ \vdots & \vdots \\ x_{k-1,1} & x_{k-1,2} \\ x_{k,1} & x_{k,2} \\ x_{k+1,1} & x_{k+1,2} \\ x_{k+2,1} & x_{k+2,2} \\ \vdots & \vdots \\ x_{n-3,1} & x_{n-3,2} \\ x_{n-2,1} & x_{n-2,2} \\ x_{n-1,1} & x_{n-1,2} \end{pmatrix}, \begin{pmatrix} [0] & [0] \\ -x_{2,2} & x_{1,2} \\ -x_{3,2} & [0] \\ -x_{4,2} & [0] \\ \vdots & \vdots \\ -x_{k-1,2} & [0] \\ -x_{k,2} & [0] \\ -x_{k+1,2} & [0] \\ -x_{k+2,2} & [0] \\ \vdots & \vdots \\ -x_{n-3,2} & [0] \\ -x_{n-2,2} & [0] \\ -x_{n-1,2} & [0] \end{pmatrix}, \begin{pmatrix} [0] & [0] \\ [0] & [0] \\ [0] & x_{1,2} \\ [0] & [0] \\ \vdots & \vdots \\ [0] & [0] \\ [0] & [0] \\ [0] & [0] \\ [0] & [0] \\ \vdots & \vdots \\ [0] & [0] \\ [0] & [0] \\ [0] & [0] \end{pmatrix}, \\
 \dots, \begin{pmatrix} [0] & [0] \\ [0] & [0] \\ [0] & [0] \\ [0] & [0] \\ \vdots & \vdots \\ [0] & x_{1,2} \\ [0] & [0] \\ [0] & [0] \\ [0] & [0] \\ \vdots & \vdots \\ [0] & [0] \\ [0] & [0] \\ [0] & [0] \end{pmatrix}, \begin{pmatrix} [0] & [0] \\ [0] & [0] \\ [0] & [0] \\ [0] & [0] \\ \vdots & \vdots \\ [0] & [0] \\ [0] & x_{1,2} \\ [0] & [0] \\ [0] & [0] \\ \vdots & \vdots \\ [0] & [0] \\ [0] & [0] \\ [0] & [0] \end{pmatrix}, \begin{pmatrix} [0] & [0] \\ [0] & [0] \\ [0] & [0] \\ [0] & [0] \\ \vdots & \vdots \\ [0] & [0] \\ [0] & [0] \\ [0] & [0] \\ [0] & [0] \\ \vdots & \vdots \\ [0] & [0] \\ [0] & [0] \\ [0] & [0] \end{pmatrix}$$

respectively, where $k = b - a$. All other strategies are equal to

$$\begin{pmatrix} [0] & [0] \\ [0] & [0] \\ [0] & [0] \\ [0] & [0] \\ \vdots & \vdots \\ [0] & [0] \\ [0] & [0] \\ [0] & [0] \\ [0] & [0] \\ \vdots & \vdots \\ [0] & [0] \\ [0] & [0] \\ [0] & [0] \end{pmatrix}.$$

By Definition 9, this profile is perfect at each player i such that $a < i < b$. \square

5.4.2 Semi-perfect strategy profiles

In this section we introduce the notion of a semi-perfect strategy profile of game G_n . Later, we show that such profiles are exactly the Nash equilibria of this game.

Definition 10 *Strategy profile*

$$\left\langle \left(\begin{pmatrix} x_{1,1}^i & x_{1,2}^i \\ x_{2,1}^i & x_{2,2}^i \\ \vdots & \vdots \\ x_{n-1,1}^i & x_{n-1,2}^i \end{pmatrix} \right) \right\rangle_{i \in \mathbb{Z}}$$

of the game G_n , where $n \geq 2$, is semi-perfect at player i if

1. $x_{1,1}^i = [0]_{n+1}$,
2. $x_{k+1,2}^i + x_{k+1,1}^{i+1} - x_{k,2}^{i-1} - x_{k,1}^i \in \{[0]_{n+1}, [1]_{n+1}\}$, for every $1 \leq k < n - 1$,
3. $x_{1,2}^{i+1} - x_{n-1,2}^{i-1} - x_{n-1,1}^i \in \{[0]_{n+1}, [1]_{n+1}\}$.

Lemma 12 *For any*

$$A = \begin{pmatrix} [0] & x_{1,2} \\ x_{2,1} & x_{2,2} \\ \vdots & \vdots \\ x_{n-1,1} & x_{n-1,2} \end{pmatrix} \in (\mathbb{Z}_{n+1} \times \mathbb{Z}_{n+1})^{n-1},$$

any

$$B = \begin{pmatrix} [0] & y_{1,2} \\ y_{2,1} & y_{2,2} \\ \vdots & \vdots \\ y_{n-1,1} & y_{n-1,2} \end{pmatrix} \in (\mathbb{Z}_{n+1} \times \mathbb{Z}_{n+1})^{n-1},$$

and any $a, b \in \mathbb{Z}$, if $|a - b| > n$, then there is a strategy profile $s = \langle s_i \rangle_{i \in \mathbb{Z}}$ of the game G_n such that

1. $s_a = A$,
2. $s_b = B$,
3. strategy profile s is semi-perfect at each player i such that $a < i < b$.

Proof. We will assume that $a < b$. The other case is similar. Thus, $b - a > n$ due to the assumption $|a - b| > n$. Let k be an integer such that $0 \leq k < n - 1$ and $k \equiv b - a \pmod{n - 1}$. We first consider case when $k \neq 1$. Since $x_{1,2}, y_{k,1} \in \mathbb{Z}_{n+1}$, by Lemma 6, there must exist a sequence z_1, z_2, \dots, z_{b-a} of equivalence classes in \mathbb{Z}_{n+1} such that

$$\begin{aligned} z_1 &= x_{1,2}, \\ z_{b-a} &= y_{k,1}, \end{aligned} \tag{5}$$

$$z_{i+1} - z_i \in \{[0]_{n+1}, [1]_{n+1}\} \quad (0 \leq i < b - a). \tag{6}$$

Consider now strategy profile $s = \langle s_i \rangle_{i \in \mathbb{Z}}$ such that strategies s_a, s_{a+1}, \dots, s_b are equal to

$$\begin{pmatrix} [0] & z_1 \\ x_{2,1} & x_{2,2} \\ x_{3,1} & x_{3,2} \\ x_{4,1} & x_{4,2} \\ \vdots & \vdots \\ x_{k-1,1} & x_{k-1,2} \\ x_{k,1} & x_{k,2} \\ x_{k+1,1} & x_{k+1,2} \\ \vdots & \vdots \\ x_{n-3,1} & x_{n-3,2} \\ x_{n-2,1} & x_{n-2,2} \\ x_{n-1,1} & x_{n-1,2} \end{pmatrix}, \begin{pmatrix} [0] & [0] \\ -x_{2,2} & [0] \\ -x_{3,2} & [0] \\ -x_{4,2} & [0] \\ \vdots & \vdots \\ -x_{k-1,2} & [0] \\ -x_{k,2} & [0] \\ -x_{k+1,2} & [0] \\ \vdots & \vdots \\ -x_{n-3,2} & [0] \\ -x_{n-2,2} & [0] \\ -x_{n-1,2} & [0] \end{pmatrix}, \begin{pmatrix} [0] & [0] \\ z_2 & [0] \\ [0] & [0] \\ [0] & [0] \\ \vdots & \vdots \\ [0] & [0] \\ [0] & [0] \\ [0] & [0] \\ \vdots & \vdots \\ [0] & [0] \\ [0] & [0] \\ [0] & [0] \end{pmatrix}, \begin{pmatrix} [0] & [0] \\ [0] & [0] \\ z_3 & [0] \\ [0] & [0] \\ \vdots & \vdots \\ [0] & [0] \\ [0] & [0] \\ [0] & [0] \\ \vdots & \vdots \\ [0] & [0] \\ [0] & [0] \\ [0] & [0] \end{pmatrix}, \dots,$$

$$\begin{pmatrix} [0] & [0] \\ [0] & [0] \\ [0] & [0] \\ \vdots & \vdots \\ [0] & [0] \\ [0] & [0] \\ \vdots & \vdots \\ [0] & [0] \\ [0] & [0] \end{pmatrix}, \begin{pmatrix} [0] & [0] \\ [0] & [0] \\ [0] & [0] \\ \vdots & \vdots \\ [0] & [0] \\ [0] & [0] \\ \vdots & \vdots \\ [0] & [0] \\ [0] & [0] \end{pmatrix}, \begin{pmatrix} [0] & z_n \\ [0] & [0] \\ [0] & [0] \\ \vdots & \vdots \\ [0] & [0] \\ [0] & [0] \\ \vdots & \vdots \\ [0] & [0] \\ [0] & [0] \end{pmatrix}, \begin{pmatrix} [0] & [0] \\ [0] & [0] \\ [0] & [0] \\ \vdots & \vdots \\ [0] & [0] \\ [0] & [0] \\ \vdots & \vdots \\ [0] & [0] \\ [0] & [0] \end{pmatrix}, \begin{pmatrix} [0] & [0] \\ z_{n+1} & [0] \\ [0] & [0] \\ \vdots & \vdots \\ [0] & [0] \\ [0] & [0] \\ \vdots & \vdots \\ [0] & [0] \\ [0] & [0] \end{pmatrix}, \\
\begin{pmatrix} [0] & [0] \\ [0] & [0] \\ z_{n+2} & [0] \\ [0] & [0] \\ \vdots & \vdots \\ [0] & [0] \\ [0] & [0] \\ \vdots & \vdots \\ [0] & [0] \\ [0] & [0] \end{pmatrix}, \dots, \begin{pmatrix} [0] & [0] \\ [0] & -y_{2,1} \\ [0] & -y_{3,1} \\ [0] & -y_{4,1} \\ \vdots & \vdots \\ z_{b-a-1} & -y_{k-1,1} \\ [0] & [0] \\ [0] & -y_{k+1,1} \\ \vdots & \vdots \\ [0] & -y_{n-3,1} \\ [0] & -y_{n-2,1} \\ [0] & -y_{n-1,1} \end{pmatrix}, \begin{pmatrix} [0] & y_{1,2} \\ y_{2,1} & y_{2,2} \\ y_{3,1} & y_{3,2} \\ y_{4,1} & y_{4,2} \\ \vdots & \vdots \\ y_{k-1,1} & y_{k-1,2} \\ z_{b-a} & y_{k,2} \\ y_{k+1,1} & y_{k+1,2} \\ \vdots & \vdots \\ y_{n-3,1} & y_{n-3,2} \\ y_{n-2,1} & y_{n-2,2} \\ y_{n-1,1} & y_{n-1,2} \end{pmatrix}$$

respectively. All other strategies are equal

$$\begin{pmatrix} [0] & [0] \\ [0] & [0] \\ [0] & [0] \\ [0] & [0] \\ \vdots & \vdots \\ [0] & [0] \\ [0] & [0] \\ [0] & [0] \\ \vdots & \vdots \\ [0] & [0] \\ [0] & [0] \\ [0] & [0] \end{pmatrix}.$$

Due to condition (6), this profile is semi-perfect at each player i such that $a < i < b$. Case $k = 1$ is similar except that equation (5) should be replaced with $z_{b-a} = y_{k,2}$. \square

Lemma 13 (right expansion) For each $a, b \in \mathbb{Z}$ such that $a < b$, if strategy profile $s = \langle s_i \rangle_{i \in \mathbb{Z}}$ is semi-perfect for each player i such that $a < i < b$, then there is a strategy profile $s' = \langle s'_i \rangle_{i \in \mathbb{Z}}$ such that

1. $s'_i = s_i$ for each i such that $a \leq i \leq b$,
2. s' is semi-perfect for each player i such that $a < i < b + 1$.

Proof. Let

$$s_{b-1} = \begin{pmatrix} [0] & x_{1,2} \\ x_{2,1} & x_{2,2} \\ x_{3,1} & x_{3,2} \\ \vdots & \vdots \\ x_{n-2,1} & x_{n-2,2} \\ x_{n-1,1} & x_{n-1,2} \end{pmatrix}, \quad s_b = \begin{pmatrix} [0] & y_{1,2} \\ y_{2,1} & y_{2,2} \\ y_{3,1} & y_{3,2} \\ \vdots & \vdots \\ y_{n-2,1} & y_{n-2,2} \\ y_{n-1,1} & y_{n-1,2} \end{pmatrix}.$$

Define s'_i to be equal to s_i for all $i \neq b + 1$ and s'_{b+1} to be

$$\begin{pmatrix} [0] & x_{n-1,2} + y_{n-1,1} \\ x_{1,2} - y_{2,2} & [0] \\ x_{2,2} + y_{2,1} - y_{3,2} & [0] \\ \vdots & \vdots \\ x_{n-3,2} + y_{n-3,1} - y_{n-2,2} & [0] \\ x_{n-2,2} + y_{n-2,1} - y_{n-1,2} & [0] \end{pmatrix}.$$

By Definition 10, strategy profile s' is perfect at player b . \(\boxtimes\)

Lemma 14 (left expansion) For each $a, b \in \mathbb{Z}$ such that $a < b$, if strategy profile $s = \langle s_i \rangle_{i \in \mathbb{Z}}$ is semi-perfect for each player i such that $a < i < b$, then there is a strategy profile $s' = \langle s'_i \rangle_{i \in \mathbb{Z}}$ such that

1. $s'_i = s_i$ for each i such that $a \leq i \leq b$,
2. s' is semi-perfect for each player i such that $a - 1 < i < b$.

Proof. Similar to the proof of Lemma 13. \(\boxtimes\)

Lemma 15 (infinite expansion) For each $a, b \in \mathbb{Z}$ such that $a < b$, if strategy profile $s = \langle s_i \rangle_{i \in \mathbb{Z}}$ is semi-perfect for each for player i such that $a < i < b$, then there is a strategy profile $s' = \langle s'_i \rangle_{i \in \mathbb{Z}}$ such that

1. $s'_i = s_i$ for each i such that $a \leq i \leq b$,
2. s' is semi-perfect for each player $i \in \mathbb{Z}$.

Proof. Follows from Lemma 13 and Lemma 14. \(\boxtimes\)

5.4.3 Properties of Nash equilibria of game G_n

Lemma 16 *For any $n \geq 3$, strategy profile e is a Nash equilibrium of the game G_n if and only if profile e is semi-perfect at each player $i \in \mathbb{Z}$.*

Proof. (\Rightarrow): If strategy profile e is not semi-perfect at player i , then one of the conditions 1.-3. from Definition 10 are not satisfied. Thus, by Definition 8, pay-off of player i under profile e is zero. Note that in this case, by Definition 8, player i alone can adjust her strategy in profile e to satisfy conditions 1.-3. of Definition 8 to increase pay-off to 1. Therefore, e is not a Nash equilibrium of the game G_n .

(\Leftarrow): Suppose that strategy profile e is semi-perfect at each player $i \in \mathbb{Z}$. Thus, this profile satisfies conditions 1.-3. from Definition 10 for each player i . Hence, by Definition 8, each player under profile e has the largest possible pay-off in the game G_n . Therefore, e is a Nash equilibrium of the game G_n . \square

Lemma 17 *For any $a \in \mathbb{Z}$ and any $n \geq 3$, if*

$$\left\langle \left(\begin{array}{cc} x_{1,1}^i & x_{1,2}^i \\ x_{2,1}^i & x_{2,2}^i \\ \vdots & \vdots \\ x_{n-1,1}^i & x_{n-1,2}^i \end{array} \right) \right\rangle_{i \in \mathbb{Z}}$$

is a Nash equilibrium of the game G_n and $x_{1,2}^a = [0]_{n+1}$, then

$$x_{k+1,2}^{a+k} + x_{k+1,1}^{a+k+1} \in \{[0], [1], [2], \dots, [k]\},$$

for each $0 \leq k \leq n-2$.

Proof. Induction on k . If $k = 0$, then $x_{k+1,2}^{a+k} = x_{1,2}^a = [0]$ due to the assumption of the lemma. At the same time, $x_{k+1,1}^{a+k+1} = x_{1,1}^{a+1} = [0]$ by Lemma 16 and item 1 of Definition 10. Thus,

$$x_{k+1,2}^{a+k} + x_{k+1,1}^{a+k+1} = [0] + [0] = [0] \in \{[0]\}.$$

For the induction step, assume that

$$x_{k+1,2}^{a+k} + x_{k+1,1}^{a+k+1} \in \{[0], [1], [2], \dots, [k]\}.$$

By Lemma 16 and item 2 of Definition 10, there is $\varepsilon \in \{[0]_{n+1}, [1]_{n+1}\}$ such that

$$x_{k+3,2}^{a+k+1} + x_{k+2,1}^{a+k+2} = x_{k+1,2}^{a+k} + x_{k+1,1}^{a+k+1} + \varepsilon.$$

Therefore,

$$x_{k+3,2}^{a+k+1} + x_{k+2,1}^{a+k+2} \in \{[0], [1], [2], \dots, [k], [k+1]\}.$$

\square

Lemma 18 For any $a \in \mathbb{Z}$ and any $n \geq 3$, if

$$\left\langle \left(\begin{array}{cc} x_{1,1}^i & x_{1,2}^i \\ x_{2,1}^i & x_{2,2}^i \\ \vdots & \vdots \\ x_{n-1,1}^i & x_{n-1,2}^i \end{array} \right) \right\rangle_{i \in \mathbb{Z}}$$

is a Nash equilibrium of the game G_n and $x_{1,2}^a = [0]_{n+1}$, then

$$x_{1,2}^{a+n} \in \{[0], [1], [2], \dots, [n-1]\}.$$

Proof. By Lemma 17,

$$x_{n-1,2}^{a+n-2} + x_{n-1,1}^{a+n-1} \in \{[0], [1], [2], \dots, [n-2]\}.$$

By Lemma 16 and item 3 of Definition 10, for $i = a + n - 1$, there is $\varepsilon \in \{[0]_{n+1}, [1]_{n+1}\}$ such that

$$x_{1,2}^{a+n} = x_{n-1,2}^{a+n-2} + x_{n-1,1}^{a+n-1} + \varepsilon.$$

Therefore, $x_{1,2}^{a+n} \in \{[0], [1], [2], \dots, [n-2], [n-1]\}$. \square

Definition 11 For any $n \geq 3$, let $f^n = \langle f_i^n \rangle_{i \in \mathbb{Z}}$ be the strategy profile of the game G_n such that

$$f_i^n = \begin{pmatrix} [0]_{n+1} & [0]_{n+1} \\ [0]_{n+1} & [0]_{n+1} \\ \vdots & \vdots \\ [0]_{n+1} & [0]_{n+1} \end{pmatrix},$$

for each $i \in \mathbb{Z}$.

Lemma 19 $f^n \in NE(G_n)$ for each $n \geq 3$.

Proof. By Lemma 16 and Definition 10. \square

Definition 12 For any $n \geq 3$, let $g^n = \langle g_i^n \rangle_{i \in \mathbb{Z}}$ be the strategy profile of the game G_n such that

$$g_i^n = \begin{pmatrix} [0]_{n+1} & [n]_{n+1} \\ [0]_{n+1} & [n]_{n+1} \\ \vdots & \vdots \\ [0]_{n+1} & [n]_{n+1} \end{pmatrix},$$

for each $i \in \mathbb{Z}$.

Lemma 20 $g^n \in NE(G_n)$ for each $n \geq 3$.

Proof. By Lemma 16 and Definition 10. \square

5.4.4 Main property of game G_n

Theorem 4 $G_n \models a \parallel b$ if and only if $|a - b| \neq n$ and $a \neq b$, where $n \geq 3$.

Proof. (\Rightarrow): Let $G_n \models a \parallel b$. First, suppose that $|a - b| = n$. Without loss of generality, assume that $b = a + n$. Due to assumption $G_n \models a \parallel b$, there must exist an equilibrium

$$\langle e_i \rangle_{i \in \mathbb{Z}} = \left\langle \left(\begin{array}{cc} x_{1,1}^i & x_{1,2}^i \\ x_{2,1}^i & x_{2,2}^i \\ \vdots & \vdots \\ x_{n-1,1}^i & x_{n-1,2}^i \end{array} \right) \right\rangle_{i \in \mathbb{Z}}$$

of the game G_n such that $e_a = f_a^n$ and $e_b = g_b^n$, where $f^n = \langle f_i^n \rangle_{i \in \mathbb{Z}}$ and $g^n = \langle g_i^n \rangle_{i \in \mathbb{Z}}$ are Nash equilibria of the game G_n defined in the previous section. Thus, $x_{1,2}^a = [0]$ and $x_{1,2}^b = [n]$, which is a contradiction to Lemma 18.

Assume now that $a = b$. Due to assumption $G_n \models a \parallel b$, there must exist an equilibrium $e = \langle e_i \rangle_{i \in \mathbb{Z}}$ of the game G_n such that $f_a^n = e_a = e_b = g_b^n$. Thus, $[0]_{n+1} = [n]_{n+1}$, which is a contradiction.

(\Leftarrow): Without loss of generality, assume that $b > a$. To prove that $G_n \models a \parallel b$, consider any two Nash equilibria

$$v = \langle v_i \rangle_{i \in \mathbb{Z}} = \left\langle \left(\begin{array}{cc} [0] & x_{1,2}^i \\ x_{2,1}^i & x_{2,2}^i \\ \vdots & \vdots \\ x_{n-1,1}^i & x_{n-1,2}^i \end{array} \right) \right\rangle_{i \in \mathbb{Z}}$$

and

$$w = \langle w_i \rangle_{i \in \mathbb{Z}} = \left\langle \left(\begin{array}{cc} [0] & y_{1,2}^i \\ y_{2,1}^i & y_{2,2}^i \\ \vdots & \vdots \\ y_{n-1,1}^i & y_{n-1,2}^i \end{array} \right) \right\rangle_{i \in \mathbb{Z}}$$

of the game G_n . Note that the upper left element in each of the above matrices is $[0]$ due to Lemma 16. We need to show that there is an equilibrium $e \in NE(G_n)$ such that e agrees with equilibrium v on the strategy of player a and with equilibrium w on the strategy of player b .

Case I: $b - a < n$. By Lemma 11, there are strategy profiles $\langle s_i \rangle_{i \in \mathbb{Z}}$ and $\langle t_i \rangle_{i \in \mathbb{Z}}$ both perfect at each player i such that $a < i < b$ and

$$s_a = \left(\begin{array}{cc} [0] & x_{1,2}^a \\ x_{2,1}^a & x_{2,2}^a \\ \vdots & \vdots \\ x_{n-1,1}^a & x_{n-1,2}^a \end{array} \right), \quad s_b = \left(\begin{array}{cc} [0]_{n+1} & [0]_{n+1} \\ [0]_{n+1} & [0]_{n+1} \\ \vdots & \vdots \\ [0]_{n+1} & [0]_{n+1} \end{array} \right),$$

$$t_a = \begin{pmatrix} [0]_{n+1} & [0]_{n+1} \\ [0]_{n+1} & [0]_{n+1} \\ \vdots & \vdots \\ [0]_{n+1} & [0]_{n+1} \end{pmatrix}, \quad t_b = \begin{pmatrix} [0] & y_{1,2}^b \\ y_{2,1}^b & y_{2,2}^b \\ \vdots & \vdots \\ y_{n-1,1}^b & y_{n-1,2}^b \end{pmatrix}.$$

By Lemma 10, strategy profile $\langle s_i + t_i \rangle_{i \in \mathbb{Z}}$ is perfect at each player i such that $a < i < b$. Thus, by Lemma 15, there is a strategy profile $\langle e_i \rangle_{i \in \mathbb{Z}}$ of the game G_n where $e_i = s_i + t_i$ for each i such that $a \leq i \leq b$ and e_i is semi-perfect for each player $i \in \mathbb{Z}$. Hence, by Lemma 16, strategy profile $\langle e_i \rangle_{i \in \mathbb{Z}}$ is a Nash equilibrium of the game G_n . We are only left to notice that

$$\begin{aligned} e_a = s_a + t_a &= \begin{pmatrix} [0] & x_{1,2}^a \\ x_{2,1}^a & x_{2,2}^a \\ \vdots & \vdots \\ x_{n-1,1}^a & x_{n-1,2}^a \end{pmatrix} + \begin{pmatrix} [0]_{n+1} & [0]_{n+1} \\ [0]_{n+1} & [0]_{n+1} \\ \vdots & \vdots \\ [0]_{n+1} & [0]_{n+1} \end{pmatrix} \\ &= \begin{pmatrix} [0] & x_{1,2}^a \\ x_{2,1}^a & x_{2,2}^a \\ \vdots & \vdots \\ x_{n-1,1}^a & x_{n-1,2}^a \end{pmatrix} = v_a \end{aligned}$$

and

$$\begin{aligned} e_b = s_b + t_b &= \begin{pmatrix} [0]_{n+1} & [0]_{n+1} \\ [0]_{n+1} & [0]_{n+1} \\ \vdots & \vdots \\ [0]_{n+1} & [0]_{n+1} \end{pmatrix} + \begin{pmatrix} [0] & y_{1,2}^b \\ y_{2,1}^b & y_{2,2}^b \\ \vdots & \vdots \\ y_{n-1,1}^b & y_{n-1,2}^b \end{pmatrix} \\ &= \begin{pmatrix} [0] & y_{1,2}^b \\ y_{2,1}^b & y_{2,2}^b \\ \vdots & \vdots \\ y_{n-1,1}^b & y_{n-1,2}^b \end{pmatrix} = w_b. \end{aligned}$$

Case II: $b - a > n$. By Lemma 12, there exists a strategy profile $\langle s_i \rangle_{i \in \mathbb{Z}}$ such that $s_a = v_a$, $s_b = w_b$, and strategy profile $\langle s_i \rangle_{i \in \mathbb{Z}}$ is semi-perfect at each player i such that $a < i < b$. By Lemma 15, there is a strategy profile $\langle e_i \rangle_{i \in \mathbb{Z}}$ semi-perfect at each player $i \in \mathbb{Z}$ such that $e_i = s_i$ for each $a \leq i \leq b$. By Lemma 16, strategy profile $\langle e_i \rangle_{i \in \mathbb{Z}}$ is a Nash equilibrium of the game G_n . We are only left to notice that $e_a = s_a = v_a$ and $e_b = s_b = w_b$. \square

5.5 Combining games together

Theorem 5 *For any $n \geq 0$, and any $a, b \in \mathbb{Z}$,*

$$G_n \Vdash a \parallel b \quad \text{if and only if} \quad |a - b| \neq n \text{ and } a \neq b.$$

Proof. See Theorem 1, Theorem 2, Theorem 3, and Theorem 4. \square

Theorem 6 For any $n \geq 0$, game G_n has at least one Nash equilibrium.

Proof. *Case I:* $n = 0$. Consider strategy profile $e = \langle e_k \rangle_{k \in \mathbb{Z}}$ such that $e_k = [0]_2$ for each $k \in \mathbb{Z}$. By Lemma 7, strategy profile e is a Nash equilibrium of the game G_0 .

Case II: $n = 1, 2$. Consider strategy profile $e = \langle e_k \rangle_{k \in \mathbb{Z}}$ such that $e_k = [0]_3$ for each $k \in \mathbb{Z}$. By Lemma 8 and Lemma 9, strategy profile e is a Nash equilibrium of the games G_1 and G_2 .

Case III: $n > 2$. By Lemma 19, strategy profile f^n is a Nash equilibrium of the game G_n . \square

5.6 Product of games

In this section we define product operation on cellular games. Informally, product is a composition of several cellular games played concurrently and independently. The pay-off of a player in the product of the games is the sum of the pay-offs in the individual games.

Definition 13 Product $\prod_{i=1}^n G^i$ of any finite family of cellular games $\{G^i\}_{i=1}^n = \{(S^i, u^i)\}_{i=1}^n$ is the cellular game $G = (S, u)$, where

1. S is Cartesian product $\prod_{i=1}^n S^i$,
2. $u(\langle x^i \rangle_{i \leq n}, \langle y^i \rangle_{i \leq n}, \langle z^i \rangle_{i \leq n}) = \sum_{i=1}^n u^i(x^i, y^i, z^i)$.

Lemma 21 If $\{G^i\}_{i=1}^n = \{(S^i, u^i)\}_{i=1}^n$ is a finite family of cellular games, then $\langle \langle e_k^i \rangle_{i \leq n} \rangle_{k \in \mathbb{Z}} \in NE(\prod_{i=1}^n G^i)$ if and only if $\langle e_k^i \rangle_{k \in \mathbb{Z}} \in NE(G^i)$ for each $i \leq n$.

Proof. (\Rightarrow) Suppose that $e^{i_0} = \langle e_k^{i_0} \rangle_{k \in \mathbb{Z}} \notin NE(G^{i_0})$ for some $i_0 \leq n$. Thus, there is $k_0 \in \mathbb{Z}$ and $s \in S^{i_0}$ such that

$$u^{i_0}(e_{k_0-1}^{i_0}, s, e_{k_0+1}^{i_0}) > u^{i_0}(e_{k_0-1}^{i_0}, e_{k_0}^{i_0}, e_{k_0+1}^{i_0}). \quad (7)$$

Let tuple \mathbf{s} be $\langle e_{k_0}^1, e_{k_0}^2, \dots, e_{k_0}^{i_0-1}, s, e_{k_0}^{i_0+1}, \dots, e_{k_0}^n \rangle$. Then, by Definition 13 and due to inequality (7),

$$\begin{aligned} u(\langle e_{k_0-1}^i \rangle_{i \leq n}, \mathbf{s}, \langle e_{k_0+1}^i \rangle_{i \leq n}) &= \sum_{i \neq i_0} u^i(e_{k_0-1}^i, e_{k_0}^i, e_{k_0+1}^i) + u^{i_0}(e_{k_0-1}^{i_0}, s, e_{k_0+1}^{i_0}) > \\ &= \sum_{i \neq i_0} u^i(e_{k_0-1}^i, e_{k_0}^i, e_{k_0+1}^i) + u^{i_0}(e_{k_0-1}^{i_0}, e_{k_0}^{i_0}, e_{k_0+1}^{i_0}) = \sum_i u^i(e_{k_0-1}^i, e_{k_0}^i, e_{k_0+1}^i) = \\ &= u(\langle e_{k_0-1}^i \rangle_{i \leq n}, \langle e_{k_0}^i \rangle_{i \leq n}, \langle e_{k_0+1}^i \rangle_{i \leq n}), \end{aligned}$$

which is a contradiction with the assumption $\langle \langle e_k^i \rangle_{i \leq n} \rangle_{k \in \mathbb{Z}} \in NE(\prod_{i=1}^n G^i)$.

(\Leftarrow) Assume now that $\langle \langle e_k^i \rangle_{i \leq n} \rangle_{k \in \mathbb{Z}} \notin NE(\prod_{i=1}^n G^i)$. Thus, there must exist $k \in \mathbb{Z}$ and $\mathbf{s} = \langle s^i \rangle_{i \leq n}$ such that

$$u(\langle e_{k_0-1}^i \rangle_{i \leq n}, \mathbf{s}, \langle e_{k_0+1}^i \rangle_{i \leq n}) > u(\langle e_{k_0-1}^i \rangle_{i \leq n}, \langle e_{k_0}^i \rangle_{i \leq n}, \langle e_{k_0+1}^i \rangle_{i \leq n}).$$

Hence, by Definition 13,

$$\sum_i u^i(e_{k_0-1}^i, s^i, e_{k_0+1}^i) > \sum_i u^i(e_{k_0-1}^i, e_{k_0}^i, e_{k_0+1}^i).$$

Thus, there must exist at least one $i_0 \leq n$ such that

$$u^{i_0}(e_{k_0-1}^{i_0}, s^{i_0}, e_{k_0+1}^{i_0}) > u^{i_0}(e_{k_0-1}^{i_0}, e_{k_0}^{i_0}, e_{k_0+1}^{i_0}),$$

which is a contradiction with the assumption $\langle e_k^{i_0} \rangle_{k \in \mathbb{Z}} \in NE(G^{i_0})$. \square

Theorem 7 *If $a, b \in \mathbb{Z}$ and each of the cellular games in family $\{G^i\}_{i=1}^n = \{(S^i, u^i)\}_{i=1}^n$ has at least one Nash equilibrium, then $\prod_{i=1}^n G^i \models a \parallel b$ if and only if $G^i \models a \parallel b$ for each $i \leq n$.*

Proof. (\Rightarrow) Let $\langle e_k^1 \rangle_{k \in \mathbb{Z}}, \langle e_k^2 \rangle_{k \in \mathbb{Z}}, \dots, \langle e_k^n \rangle_{k \in \mathbb{Z}}$ be Nash equilibria of games G^1, G^2, \dots, G^n that exist by the assumption of the theorem. Consider any $i_0 \leq n$ and any two equilibria $\langle f_k \rangle_{k \in \mathbb{Z}}$ and $\langle g_k \rangle_{k \in \mathbb{Z}}$ of the game G^{i_0} . We need to show that there exists Nash equilibrium $h = \langle h_k \rangle_{k \in \mathbb{Z}}$ of the game G^{i_0} such that $h_a = f_a$ and $h_b = g_b$. Indeed, by Lemma 21,

$$\langle \langle e_k^1, e_k^2, \dots, e_k^{i_0-1}, f_k, e_k^{i_0+1}, \dots, e_k^n \rangle \rangle_{k \in \mathbb{Z}}$$

and

$$\langle \langle e_k^1, e_k^2, \dots, e_k^{i_0-1}, g_k, e_k^{i_0+1}, \dots, e_k^n \rangle \rangle_{k \in \mathbb{Z}}$$

are Nash equilibria of the game $\prod_{i=1}^n G^i$. Hence, by the assumption $\prod_{i=1}^n G^i \models a \parallel b$, there exists a Nash equilibrium

$$\langle \langle w_k^1, w_k^2, \dots, w_k^{i_0-1}, w_k^{i_0}, w_k^{i_0+1}, \dots, w_k^n \rangle \rangle_{k \in \mathbb{Z}}$$

of the game $\prod_{i=1}^n G^i$ such that

$$\langle w_a^1, \dots, w_a^{i_0-1}, w_a^{i_0}, w_a^{i_0+1}, \dots, w_a^n \rangle = \langle e_a^1, \dots, e_a^{i_0-1}, f_a, e_a^{i_0+1}, \dots, e_a^n \rangle$$

and

$$\langle w_b^1, \dots, w_b^{i_0-1}, w_b^{i_0}, w_b^{i_0+1}, \dots, w_b^n \rangle = \langle e_b^1, \dots, e_b^{i_0-1}, g_b, e_b^{i_0+1}, \dots, e_b^n \rangle.$$

In particular, $w_a^{i_0} = f_a$ and $w_b^{i_0} = g_b$. At the same time, by Theorem 21, $\langle w_k^{i_0} \rangle_{k \in \mathbb{Z}}$ is a Nash equilibrium of game G^{i_0} . Choose h to be equilibrium $\langle w_k^{i_0} \rangle_{k \in \mathbb{Z}}$.

(\Leftarrow) Let $\langle \langle f_k^1, f_k^2, \dots, f_k^n \rangle \rangle_{k \in \mathbb{Z}}$ and $\langle \langle g_k^1, g_k^2, \dots, g_k^n \rangle \rangle_{k \in \mathbb{Z}}$ be two Nash equilibria of the game $\prod_{i=1}^n G^i$. We will prove that there is a Nash equilibrium $\langle \langle h_k^1, h_k^2, \dots, h_k^n \rangle \rangle_{k \in \mathbb{Z}}$ of game $\prod_{i=1}^n G^i$ such that

$$\langle h_a^1, h_a^2, \dots, h_a^n \rangle = \langle f_a^1, f_a^2, \dots, f_a^n \rangle$$

and

$$\langle h_b^1, h_b^2, \dots, h_b^n \rangle = \langle g_b^1, g_b^2, \dots, g_b^n \rangle.$$

Indeed, by Lemma 21, strategy profiles $\langle f_k^i \rangle_{k \in \mathbb{Z}}$ and $\langle g_k^i \rangle_{k \in \mathbb{Z}}$ are Nash equilibria of game G^i for each $i \leq n$. Thus, due to the assumption of the theorem, for each $i \leq n$ there exists Nash equilibrium $\langle h_k^i \rangle_{k \in \mathbb{Z}}$ of the game G^i such that $h_a^i = f_a^i$ and $h_b^i = g_b^i$ for each $i \leq n$. By Theorem 21, $\langle \langle h_k^1, h_k^2, \dots, h_k^n \rangle \rangle_{k \in \mathbb{Z}}$ is a Nash equilibrium of the game $\prod_{i=1}^n G^i$. \square

5.7 Game G_∞

Definition 14 Game G_∞ is pair $(\{0\}, 0)$, whose first element is the single-element set containing number 0 and whose second component is the constant function equal to 0.

Lemma 22 $G_\infty \models a \parallel b$ for each $a, b \in \mathbb{Z}$.

Proof. Game G_∞ has a unique strategy profile, which is also the unique Nash equilibrium of this game. \square

5.8 Completeness: final steps

We are now ready to state and prove the completeness theorem for our logical system.

Theorem 8 (completeness) For each formula $\varphi \in \Phi$, if $G \models \varphi$ for each cellular game G , then $\vdash \varphi$.

Proof. Suppose that $\not\vdash \varphi$. Let M be any maximal consistent subset of Φ such that $\neg\varphi \in M$. Thus, $\varphi \notin M$ due to assumption of the consistency of M . There are two cases that we consider separately.

Case I: $0 \parallel 0 \in M$.

Lemma 23 $\psi \in M$ if and only if $G_\infty \models \psi$ for each $\psi \in \Phi$.

Proof. Induction on structural complexity of formula ψ . If φ is \perp , then $\psi \notin M$ due to consistency of set M . At the same time, $G_\infty \not\models \psi$ by Definition 4.

Suppose now that ψ is formula $a \parallel b$. By Reflexivity axiom, $0 \parallel 0 \rightarrow a \parallel b$. Thus, $a \parallel b \in M$, by the assumption $0 \parallel 0 \in M$ and due to maximality of set M . At the same time, $G_\infty \models a \parallel b$ by Lemma 22.

Finally, case when ψ is an implication $\sigma \rightarrow \tau$ follows in the standard way from maximality and consistency of set M and Definition 4. \square

To finish the proof of the theorem, note that $\varphi \notin M$ implies, by the above lemma, that $G_\infty \not\models \varphi$.

Case II: $0 \parallel 0 \notin M$. Let $Sub(\varphi)$ be the finite set of all $(p, q) \in \mathbb{Z}^2$ such that $p \parallel q$ is a subformula of φ . Define game G to be

$$\prod \{G_{|p-q|} \mid (p, q) \in Sub(\varphi) \text{ and } p \parallel q \notin M\}.$$

Lemma 24 *For each subformula ψ of formula φ ,*

$$\psi \in M \quad \text{if and only if} \quad G \models \psi.$$

Proof. Induction on structural complexity of formula ψ . If φ is \perp , then $\psi \notin M$ due to consistency of set M . At the same time, $G \not\models \psi$ by Definition 4.

Next assume that ψ is an atomic formula $a \parallel b$.

(\Rightarrow) : Suppose first that $a \parallel b \in M$ and $G \not\models a \parallel b$. Thus, by Theorem 7 and Theorem 6, there must exist $(p, q) \in Sub(\varphi)$ such that $p \parallel q \notin M$ and $G_{|p-q|} \not\models a \parallel b$. Hence, by Theorem 5, either $a = b$ or $|p - q| = |a - b|$. If $a = b$, then assumption $a \parallel b \in M$ implies that $0 \parallel 0 \in M$ due to Homogeneity axiom and maximality of set M . The latter, however, is contradiction with the assumption of the case that we consider. If $|p - q| = |a - b|$, then $a \parallel b \in M$ implies $p \parallel q \in M$ by Lemma 2. The latter is a contradiction with the choice of pair (p, q) .

(\Leftarrow) : Suppose that $a \parallel b \notin M$. Thus, $G_{|a-b|} \not\models a \parallel b$ by Theorem 5. Hence, $G \not\models a \parallel b$ by Theorem 7 and Theorem 6.

Case when ψ is an implication $\sigma \rightarrow \tau$ follows in the standard way from maximality and consistency of set M and Definition 4. \square

To finish the proof of the theorem, note that $\varphi \notin M$ implies, by Lemma 24, that $G \not\models \varphi$. \square

6 Conclusion

In this article we have shown the existence of cellular games in which Nash equilibria are interchangeable for nearby players, but not interchangeable for far-away players. We also gave complete axiomatization of all propositional properties of interchangeability of cellular games. Possible next step in this work could be the axiomatization of properties of Nash equilibria for two-dimensional or even circular cellular games. Circular economies have been studied in the economics literature before [8].

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