Budget-Constrained Knowledge in Multiagent Systems

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ABSTRACT

The paper introduces a modal logical system for reasoning about knowledge in which information available to agents might be constrained by the available budget. Although the system lacks an equivalent of the standard Negative Introspection axiom from epistemic logic S5, it is proven to be sound and complete with respect to an S5-like Kripke semantics.

Categories and Subject Descriptors

I.2.11 [**Distributed Artificial Intelligence**]: Multiagent systems; F.4.1 [**Mathematical Logic**]

Keywords

modal logic, epistemology, axiomatization, completeness

General Terms

Theory, Economics

1. INTRODUCTION

There are many examples where large amount of data might be potentially accessible to an agent, but the agent is constrained by the available budget.

For instance, a recent ruling of the European Union Court of Justice [12] and the proposed EU regulation on "right to be forgotten"/"right to erasure" emphasize that this right is not absolute. The related European Commission memo [3] explicitly states that "... the right to be forgotten cannot amount to a right to re-write or erase history. Neither must the right to be forgotten take precedence over freedom of expression or freedom of the media. The right to be forgotten includes an explicit provision that ensures it does not encroach on the freedom of expression and information."

To balance individual privacy with freedom of media, these regulations, for example, require online search engines to remove links to certain newspaper articles from its search results, but they do not require libraries to remove the newspapers from their archives. As a result, the regulations enforce individual privacy not by completely eliminating access to certain information, but, instead, by making such access prohibitively time consuming or prohibitively expensive.

In this paper we study epistemic properties of budget-constrained knowledge. Consider a more specific example of such a situation based on Figure 1 that lists current criminal background check fees in several states in the US [11, 15, 14, 13, 2, 10, 16].

State	Fee (USD)
California	32
Florida	24
Hawaii	30
Iowa	15
Maryland	18
Pennsylvania	10
Virginia	15

Figure 1: Background Check Fees.

Let a hypothetical person John have criminal records in Florida and Maryland, but not in any other state. Any agent a who has at least 18 dollars has a chance to learn that John has criminal records in one of the states. We formally denote this as

$$\Box_a^{18}(\text{"John has criminal records in one of the states."}).$$

At the same time, if agent a has at least 42 dollars, then she could learn that John has at least two different criminal records:

$$\Box_a^{42}(\text{"John has at least two criminal records."}).$$

Furthermore, if agent a pays Florida and Maryland fees for background check, then this agent not only knows that John has at least two criminal records, but also learns that she can find out about this by paying only 42 dollars:

$$\Box_a^{42}\Box_a^{42}(\text{"John has at least two criminal records."}).$$

Note that no agent is able to discover that John has two criminal records with only 41 dollars:

$$\neg\Box_a^{41}(\text{"John has at least two criminal records."}).$$

The above statement is an example of a privacy claim about John's criminal records, that the "right to be forgotten" legislation is designed to provide.

Of course, if John's criminal records are different or if the states decide to change background check fees, then some of the above statements might no longer be true. In this paper we study properties of budget-constrained knowledge

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that are universally true. An example of such properties is the monotonicity principle: $\Box^5_a\varphi \to \Box^7_a\varphi$. That is, more money buys you at least as much information as less money does. A more interesting example of such universal principle is the budget-constrained version of the standard modal Distributivity axiom:

$$\Box^c_a(\varphi \to \psi) \to (\Box^d_a\varphi \to \Box^{c+d}_a\psi). \tag{1}$$

In other words, if agent a can learn $\varphi \to \psi$ by spending at most c and she can learn φ by spending at most d, then she can learn ψ by spending at most $c+d$. The budget-constrained versions of Truth axiom and Positive Introspection axiom from epistemic logic [4, 7] are also true:

$$\Box^c_a\varphi \to \varphi,$$

$$\Box^c_a\varphi \to \Box^c_a\Box^c_a\varphi. \tag{2}$$

Negative Introspection. It is interesting to point out that budget-constrained version of Negative Introspection axiom from epistemic logic

$$\neg\Box^c_a\varphi \to \Box^c_a\neg\Box^c_a\varphi. \tag{3}$$

is not generally true. Indeed, suppose that another hypothetical person, say Frank, has no criminal records in any of the above states. Let statement φ be "Frank does have criminal records in one of the above states". Note that $\neg\Box^{100}_a\varphi$ is true because even for \$100 agent a can not find out something which simply is not true. At the same time, $\Box^{100}_a\neg\Box^{100}_a\varphi$ is false because the only way to learn $\neg\Box^{100}_a\varphi$ is to obtain criminal records from all states, which can not be done on \$100 budget. Thus, statement (3), for $c = 100$, is false.

The example above appears to rely on the fact that $c > 0$. One might wonder if the Negative Introspection axiom is true for zero-cost knowledge:

$$\neg\Box^0_a\varphi \to \Box^0_a\neg\Box^0_a\varphi. \tag{4}$$

The answer to this question is not trivial. Whether property (4) is universally true depends on exact details of the formal semantics definition. To illustrate the issue at hand, consider famous Hilbert Grand Hotel that has infinitely many rooms. Let us first assume that it costs one dollar to check if any particular room is occupied or not and that currently several rooms in this hotel are vacant. In this case any agent a, if she is lucky enough, spends one dollar and happens to check the room that is vacant. Then she learns that the hotel has vacancies:

$$\Box^1_a(\text{"The Grand Hotel has vacancies."}).$$

Now consider the case when the hotel is actually full and it costs nothing to check if any particular room is occupied. Note that in this situation statement

$$\neg\Box^0_a(\text{"The Grand Hotel has vacancies."}) \tag{5}$$

is true, because agent a can not learn something which is false. However, the only feasible way for any agent to learn that statement (5) is true is to open all doors and to see that the Grant Hotel has no vacancies. Let us consider the question whether statement

$$\Box^0_a\neg\Box^0_a(\text{"The Grand Hotel has vacancies."}) \tag{6}$$

is true. In other words, should we allow the agent to check infinitely many rooms and to make certain conclusion only after getting all the results? In this paper we assume that any agent can take only finitely many actions before making a logical conclusion. Thus, from our point of view, statement (6) is false. Hence, implication

$$\neg\Box^0_a(\text{"The Grand Hotel has vacancies."}) \to$$
$$\Box^0_a\neg\Box^0_a(\text{"The Grand Hotel has vacancies."})$$

is false, which shows that Negative Introspection principle is not always true even for zero-cost knowledge. We will further analyze this example after we give formal semantics of our logical system.

Multiagent Case. The logical system proposed in this paper supports reasoning about budget-constrained knowledge in multiagent systems. For example, the state of Virginia has lower background check fee if the request is for the purpose of volunteering for a non-profit organization. The current fee in this case is only eight dollars instead of fifteen [16]. Suppose that John is simultaneously applying for an internship with a for-profit organization (agent a) and for volunteering for a non-profit organization (agent b). Thus,

$$\neg\Box^{10}_a(\text{"John has no criminal records in Virginia"})$$

and

$$\Box^{10}_b(\text{"John has no criminal records in Virginia"}).$$

If agent a pays 15 dollars, then she learns that John has no criminal records in Virginia. Once she knows that this statement is true, she would also know that agent b may learn the same information for just eight dollars:

$$\Box^{15}_a\Box^8_b(\text{"John has no criminal records in Virginia"}).$$

Related Work. The idea to add numerical labels to epistemic modality goes back to Van der Hoek and Meyer's work [18] on graded modalities in epistemic logic. The numerical labels in their system, however, count the number of exceptions in which the statement is false, not the budget required to learn that statement is true. As a result, their axiomatic system is very different from the one proposed here.

The most popular logical system for reasoning about resource constraints is linear logic [6]. It treats propositional variables as resources without introducing any additional labeled modalities. Different from our system, linear logic reasons only about resources and does not deal with epistemology.

Our logical system is related to Artëmov's logic of justification [1]. Although logic of justification does not consider the cost of justification, the Distributivity axiom (1) could be viewed as a budget-constrained version of Application axiom from justification logic. Similarly, budget constrained version of Positive Introspection axiom (2) is related to Proof Checker axiom in justification logic: the costs of checking the proof, at least in our setting, is the same as the cost of the proof itself. Whereas justification logic deals with justifications syntactically, our logical system deals with them semantically. The superscript of the modal operator refers only to the cost of a justification. Thus, our logical system could be viewed, to some degree, as a more abstract version of justification logic. Note, however, that the formal semantics for budget-constrained knowledge that we propose in this paper is very different from Fitting's [5] semantics for justification logic. While Fitting's semantics extends S4 models with admissible evidence function, our S5-like semantics treats each justification as an equivalence relation.

Our system is also related to Moss and Parikh's work on topological semantics for epistemic logic [8], in which they developed a logical system that takes into account "effort" needed to acquire new knowledge. Unlike ours, their system treats efforts topologically without assigning them numerical costs.

Finally, in terms of a modality indexed with a real number subscript, this work is related to the author's article [9] on logic of confidence. In spite of this syntactic similarity, logic of confidence contains Negative Introspection axiom. Moreover, Distributivity axioms in these two systems are different.

Paper Structure. Section 2 introduces the syntax and the semantics of our logical system for budget-constrained knowledge. Formally equipped, we further analyze the Grand Hotel example and prove that statement (5) does not imply statement (6). The logical system is presented in Section 3 and a couple of example properties of budget-constrained knowledge followed from the logical system is shown in Section 4. In Section 5 and Section 6, the soundness and the completeness of the logical system are established. Finally, Section 7 discusses potential extensions to our work when our assumption of finitely many actions is relaxed to at most countably many actions, when one considers distributed knowledge with constrained budget, and when the budget becomes a function that relates time and money.

2. SYNTAX AND SEMANTICS

In this section we introduce formal syntax and semantics of our logical system. We assume that $\mathcal{A}$ is an arbitrary (possibly infinite) set of agents and $\mathcal{P}$ is a set of atomic propositions.

Definition 1. *Let Φ be the minimal set of formulas such that*

1. $\mathcal{P} \subseteq \Phi$,
2. *if* $\varphi \in \Phi$, *then* $\neg\varphi \in \Phi$,
3. *if* $\varphi, \psi \in \Phi$, *then* $\varphi \to \psi \in \Phi$,
4. *if* $\varphi \in \Phi$, *then* $\Box^c_a\varphi \in \Phi$ *where* $a \in \mathcal{A}$ *and* $c \in [0, +\infty)$.

In the introduction we have seen that the cost of knowledge comes from costs associated with obtaining justifications for this knowledge. In some situations one might find it more natural to talk about costs of "efforts" [8], costs of "actions", or costs of "evidences". In our formal semantics we assume an existence of a set of justifications J_a for each agent $a \in A$. For technical convenience we assume that sets $\{J_a\}_{a\in\mathcal{A}}$ are disjoint for different agents a. The cost of justifications is captured by a cost function $\|j\|$ that maps each $j \in J_a$ into a non-negative real value. This function is not agent-dependent because any justification j uniquely determines the agent. Note that the same knowledge has different costs to different agents in the background check fees example. To account for this, in our model we assume that different agents have different sets of justifications. The difference in knowledge costs to different agents comes from the fact that they use agent-specific justifications for the same knowledge.

In our semantics, any justification $j \in J_a$ has an associated indistinguishability relation $\sim_j$. Informally, $u \sim_j w$ means that agent a can not distinguish epistemic worlds u and w with justification j. As mentioned in the introduction, this approach to capturing justifications, natural from epistemic logic point of view, is very different from Fitting's [5] models for justification logic.

Finally, the definition of formal semantics given below also includes function π that specifies, for each atomic proposition, the set of epistemic worlds in which the proposition is satisfied.

Definition 2. *A budget-constrained epistemic model is a tuple* $(W, \{J_a\}_{a\in\mathcal{A}}, \{\{\sim_j\}_{j\in J_a}\}_{a\in\mathcal{A}}, \|\cdot\|, \pi)$, *where*

1. *W is a set (of "epistemic worlds"),*
2. *$\{J_a\}_{a\in\mathcal{A}}$ is family of disjoint sets of "justifications",*
3. *$\sim_j$ is an "indistinguishability" relation on W associated with justification $j \in J_a$ of an agent $a \in \mathcal{A}$,*
4. *$\|\cdot\|$ is the cost function that maps each justification $j \in \bigcup_{a\in\mathcal{A}} J_a$ into its price $\|j\| \in [0, +\infty)$,*
5. *π is a function that maps each $p \in \mathcal{P}$ into a subset of W.*

For any $J \subseteq J_a$, we write $u \sim_J w$ if $u \sim_j w$ for each $j \in J$. Informally, $u \sim_J w$ means that agent a can not distinguish epistemic worlds u and w while possessing all of the justifications in set J.

Lemma 1. *For any $J \subseteq J_a$, relation $\sim_J$ is an equivalence relation on the set of epistemic worlds W.* $\boxtimes$

Lemma 2. *If $J \subseteq J' \subseteq J_a$, then equivalence relation $\sim_{J'}$ is at least as strong as equivalence relation $\sim_J$.* $\boxtimes$

Recall from the introduction that, informally, $w \Vdash \Box^c_a\varphi$ means that in epistemic world w agent a has a *possibility* to learn φ with budget c. Below we capture this informal definition by saying that $\Box^c_a\varphi$ is satisfied in epistemic world w if there is a set of justifications J with a total cost of no more than c such that φ is satisfied in all epistemic worlds that are J-indistinguishable from w.

Definition 3. *Satisfiability relation $\Vdash$ between epistemic worlds in W and all formulas in Φ is defined as follows:*

1. $w \Vdash p$ *if* $w \in \pi(p)$, *where* $p \in \mathcal{P}$,
2. $w \Vdash \neg\varphi$ *if* $w \nVdash \varphi$,
3. $w \Vdash \varphi \to \psi$ *if* $w \nVdash \varphi$ *or* $w \Vdash \psi$,
4. $w \Vdash \Box^c_a\varphi$ *if there is a finite set* $J \subseteq J_a$ *such that*
   (a) $\sum_{j\in J} \|j\| \leq c$,
   (b) *for any* $u \in W$, *if* $u \sim_J w$, *then* $u \Vdash \varphi$.

Grand Hotel Example. We illustrate Definitions 2 and 3 using the Hilbert's Grand Hotel example from the introduction. The set of agents in this example consists of a single agent a. An epistemic world w is a possible hotel occupation state which can be represented as a Boolean function on hotel room numbers. That is, $w : \mathbb{N} \mapsto \{occupied, available\}$.

In this example, a justification could be thought of as an action of opening a single door in the hotel to check if the room is occupied. Thus, the set of all justifications can be identified with the set of room numbers $\mathbb{N}$. Such a justification j can not distinguish epistemic worlds $u : \mathbb{N} \mapsto \{occupied, available\}$ and $w : \mathbb{N} \mapsto \{occupied, available\}$ if $u(j) = w(j)$. In other words, $u \sim_j w$ if and only if $u(j) = w(j)$.

Suppose that the cost of opening any door is zero: $\|j\| = 0$ for each $j \in \mathbb{N}$ and that in epistemic world w_0 the hotel is full: $w_0(j) = occupied$ for each $j \in \mathbb{N}$.

Finally, let valuation $\pi(p_0)$ for some atomic proposition p_0 be defined to be the set of all epistemic worlds in which at least one room is vacant:

$$\pi(p_0) = \{w \in W \mid \exists n \in \mathbb{N}\ (w(n) = available)\}.$$

Informally, proposition p_0 is statement "The Grand Hotel has vacancies". For the sake of clarity, in what follows, we write "The Grand Hotel has vacancies" instead of atomic proposition p_0. The truth of statements (5) and (6) is formally shown below.

Proposition 1.

$$w_0 \Vdash \neg\Box^0_a(\text{“The Grand Hotel has vacancies.”}).$$

Proof. Assume the opposite. Thus, there must exist a finite set $J \subseteq \mathbb{N}$ such that $\sum_{j\in J} \|j\| = 0$ and

$$u \Vdash \text{“The Grand Hotel has vacancies.”}$$

for each epistemic world u where $u \sim_J w_0$. By Lemma 1, $w_0 \sim_J w_0$. Thus, $w_0 \Vdash$ "The Grand Hotel has vacancies.", which contradicts the choice of world w_0. ⊠

Proposition 2.

$$w_0 \nVdash \Box^0_a\neg\Box^0_a(\text{“The Grand Hotel has vacancies.”}).$$

Proof. Assume the opposite. Thus, there must exist a *finite* set $J \subseteq \mathbb{N}$ such that $\sum_{j\in J} \|j\| = 0$ and

$$u \Vdash \neg\Box^0_a(\text{“The Grand Hotel has vacancies.”})$$

for each epistemic world u where $u \sim_J w_0$. Since set J is finite, there must exist $n \in \mathbb{N} \setminus J$. Consider epistemic world u_0 such that

$$u_0(j) = \begin{cases} available, & \text{if } j = n, \\ occupied, & \text{otherwise.} \end{cases}$$

Note that $u_0 \sim_J w_0$ because $n \notin J$. Hence,

$$u_0 \Vdash \neg\Box^0_a(\text{“The Grand Hotel has vacancies.”}). \tag{7}$$

Consider set $J' = \{n\}$. Note that $\sum_{j\in J'} \|j\| = \|n\| = 0$. Thus, by Definition 3, statement (7) implies that there exists epistemic world v_0 such that $v_0 \sim_{J'} u_0$ and

$$v_0 \nVdash \text{“The Grand Hotel has vacancies.”}. \tag{8}$$

Finally, note that $v_0 \sim_{J'} u_0$ implies that $v_0(n) = u_0(n) = available$, which is a contradiction to statement (8). ⊠

3. AXIOMS

Our logical system, in addition to all propositional tautologies in language Φ, contains the following axioms:

1. Truth: $\Box^c_a\varphi \to \varphi$,
2. Positive Introspection: $\Box^c_a\varphi \to \Box^c_a\Box^c_a\varphi$,
3. Distributivity: $\Box^c_a(\varphi \to \psi) \to (\Box^d_a\varphi \to \Box^{c+d}_a\psi)$.

We write $\vdash \varphi$ if formula φ is provable from the propositional tautologies and the above axioms using Modus Ponens and Necessitation inference rules:

$$\frac{\varphi,\quad \varphi \to \psi}{\psi} \qquad\qquad \frac{\varphi}{\Box^c_a\varphi}.$$

We write $X \vdash \varphi$ if formula φ is provable from propositional tautologies, the above axioms, and the additional set of axioms X using only Modus Ponens inference rule.

4. EXAMPLES

The soundness of our logical system will be established in the next section. In this section we show how this system can be used to prove two additional properties of budget-constrained knowledge. We later use these results in the proof of completeness.

The first property is monotonicity of budget-constrained knowledge. Informally, it says that anything that can be learned on a smaller budget also can be learned with a larger budget.

Lemma 3. *$\vdash \Box^c_a\varphi \to \Box^d_a\varphi$, where $c \le d$.*

Proof. Note that $\varphi \to \varphi$ is a propositional tautology. Thus, $\vdash \Box^{d-c}_a(\varphi \to \varphi)$ by Necessitation rule. Note also that

$$\vdash \Box^{d-c}(\varphi \to \varphi) \to (\Box^c\varphi \to \Box^d\varphi)$$

is an instance of Distributivity axiom. Thus, from the two statements above, by Modus Ponens, $\vdash \Box^c_a\varphi \to \Box^d_a\varphi$. ⊠

The property expressed by the lemma below is something that, at first glance, might look as a more general version of Distributivity axiom. However, as we show below, it is provable from Positive Introspection axiom and the regular version of Distributivity axiom.

Lemma 4. $\vdash \Box^c_a(\Box^d_a\varphi \to \psi) \to (\Box^d_a\varphi \to \Box^{c+d}_a\psi)$.

Proof. By Distributivity axiom,

$$\vdash \Box^c_a(\Box^d_a\varphi \to \psi) \to (\Box^d_a\Box^d_a\varphi \to \Box^{c+d}_a\psi).$$

By Modus Ponens rule applied twice,

$$\Box^c_a(\Box^d_a\varphi \to \psi), \Box^d_a\Box^d_a\varphi \vdash \Box^{c+d}_a\psi.$$

By Positive Introspection axiom,

$$\Box^c_a(\Box^d_a\varphi \to \psi), \Box^d_a\varphi \vdash \Box^{c+d}_a\psi.$$

By Deduction theorem for propositional logic,

$$\vdash \Box^c_a(\Box^d_a\varphi \to \psi) \to (\Box^d_a\varphi \to \Box^{c+d}_a\psi).$$

⊠

5. SOUNDNESS

In this section we establish the soundness of our logical system with respect to the semantics given in Definition 3. The soundness of propositional tautologies and of Modus Ponens inference rule is straightforward. Below we prove the soundness of each of the remaining axioms and of Necessitation inference rule as a separate lemma. We assume that w is an arbitrary epistemic world of a budget-constrained epistemic model $(W, \{J_a\}_{a\in\mathcal{A}}, \{\{\sim_j\}_{j\in J_a}\}_{a\in\mathcal{A}}, \|\cdot\|, \pi)$.

LEMMA 5. *If $w \Vdash \Box^c_a\varphi$, then $w \Vdash \varphi$.*

PROOF. Suppose that $w \Vdash \Box^c_a\varphi$. Thus, there is a set $J \subseteq J_a$ such that $\sum_{j\in J}\|j\| \le c$ and $u \Vdash \varphi$ for each u such that $u \sim_J w$. Note that $w \sim_J w$ by Lemma 1. Therefore, $w \Vdash \varphi$. ⊠

LEMMA 6. *If $w \Vdash \Box^c_a\varphi$, then $w \Vdash \Box^c_a\Box^c_a\varphi$.*

PROOF. By assumption $w \Vdash \Box^c_a\varphi$, there must exist a set $J \subseteq J_a$ such that $\sum_{j\in J}\|j\| \le c$ and $u \Vdash \varphi$ for each $u \in W$ such that $u \sim_J w$. To prove $w \Vdash \Box^c_a\Box^c_a\varphi$, it is sufficient to show that $u \Vdash \Box^c_a\varphi$ for each $u \in W$ such that $u \sim_J w$. To show the latter, it is enough to justify that $v \Vdash \varphi$ for each $v \in W$ such that $v \sim_J u$. Indeed, consider any $v \in W$ such that $v \sim_J u$. By the assumption $u \sim_J w$ and Lemma 1, we have $v \sim_J w$. This, by the choice of the set J, implies that $v \Vdash \varphi$. ⊠

LEMMA 7. *If $w \Vdash \Box^c_a(\varphi \to \psi)$ and $w \Vdash \Box^d_a\varphi$, then $w \Vdash \Box^{c+d}_a\psi$.*

PROOF. Suppose that $w \Vdash \Box^c_a(\varphi \to \psi)$ and $w \Vdash \Box^d_a\varphi$. By the first assumption, there exists a set $J_1 \subseteq J_a$ such that $\sum_{j\in J_1}\|j\| \le c$ and $u \sim_{J_1} w$ implies $u \Vdash \varphi \to \psi$ for each $u \in W$. By the second assumption, there exists a set $J_2 \subseteq J_a$ such that $\sum_{j\in J_2}\|j\| \le d$ and $u \sim_{J_2} w$ implies $u \Vdash \varphi$ for each $u \in W$. Let $J = J_1 \cup J_2$. Then

$$\sum_{j\in J}\|j\| \le \sum_{j\in J_1}\|j\| + \sum_{j\in J_2}\|j\| \le c + d$$

and equivalence relation $\sim_J$, by Lemma 2, is at least as strong as both of the equivalence relations $\sim_{J_1}$ and $\sim_{J_2}$. Hence, $u \Vdash \varphi \to \psi$ and $u \Vdash \varphi$ for each $u \in W$ such that $u \sim_J w$. Thus, by Definition 3, $u \Vdash \psi$ for each $u \in W$ such that $u \sim_J w$. Therefore, again by Definition 3, $w \Vdash \Box^{c+d}_a\psi$. ⊠

LEMMA 8. *For any formula $\varphi \in \Phi$, if $w' \Vdash \varphi$ for each epistemic world $w' \in W'$ of each budget-constrained epistemic model $(W', \{J'_a\}_{a\in\mathcal{A}}, \{\{\sim'_j\}_{j\in J_a}\}_{a\in\mathcal{A}}, \|\cdot\|', \pi')$, then $w \Vdash \Box^c_a\varphi$.*

PROOF. Assume that $w \nVdash \Box^c_a\varphi$. Let $J = \varnothing \subseteq J_a$. Note that $\sum_{j\in J}\|j\| = 0 \le c$. Thus, by assumption $w \nVdash \Box^c_a\varphi$, there must exist $w' \in W$ such that $w' \sim_J w$ and $w' \nVdash \varphi$. The latter contradicts the assumption of the lemma. ⊠

6. COMPLETENESS

In this section we prove the completeness of our logical system with respect to the formal semantics given in Definition 3. This proof significantly deviates from the standard completeness proof techniques for modal logic systems due to the challenge that we describe below.

S5-like Semantics for S4-like Logic. If one ignores budget superscript on modalities in the language of our logical system, then the system is reduced to the multiagent S4 modal logic. The standard Kripke semantics for S4 is using Kripke frames with reflexive and transitive accessibility relations. However, the budget-constrained epistemic semantics in Definition 2, just like the standard S5 epistemic logic Kripke semantics, assumes that the indistinguishibility relation is not only reflexive and transitive, but also symmetric.

One might think that it would be possible to easily adopt the standard canonical Kripke model construction to our logical system. Unfortunately, in the absence of Negative Introspection axiom, this construction would yield nonsymmetric accessibility relations as it does in the case of S4 models.

The main challenge in the proof of the completeness of our logical system is to construct a Kripke-like model with symmetric accessibility relations in the absence of Negative Introspection axiom.

To overcome this challenge, we add extra information to each epistemic world. In the standard canonical model construction, an epistemic world is simply a maximal consistent set of formulas. In our construction, an epistemic world is a pair (X, β), where X is a maximal consistent set of formulas and β is a function of the type specified in Definition 4. Function β can be thought of as an array of "flags" that are used to fine-tune indistinguishibility relations between epistemic worlds. The details of this "fine-tuning" are given in Definition 6.

We are now ready to define "canonical" budget-constrained epistemic model $(W, \{J_a\}_{a\in\mathcal{A}}, \{\{\sim_j\}_{j\in J_a}\}_{a\in\mathcal{A}}, \|\cdot\|, \pi)$.

DEFINITION 4. *The set of epistemic worlds W consists of all pairs (X, β), where X is a maximal consistent subset of Φ and β is a function that maps each tuple (c, φ), where $c \in [0, +\infty)$ and $\varphi \in \Phi$, into an integer value $\beta(c, \varphi) \in \mathbb{Z}$.*

DEFINITION 5. *For each agent $a \in \mathcal{A}$, the set of justifications J_a consists of all tuples of the form (a, b, c, φ), where $b \in \mathbb{Z}$, $c \in [0, +\infty)$, and $\varphi \in \Phi$.*

Informally, tuple (a, b, c, φ) justifies to agent a at cost c that statement φ is true. The component b of the tuple is a part of "fine-tuning" mechanism whose purpose is clarified in the next definition.

DEFINITION 6. *For any $(X_1, \beta_1) \in W$, any $(X_2, \beta_2) \in W$ and any $(a, b, c, \varphi) \in J_a$, let $(X_1, \beta_1) \sim_{(a,b,c,\varphi)} (X_2, \beta_2)$ if the following two conditions are either both true or both false:*

1. *$\Box^c_a\varphi \in X_1$ and $\beta_1(c, \varphi) = b$,*
2. *$\Box^c_a\varphi \in X_2$ and $\beta_2(c, \varphi) = b$.*

LEMMA 9. *For any $(a, b, c, \varphi) \in J_a$, relation $\sim_{(a,b,c,\varphi)}$ is an equivalence relation on set W.*

PROOF. The statement of the lemma follows from Definition 6 because biconditional is an equivalence relation on propositional formulas. ⊠

DEFINITION 7. *Let $\|(a, b, c, \varphi)\| = c$.*

In the standard construction of a canonical Kripke model, an atomic proposition is satisfied in an epistemic world if the maximal consistent set representing this world contains this proposition. We adopt the same approach in our construction without taking function β into consideration.

Definition 8. *For any atomic proposition $p \in \mathcal{P}$, let $\pi(p)$ be the set $\{(X,\beta) \in W \mid p \in X\}$.*

The next two lemmas are at the core of the proof of the completeness theorem. They both have analogy in the standard completeness proofs for modal logics such as S4 and S5. In the case of these logics, we normally need to show that if a maximal consistent set representing world w contains formula $\Box\psi$, then ψ belongs to the maximal consistent set representing any world accessible from world w. In our case, such a property is stated as follows.

Lemma 10. *If $(X,\beta) \in W$ and $\Box^c_a\psi \in X$, then $\psi \in Y$ for each $(Y,\gamma) \in W$ such that $(Y,\gamma) \sim_{(a,\beta(c,\psi),c,\psi)} (X,\beta)$.*

Proof. Consider any $(Y,\gamma) \in W$ such that

$$(Y,\gamma) \sim_{(a,\beta(c,\psi),c,\psi)} (X,\beta).$$

Then, by Definition 6, $\Box^c_a\psi \in Y$. Hence, $Y \vdash \psi$ by Truth axiom. Therefore, $\psi \in Y$ due to maximality of set Y. ⊠

As mentioned earlier, tuple (a,b,c,ψ) justifies to agent a at cost c that statement ψ is true, where component b serves as a part of the "fine-tuning" mechanism. The above lemma clarifies the details of this mechanism. Namely, in epistemic world (X,β), such a justification is tuple $(a,\beta(c,\psi),c,\psi)$.

In a proof of completeness for classical modal logics such as S4 and S5, one shows that if formula $\Box\psi$ does not belong to the maximal consistent set of formulas representing world w, then there is a world u, accessible from world w, such that the maximal consistent set of formulas representing u does not contain formula ψ. Below is the corresponding statement in our proof of completeness.

Lemma 11. *For any $(X,\beta) \in W$, any $\Box^d_a\psi \notin X$, and any finite $J \subseteq J_a$ such that $\sum_{j\in J}\|j\| \le d$, there is $(Y,\gamma) \in W$ such that $(Y,\gamma) \sim_J (X,\beta)$ and $\psi \notin Y$.*

Proof. Define

$$\gamma(c,\varphi) = \begin{cases} \beta(c,\varphi), & \text{if } (a,\beta(c,\varphi),c,\varphi) \in J \text{ and } \Box^c_a\varphi \in X,\\ \omega, & \text{otherwise,}\end{cases}$$

where ω is an arbitrary integer such that $(a,\omega,c,\varphi) \notin J$. Such ω exists due to the finiteness of J. We next show that set

$$Y_0 \;=\; \{\neg\psi\} \cup \{\Box^c_a\varphi \mid \Box^c_a\varphi \in X, (a,\beta(c,\varphi),c,\varphi) \in J\}$$

is consistent. Assuming the opposite, there are formulas $\Box^{c_1}_a\varphi_1,\dots,\Box^{c_n}_a\varphi_n \in X$ such that $(a,\beta(c_i,\varphi_i),c_i,\varphi_i) \in J$ for each $i \le n$ and $\Box^{c_1}_a\varphi_1,\dots,\Box^{c_n}_a\varphi_n \vdash \psi$. Then, by Deduction theorem for propositional logic,

$$\vdash \Box^{c_1}_a\varphi_1 \to (\Box^{c_2}_a\varphi_2 \to \dots(\Box^{c_n}_a\varphi_n \to \psi)\dots).$$

By Necessitation rule,

$$\vdash \Box^0_a(\Box^{c_1}_a\varphi_1 \to (\Box^{c_2}_a\varphi_2 \to \dots(\Box^{c_n}_a\varphi_n \to \psi)\dots)).$$

By Lemma 4,

$$\vdash \Box^{c_1}_a\varphi_1 \to \Box^{c_1}_a(\Box^{c_2}_a\varphi_2 \to \dots(\Box^{c_n}_a\varphi_n \to \psi)\dots).$$

Note that $\Box^{c_1}_a\varphi_1 \in X$ by the choice of set Y_0. Thus, by Modus Ponens rule,

$$X \vdash \Box^{c_1}_a(\Box^{c_2}_a\varphi_2 \to \dots(\Box^{c_n}_a\varphi_n \to \psi)\dots).$$

Again by Lemma 4 and Modus Ponens inference rule,

$$X \vdash \Box^{c_2}_a\varphi_2 \to \Box^{c_1+c_2}_a(\Box^{c_3}_a\varphi_3 \to \dots(\Box^{c_n}_a\varphi_n \to \psi)\dots).$$

Note that $\Box^{c_2}_a\varphi_2 \in X$ again by the choice of set Y_0. Thus, by Modus Ponens rule,

$$X \vdash \Box^{c_1+c_2}_a(\Box^{c_3}_a\varphi_3 \to \dots(\Box^{c_n}_a\varphi_n \to \psi)\dots).$$

By repeating the last two steps $(n-2)$ times,

$$X \vdash \Box^{c_1+c_2+\dots+c_n}_a\psi.$$

Thus, $X \vdash \Box^d_a\psi$ by Lemma 3 and due to

$$\sum_i c_i = \sum_i \|(a,\beta(c_i,\varphi_i),c_i,\varphi_i)\| \le \sum_{j\in J}\|j\| \le d,$$

which is a contradiction to the assumption $\neg\Box^d_a\psi \in X$ and the consistency of the set X. Therefore, set Y_0 is consistent. Let Y be any maximal consistent extension of the set Y_0.

Note that $\neg\psi \in Y_0 \subseteq Y$ by the choice of Y_0 and Y. Thus, $\psi \notin Y$ due to consistency of the set Y. We are only left to show that $(X,\beta) \sim_J (Y,\gamma)$, which, by Definition 6, follows from the following claim:

Claim 1. *For each $(a,b,c,\varphi) \in J$, the following two conditions are either both true or both false:*

1. *$\Box^c_a\varphi \in X$ and $\beta(c,\varphi) = b$,*
2. *$\Box^c_a\varphi \in Y$ and $\gamma(c,\varphi) = b$.*

Proof. $1) \Rightarrow 2)$: Suppose that $\Box^c_a\varphi \in X$ and $\beta(c,\varphi) = b$. Thus, $\Box^c_a\varphi \in Y_0$ by the choice of Y_0. Hence, $\Box^c_a\varphi \in Y$ because Y is an extension of Y_0. Note also that $\Box^c_a\varphi \in X$ and $\beta(c,\varphi) = b$ implies, by the choice of γ, that $\gamma(c,\varphi) = \beta(c,\varphi)$. Therefore, $\gamma(c,\varphi) = \beta(c,\varphi) = b$.

$2) \Rightarrow 1)$: First, assume that $\Box^c_a\varphi \notin X$. Then, $\gamma(c,\varphi) = \omega$ where $(a,\omega,c,\varphi) \notin J$. Recall that $(a,b,c,\varphi) \in J$. Thus, $\omega \ne b$. Therefore, $\gamma(c,\varphi) \ne b$.

Next, suppose that $\Box^c_a\varphi \in X$ and $\beta(c,\varphi) \ne b$. There are two cases to consider. If $(a,\beta(c,\varphi),c,\varphi) \in J$, then $\gamma(c,\varphi) = \beta(c,\varphi)$, by the definition of function γ. Therefore, $\gamma(c,\varphi) = \beta(c,\varphi) \ne b$. Otherwise, suppose that $(a,\beta(c,\varphi),c,\varphi) \notin J$. Then, $\gamma(c,\varphi) = \omega$ where $(a,\omega,c,\varphi) \notin J$. Recall that $(a,b,c,\varphi) \in J$. Thus, $\omega \ne b$. Therefore, $\gamma(c,\varphi) \ne b$. ⊠

This concludes the proof of the lemma. ⊠

Next, we use Lemma 10 and Lemma 11 to show that a formula is satisfied in an epistemic world if and only if it belongs to the maximal consistent set of this world.

Lemma 12. *$(X,\beta) \Vdash \varphi$ if and only if $\varphi \in X$, for each $\varphi \in \Phi$ and each $(X,\beta) \in W$.*

Proof. We prove the lemma by induction on the structural complexity of formula φ. If φ is an atomic proposition, then the required follows from Definition 8. Cases of φ having form $\psi_1 \to \psi_2$ and $\neg\psi$ follow from Definition 3 and the maximality and consistency of set X in the standard way. Now suppose that φ has the form $\Box^c_a\psi$, for some $a \in \mathcal{A}$, $c \in [0,+\infty)$, and $\psi \in \Phi$.

$(\Rightarrow)$: Assume that $\Box^c_a\psi \notin X$. Consider any finite set $J \subseteq J_a$ such that $\sum_{j\in J}\|j\| \le c$. It is sufficient to show that there is $(Y,\gamma) \in W$ such that $(Y,\gamma) \sim_J (X,\beta)$ and $(Y,\gamma) \nVdash \psi$. Indeed, by Lemma 11, there is $(Y,\gamma) \in W$ such that $(Y,\gamma) \sim_J (X,\beta)$ and $\psi \notin Y$. Therefore, by the induction hypothesis, $(Y,\gamma) \nVdash \psi$.

$(\Leftarrow)$: If $\Box^c_a\psi \in X$, then by Definition 3, it is sufficient to show that $(Y,\gamma) \Vdash \psi$ for each $(Y,\gamma) \in W$ such that

$(X,\beta) \sim_{(a,\beta(c,\psi),c,\psi)} (Y,\gamma)$. Indeed, by Lemma 10, $\psi \in Y$ for each $(Y,\gamma) \in W$ such that $(Y,\gamma) \sim_{(a,\beta(c,\psi),c,\psi)} (X,\beta)$. Hence, by the induction hypothesis, $(Y,\gamma) \Vdash \psi$ for each $(Y,\gamma) \in W$ such that $(Y,\gamma) \sim_{(a,\beta(c,\psi),c,\psi)} (X,\beta)$. ⊠

We are now ready to state and prove the completeness theorem for our logical system.

THEOREM 1. *If $w \Vdash \varphi$ for each $w \in W$ of each budget-constrained model $(W, \{J_a\}_{a\in\mathcal{A}}, \{\{\sim_j\}_{j\in J_a}\}_{a\in\mathcal{A}}, \|\cdot\|, \pi)$, then $\vdash \varphi$.*

PROOF. Suppose that $\nvdash \varphi$. Let X be any maximal consistent subset of Φ containing formula $\neg\varphi$. Consider function β such that $\beta(c,\psi) = 0$ for each $c \in [0,+\infty)$ and each $\psi \in \Phi$. Note that $\neg\varphi \in X$ implies that $\varphi \notin X$ due to consistency of the set X. Therefore, $(X,\beta) \nVdash \varphi$, by Lemma 12. ⊠

7. CONCLUSION AND DISCUSSION

In this paper we introduced a formal logical system for reasoning about budget constrained knowledge. We provided a formal semantics for this system in terms of justifications with costs and proved the soundness and the completeness of our logical system with respect to this semantics. There are several issues related to this work that we want to discuss in the conclusion.

Infinite Sets of Justifications. In Definition 3, we have assumed that subset J is finite. Note that this assumption could be potentially weakened to subset J being *at most countable*. If the set J is countable, then the sum of costs in Definition 3 becomes an infinite series. Since all terms in this series are non-negative, if such series converges, then it converges absolutely and, thus, the value of the sum does not depend on the order of the terms [17, p. 78]. This means that semantics given by Definition 3 is well-defined for at most countable sets J. However, the proof of the completeness in given in Section 6 of this paper is no longer valid. Indeed, existence of integer ω in proof of Lemma 11 relies on the assumption of finiteness of set J. Furthermore, our logical system, although still sound, is no longer complete with respect to the semantics of at most countable sets of justifications, because Negative Introspection principle becomes true for zero-cost knowledge: $\neg\Box^0_a\varphi \to \Box^0_a\neg\Box^0_a\varphi$. We prove the soundness of this principle in such a setting in the following lemma:

LEMMA 13. *If $w \Vdash \neg\Box^0_a\varphi$, then $w \Vdash \Box^0_a\neg\Box^0_a\varphi$.*

PROOF. Consider set $J = \{j \in J_a \mid \|j\| = 0\}$. Thus, $\sum_{j\in J}\|j\| = 0$. By Definition 3, it is sufficient to show that for any $u \in W$ if $w \sim_J u$, then $u \Vdash \neg\Box^0_a\varphi$. We will prove this by contradiction. Consider any $u \in W$ such that $w \sim_J u$ and suppose that $u \Vdash \Box^0_a\varphi$. Thus, there is $J' \subseteq J_a$ such that $\sum_{j\in J'}\|j\| = 0$ and for each $v \in W$, if $u \sim_{J'} v$, then $v \Vdash \varphi$. Note that $\sum_{j\in J'}\|j\| = 0$ implies that $J' \subseteq J$.

Note that $\sum_{j\in J'}\|j\| = 0$ by the choice of J'. Thus, by Definition 3, the assumption $w \Vdash \neg\Box^0_a\varphi$ implies that there is $v_0 \in W$ such that $w \sim_{J'} v_0$ and $v_0 \Vdash \neg\varphi$. Recall that $w \sim_J u$ and that $J' \subseteq J$. Hence, $w \sim_{J'} u$. Thus, by Lemma 1, $u \sim_{J'} v_0$ and $v_0 \Vdash \neg\varphi$, which is a contradiction to our result from the previous paragraph. ⊠

However, under this modified semantics the negative introspection principle remains false for non-zero costs knowledge. To construct a specific example when this principle is false, consider again Hilbert's Grand Hotel example from the introduction, yet assuming that it costs one Turkish lira to open any door in the hotel: $\|j\| = 1$ for each $j \in \mathbb{N}$. In this case, if w_0 is, as before, the epistemic world in which the hotel has no vacancies, then

$$w_0 \nVdash \neg\Box^c_a(\text{“The Grand Hotel has vacancies.”}) \to \Box^c_a\neg\Box^c_a(\text{“The Grand Hotel has vacancies.”})$$

for each $c > 0$. The proof is very similar to the proofs of Proposition 1 and Proposition 2. In the proof of the second proposition, set J is still finite because $\sum_{j\in J}\|j\| \le c$ and, by our assumption, the cost of each justification is one lira.

Budget-Constrained Distributed Knowledge. In this paper each modality is labeled by only one agent. One can consider modality $\Box^c_A\varphi$, where A is a group of agents, meaning that, at cost no more than c, the group of agent A can obtain the distributed knowledge [4] of statement φ. Different from the single-agent knowledge case, the exact properties of budget-constrained distributed knowledge depend on the specific type of the constraints. Namely, for financial constraints we have:

$$\Box^{c_1}_a(\varphi \to \psi) \to (\Box^{c_2}_b\varphi \to \Box^{c_1+c_2}_{a,b}\psi),$$

because it is natural to assume that if agent a needs c_1 Turkish liras to learn $\varphi \to \psi$ and agent b needs c_2 liras to discover φ, then they together “distributively” know ψ after spending at most $c_1 + c_2$ liras.

However, for time constraints we have

$$\Box^{c_1}_a(\varphi \to \psi) \to (\Box^{c_2}_b\varphi \to \Box^{\max\{c_1,c_2\}}_{a,b}\psi),$$

assuming that the two agents can work simultaneously.

Production Possibility Frontiers. So far, we have been assuming that we either deal with a financial constraint or a time constraint. In many cases goods are priced based on delivery time. That is, a good could be obtained slow but cheap or faster but at a higher cost. Such a constraint could be described by a function that show how fast the good can be obtained within the given budget. In economics, graphs of such functions (see Figure 2) are called production possibility frontiers. For any production possibility frontier f, by $\Box^f_a\varphi$ we mean the statement that for any given cost c, agent a can learn φ in time $f(c)$.

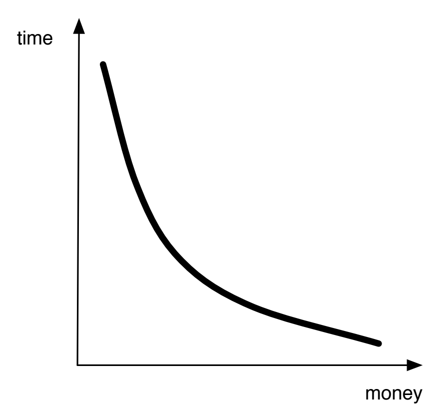

Figure 2: Production Possibility Frontier.

Suppose now that $\Box_a^f(\varphi \to \psi)$ and $\Box_a^g\varphi$, where f and g are two production possibility frontiers. To find function h such that $\Box_a^h\psi$, let us first fix any argument c of this function and decide what is the minimal time that agent a needs to learn both $\varphi \to \psi$ and φ. The agent can spend some part of this budget, say x, to learn $\varphi \to \psi$ in time $f(x)$ and the rest of the budget to learn ψ in time $g(c-x)$. Thus, $h(c) = \min_{0\le x\le c}\{f(x)+g(c-x)\}$. In other words,

$$\Box_a^f(\varphi \to \psi) \to (\Box_a^g\varphi \to \Box_a^{f*g}\psi),$$

where

$$(f*g)(x) = \min_{0\le x\le c}\{f(x)+g(c-x)\}.$$

Complete axiomatization of knowledge constrained by production possibility frontiers remains an open problem.

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