Marketing Impact on Diffusion in Social Networks

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Abstract

The paper proposes a way to add marketing into the standard threshold model of social networks. Within this framework, the paper studies logical properties of the influence relation between sets of agents in social networks. Two different forms of this relation are considered: one for promotional marketing and the other for preventive marketing. In each case a sound and complete logical system describing properties of the influence relation is proposed. Both systems could be viewed as extensions of Armstrong's axioms of functional dependency from the database theory.

1 Introduction

1.1 Social Networks

In this paper we study how diffusion in social networks could be affected by marketing. Diffusion happens when a product or a social norm is initially adopted by a small group of agents who later influence their peers to adopt the same product. The peers influence their peers, and so on. There are two most commonly used models of diffusion: the cascading model and the threshold model. In the cascading model [14; 9] the behaviour of agents is random and the peer influence manifests itself in a change of a probability of an agent to adopt the product. In the threshold model [18; 10; 8; 1], originally introduced by Granovetter [7] and Schelling [15], the behavior of the agents is deterministic. Other models of diffusion, such as propositional opinion diffusion model [6], have also been studied.

The focus of this paper is on the threshold model of diffusion of a given product. In this model, there is a threshold value $\theta(a)$ associated with each agent a and an influence value $w(a, b)$ associated with each pair of agents a and b. Informally, the threshold value $\theta(a)$ represents the resistance of agent a to adoption of the product and the influence value $w(a, b)$ represents the peer pressure that agent a puts on agent b upon adopting the product. If the total peer pressure from the set of agents A who have already adopted the product on an agent b is no less than the threshold value $\theta(b)$, i.e.,

$$\sum_{a\in A} w(a, b) \geq \theta(b), \quad (1)$$

then agent b also adopts the product.

1.2 Influence Relation

We say that a set of agents A influences a set of agents B if the social network is such that an adoption of the product by all agents in set A will unavoidably lead to an adoption of the product by all agents in set B. Note that it is not important how original adoption of the product by agents in set A happens. For example, agents in set A can receive and start using free samples of the product. Also, agents in set A can influence agents in set B indirectly. If agents in set A put enough peer pressure on some other agents to adopt the product, who in turn put enough peer pressure on the agents in set B to adopt the product, we still say that set A influences set B. We denote this influence relation by $A \rhd B$.

In this paper we focus on universal principles of influence that are true for all social networks. The set of such principles for a fixed distribution of influence values has been studied by Azimipour and Naumov [3], who provided a complete axiomatization of these principles that consists of the following three axioms of influence:

1. Reflexivity: $A \rhd B$, where $B \subseteq A$,
2. Augmentation: $A \rhd B \to (A, C \rhd B, C)$,
3. Transitivity: $A \rhd B \to (B \rhd C \to A \rhd C)$,

and an additional fourth axiom describing a property specific to the fixed distribution of influence values. In these axioms, A, B denotes the union of sets A and B. The three axioms above were originally proposed by Armstrong [2] to describe functional dependence relation in database theory. They became known in database literature as Armstrong's axioms [5, p. 81]. Väänänen proposed a first order version of these principles [16]. Beeri, Fagin, and Howard [4] suggested a variation of Armstrong's axioms that describes properties of multi-valued dependence. Naumov and Nicholls [11] proposed another variation of these axioms that describes a rationally functional dependence.

There have been at least two different attempts to enrich the language of Armstring's axioms by introducing an additional parameter to the functional dependence relation. Väänänen [17] studied approximate dependence relation $A \rhd_p B$, where p refers to the fraction of "exceptions" in which functional dependence does not hold. Naumov and Tao [12] interpreted relation $A \rhd_p B$ as "knowing values of database attributes A and having an additional budget p one can reconstruct the values of attributes in set B". In this paper, we interpret $A \rhd_p B$ as the influence relation in social networks with parameter p referring to the available marketing budget to either promote or prevent influence.

1.3 Marketing Impact

We propose an extension of the threshold model that incorporates marketing. This is done by representing a marketing campaign as a non-negative spending function s, where $s(b)$ specifies the amount of money spent on marketing the product to agent b. In addition, we associate a value $\lambda(b)$ with each agent b, which we call the *propensity* of agent b. This value represents the resistance of agent b to marketing. The higher the value of the propensity is, the more responsive the agent is to the marketing. We modify formula (1) to say that agent b adopts the product if the total sum of the marketing pressure and the peer pressure from the set of agents who have already adopted the product is no less than the threshold value:

$$\lambda(b) \cdot s(b) + \sum_{a \in A} w(a,b) \geq \theta(b). \tag{2}$$

In the first part of this paper we assume that the goal of marketing is to *promote* the adoption of the product. In the second part of the paper we investigate marketing campaigns designed to *prevent* adoption of the product. An example of the second type of campaign is an anti-smoking advertisement campaign. In either of these two cases, the same equation (2) describes the condition under which the product is adopted by agent b.

Note that most people would be more likely to buy a product when they are exposed to a promotional marketing campaign. That is, in case of promotional marketing, the value of the propensity is usually positive. On the other hand, people are usually less likely to buy a product or to adopt a social norm after being exposed to preventive marketing. In other words, in the preventive marketing setting, the value of the propensity is usually negative. However, our framework is general enough to allow for the propensity value to be either positive or negative in both of these cases.

While studying the marketing that promotes adoption of the product, we interpret predicate $A \rhd_p B$ as "there is a marketing campaign with budget no more than p that guarantees that the set of agents A will influence the set of agents B". As we show, the following three modified Armstrong's axioms give a sound and complete axiomatization of universal propositional properties of this relation:

1. Reflexivity: $A \rhd_p B$, where $B \subseteq A$,
2. Augmentation: $A \rhd_p B \to A, C \rhd_p B, C$,
3. Transitivity: $A \rhd_p B \to (B \rhd_q C \to A \rhd_{p+q} C)$.

These axioms are identical to Naumov and Tao [12] axioms of budget-constrained functional dependence.

In the case of marketing that aims to *prevent* the influence, one would naturally be interested in considering relation "there is a marketing campaign with budget no more than p that guarantees that the set of agents A will *not* influence the set of agents B". Equivalently, one can study the properties of the negation of this relation, or, in other words, the properties of the relation "for *any* preventive marketing campaign with budget no more than p, the set of agents A is able to influence the set of agents B". We have chosen to study the latter relation because the axiomatic system for this relation is more elegant. The following four axioms give a sound and complete axiomatization of the latter relation:

1. Reflexivity: $A \rhd_p B$, where $B \subseteq A$,
2. Augmentation: $A \rhd_p B \to A, C \rhd_p B, C$,
3. Transitivity: $A \rhd_p B \to (B \rhd_p C \to A \rhd_p C)$,
4. Monotonicity: $A \rhd_p B \to A \rhd_q B$, where $q \leq p$.

The difference between the axiomatic systems for promotional marketing and preventive marketing is in transitivity and monotonicity axioms. Both systems include a form of transitivity axiom, but these forms are different and not equivalent. The system for preventive marketing contains a form of monotonicity axiom. For promotional marketing, the following form of monotonicity axiom is true and provable, as is shown in Proposition 1:

$$A \rhd_p B \to A \rhd_q B, \text{ where } p \leq q.$$

Both of the above axiomatic systems differ from Väänänen [17] axiomatization of approximate functional dependence.

The paper is organized as follows. In Section 2, we give formal definitions of a social network and of a diffusion in such networks. We also prove basic properties of diffusion used later in the paper. This section of the paper is common to both promotional and preventive marketing. In Section 3, we introduce semantics of promotional marketing, give axioms of our logical system for promotional marketing, prove the soundness, and state the completeness of this logical system. In Section 4, we do the same for preventive marketing. The proofs of the completeness can be found in the full version of this paper [13]. They consist of constructing a canonical social network that satisfies a given unprovable formula. Section 5 concludes the paper.

2 Social Networks

As discussed in the introduction, the threshold model of a social network is specified by a non-negative influence value between any pair of agents in the network and by a threshold value for each agent. Additionally, each agent is assigned a propensity value that specifies the resistance of the agent to marketing. The value of the propensity could be positive, zero, or negative. We assume that the set of agents is finite.

Definition 1 *A social network is a tuple $(\mathcal{A}, w, \lambda, \theta)$, where (i) Set $\mathcal{A}$ is a finite set of agents. (ii) Function w maps $\mathcal{A} \times \mathcal{A}$ into the set of non-negative real numbers. The value $w(a,b)$ represents the "influence" of agent a on agent b. (iii) Function λ maps $\mathcal{A}$ into real numbers. The value of $\lambda(a)$ represents the "propensity" of an agent a to marketing. (iv) "Threshold" function θ maps $\mathcal{A}$ into the real numbers.*

In Figure 1 that illustrates Definition 1, the set of agents is the set $\{u, v, w, t, x, y, z\}$. The influence value $w(a,b)$ is specified by the label on the directed edge from a to b.

The edges for which the influence value is zero are omitted. Threshold and propensity values are shown next to each agent.

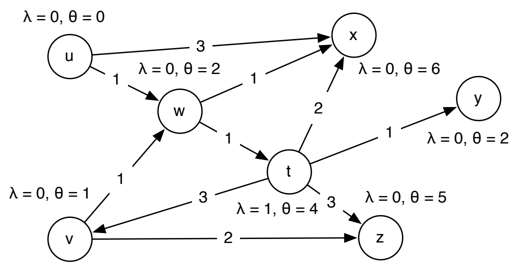

Figure 1: A Social Network.

We describe a marketing campaign by specifying "spending" on advertisement to each agent in the social network.

Definition 2 *For any social network $(\mathcal{A}, w, \lambda, \theta)$, a spending function is an arbitrary function from set $\mathcal{A}$ into non-negative real numbers.*

The following is an example of a spending function for the social network depicted in Figure 1. This function specifies a marketing campaign targeting exclusively agent t.

$$s(a) = \begin{cases} 3, & \text{if } a = t, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Definition 3 *For any social network $(\mathcal{A}, w, \lambda, \theta)$ and any spending function s, let $\|s\| = \sum_{a \in \mathcal{A}} s(a)$.*

For the function defined by equation (3), we have $\|s\| = 3$.

Next we formally define the diffusion in social network under marketing campaign specified by a spending function s. Suppose that initially the product is adopted by a set of agent A. We recursively define the *diffusion chain* of sets of agents $A = A_s^0 \subseteq A_s^1 \subseteq A_s^2 \subseteq A_s^3 \subseteq \dots$, where A_s^k is the set of agents who have adopted the product on or before the k-th step of the diffusion.

Definition 4 *For any given social network $(\mathcal{A}, w, \lambda, \theta)$, any spending function s, and any subset $A \subseteq \mathcal{A}$, let set A_s^n be recursively defined as follows: $A_s^0 = A$ and $A_s^{n+1} = A_s^n \cup \left\{ b \in \mathcal{A} \mid \lambda(b) \cdot s(b) + \sum_{a \in A_s^n} w(a, b) \geq \theta(b) \right\}$.*

For example, consider again the social network depicted in Figure 1. Let A be the set $\{v\}$ and s be the spending function defined by equation (3). Note that the threshold value of agent u in this network is zero and, thus, it will adopt the product without any peer or marketing pressure. For the other agents in this network, the combination of the marketing pressure specified by the marketing function s and the peer pressure from agent v is not enough to adopt the product. Thus, $A_s^1 = \{v, u\}$. Once agent v and agent u both adopt the product, their combined peer pressure on agent w reaches the threshold value of w and agent w also adopts the product. No other agent is experiencing enough pressure to adopt the product at this point. Hence, $A_s^2 = \{v, u, w\}$. Next, agent t will adopt the product due to the combination of the peer pressure from agent w and the marketing pressure specified by the spending function s, and so on. This process is illustrated in Figure 2.

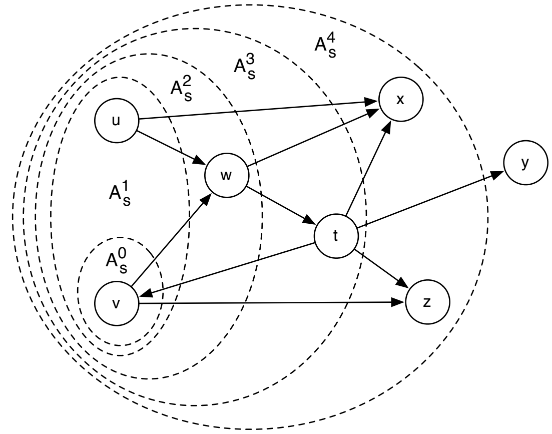

Figure 2: Diffusion Chain $A_s^1 \subseteq A_s^2 \subseteq A_s^2 \subseteq A_s^4$.

Corollary 1 *$(A_s^n)_s^k = A_s^{n+k}$ for each social network $(\mathcal{A}, w, \lambda, \theta)$, each $n, k \geq 0$, each set $A \subseteq \mathcal{A}$, and each spending function s.*

Definition 5 $A_s^* = \bigcup_{n \geq 0} A_s^n$.

Corollary 2 *$A \subseteq A_s^*$ for each social network $(\mathcal{A}, w, \lambda, \theta)$, each spending function s, and each subset $A \subseteq \mathcal{A}$.*

In the rest of this section we establish technical properties of the chain $\{A_s^n\}_{n \geq 0}$ and the set A_s^* that are used later. The first of these properties is a corollary that follows from the assumption of the finiteness of set $\mathcal{A}$ in Definition 1.

Corollary 3 *For any social network $(\mathcal{A}, w, \lambda, \theta)$, any subset A of $\mathcal{A}$ and any spending function s, there is $n \geq 0$ such that $A_s^* = A_s^n$.*

Next, we prove that A_s^* is an idempotent operator.

Lemma 1 *$(A_s^*)_s^* \subseteq A_s^*$ for each social network $(\mathcal{A}, w, \lambda, \theta)$, each spending function s, and each subsets A of $\mathcal{A}$.*

Proof. By Corollary 3, there is $n \geq 0$ such that $A_s^* = A_s^n$. By the same corollary, there also is $k \geq 0$ such that $(A_s^n)_s^* = (A_s^n)_s^k$. Thus, by Corollary 1, $(A_s^*)_s^* = (A_s^n)_s^* = (A_s^n)_s^k = A_s^{n+k}$. Therefore, $(A_s^*)_s^* \subseteq A_s^*$ by Definition 5. ⊠

We now show that any set of agents influences at least as many agents as any of its subsets, given the same fixed spending function. This claim is formally stated as Corollary 4 that follows from the next lemma:

Lemma 2 *If $A \subseteq B$, then $A_s^k \subseteq B_s^k$, for each social network $(\mathcal{A}, w, \lambda, \theta)$, each spending function s, each $k \geq 0$, and all subsets A and B of $\mathcal{A}$.*

Proof. We prove the statement of the lemma by induction on k. If $k = 0$, then $A_s^0 = A \subseteq B = B_s^0$ by Definition 4.

Suppose that $A_s^k \subseteq B_s^k$. Let $x \in A_s^{k+1}$. It suffices to show that $x \in B_s^{k+1}$. Indeed, by Definition 4, assumption $x \in A_s^{k+1}$ implies that either $x \in A_s^k$ or $\lambda(x) \cdot s(x) + \sum_{a \in A_s^k} w(a, x) \geq \theta(x)$. When $x \in A_s^k$, by the induction hypothesis, $x \in A_s^k \subseteq B_s^k$. Thus, $x \in B_s^k$. Therefore, $x \in B_s^{k+1}$ by Definition 4.

When $\lambda(x)\cdot s(x)+\sum_{a\in A^k_s} w(a,x)\ge\theta(x)$, due to the assumption $A^k\subseteq B^k$,

$$\lambda(x)\cdot s(x)+\sum_{b\in B^k_s} w(b,x)\ge\lambda(x)\cdot s(x)+\sum_{a\in A^k_s} w(a,x)\ge\theta(x).$$

Therefore, $x\in B^{k+1}_s$ by Definition 4. ⊠

Corollary 4 *If $A\subseteq B$, then $A^*_s\subseteq B^*_s$, for each social network $(\mathcal{A},w,\lambda,\theta)$, each spending function s, and all subsets A and B of $\mathcal{A}$.*

Next, we establish that the influence of the union of two sets of agents is at least as strong as the combination of the influence of these two sets.

Lemma 3 $A^*_s\cup B^*_s\subseteq(A\cup B)^*_s$*, for each social network $(\mathcal{A},w,\lambda,\theta)$, each spending function s, and all subsets A and B of $\mathcal{A}$.*

Proof. Note that $A\subseteq A\cup B$ and $B\subseteq A\cup B$. Thus, $A^*_s\subseteq(A\cup B)^*_s$ and $B^*_s\subseteq(A\cup B)^*_s$ by Corollary 4. Therefore, $A^*_s\cup B^*_s\subseteq(A\cup B)^*_s$. ⊠

One might intuitively think that the result of two consecutive marketing campaigns can not be more effective than the combined campaign, or, in other terms, that $(A^*_{s_1})^*_{s_2}\subseteq A^*_{s_1+s_2}$. More careful analysis shows that this claim is true only if all agents have non-negative propensity. However, this property can be restated in the form which is true for negative propensity as well. To do this, we introduce a binary operation $\oplus_\lambda$ on spending functions.

Definition 6 *For any two spending functions s_1 and s_2 and any propensity function λ, let $s_1\oplus_\lambda s_2$ be spending function such that for each agent a,*

$$(s_1\oplus_\lambda s_2)(a)=\begin{cases}s_1(a)+s_2(a), & \textit{if } \lambda(a)\ge 0,\\ 0, & \textit{otherwise.}\end{cases}$$

The desired property, expressed in terms of operation $\oplus_\lambda$, is stated later as Lemma 6. We start with two auxiliary observations.

Lemma 4 $\lambda(b)\cdot s_1(b)\le\lambda(b)\cdot(s_1\oplus_\lambda s_2)(b)$ *for any social network $(\mathcal{A},w,\lambda,\theta)$, any agent $b\in\mathcal{A}$, and any two spending functions s_1 and s_2.*

Proof. We consider the following two cases separately:
Case I: $\lambda(b)\ge 0$. In this case by Definition 6 and because $s_2(b)\ge 0$ due to Definition 2, we have $s_1(b)\le s_1(b)+s_2(b)=(s_1\oplus_\lambda s_2)(b)$. Therefore, $\lambda(b)\cdot s_1(b)\le\lambda(b)\cdot(s_1\oplus_\lambda s_2)(b)$ by the assumption $\lambda(b)\ge 0$.
Case II: $\lambda(b)<0$. By Definition 6 and because $s_1(b)\ge 0$ due to Definition 2, we have $s_1(b)\ge 0=(s_1\oplus_\lambda s_2)(b)$. Therefore, $\lambda(b)\cdot s_1(b)\le\lambda(b)\cdot(s_1\oplus_\lambda s_2)(b)$ by the assumption $\lambda(b)<0$. ⊠

Now we prove that the spending function $s_1\oplus_\lambda s_2$ is at least as effective as s_1.

Lemma 5 $A^n_{s_1}\subseteq A^n_{s_1\oplus_\lambda s_2}$*, for any social network $(\mathcal{A},w,\lambda,\theta)$, any set $A\subseteq\mathcal{A}$, any $n\ge 0$, any propensity function λ, and any two spending function s_1 and s_2.*

Proof. We show the lemma by induction on n. If $n=0$, then, by Definition 4, $A^0_{s_1}=A=A^0_{s_1\oplus_\lambda s_2}$. Suppose that $A^n_{s_1}\subseteq A^n_{s_1\oplus_\lambda s_2}$. We need to show that $A^{n+1}_{s_1}\subseteq A^{n+1}_{s_1\oplus_\lambda s_2}$. Indeed, by Definition 4, Lemma 4, and the induction hypothesis, set $A^{n+1}_{s_1}$ is equal to

$$\left\{b\in\mathcal{A}\mid\lambda(b)\cdot s_1(b)+\sum_{a\in A^n_{s_1}} w(a,b)\ge\theta(b)\right\}\cup A^n_{s_1}\subseteq$$

$$\left\{b\in\mathcal{A}\mid\lambda(b)\cdot(s_1\oplus_\lambda s_2)(b)+\sum_{a\in A^n_{s_1\oplus_\lambda s_2}} w(a,b)\ge\theta(b)\right\}$$

$\cup A^n_{s_1\oplus_\lambda s_2}=A^{n+1}_{s_1\oplus_\lambda s_2}$.

⊠

Finally, we are ready to state and prove that a marketing campaign with spending function $s_1\oplus_\lambda s_2$ is at least as effective as a sequential combination of two marketing campaigns with spending functions s_1 and s_2. This property is used in Lemma 9 to prove the soundness of Transitivity axiom for promotional marketing.

Lemma 6 $(A^*_{s_1})^*_{s_2}\subseteq A^*_{s_1\oplus_\lambda s_2}$*, for any social network $(\mathcal{A},w,\lambda,\theta)$, any set $A\subseteq\mathcal{A}$, any propensity function λ, and any two spending function s_1 and s_2.*

Proof. By Corollary 3, there are $n_1,n_2\ge 0$ such that $A^*_{s_1}=A^{n_1}_{s_1}$ and $(A^{n_1}_{s_1})^*_{s_2}=(A^{n_1}_{s_1})^{n_2}_{s_2}$. Thus,

$$\begin{aligned}(A^*_{s_1})^*_{s_2}&=(A^{n_1}_{s_1})^{n_2}_{s_2}\\ &\subseteq(A^{n_1}_{s_1\oplus_\lambda s_2})^{n_2}_{s_2} && \text{by Lemma 5 and Lemma 2}\\ &\subseteq(A^{n_1}_{s_1\oplus_\lambda s_2})^{n_2}_{s_2\oplus_\lambda s_1} && \text{by Lemma 5}\\ &\subseteq(A^{n_1}_{s_1\oplus_\lambda s_2})^{n_2}_{s_1\oplus_\lambda s_2} && \text{by Definition 6}\\ &\subseteq A^{n_1+n_2}_{s_1\oplus_\lambda s_2} && \text{by Corollary 1}\\ &\subseteq A^*_{s_1\oplus_\lambda s_2} && \text{by Definition 5.}\end{aligned}$$

⊠

3 Logic of Promotional Marketing

There are two logical systems that we study in this paper. In this section we introduce a logical system for the marketing aiming to promote influence, prove its soundness, and state its completeness. In the next section we do the same for the marketing aiming to prevent influence.

3.1 Syntax and Semantics

We start by defining the syntax of our logical systems. The logic of promotional marketing and the logic of preventive marketing use the same language $\Phi(\mathcal{A})$, but different semantics.

Definition 7 *For any finite set $\mathcal{A}$, let $\Phi(\mathcal{A})$ be the minimum set of formulas such that*

1. *$A\rhd_p B\in\Phi(\mathcal{A})$ for all subsets A and B of set $\mathcal{A}$ and all non-negative real numbers p,*

2. $\neg\varphi \in \Phi(\mathcal{A})$ *for all* $\varphi \in \Phi(\mathcal{A})$,
3. $\varphi \to \psi \in \Phi(\mathcal{A})$ *for all* $\varphi, \psi \in \Phi(\mathcal{A})$.

The next definition is the key definition of this section. Its item 1 specifies the influence relation in a social network with a fixed marketing budget.

Definition 8 *For any social network N with the set of agents $\mathcal{A}$ and any formula $\varphi \in \Phi(\mathcal{A})$, we define the satisfiability relation $N \vDash \varphi$ as follows:*

1. *$N \vDash A \rhd_p B$ if $B \subseteq A^*_s$ for some spending function s such that $\|s\| \le p$,*
2. *$N \vDash \neg\psi$ if $N \nvDash \psi$,*
3. *$N \vDash \psi \to \chi$ if $N \nvDash \psi$ or $N \vDash \chi$.*

For example, as we have seen in the introduction, for social network N depicted in Figure 1, we have $\{x, z\} \subseteq \{v\}^*_s$, where spending function s is defined by equation (3). Thus, $N \vDash \{v\} \rhd_3 \{x, z\}$. Through the rest of the paper we omit curly braces from the formulas like this and write them simply as $N \vDash v \rhd_3 x, z$.

3.2 Axioms

Let $\mathcal{A}$ be any fixed finite set of agents. Our logical system for promotional influence, in addition to propositional tautologies in language $\Phi(\mathcal{A})$, contains the following axioms:

1. Reflexivity: $A \rhd_p B$, where $B \subseteq A$,
2. Augmentation: $A \rhd_p B \to A, C \rhd_p B, C$,
3. Transitivity: $A \rhd_p B \to (B \rhd_q C \to A \rhd_{p+q} C)$.

We write $\vdash \varphi$ if formula $\varphi \in \Phi(\mathcal{A})$ is derivable in this logical system using Modus Ponens inference rule.

3.3 Examples

In this section we give two examples of formal proofs in our system. We start with a form of the monotonicity statement from the introduction. As the next lemma shows, this statement is provable in our logic of promotional marketing when $p \le q$:

Proposition 1 $\vdash A \rhd_p B \to A \rhd_q B$, *where* $p \le q$.

Proof. By Transitivity axiom, $\vdash A \rhd_{q-p} A \to (A \rhd_p B \to A \rhd_q B)$. At the same time, $\vdash A \rhd_{q-p} A$ by Reflexivity axiom. Thus, $\vdash A \rhd_p B \to A \rhd_q B$ by Modus Ponens inference rule. ⊠

Proposition 2 $\vdash A \rhd_p B \to (A \rhd_q C \to A \rhd_{p+q} B, C)$.

Proof. By Augmentation axiom,

$$\vdash A \rhd_p B \to A \rhd_p A, B \quad (4)$$

and

$$\vdash A \rhd_q C \to A, B \rhd_q B, C. \quad (5)$$

By Transitivity axiom,

$$\vdash A \rhd_p A, B \to (A, B \rhd_q B, C \to A \rhd_{p+q} B, C). \quad (6)$$

The statement of the lemma follows from statements (4), (5), and (6) by the laws of propositional logic. ⊠

3.4 Soundness and Completeness

In this section we prove the soundness and state the completeness of the logic for promotional marketing.

Theorem 1 *For any finite set $\mathcal{A}$ and any $\varphi \in \Phi(\mathcal{A})$, if $\vdash \varphi$, then $N \vDash \varphi$ for each social network $N = (\mathcal{A}, w, \lambda, \theta)$.*

The soundness of propositional tautologies and of Modus Ponens inference rule is straightforward. Below we show the soundness of each of the remaining axioms as a separate lemma.

Lemma 7 *$N \vDash A \rhd_p B$, for any social network $N = (\mathcal{A}, w, \lambda, \theta)$ and any subsets A and B of $\mathcal{A}$ such that $B \subseteq A$.*

Proof. Let s be the spending function equal to 0 on each $a \in \mathcal{A}$. Thus, $\|s\| = 0 \le p$ by Definition 3. At the same time, $B \subseteq A \subseteq A^*_s$ by Corollary 2. Therefore, $N \vDash A \rhd_p B$ by Definition 8. ⊠

Lemma 8 *If $N \vDash A \rhd_p B$, then $N \vDash A, C \rhd_p B, C$, for each social network $N = (\mathcal{A}, w, \lambda, \theta)$ and all subsets A, B, and C of $\mathcal{A}$.*

Proof. Suppose that $N \vDash A \rhd_p B$. Thus, by Definition 8, there is a spending function s such that $\|s\| \le p$ and $B \subseteq A^*_s$. Note that $C \subseteq C^*_s$ by Corollary 2. Thus, $B \cup C \subseteq A^*_s \cup C^*_s \subseteq (A \cup C)^*_s$ by Lemma 3. Therefore, $N \vDash A, C \rhd_p B, C$, by Definition 8. ⊠

Lemma 9 *For any social network $N = (\mathcal{A}, w, \lambda, \theta)$, if $N \vDash A \rhd_p B$ and $N \vDash B \rhd_q C$, then $N \vDash A \rhd_{p+q} C$.*

Proof. By Definition 8, assumption $N \vDash B \rhd_q C$ implies that there is a spending function s_1 such that $\|s_1\| \le q$ and $C \subseteq B^*_{s_1}$.

Similarly, assumption $N \vDash A \rhd_p B$ implies that there is a spending function s_2 such that $\|s_2\| \le p$ and $B \subseteq A^*_{s_2}$. Hence, $B^*_{s_1} \subseteq (A^*_{s_2})^*_{s_1}$ by Corollary 4. Thus, $B^*_{s_1} \subseteq A^*_{s_1 \oplus_\lambda s_2}$ by Lemma 6.

It follows that $C \subseteq B^*_{s_1} \subseteq A^*_{s_1 \oplus_\lambda s_2}$. At the same time, $\|s_1 \oplus_\lambda s_2\| \le \|s_1\| + \|s_2\| \le p + q$, by Definition 6. Therefore, $N \vDash A \rhd_{p+q} C$ by Definition 8. ⊠

This concludes the proof of the soundness of the logical system. The proof of the completeness can be found in the full version of this paper [13].

Theorem 2 *If $\nvdash \varphi$, then there is $N(\mathcal{A}, w, \lambda, \theta)$ such that $\varphi \in \Phi(\mathcal{A})$ and $N \nvDash \varphi$.*

4 Logic of Preventive Marketing

In this section we study the impact of preventive marketing on influence in social networks. Our definition of a social network given in Definition 1 and the language $\Phi(\mathcal{A})$ remain the same. As it has been discussed in the introduction, we only modify the meaning of the influence relation $A \rhd_p B$ to be "for any preventive marketing campaign with budget no more than p, the set of agents A is able to influence the set of agents B". The latter is formally captured in item 1 of Definition 9.

Definition 9 *For any social network N with the set of agents $\mathcal{A}$ and any formula $\varphi \in \Phi(\mathcal{A})$, we define satisfiability relation $N \vDash \varphi$ as follows:*

1. *$N \vDash A \rhd_p B$ if $B \subseteq A^*_s$ for each spending function s such that $\|s\| \leq p$,*
2. *$N \vDash \neg\psi$ if $N \nvDash \psi$,*
3. *$N \vDash \psi \to \chi$ if $N \nvDash \psi$ or $N \vDash \chi$.*

Note the significant difference between the above definition and the similar Definition 8 for promotional marketing. Item 1 of Definition 9 has a universal quantifier over spending functions and corresponding part of Definition 8 has an existential quantifier over spending functions.

4.1 Axioms

Let $\mathcal{A}$ be any fixed finite set of agents. Our logical system for influence with preventive marketing, in addition to propositional tautologies in language $\Phi(\mathcal{A})$, contains the following axioms:

1. Reflexivity: $A \rhd_p B$, where $B \subseteq A$,
2. Augmentation: $A \rhd_p B \to A, C \rhd_p B, C$,
3. Transitivity: $A \rhd_p B \to (B \rhd_p C \to A \rhd_p C)$,
4. Monotonicity: $A \rhd_p B \to A \rhd_q B$, where $q \leq p$.

Just like in the case of promotional marketing, we write $\vdash \varphi$ if formula $\varphi \in \Phi(\mathcal{A})$ is derivable in our logical system using Modus Ponens inference rule.

4.2 Example

In this section we give an example of a formal proof in our system. First, we show a preventive marketing analogy of Proposition 2:

Proposition 3 $\vdash A \rhd_p B \to (A \rhd_p C \to A \rhd_p B, C)$.

Proof. By Augmentation axiom,

$$\vdash A \rhd_p B \to A \rhd_p A, B \tag{7}$$

and

$$\vdash A \rhd_p C \to A, B \rhd_p B, C. \tag{8}$$

By Transitivity axiom,

$$\vdash A \rhd_p A, B \to (A, B \rhd_p B, C \to A \rhd_p B, C). \tag{9}$$

The statement of the lemma follows from statements (7), (8), and (9) by the laws of the propositional logic. ⊠

4.3 Soundness and Completeness

In this section we prove the soundness and state the completeness of the logic for preventive marketing.

Theorem 3 *For any finite set $\mathcal{A}$ and any $\varphi \in \Phi(\mathcal{A})$, if $\vdash \varphi$, then $N \vDash \varphi$ for each social network $N = (\mathcal{A}, w, \lambda, \theta)$.*

The soundness of propositional tautologies and of Modus Ponens inference rule is straightforward. Below we show the soundness of each of the remaining axioms as a separate lemma.

Lemma 10 *$N \vDash A \rhd_p B$, for any social network $N = (\mathcal{A}, w, \lambda, \theta)$ and any subsets A and B of $\mathcal{A}$ such that $B \subseteq A$.*

Proof. Let s be any spending function. By Definition 9, it suffices to show that $B \subseteq A^*_s$. Indeed, $A \subseteq A^*_s$ by Corollary 2. Therefore, $B \subseteq A^*_s$ due to the assumption $B \subseteq A$ of the lemma. ⊠

Lemma 11 *If $N \vDash A \rhd_p B$, then $N \vDash A, C \rhd_p B, C$, for each social network $N = (\mathcal{A}, w, \lambda, \theta)$ and all subsets A, B, and C of $\mathcal{A}$.*

Proof. Suppose that $N \vDash A \rhd_p B$. Consider any spending function s such that $\|s\| \leq p$. It suffices to show that $B \cup C \subseteq (A \cup C)^*_s$. Indeed, assumption $N \vDash A \rhd_p B$ implies that $B \subseteq A^*_s$ by Definition 9. At the same time, $C \subseteq C^*_s$ by Corollary 2. Therefore, $B \cup C \subseteq A^*_s \cup C^*_s \subseteq (A \cup C)^*_s$, by Lemma 3. ⊠

Lemma 12 *If $N \vDash A \rhd_p B$ and $N \vDash B \rhd_p C$, then $N \vDash A \rhd_p C$, for each social network $N = (\mathcal{A}, w, \lambda, \theta)$ and all subsets A, B, and C of $\mathcal{A}$.*

Proof. Suppose that $N \vDash A \rhd_p B$ and $N \vDash B \rhd_p C$. Consider any spending function s such that $\|s\| \leq p$. By Definition 9, it suffices to show that $C \subseteq A^*_s$.

Note that assumption $N \vDash A \rhd_p B$, by Definition 9, imply that $B \subseteq A^*_s$. Thus, $B^*_s \subseteq (A^*_s)^*_s$ by Corollary 4. At the same time, assumption $N \vDash B \rhd_p C$ implies that $C \subseteq B^*_s$ by Definition 9. Hence, $C \subseteq (A^*_s)^*_s$. Therefore, $C \subseteq A^*_s$ by Lemma 1. ⊠

Lemma 13 *If $N \vDash A \rhd_p B$, then $N \vDash A \rhd_q B$, for each $q \leq p$, each each social network $N = (\mathcal{A}, w, \lambda, \theta)$ and all subsets A and B of $\mathcal{A}$.*

Proof. Consider any spending function s such that $\|s\| \leq q$. By Definition 9, it suffices to show that $B \subseteq A^*_s$. To prove this, note that $\|s\| \leq q \leq p$. Thus, $B \subseteq A^*_s$ due to Definition 9 and the assumption $N \vDash A \rhd_p B$ of the lemma. ⊠

This concludes the proof of the soundness of our logical system for preventive marketing. The proof of the completeness can be found in the full version of this paper [13].

Theorem 4 *If $\nvdash \varphi$, then there is $N(\mathcal{A}, w, \lambda, \theta)$ such that $\varphi \in \Phi(\mathcal{A})$ and $N \nvDash \varphi$.*

5 Conclusion

In this paper we have suggested a way of adding marketing to the standard threshold model of diffusion in social networks. The model is general enough to simulate both promotional and preventive marketing. We have also defined sound and complete formal logical systems for reasoning about influence relation in social networks with marketing of these two types. Both systems are based on Armstrong's axioms from the database theory. A possible extension of this work is an analysis of the computational complexity of this model.

References

[1] Krzysztof R Apt and Evangelos Markakis. Social networks with competing products. *Fundamenta Informaticae*, 129(3):225–250, 2014.

[2] W. W. Armstrong. Dependency structures of data base relationships. In *Information processing 74 (Proc. IFIP Congress, Stockholm, 1974)*, pages 580–583. North-Holland, Amsterdam, 1974.

[3] Sanaz Azimipour and Pavel Naumov. Lighthouse principle for diffusion in social networks. *arXiv preprint arXiv:1601.04098*, 2016.

[4] Catriel Beeri, Ronald Fagin, and John H. Howard. A complete axiomatization for functional and multivalued dependencies in database relations. In *SIGMOD '77: Proceedings of the 1977 ACM SIGMOD international conference on Management of data*, pages 47–61, New York, NY, USA, 1977. ACM.

[5] Hector Garcia-Molina, Jeffrey Ullman, and Jennifer Widom. *Database Systems: The Complete Book*. Prentice-Hall, second edition, 2009.

[6] Umberto Grandi, Emiliano Lorini, and Laurent Perrussel. Propositional opinion diffusion. In *Proceedings of the 2015 International Conference on Autonomous Agents and Multiagent Systems*, pages 989–997. International Foundation for Autonomous Agents and Multiagent Systems, 2015.

[7] Mark Granovetter. Threshold models of collective behavior. *American journal of sociology*, pages 1420–1443, 1978.

[8] David Kempe, Jon Kleinberg, and Éva Tardos. Maximizing the spread of influence through a social network. In *Proceedings of the ninth ACM SIGKDD international conference on Knowledge discovery and data mining*, pages 137–146. ACM, 2003.

[9] David Kempe, Jon Kleinberg, and Éva Tardos. Influential nodes in a diffusion model for social networks. In *Automata, languages and programming*, pages 1127–1138. Springer, 2005.

[10] Michael W Macy. Chains of cooperation: Threshold effects in collective action. *American Sociological Review*, pages 730–747, 1991.

[11] Pavel Naumov and Brittany Nicholls. Rationally functional dependence. *Journal of Philosophical Logic*, 43(2-3):603–616, 2014.

[12] Pavel Naumov and Jia Tao. The budget-constrained functional dependency. *arXiv preprint arXiv:1507.05964*, 2015.

[13] Pavel Naumov and Jia Tao. Marketing impact on diffusion in social networks. *arXiv preprint arXiv:1601.05283*, 2016.

[14] Kazumi Saito, Ryohei Nakano, and Masahiro Kimura. Prediction of information diffusion probabilities for independent cascade model. In Ignac Lovrek, Robert J. Howlett, and Lakhmi C. Jain, editors, *Knowledge-Based Intelligent Information and Engineering Systems*, volume 5179 of *Lecture Notes in Computer Science*, pages 67–75. Springer Berlin Heidelberg, 2008.

[15] Thomas Schelling. *Micromotives and Macrobehavior*. Norton, 1978.

[16] Jouko Väänänen. *Dependence logic: A new approach to independence friendly logic*, volume 70. Cambridge University Press, 2007.

[17] Jouko Väänänen. The logic of approximate dependence. *arXiv preprint arXiv:1408.4437*, 2014.

[18] Thomas W Valente. Social network thresholds in the diffusion of innovations. *Social networks*, 18(1):69–89, 1996.