Marketing Impact on Diffusion in Social Networks

Pavel Naumov

Vassar College, Poughkeepsie, New York, USA

Jia Tao

The College of New Jersey, Ewing, New Jersey, USA

Abstract

The article proposes a way to add marketing into the standard threshold model of social networks. Within this framework, the article studies logical properties of the influence relation between sets of agents in social networks. Two different forms of this relation are considered: one for promotional marketing and the other for preventive marketing. In each case a sound and complete logical system describing properties of the influence relation is proposed. Both systems could be viewed as extensions of Armstrong's axioms of functional dependency from the database theory.

1. Introduction

1.1. Social Networks

In this article we study how diffusion in social networks could be affected by marketing. Diffusion happens when a product or a social norm is initially adopted by a small group of agents who later influence their peers to adopt the same product. The peers influence their peers, and so on. There are two most commonly used models of diffusion: the cascading model and the threshold model. In the cascading model [18, 12] the behaviour of agents is random and the peer influence manifests itself in a change of a probability of an agent to adopt the product. In the threshold model [24, 14, 11, 1], originally introduced by Granovetter [10] and Schelling [19], the behavior of

Email addresses: `pgn2@cornell.edu` (Pavel Naumov), `taoj@tcnj.edu` (Jia Tao)

the agents is deterministic. Other models of diffusion, such as propositional opinion diffusion model [9], have also been studied.

The focus of this article is on the threshold model of diffusion of a given product. In this model, there is a threshold value $\theta(a)$ associated with each agent a and an influence value $w(a, b)$ associated with each pair of agents a and b. Informally, the threshold value $\theta(a)$ represents the resistance of agent a to adoption of the product and the influence value $w(a, b)$ represents the peer pressure that agent a puts on agent b upon adopting the product. If the total peer pressure from the set of agents A who have already adopted the product on an agent b is no less than the threshold value $\theta(b)$, i.e.,

$$\sum_{a \in A} w(a, b) \geq \theta(b), \tag{1}$$

then agent b also adopts the product.

1.2. Influence Relation

We say that a set of agents A influences a set of agents B if the social network is such that an adoption of the product by all agents in set A will unavoidably lead to an adoption of the product by all agents in set B. Note that it is not important how original adoption of the product by agents in set A happens. For example, agents in set A can receive and start using free samples of the product. Also, agents in set A can influence agents in set B indirectly. If agents in set A put enough peer pressure on some other agents to adopt the product, who in turn put enough peer pressure on the agents in set B to adopt the product, we still say that set A influences set B. We denote this influence relation by $A \rhd B$.

In this article we focus on universal principles of influence that are true for all social networks. The set of such principles for a fixed distribution of influence values has been studied by Azimipour and Naumov [3], who provided a complete axiomatization of these principles that consists of the following three axioms of influence:

1. Reflexivity: $A \rhd B$, where $B \subseteq A$,
2. Augmentation: $A \rhd B \to (A, C \rhd B, C)$,
3. Transitivity: $A \rhd B \to (B \rhd C \to A \rhd C)$,

and an additional fourth axiom describing a property specific to the fixed distribution of influence values. In these axioms, A, B denotes the union of sets A and B. The three axioms above were originally proposed by Armstrong [2] to describe functional dependence relation in database theory. They became known in database literature as Armstrong's axioms [8, p. 81]. Väänänen proposed a first order version of these principles [22]. Beeri, Fagin, and Howard [5] suggested a variation of Armstrong's axioms that describes properties of multi-valued dependence. Naumov and Nicholls [15] proposed another variation of these axioms that describes a rationally functional dependence.

There have been at least two different attempts to enrich the language of Armstrong's axioms by introducing an additional parameter to the functional dependence relation. Väänänen [23] studied approximate dependence relation $A \rhd_p B$, where p refers to the fraction of "exceptions" in which functional dependence does not hold. Naumov and Tao [16] interpreted relation $A \rhd_p B$ as "knowing values of database attributes A and having an additional budget p one can reconstruct the values of attributes in set B". In this article we interpret $A \rhd_p B$ as the influence relation in social networks with parameter p referring to the available marketing budget to either promote or prevent influence.

1.3. Marketing Impact

We propose an extension of the threshold model that incorporates marketing. This is done by representing a marketing campaign as a non-negative spending function s, where $s(b)$ specifies the amount of money spent on marketing the product to agent b. In addition, we associate a value $\lambda(b)$ with each agent b, which we call the *receptivity* of agent b. This value represents the responsiveness of agent b to marketing. The higher the value of the receptivity is, the more responsive the agent is to the marketing. We modify formula (1) to say that agent b adopts the product if the total sum of the marketing pressure and the peer pressure from the set of agents who have already adopted the product is no less than the threshold value:

$$\lambda(b) \cdot s(b) + \sum_{a \in A} w(a, b) \geq \theta(b). \tag{2}$$

In the first part of this article we assume that the goal of marketing is to *promote* the adoption of the product. In the second part of the article we investigate marketing campaigns designed to *prevent* adoption of the product. An example of the second type of campaign is an anti-smoking advertisement campaign. In either of these two cases, the same equation (2) describes the condition under which the product is adopted by agent b.

Note that most people would be more likely to buy a product when they are exposed to a promotional marketing campaign. That is, in case of promotional marketing, the value of the receptivity is usually positive. On the other hand, people are usually less likely to buy a product or to adopt a social norm after being exposed to preventive marketing. In other words, in the preventive marketing setting, the value of the receptivity is usually negative. However, our framework is general enough to allow for the receptivity value to be either positive or negative in both of these cases.

While studying the marketing that promotes adoption of the product, we interpret predicate $A \rhd_p B$ as "there is a marketing campaign with budget no greater than p that guarantees that the set of agents A will influence the set of agents B". When $p = 0$, relation $A \rhd_p B$ becomes the influence relation $A \rhd B$ discussed above. On the other hand, statement $\neg(\varnothing \rhd_p B)$ means that there is no marketing campaign with budget p that alone can influence all agents in set B. As we show, the following three modified Armstrong's axioms give a sound and complete axiomatization of universal propositional properties of the relation $A \rhd_p B$:

1. Reflexivity: $A \rhd_p B$, where $B \subseteq A$,
2. Augmentation: $A \rhd_p B \to A, C \rhd_p B, C$,
3. Transitivity: $A \rhd_p B \to (B \rhd_q C \to A \rhd_{p+q} C)$.

These axioms are identical to the axioms of budget-constrained functional dependence proposed by Naumov and Tao [16]. Informally, the first of these axioms captures the fact that every set of agents vacuously influences itself no matter what the budget is. The second axiom says that if set A can influence set B on a given budget, then any of its supersets can also influence B on the same budget. Finally, the last axiom states that two marketing campaigns, whose existences are expressed by statements $A \rhd_p B$ and $B \rhd_q C$, could be combined into one single campaign that achieves $A \rhd_{p+q} C$.

In the case of marketing that aims to *prevent* the influence, one would naturally be interested in considering relation "there is a marketing campaign with budget no greater than p that guarantees that the set of agents A will *not* influence the set of agents B". Equivalently, one can study the properties of the negation of this relation, or, in other words, the properties of the relation "for *any* preventive marketing campaign with budget no greater than p, the set of agents A is able to influence the set of agents B". We have chosen to study the latter relation because the axiomatic system for this relation is more elegant. In this article we show that the following four axioms give a sound and complete axiomatization of the latter relation:

1. Reflexivity: $A \rhd_p B$, where $B \subseteq A$,
2. Augmentation: $A \rhd_p B \to A, C \rhd_p B, C$,
3. Transitivity: $A \rhd_p B \to (B \rhd_p C \to A \rhd_p C)$,
4. Monotonicity: $A \rhd_p B \to A \rhd_q B$, where $q \leq p$.

The difference between the axiomatic systems for promotional marketing and preventive marketing is in transitivity and monotonicity axioms. Both systems include a form of transitivity axiom, but these forms are different and not equivalent. The system for preventive marketing contains a form of monotonicity axiom. For promotional marketing, the following form of monotonicity axiom is true and provable, as is shown in Lemma 7:

$$A \rhd_p B \to A \rhd_q B, \text{ where } p \leq q.$$

Both of the above axiomatic systems differ from Väänänen [23] axiomatization of approximate functional dependence.

There are several other logical frameworks for reasoning about diffusion in social networks. Facebook Logic proposed by Seligman, Liu, and Girard [20] captures properties of epistemic social networks in modal language. In papers [21, 13], the approach was further developed by introducing dynamic friendship relations. Christoff and Hansen [6] simplified Seligman, Liu, and Girard setting and gave a complete axiomatization of the logical system for this new setting. Minimal Threshold Influence Logic proposed by Christoff and Rendsvig [7] uses modal language to capture dynamic of diffusion in a threshold model and a complete axiomatization of this logic was given. Baltag, Christoff, Rendsvig, and Smets [4] discussed logics for informed update and prediction update. The languages of the systems described above are significantly different from ours and, as a result, neither of these systems deals with marketing in social networks.

The article is organized as follows. In Section 2, we give formal definitions of a social network and of a diffusion in such networks. We also prove basic properties of diffusion used later in the article. This section of the article is common to both promotional and preventive marketing. In Section 3, we introduce semantics of promotional marketing, give axioms of our logical system for promotional marketing and prove the soundness and the completeness of this logical system. In Section 4, we do the same for preventive marketing. Section 5 concludes the article. A preliminary version of this work without proofs of completeness appeared as [17].

2. Social Networks

As discussed in the introduction, the threshold model of a social network is specified by a non-negative influence value between any pair of agents in the network and by a threshold value for each agent. Additionally, each agent is assigned a receptivity value that specifies the responsiveness of the agent to marketing. The value of the receptivity could be positive, zero, or negative. We assume that the set of agents is finite.

Definition 1. *A social network is a tuple* $(\mathcal{A}, w, \lambda, \theta)$*, where*

1. *Set* $\mathcal{A}$ *is a finite set of agents.*
2. *Function* w *maps* $\mathcal{A} \times \mathcal{A}$ *into the set of non-negative real numbers. The value* $w(a, b)$ *represents the "influence" of agent* a *on agent* b*.*
3. *Function* λ *maps* $\mathcal{A}$ *into real numbers. The value of* $\lambda(a)$ *represents the "receptivity" of an agent* a *to marketing.*
4. *"Threshold" function* θ *maps* $\mathcal{A}$ *into the set of real numbers.*

Figure 1 illustrates Definition 1. In this figure, the set of agents is the set $\{u, v, w, t, x, y, z\}$. The influence value $w(a, b)$ is specified by the label on the directed edge from a to b. The edges for which the influence value is zero are omitted. Threshold and receptivity values are shown next to each agent.

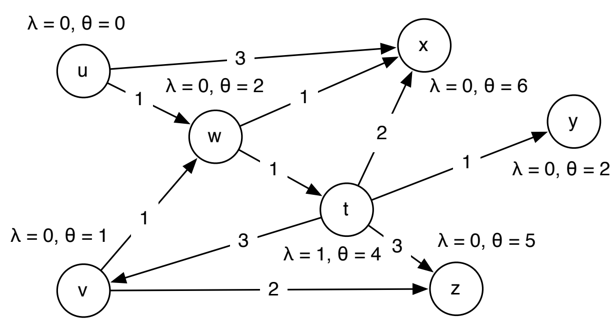

Figure 1: A Social Network.

We describe a marketing campaign by specifying "spending" on advertisement to each agent in the social network.

Definition 2. *For any social network $(\mathcal{A}, w, \lambda, \theta)$, a spending function is an arbitrary function from set $\mathcal{A}$ into non-negative real numbers.*

The following is an example of a spending function for the social network depicted in Figure 1. This function specifies a marketing campaign targeting exclusively agent t.

$$s(a) = \begin{cases} 3, & \text{if } a = t, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Definition 3. *For any social network $(\mathcal{A}, w, \lambda, \theta)$ and any spending function s, let $\|s\| = \sum_{a \in \mathcal{A}} s(a)$.*

For the spending function defined by equation (3), we have $\|s\| = 3$.

Next we formally define the diffusion in social network under marketing campaign specified by a spending function s. Suppose that initially the product is adopted by a set of agent A. We recursively define the *diffusion chain* of sets of agents

$$A = A_s^0 \subseteq A_s^1 \subseteq A_s^2 \subseteq A_s^3 \subseteq \dots,$$

where A_s^k is the set of agents who have adopted the product on or before the k-th step of the diffusion.

Definition 4. *For any given social network $(\mathcal{A}, w, \lambda, \theta)$, any spending function s, and any subset $A \subseteq \mathcal{A}$, let set A_s^n be recursively defined as follows:*

1. $A_s^0 = A$,
2. $A_s^{n+1} = A_s^n \cup \left\{ b \in \mathcal{A} \mid \lambda(b) \cdot s(b) + \sum_{a \in A_s^n} w(a, b) \geq \theta(b) \right\}$.

For example, consider again the social network depicted in Figure 1. Let A be the set $\{v\}$ and s be the spending function defined by equation (3). Note that the threshold value of agent u in this network is zero and, thus, it will adopt the product without any peer or marketing pressure. For the other agents in this network, the combination of the marketing pressure specified by the marketing function s and the peer pressure from agent v is not enough to adopt the product. Thus, $A_s^1 = \{v, u\}$. Once agent v and agent u both adopt the product, their combined peer pressure on agent w reaches the threshold value of w and agent w also adopts the product. No other agent is experiencing enough pressure to adopt the product at this point. Hence,

$A_s^2 = \{v, u, w\}$. Next, agent t will adopt the product due to the combination of the peer pressure from agent w and the marketing pressure specified by the spending function s, and so on. This diffusion process is illustrated in Figure 2.

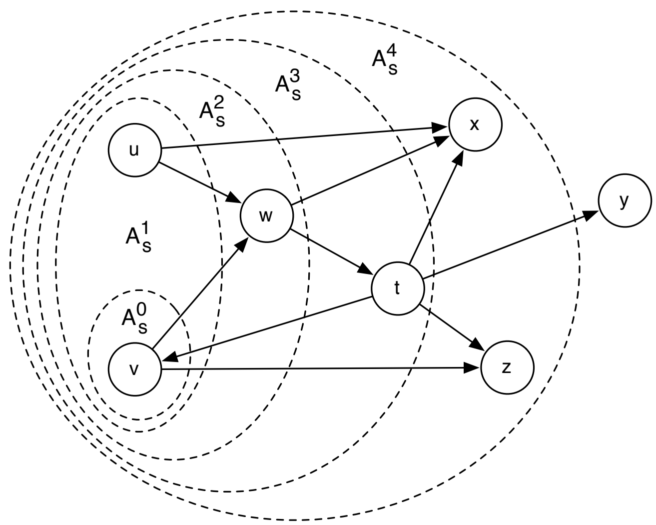

Figure 2: Diffusion Chain $A_s^1 \subseteq A_s^2 \subseteq A_s^2 \subseteq A_s^4$.

Corollary 1. $(A_s^n)_s^k = A_s^{n+k}$ *for each social network* $(\mathcal{A}, w, \lambda, \theta)$, *each* $n, k \geq 0$, *each set* $A \subseteq \mathcal{A}$, *and each spending function* s.

Definition 5. $A_s^* = \bigcup_{n \geq 0} A_s^n$.

Corollary 2. $A \subseteq A_s^*$ *for each social network* $(\mathcal{A}, w, \lambda, \theta)$, *each spending function* s, *and each subset* $A \subseteq \mathcal{A}$.

In the rest of this section we establish technical properties of the chain $\{A_s^n\}_{n \geq 0}$ and the set A_s^* that are used later. The first of these properties is a corollary that follows from the assumption of the finiteness of set $\mathcal{A}$ in Definition 1.

Corollary 3. *For any social network* $(\mathcal{A}, w, \lambda, \theta)$, *any subset* A *of* $\mathcal{A}$ *and any spending function* s, *there is* $n \geq 0$ *such that* $A_s^* = A_s^n$.

Next, we prove that A_s^* is an idempotent operator.

Lemma 1. *$(A^*_s)^*_s \subseteq A^*_s$ for each social network $(\mathcal{A}, w, \lambda, \theta)$, each spending function s, and each subsets A of $\mathcal{A}$.*

Proof. By Corollary 3, there is $n \geq 0$ such that $A^*_s = A^n_s$. By the same corollary, there also is $k \geq 0$ such that $(A^n_s)^*_s = (A^n_s)^k_s$. Thus, by Corollary 1,

$$(A^*_s)^*_s = (A^n_s)^*_s = (A^n_s)^k_s = A^{n+k}_s.$$

Therefore, $(A^*_s)^*_s \subseteq A^*_s$ by Definition 5. ⊠

We now show that any set of agents influences at least as many agents as any of its subsets, given the same fixed spending function. This claim is formally stated as Corollary 4 that follows from the next lemma:

Lemma 2. *If $A \subseteq B$, then $A^k_s \subseteq B^k_s$, for each social network $(\mathcal{A}, w, \lambda, \theta)$, each spending function s, each $k \geq 0$, and all subsets A and B of $\mathcal{A}$.*

Proof. We prove the statement of the lemma by induction on k. If $k = 0$, then $A^0_s = A \subseteq B = B^0_s$ by Definition 4.

Suppose that $A^k_s \subseteq B^k_s$. Let $x \in A^{k+1}_s$. It suffices to show that $x \in B^{k+1}_s$. Indeed, by Definition 4, assumption $x \in A^{k+1}_s$ implies that either $x \in A^k_s$ or $\lambda(x) \cdot s(x) + \sum_{a \in A^k_s} w(a, x) \geq \theta(x)$. When $x \in A^k_s$, by the induction hypothesis, $x \in A^k_s \subseteq B^k_s$. Thus, $x \in B^k_s$. Therefore, $x \in B^{k+1}_s$ by Definition 4.

When $\lambda(x) \cdot s(x) + \sum_{a \in A^k_s} w(a, x) \geq \theta(x)$, due to the assumption $A^k \subseteq B^k$,

$$\lambda(x) \cdot s(x) + \sum_{b \in B^k_s} w(b, x) \geq \lambda(x) \cdot s(x) + \sum_{a \in A^k_s} w(a, x) \geq \theta(x).$$

Therefore, $x \in B^{k+1}_s$ by Definition 4. ⊠

Corollary 4. *If $A \subseteq B$, then $A^*_s \subseteq B^*_s$, for each social network $(\mathcal{A}, w, \lambda, \theta)$, each spending function s, and all subsets A and B of $\mathcal{A}$.*

Next, we establish that the influence of the union of two sets of agents is at least as strong as the combination of the influence of these two sets.

Lemma 3. *$A^*_s \cup B^*_s \subseteq (A \cup B)^*_s$, for each social network $(\mathcal{A}, w, \lambda, \theta)$, each spending function s, and all subsets A and B of $\mathcal{A}$.*

Proof. Note that $A \subseteq A \cup B$ and $B \subseteq A \cup B$. Thus, $A^*_s \subseteq (A \cup B)^*_s$ and $B^*_s \subseteq (A \cup B)^*_s$ by Corollary 4. Therefore, $A^*_s \cup B^*_s \subseteq (A \cup B)^*_s$. ⊠

One might intuitively think that the result of two consecutive marketing campaigns can not be more effective than the combined campaign, or, in other terms, that $(A^*_{s_1})^*_{s_2} \subseteq A^*_{s_1+s_2}$. More careful analysis shows that this claim is true only if all agents have non-negative receptivity. However, this property can be restated in the form which is true for negative receptivity as well. To do this, we introduce a binary operation $\oplus_\lambda$ on spending functions.

Definition 6. *For any two spending functions s_1 and s_2 and any receptivity function λ, let $s_1 \oplus_\lambda s_2$ be spending function such that for each agent a,*

$$(s_1 \oplus_\lambda s_2)(a) = \begin{cases} s_1(a) + s_2(a), & \text{if } \lambda(a) \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

The desired property, expressed in terms of operation $\oplus_\lambda$, is stated later as Lemma 6. We start with two auxiliary observations.

Lemma 4. $\lambda(b) \cdot s_1(b) \leq \lambda(b) \cdot (s_1 \oplus_\lambda s_2)(b)$ *for any social network $(\mathcal{A}, w, \lambda, \theta)$, any agent $b \in \mathcal{A}$, and any two spending functions s_1 and s_2.*

Proof. We consider the following two cases separately:
Case I: $\lambda(b) \geq 0$. In this case by Definition 6 and because $s_2(b) \geq 0$ due to Definition 2, we have $s_1(b) \leq s_1(b) + s_2(b) = (s_1 \oplus_\lambda s_2)(b)$. Therefore, $\lambda(b) \cdot s_1(b) \leq \lambda(b) \cdot (s_1 \oplus_\lambda s_2)(b)$ by the assumption $\lambda(b) \geq 0$.
Case II: $\lambda(b) < 0$. In this case by Definition 6 and because $s_1(b) \geq 0$ due to Definition 2, we have $s_1(b) \geq 0 = (s_1 \oplus_\lambda s_2)(b)$. Therefore, $\lambda(b) \cdot s_1(b) \leq \lambda(b) \cdot (s_1 \oplus_\lambda s_2)(b)$ by the assumption $\lambda(b) < 0$. ⊠

Now we prove that the spending function $s_1 \oplus_\lambda s_2$ is at least as effective as s_1.

Lemma 5. $A^n_{s_1} \subseteq A^n_{s_1 \oplus_\lambda s_2}$*, for any social network $(\mathcal{A}, w, \lambda, \theta)$, any set $A \subseteq \mathcal{A}$, any $n \geq 0$, any receptivity function λ, and any two spending function s_1 and s_2.*

Proof. We show the lemma by induction on n. If $n = 0$, then, by Definition 4, $A^0_{s_1} = A = A^0_{s_1 \oplus_\lambda s_2}$. Suppose that $A^n_{s_1} \subseteq A^n_{s_1 \oplus_\lambda s_2}$. We need to show that $A^{n+1}_{s_1} \subseteq A^{n+1}_{s_1 \oplus_\lambda s_2}$. Indeed, by Definition 4, Lemma 4, and the induction hypothesis,

$$\begin{aligned} A^{n+1}_{s_1} &= A^n_{s_1} \cup \left\{ b \in \mathcal{A} \mid \lambda(b) \cdot s_1(b) + \sum_{a \in A^n_{s_1}} w(a,b) \geq \theta(b) \right\} \\ &\subseteq A^n_{s_1 \oplus_\lambda s_2} \cup \left\{ b \in \mathcal{A} \mid \lambda(b) \cdot (s_1 \oplus_\lambda s_2)(b) + \sum_{a \in A^n_{s_1 \oplus_\lambda s_2}} w(a,b) \geq \theta(b) \right\} \\ &= A^{n+1}_{s_1 \oplus_\lambda s_2}. \end{aligned}$$

⊠

Finally, we are ready to state and prove that a marketing campaign with spending function $s_1 \oplus_\lambda s_2$ is at least as effective as a sequential combination of two marketing campaigns with spending functions s_1 and s_2. This property is used in Lemma 12 to prove the soundness of Transitivity axiom for promotional marketing.

Lemma 6. *$(A^*_{s_1})^*_{s_2} \subseteq A^*_{s_1 \oplus_\lambda s_2}$, for any social network $(\mathcal{A}, w, \lambda, \theta)$, any set $A \subseteq \mathcal{A}$, any receptivity function λ, and any two spending function s_1 and s_2.*

Proof. By Corollary 3, there are $n_1, n_2 \geq 0$ such that $A^*_{s_1} = A^{n_1}_{s_1}$ and $(A^{n_1}_{s_1})^*_{s_2} = (A^{n_1}_{s_1})^{n_2}_{s_2}$. Thus,

$$\begin{aligned} (A^*_{s_1})^*_{s_2} &= (A^{n_1}_{s_1})^{n_2}_{s_2} & \\ &\subseteq (A^{n_1}_{s_1 \oplus_\lambda s_2})^{n_2}_{s_2} & \text{by Lemma 5 and Lemma 2} \\ &\subseteq (A^{n_1}_{s_1 \oplus_\lambda s_2})^{n_2}_{s_2 \oplus_\lambda s_1} & \text{by Lemma 5} \\ &\subseteq (A^{n_1}_{s_1 \oplus_\lambda s_2})^{n_2}_{s_1 \oplus_\lambda s_2} & \text{by Definition 6} \\ &\subseteq A^{n_1+n_2}_{s_1 \oplus_\lambda s_2} & \text{by Corollary 1} \\ &\subseteq A^*_{s_1 \oplus_\lambda s_2} & \text{by Definition 5.} \end{aligned}$$

⊠

3. Logic of Promotional Marketing

There are two logical systems that we study in this article. In this section we introduce a logical system for the marketing aiming to promote influence and prove its soundness and completeness. In the next section we do the same for the marketing aiming to prevent influence.

3.1. Syntax and Semantics

We start by defining the syntax of our logical systems. The logic of promotional marketing and the logic of preventive marketing use the same language $\Phi(\mathcal{A})$, but different semantics.

Definition 7. *For any finite set $\mathcal{A}$, let $\Phi(\mathcal{A})$ be the minimum set of formulas such that*

1. $A \rhd_p B \in \Phi(\mathcal{A})$ *for all subsets A and B of set $\mathcal{A}$ and all non-negative real numbers p,*
2. $\neg\phi \in \Phi(\mathcal{A})$ *for all $\phi \in \Phi(\mathcal{A})$,*
3. $\phi \to \psi \in \Phi(\mathcal{A})$ *for all $\phi, \psi \in \Phi(\mathcal{A})$.*

The next definition is the key definition of this section. Its item 1 specifies the influence relation in a social network with a fixed marketing budget.

Definition 8. *For any social network N with the set of agents $\mathcal{A}$ and any formula $\phi \in \Phi(\mathcal{A})$, we define satisfiability relation $N \vDash \phi$ as follows:*

1. $N \vDash A \rhd_p B$ *if $B \subseteq A^*_s$ for some spending function s such that $\|s\| \leq p$,*
2. $N \vDash \neg\psi$ *if $N \nvDash \psi$,*
3. $N \vDash \psi \to \chi$ *if $N \nvDash \psi$ or $N \vDash \chi$.*

For example, as we have seen in the introduction, for social network N depicted in Figure 1, we have $\{x, z\} \subseteq \{v\}^*_s$, where spending function s is defined by equation (3). Thus, $N \vDash \{v\} \rhd_3 \{x, z\}$. Through the rest of the article we omit curly braces from the formulas like this and write them simply as $N \vDash v \rhd_3 x, z$.

3.2. Axioms

Let $\mathcal{A}$ be any fixed finite set of agents. Our logical system for promotional influence, in addition to propositional tautologies in language $\Phi(\mathcal{A})$, contains the following axioms:

1. Reflexivity: $A \rhd_p B$, where $B \subseteq A$,
2. Augmentation: $A \rhd_p B \to A, C \rhd_p B, C$,
3. Transitivity: $A \rhd_p B \to (B \rhd_q C \to A \rhd_{p+q} C)$.

We write $\vdash \phi$ if formula $\phi \in \Phi(\mathcal{A})$ is derivable in this logical system using Modus Ponens inference rule. We write $X \vdash \phi$ if formula ϕ is derivable using an additional set of axioms $X \subseteq \Phi(\mathcal{A})$.

3.3. Examples

The soundness and the completeness of our logical system will be shown later. In this section we give several examples of formal proofs in our system. We start with a form of the monotonicity statement from the introduction. As the next lemma shows, this statement is provable in our logic of promotional marketing when $p \le q$:

Lemma 7. $\vdash A \rhd_p B \to A \rhd_q B$, *where* $p \le q$.

Proof. By Transitivity axiom, $\vdash A \rhd_{q-p} A \to (A \rhd_p B \to A \rhd_q B)$. At the same time, $\vdash A \rhd_{q-p} A$ by Reflexivity axiom. Thus, $\vdash A \rhd_p B \to A \rhd_q B$ by Modus Ponens inference rule. ⊠

Lemma 8. $\vdash A \rhd_p B \to (A \rhd_q C \to A \rhd_{p+q} B, C)$.

Proof. By Augmentation axiom,

$$\vdash A \rhd_p B \to A \rhd_p A, B \tag{4}$$

and

$$\vdash A \rhd_q C \to A, B \rhd_q B, C. \tag{5}$$

By Transitivity axiom,

$$\vdash A \rhd_p A, B \to (A, B \rhd_q B, C \to A \rhd_{p+q} B, C). \tag{6}$$

The statement of the lemma follows from statements (4), (5), and (6) by the laws of propositional logic. ⊠

The next lemma will be used later in the proof of the completeness.

Lemma 9. *Let X be a subset of $\Phi(\mathcal{A})$, m be a non-negative integer number, sets $A, B_1, \dots, B_m$ be subsets of $\mathcal{A}$, and $p_1, \dots, p_m$ be non-negative real numbers. If $X \vdash A \rhd_{p_i} B_i$ for all $1 \leq i \leq m$, then $X \vdash A \rhd_q \bigcup_{i=1}^m B_i$, where $q = \sum_{i=1}^m p_i$.*

Proof. We prove the lemma by induction on m. If $m = 0$, then we need to show that $X \vdash A \rhd_0 \varnothing$, which is an instance of Reflexivity axiom.

Suppose that $X \vdash A \rhd_{q'} \bigcup_{i=1}^{m-1} B_i$, where $q' = \sum_{i=1}^{m-1} p_i$. Since $X \vdash A \rhd_{p_m} B_m$ due to the assumption of the lemma, by Lemma 8, $X \vdash A \rhd_q \bigcup_{i=1}^m B_i$, where $q = \sum_{i=1}^m p_i$. ⊠

3.4. Soundness

In this section we prove the soundness of the logic for promotional marketing.

Theorem 1. *For any finite set $\mathcal{A}$ and any $\phi \in \Phi(\mathcal{A})$, if $\vdash \phi$, then $N \vDash \phi$ for each social network $N = (\mathcal{A}, w, \lambda, \theta)$.*

The soundness of propositional tautologies and of Modus Ponens inference rule is straightforward. Below we show the soundness of each of the remaining axioms as a separate lemma.

Lemma 10. *$N \vDash A \rhd_p B$, for any social network $N = (\mathcal{A}, w, \lambda, \theta)$ and any subsets A and B of $\mathcal{A}$ such that $B \subseteq A$.*

Proof. Let s be the spending function equal to 0 on each $a \in \mathcal{A}$. Thus, $\|s\| = 0 \leq p$ by Definition 3. At the same time, $B \subseteq A \subseteq A_s^*$ by Corollary 2. Therefore, $N \vDash A \rhd_p B$ by Definition 8. ⊠

Lemma 11. *If $N \vDash A \rhd_p B$, then $N \vDash A, C \rhd_p B, C$, for each social network $N = (\mathcal{A}, w, \lambda, \theta)$ and all subsets A, B, and C of $\mathcal{A}$.*

Proof. Suppose that $N \vDash A \rhd_p B$. Thus, by Definition 8, there is a spending function s such that $\|s\| \leq p$ and $B \subseteq A^*_s$. Note that $C \subseteq C^*_s$ by Corollary 2. Thus, $B \cup C \subseteq A^*_s \cup C^*_s \subseteq (A \cup C)^*_s$ by Lemma 3. Therefore, $N \vDash A, C \rhd_p B, C$, by Definition 8. ⊠

Lemma 12. *For any social network $N = (\mathcal{A}, w, \lambda, \theta)$, if $N \vDash A \rhd_p B$ and $N \vDash B \rhd_q C$, then $N \vDash A \rhd_{p+q} C$.*

Proof. By Definition 8, assumption $N \vDash B \rhd_q C$ implies that there is a spending function s_1 such that $\|s_1\| \leq q$ and $C \subseteq B^*_{s_1}$.

Similarly, assumption $N \vDash A \rhd_p B$ implies that there is a spending function s_2 such that $\|s_2\| \leq p$ and $B \subseteq A^*_{s_2}$. Hence, $B^*_{s_1} \subseteq (A^*_{s_2})^*_{s_1}$ by Corollary 4. Thus, $B^*_{s_1} \subseteq A^*_{s_1 \oplus_\lambda s_2}$ by Lemma 6.

It follows that $C \subseteq B^*_{s_1} \subseteq A^*_{s_1 \oplus_\lambda s_2}$. At the same time, $\|s_1 \oplus_\lambda s_2\| \leq \|s_1\| + \|s_2\| \leq p + q$, by Definition 6. Therefore, $N \vDash A \rhd_{p+q} C$ by Definition 8. ⊠

This concludes the proof of the soundness of our logical system for promotional marketing.

3.5. Completeness

We now show the completeness of our logical system for promotional marketing. This result is formally stated as Theorem 2 in the end of this section. As usual, at the core of the proof of the completeness is a construction of a canonical model. In our case, the role of a canonical model is played by the canonical social network.

Let $\mathcal{A}_0$ be any finite set and $X = \{A_i \rhd_{p_i} B_i\}_{i \leq m}$ be any finite set of atomic formulas in language $\Phi(\mathcal{A}_0)$. We now proceed to define the canonical social network $N_X = (\mathcal{A}, w, \lambda, \theta)$. An example of the canonical network for set X consisting of formula $a, c \rhd_1 d$, formula $b, c \rhd_2 a$, and formula $a, b \rhd_3 c$ is depicted in Figure 3. We associate two new agents α_i and β_i with each formula $A_i \rhd_{p_i} B_i \in X$. We assume that agents $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m$ are distinct and that they do not belong to set $\mathcal{A}_0$.

Definition 9. $\mathcal{A} = \mathcal{A}_0 \cup \{\alpha_i\}_{i \leq m} \cup \{\beta_i\}_{i \leq m}$.

In social network N_X only agents $\{\alpha_i\}_{i \leq m}$ are responsive to promotional marketing. We formally capture this through the following definition of function λ:

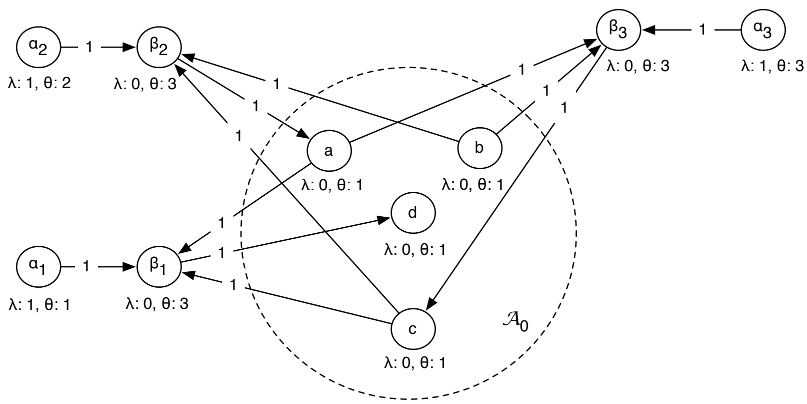

Figure 3: The canonical social network N_X for set X consisting of formula $a, c \rhd_1 d$, formula $b, c \rhd_2 a$, and formula $a, b \rhd_3 c$.

Definition 10. *For any $a \in \mathcal{A}$,*

$$\lambda(a) = \begin{cases} 1, & \text{if } a = \alpha_i \text{ for some } i \leq m, \\ 0, & \text{otherwise.} \end{cases}$$

We assume that for each $i \leq m$, all agents in set A_i as well as agent α_i put peer pressure on agent β_i once they adopt the product. In addition, upon adopting the product, agent β_i puts peer pressure on each agent in set B_i. Besides that, no agent can put peer pressure on any other agent in this network. We formally capture this in Definition 11.

Definition 11.

$$w(a, b) = \begin{cases} 1, & \text{if } a \in A_i \cup \{\alpha_i\} \text{ and } b = \beta_i \text{ for some } i \leq m, \\ 1, & \text{if } a = \beta_i \text{ and } b \in B_i \text{ for some } i \leq m, \\ 0, & \text{otherwise.} \end{cases}$$

Before continuing with the definition of the social network N_X, we state and prove a property of this network that follows from Definition 11. We show that in order to put peer pressure of at least $|A_i| + 1$ on agent β_i, one needs to influence agent α_i and all of the agents in set A_i.

Lemma 13. *If $\sum_{a \in A_s^n} w(a, \beta_i) \geq |A_i| + 1$, then $\alpha_i \in A_s^n$ and $A_i \subseteq A_s^n$.*

Proof. By Definition 11, $w(a, \beta_i) = 1$ if $a \in A_i \cup \{\alpha_i\}$ and $w(a, \beta_i) = 0$ for all other $a \in \mathcal{A}$. Thus, inequality $\sum_{a \in A_s^n} w(a, \beta_i) \geq |A_i| + 1$ implies that $A_i \cup \{\alpha_i\} \subseteq A_s^n$. Therefore, $\alpha_i \in A_s^n$ and $A_i \subseteq A_s^n$. ⊠

We are now ready to define the threshold value function θ for the social network N_X. Recall that according to Definition 11 and Definition 10, no agent can put peer pressure on agent α_i, but agent α_i is responsive to promotional marketing. We set the threshold value $\theta(\alpha_i)$ to p_i so that this agent can only be influenced by a marketing campaign with budget at least p_i. We set value $\theta(\beta_i)$ high enough to guarantee (see Lemma 13) that agent α_i and each agent in set A_i adopt the product before agent β_i is influenced. Threshold values of all agents in set $\mathcal{A}_0$ are set to 1.

Definition 12.

$$\theta(a) = \begin{cases} p_i, & \textit{if } a = \alpha_i \textit{ for some } i \leq m, \\ |A_i| + 1, & \textit{if } a = \beta_i \textit{ for some } i \leq m, \\ 1, & \textit{otherwise.} \end{cases}$$

This concludes the definition of the canonical social network $N_X = (\mathcal{A}, w, \lambda, \theta)$.

Recall that Figure 3 depicts the canonical social network N_X for set X consisting of formula $a, c \rhd_1 d$, formula $b, c \rhd_2 a$, and formula $a, b \rhd_3 c$. Note that formula $a, c \rhd_1 d$, formula $b, c \rhd_2 a$, and formula $a, b \rhd_3 c$ are all satisfied in the canonical network depicted in Figure 3. For example, for the formula $a, c \rhd_1 d$, let spending function s be such that it spends 1 on agent α_i and nothing on all other agents. Thus, $\{a, c\}_s^1 = \{a, c, \alpha_1\}$. Once α_1 adopts the product, the total peer pressure on agent β_1 becomes 3 and it too adopts the product: $\{a, c\}_s^2 = \{a, c, \alpha_1, \beta_1\}$. Finally, upon adopting of the product, agent β_1 alone puts enough pressure on agent d to also adopt the product: $\{a, c\}_s^3 = \{a, c, \alpha_1, \beta_1, d\}$. Thus, formula $a, c \rhd_1 d$ is satisfied in this network.

The next lemma generalizes the observation made in the previous paragraph to a claim that all formulas from set $X = \{A_i \rhd_{p_i} B_i\}_{i \leq m}$ are satisfied in the canonical network N_X.

Lemma 14. *$N_X \vDash A_i \rhd_{p_i} B_i$ for each $i \leq m$.*

Proof. Consider any $i \leq m$. Let s be a spending function such that

$$s(a) = \begin{cases} p_i, & \text{if } a = \alpha_i, \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

Then, by Definition 10, Definition 11, and Definition 12,

$$\lambda(\alpha_i) \cdot s(\alpha_i) + \sum_{a \in A_i} w(a, \alpha_i) = 1 \cdot p_i + \sum_{a \in A_i} 0 = p_i = \theta(\alpha_i).$$

Thus, $\alpha_i \in (A_i)^1_s$ by Definition 4. Hence, by Definition 10, Definition 11, and Definition 12,

$$\begin{aligned} \lambda(\beta_i) \cdot s(\beta_i) + \sum_{a \in (A_i)^1_s} w(a, \beta_i) &\geq \lambda(\beta_i) \cdot s(\beta_i) + w(\alpha_i, \beta_i) + \sum_{a \in A_i} w(a, \beta_i) \\ &\geq 0 \cdot 0 + 1 + |A_i| = 1 + |A_i| = \theta(\beta_i). \end{aligned}$$

Thus, $\beta_i \in (A_i)^2_s$ by Definition 4. Finally, for each $b \in B_i$, by Definition 10, Definition 11, and Definition 12,

$$\lambda(b) \cdot s(b) + \sum_{a \in (A_i)^2_s} w(a, b) \geq 0 \cdot 0 + w(\beta_i, b) = w(\beta_i, b) = 1 = \theta(b).$$

Hence, $b \in (A_i)^3_s$ by Definition 4. Thus, $b \in (A_i)^*_s$ by Definition 5 for each $b \in B_i$. Then, $B_i \subseteq A^*_s$. Note that $\|s\| = p_i$ due to definition (7). Therefore, $N_X \vDash A_i \rhd_{p_i} B_i$ by Definition 8. ⊠

Our next important result is the converse of Lemma 14 stated later as Lemma 22. In preparation for its, we make several technical observations about the social network N_X. First, we prove that, for each $i \leq m$, agent β_i can not be influenced without agent α_i being influenced as well.

Lemma 15. *If $\beta_i \in A^n_s$, then $\alpha_i \in A^n_s$, for each $A \subseteq \mathcal{A}_0$, each $i \leq m$, and each $n \geq 0$.*

Proof. Suppose $\beta_i \in A^n_s$. Let k be the smallest integer such that $0 \leq k \leq n$ and $\beta_i \in A^k_s$.

If $k = 0$, then $\beta_i \in A^0_s = A$ by Definition 4. Thus, $\beta_i \in \mathcal{A}_0$ due to the assumption $A \subseteq \mathcal{A}_0$, which contradicts the choice of $\beta_1, \dots, \beta_m$. Therefore, the lemma is vacuously true.

Suppose that $k > 0$. Since $k > 0$ is the smallest integer such that $\beta_i \in A^k_s$, it must be that $\beta_i \in A^k_s \setminus A^{k-1}_s$. Thus, by Definition 4,

$$\lambda(\beta_i) \cdot s(\beta_i) + \sum_{a \in A^{k-1}_s} w(a, \beta_i) \geq \theta(\beta_i).$$

By Definition 10, $\lambda(\beta_i) = 0$. By Definition 12, $\theta(\beta_i) = |A_i| + 1$. Thus,

$$\sum_{a\in A_s^{k-1}} w(a, \beta_i) \geq |A_i| + 1.$$

Thus, $\alpha_i \in A_s^{k-1}$ by Lemma 13. Hence, $\alpha_i \in A_s^{k-1}$. Therefore, $\alpha_i \in A_s^n$ by Definition 4 and since $k - 1 < k \leq n$. ⊠

The next lemma shows that the only way to influence agent α_i is to spend at least p_i on promotional marketing to this agent.

Lemma 16. *If $\alpha_i \in A_s^n$, then $s(\alpha_i) \geq p_i$, for each $A \subseteq \mathcal{A}_0$.*

Proof. Suppose that $\alpha_i \in A_s^n$. Note that $\alpha_i \notin \mathcal{A}_0 \supseteq A = A_s^0$ by the choice of $\alpha_1, \dots, \alpha_m$. Thus, by Definition 4, there is $k < n$ such that

$$\lambda(\alpha_i) \cdot\ s(\alpha_i) + \sum_{a\in A_s^k} w(a, \alpha_i) \geq \theta(\alpha_i).$$

By Definition 11, $w(a, \alpha_i) = 0$ for each $a \in \mathcal{A}$. Hence, $\lambda(\alpha_i) \cdot\ s(\alpha_i) \geq \theta(\alpha_i)$. By Definition 10, $\lambda(\alpha_i) = 1$. By Definition 12, $\theta(\alpha_i) = p_i$. Therefore, $s(\alpha_i) \geq p_i$. ⊠

Lemma 17. *For each $n \geq 0$ and each subset A of $\mathcal{A}_0$, if $\beta_i \in A_s^{n+1} \setminus A_s^n$, then $X \vdash (A_s^n \cap \mathcal{A}_0) \rhd_{p_i} B_i$.*

Proof. By Definition 4, assumption $\beta_i \in A_s^{n+1} \setminus A_s^n$ implies that

$$\lambda(\beta_i) \cdot s(\beta_i) + \sum_{a\in A_s^n} w(a, \beta_i) \geq \theta(\beta_i).$$

By Definition 10, $\lambda(\beta_i) = 0$. Thus, $\sum_{a\in A_s^n} w(a, \beta_i) \geq \theta(\beta_i)$. Hence, by Definition 12, $\sum_{a\in A_s^n} w(a, \beta_i) \geq |A_i|+1$. Thus, $A_i \subseteq A_s^n$ by Lemma 13. Recall that $A_i \subseteq \mathcal{A}_0$ by the choice of set X. Hence, $A_i \subseteq A_s^n \cap \mathcal{A}_0$. Then, $\vdash (A_s^n \cap \mathcal{A}_0) \rhd_0 A_i$ by Reflexivity axiom. Recall that $A_i \rhd_{p_i} B_i \in X$. Thus, $X \vdash (A_s^n \cap \mathcal{A}_0) \rhd_{p_i} B_i$ by Transitivity axiom. ⊠

Lemma 18. *$X \vdash (A_s^n \cap \mathcal{A}_0) \rhd_q \bigcup_{\beta_i\in A_s^{n+1}\setminus A_s^n} B_i$, where $q = \sum_{\beta_i\in A_s^{n+1}\setminus A_s^n} p_i$.*

Proof. The statement of the lemma follows from Lemma 17 and Lemma 9. ⊠

Lemma 19. *$X \vdash (A_s^n \cap \mathcal{A}_0) \rhd_q ((A_s^{n+2} \setminus A_s^{n+1}) \cap \mathcal{A}_0)$, where $q = \sum_{\beta_i \in A_s^{n+1} \setminus A_s^n} p_i$, for each subset A of $\mathcal{A}_0$, each spending function s, and each $n \geq 0$.*

Proof. By Definition 4,

$$\begin{aligned} (A_s^{n+2} \setminus A_s^{n+1}) \cap \mathcal{A}_0 &= \left\{ b \in \mathcal{A}_0 \,\middle|\, \lambda(b) \cdot s(b) + \sum_{a \in A_s^{n+1}} w(a,b) \geq \theta(b) \right\} \\ &\setminus \left\{ b \in \mathcal{A}_0 \,\middle|\, \lambda(b) \cdot s(b) + \sum_{a \in A_s^{n}} w(a,b) \geq \theta(b) \right\}. \end{aligned}$$

By Definition 10, $\lambda(b) = 0$ for all $b \in \mathcal{A}_0$. By Definition 12, $\theta(b) = 1$ for all $b \in \mathcal{A}_0$. Thus,

$$\begin{aligned} & (A_s^{n+2} \setminus A_s^{n+1}) \cap \mathcal{A}_0 \\ = \quad & \left\{ b \in \mathcal{A}_0 \,\middle|\, \sum_{a \in A_s^{n+1}} w(a,b) \geq 1 \right\} \setminus \left\{ b \in \mathcal{A}_0 \,\middle|\, \sum_{a \in A_s^{n}} w(a,b) \geq 1 \right\}. \end{aligned}$$

Since $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots \beta_m \notin \mathcal{A}_0$, by Definition 11, for each $b \in \mathcal{A}_0$, we have $w(a,b) \neq 0$ only if $a = \beta_i$ and $b \in B_i$ for some $i \leq m$, in which case $w(a,b) = 1$. Hence,

$$(A_s^{n+2} \setminus A_s^{n+1}) \cap \mathcal{A}_0 = \bigcup_{\beta_i \in A_s^{n+1} \setminus A_s^n} B_i.$$

Thus, to finish the proof of the lemma, it is sufficient to show that

$$X \vdash (A_s^n \cap \mathcal{A}_0) \rhd_q \bigcup_{\beta_i \in A_s^{n+1} \setminus A_s^n} B_i,$$

where $q = \sum_{\beta_i \in A_s^{n+1} \setminus A_s^n} p_i$, which follows from Lemma 18. ⊠

Lemma 20. *$X \vdash A \rhd_q (A_s^1 \cap \mathcal{A}_0)$, for each subset A of $\mathcal{A}_0$, each spending function s, and each non-negative real number q.*

Proof. By Definition 4,

$$A_s^1 \cap \mathcal{A}_0 \;=\; (A \cap \mathcal{A}_0) \cup \left\{ b \in \mathcal{A}_0 \;\middle|\; \lambda(b) \cdot s(b) + \sum_{a \in A} w(a,b) \geq \theta(b) \right\}.$$

By Definition 10, $\lambda(b) = 0$ for all $b \in \mathcal{A}_0$. By Definition 12, $\theta(b) = 1$ for all $b \in \mathcal{A}_0$. Thus,

$$A_s^1 \cap \mathcal{A}_0 \;=\; (A \cap \mathcal{A}_0) \cup \left\{ b \in \mathcal{A}_0 \;\middle|\; \sum_{a \in A} w(a,b) \geq 1 \right\}.$$

By Definition 11, $w(a,b) = 0$ for all $a \in A \subseteq \mathcal{A}_0$ and all $b \in \mathcal{A}_0$. Thus, the set

$$\left\{ b \in \mathcal{A}_0 \;\middle|\; \sum_{a \in A} w(a,b) \geq 1 \right\}$$

is empty. Hence, $A_s^1 \cap \mathcal{A}_0 = A \cap \mathcal{A}_0$. Therefore, $X \vdash A \rhd_q (A_s^1 \cap \mathcal{A}_0)$ by Reflexivity axiom. ⊠

Lemma 21. *$X \vdash A \rhd_q (A_s^{n+1} \cap \mathcal{A}_0)$, where $q = \sum_{\beta_i \in A_s^n} p_i$, for each subset A of $\mathcal{A}_0$, each spending function s, and each $n \geq 0$.*

Proof. We prove this lemma by induction on n. If $n = 0$, then the required follows from Lemma 20.

Assume now that

$$X \vdash A \rhd_q (A_s^{n+1} \cap \mathcal{A}_0), \tag{8}$$

where $q = \sum_{\beta_i \in A_s^n} p_i$.

Note that $A_s^n \subseteq A_s^{n+1}$ by Definition 4. Hence, $\vdash (A_s^{n+1} \cap \mathcal{A}_0) \rhd_0 (A_s^n \cap \mathcal{A}_0)$ by Reflexivity axiom. At the same time, by Lemma 19,

$$X \vdash (A_s^n \cap \mathcal{A}_0) \rhd_r ((A_s^{n+2} \setminus A_s^{n+1}) \cap \mathcal{A}_0),$$

where $r = \sum_{\beta_i \in A_s^{n+1} \setminus A_s^n} p_i$. Thus, by Transitivity axiom,

$$X \vdash (A_s^{n+1} \cap \mathcal{A}_0) \rhd_r ((A_s^{n+2} \setminus A_s^{n+1}) \cap \mathcal{A}_0).$$

Then, by Augmentation axiom,

$$X \vdash (A_s^{n+1} \cap \mathcal{A}_0), (A_s^{n+1} \cap \mathcal{A}_0) \rhd_r ((A_s^{n+2} \setminus A_s^{n+1}) \cap \mathcal{A}_0), (A_s^{n+1} \cap \mathcal{A}_0).$$

In other words,

$$X \vdash (A_s^{n+1} \cap \mathcal{A}_0) \rhd_r (A_s^{n+2} \cap \mathcal{A}_0). \tag{9}$$

Therefore, by Transitivity axiom from statement (8) and (9) we can conclude $X \vdash A \rhd_{q'} (A_s^{n+2} \cap \mathcal{A}_0)$, where $q' = q + r = \sum_{\beta_i \in A_s^{n+1}} p_i$. ⊠

We are now ready to prove the converse of Lemma 14.

Lemma 22. *If $N_X \vDash A \rhd_p B$, then $X \vdash A \rhd_p B$, for each subsets A and B of $\mathcal{A}_0$ and each non-negative real number p.*

Proof. Suppose that $N_X \vDash A \rhd_p B$. By Definition 8, there is a spending function s such that $\|s\| \le p$ and $B \subseteq A_s^*$. Thus, by Corollary 3, there is $n \ge 0$ such that $B \subseteq A_s^n$. By Definition 4, $A_s^n \subseteq A_s^{n+1}$. Thus, $B \subseteq A_s^{n+1}$. Since B is a subset of $\mathcal{A}_0$, we have $B \subseteq A_s^{n+1} \cap \mathcal{A}_0$. Hence, $\vdash (A^{n+1} \cap \mathcal{A}_0) \rhd_0 B$ by Reflexivity axiom. Then, from Transitivity axiom and Lemma 21, we have $X \vdash A \rhd_q B$, where $q = \sum_{\beta_i \in A_s^n} p_i$.

Note that $\sum_{\beta_i \in A_s^n} p_i \le \sum_{\alpha_i \in A_s^n} p_i$ by Lemma 15 and $\sum_{\alpha_i \in A_s^n} p_i \le \sum_{s(\alpha_i) \ge p_i} p_i$ by Lemma 16. Thus, taking into account Definition 3,

$$q = \sum_{\beta_i \in A_s^n} p_i \le \sum_{\alpha_i \in A_s^n} p_i \le \sum_{s(\alpha_i) \ge p_i} p_i \le \sum_{s(\alpha_i) \ge p_i} s(\alpha_i) \le \sum_{a \in \mathcal{A}} s(a) = \|s\| \le p.$$

Hence, $q \le p$. Then, $\vdash B \rhd_{p-q} B$ by Reflexivity axiom. Finally, $X \vdash A \rhd_q B$ and $\vdash B \rhd_{p-q} B$, by Transitivity axiom, imply that $X \vdash A \rhd_p B$. ⊠

We conclude this section by stating and proving the completeness theorem for promotional marketing.

Theorem 2. *If $\nvdash \phi$, then there exists social network $N = (\mathcal{A}, w, \lambda, \theta)$ such that $\phi \in \Phi(\mathcal{A})$ and $N \nvDash \phi$.*

Proof. Suppose that $\nvdash \phi$. Let M be any maximal consistent subset of

$$\{\psi, \neg\psi \mid \psi \text{ is a subformula of } \neg\phi\}$$

such that $\neg\phi \in M$. Let X be the set of all atomic formulas of the form $A \rhd_p B$ in set M. To finish the proof of the theorem, we first establish the following lemma:

Lemma 23. *$\psi \in M$ if and only if $N_X \vDash \psi$ for each subformula ψ of $\neg\phi$.*

Proof. We prove the lemma by induction on the structural complexity of formula ψ. In the base case, suppose that ψ is $A \rhd_p B$.
($\Rightarrow$) If $A \rhd_p B \in M$, then $A \rhd_p B \in X$ by the choice of set X. Thus, $N_X \vDash A \rhd_p B$ by Lemma 14.
($\Leftarrow$) If $N_X \vDash A \rhd_p B$, then $X \vdash A \rhd_p B$ by Lemma 22. Thus, $M \vdash A \rhd_p B$. Hence, by the maximality of set M, we have $A \rhd_p B \in M$ since $A \rhd_p B$ is a subformula of $\neg\phi$.

The induction step follows from the induction hypothesis, the maximality and the consistency of set M and Definition 8 in the standard way. ⊠

To finish the proof of the theorem, note that $\neg\phi \in M$ by the choice of set M. Thus, $N_X \vDash \neg\psi$ by Lemma 23. Therefore, $N_X \nvDash \psi$ by Definition 8. ⊠

4. Logic of Preventive Marketing

In this section we study the impact of preventive marketing on influence in social networks. Our definition of a social network given in Definition 1 and the language $\Phi(\mathcal{A})$ remain the same. As it has been discussed in the introduction, we only modify the meaning of the influence relation $A \rhd_p B$ to be "for any preventive marketing campaign with budget no greater than p, the set of agents A is able to influence the set of agents B". The latter is formally captured in item 1 of Definition 13.

Definition 13. *For any social network N with the set of agents $\mathcal{A}$ and any formula $\phi \in \Phi(\mathcal{A})$, we define the satisfiability relation $N \vDash \phi$ as follows:*

1. *$N \vDash A \rhd_p B$ if $B \subseteq A^*_s$ for each spending function s such that $\|s\| \leq p$,*
2. *$N \vDash \neg\psi$ if $N \nvDash \psi$,*
3. *$N \vDash \psi \to \chi$ if $N \nvDash \psi$ or $N \vDash \chi$.*

Note the significant difference between the above definition and the similar Definition 8 for promotional marketing. Item 1 of Definition 13 has a universal quantifier over spending functions and corresponding part of Definition 8 has an existential quantifier over spending functions.

4.1. Axioms

Let $\mathcal{A}$ be any fixed finite set of agents. Our logical system for influence with preventive marketing, in addition to propositional tautologies in language $\Phi(\mathcal{A})$, contains the following axioms:

1. Reflexivity: $A \rhd_p B$, where $B \subseteq A$,
2. Augmentation: $A \rhd_p B \to A, C \rhd_p B, C$,
3. Transitivity: $A \rhd_p B \to (B \rhd_p C \to A \rhd_p C)$,
4. Monotonicity: $A \rhd_p B \to A \rhd_q B$, where $q \leq p$.

Just like in the case of promotional marketing, we write $\vdash \phi$ if formula $\phi \in \Phi(\mathcal{A})$ is derivable in our logical system using Modus Ponens inference rule. We write $X \vdash \phi$ if formula ϕ is derivable using an additional set of axioms $X \subseteq \Phi(\mathcal{A})$.

4.2. Example

The soundness and the completeness of our logical system will be shown later. In this section we give two examples of formal proofs in our system. First, we show a preventive marketing analogy of Lemma 8:

Lemma 24. $\vdash A \rhd_p B \to (A \rhd_p C \to A \rhd_p B, C)$.

Proof. By Augmentation axiom,

$$\vdash A \rhd_p B \to A \rhd_p A, B \tag{10}$$

and

$$\vdash A \rhd_p C \to A, B \rhd_p B, C. \tag{11}$$

By Transitivity axiom,

$$\vdash A \rhd_p A, B \to (A, B \rhd_p B, C \to A \rhd_p B, C). \tag{12}$$

The statement of the lemma follows from statements (10), (11), and (12) by the laws of the propositional logic. ⊠

Next, we show an auxiliary lemma that is used later in the proof of completeness.

Lemma 25. *If $X \vdash B \rhd_p c$ for each $c \in C$, then $X \vdash B \rhd_p C$, where B and C are subsets of $\mathcal{A}_0$ and $p \geq 0$.*

Proof. We prove the lemma by induction on the size of set C.
Base Case: $X \vdash B \rhd_p \varnothing$ by Reflexivity axiom.
Induction Step: Assume that $X \vdash B \rhd_p C$. Let c be any element of $\mathcal{A}_0 \setminus C$ such that $X \vdash B \rhd_p c$. We need to show that $X \vdash B \rhd_p C \cup \{c\}$. By Augmentation axiom,

$$X \vdash B \cup \{c\} \rhd_p C \cup \{c\}. \tag{13}$$

Recall that $X \vdash B \rhd_p c$. Again by Augmentation axiom, $X \vdash B \rhd_p B \cup \{c\}$. Hence, $X \vdash B \rhd_p C \cup \{c\}$, due to (13) and Transitivity axiom. ⊠

4.3. Soundness

In this section we prove the soundness of the logic for preventive marketing.

Theorem 3. *For any finite set $\mathcal{A}$ and any $\phi \in \Phi(\mathcal{A})$, if $\vdash \phi$, then $N \vDash \phi$ for each social network $N = (\mathcal{A}, w, \lambda, \theta)$.*

The soundness of propositional tautologies and of Modus Ponens inference rule is straightforward. Below we show the soundness of each of the remaining axioms as a separate lemma.

Lemma 26. *$N \vDash A \rhd_p B$, for any social network $N = (\mathcal{A}, w, \lambda, \theta)$ and any subsets A and B of $\mathcal{A}$ such that $B \subseteq A$.*

Proof. Let s be any spending function. By Definition 13, it suffices to show that $B \subseteq A^*_s$. Indeed, $A \subseteq A^*_s$ by Corollary 2. Therefore, $B \subseteq A^*_s$ due to the assumption $B \subseteq A$ of the lemma. ⊠

Lemma 27. *If $N \vDash A \rhd_p B$, then $N \vDash A, C \rhd_p B, C$, for each social network $N = (\mathcal{A}, w, \lambda, \theta)$ and all subsets A, B, and C of $\mathcal{A}$.*

Proof. Suppose that $N \vDash A \rhd_p B$. Consider any spending function s such that $\|s\| \le p$. It suffices to show that $B \cup C \subseteq (A \cup C)^*_s$. Indeed, assumption $N \vDash A \rhd_p B$ implies that $B \subseteq A^*_s$ by Definition 13. At the same time, $C \subseteq C^*_s$ by Corollary 2. Therefore, $B \cup C \subseteq A^*_s \cup C^*_s \subseteq (A \cup C)^*_s$, by Lemma 3. ⊠

Lemma 28. *If $N \vDash A \rhd_p B$ and $N \vDash B \rhd_p C$, then $N \vDash A \rhd_p C$, for each social network $N = (\mathcal{A}, w, \lambda, \theta)$ and all subsets A, B, and C of $\mathcal{A}$.*

Proof. Suppose that $N \vDash A \rhd_p B$ and $N \vDash B \rhd_p C$. Consider any spending function s such that $\|s\| \leq p$. By Definition 13, it suffices to show that $C \subseteq A^*_s$.

Note that assumption $N \vDash A \rhd_p B$, by Definition 13, imply that $B \subseteq A^*_s$. Thus, $B^*_s \subseteq (A^*_s)^*_s$ by Corollary 4. At the same time, assumption $N \vDash B \rhd_p C$ implies that $C \subseteq B^*_s$ by Definition 13. Hence, $C \subseteq (A^*_s)^*_s$. Therefore, $C \subseteq A^*_s$ by Lemma 1. ⊠

Lemma 29. *If $N \vDash A \rhd_p B$, then $N \vDash A \rhd_q B$, for each $q \leq p$, each each social network $N = (\mathcal{A}, w, \lambda, \theta)$ and all subsets A and B of $\mathcal{A}$.*

Proof. Consider any spending function s such that $\|s\| \leq q$. By Definition 13, it suffices to show that $B \subseteq A^*_s$. To prove this, note that $\|s\| \leq q \leq p$. Thus, $B \subseteq A^*_s$ due to Definition 13 and the assumption $N \vDash A \rhd_p B$ of the lemma. ⊠

This concludes the proof of the soundness of our logical system for preventive marketing.

4.4. Completeness

The rest of this section contains the proof of the following result.

Theorem 4. *If $\nvdash \phi$, then there is a social network $N = (\mathcal{A}, w, \lambda, \theta)$ such that $\phi \in \Phi(\mathcal{A})$ and $N \nvDash \phi$.*

Suppose that $\nvdash \phi$. It suffices to construct a "canonical" social network $N = (\mathcal{A}, w, \lambda, \theta)$ such that $N \nvDash \phi$. Define $P \subset \mathbb{R}$ to be the finite set of all subscripts that appear in formula ϕ. Let $\varepsilon > 0$ be such that $|p_1 - p_2| > \varepsilon$ for all $p_1, p_2 \in P$ where $p_1 \neq p_2$. Let $\mathcal{A}_0$ be the finite set of all agents that appear in formula ϕ and X be a maximal consistent subset of $\Phi(\mathcal{A}_0)$ containing formula $\neg\phi$.

In Section 2, we have introduced closures A^k_s and A^*_s of a set of agents A. Both of these closures are *semantic* in the sense that they are defined in terms of a given social network. We are about to introduce another closure A^+_p that will be used to construct the canonical social network N. Unlike closures A^k_s and A^*_s, closure the A^+_p is *syntactic* because it is defined in terms of provability of certain statements in our logical system.

Definition 14. $A_p^+ = \{a \in \mathcal{A}_0 \mid X \vdash A \rhd_p a\}$, *for any set of agents* $A \subseteq \mathcal{A}_0$ *and any* $p \geq 0$.

Lemma 30. $X \vdash A \rhd_p A_p^+$, *for any* $A \subseteq \mathcal{A}_0$ *and any* $p \geq 0$.

Proof. The statement of the lemma follows from Definition 14 and Lemma 25. ⊠

Generally speaking, it is possible that $A_p^+ = A_q^+$ for some p and q such that $p \neq q$. In the construction of the canonical social network N it will be convenient to distinguish closures A_p^+ for different values of parameter p. In such situations, instead of closure A_p^+ we consider labeled closure, formally defined as pair (A_p^+, p).

Definition 15. *Let* $\mathbb{L} = \{(A_p^+, p) \mid A \subseteq \mathcal{A}_0, p \in P\}$.

Next we define the canonical network $N = (\mathcal{A}, w, \lambda, \theta)$. Besides agents in set $\mathcal{A}_0$, our social network also has two additional agents for each $\ell \in \mathbb{L}$. By analogy with the canonical social network N_X from the proof of completeness for promotional marking, we call these additional agents $\alpha(\ell)$ and $\beta(\ell)$.

Definition 16. $\mathcal{A} = \mathcal{A}_0 \cup \{\alpha(\ell), \beta(\ell) \mid \ell \in \mathbb{L}\}$.

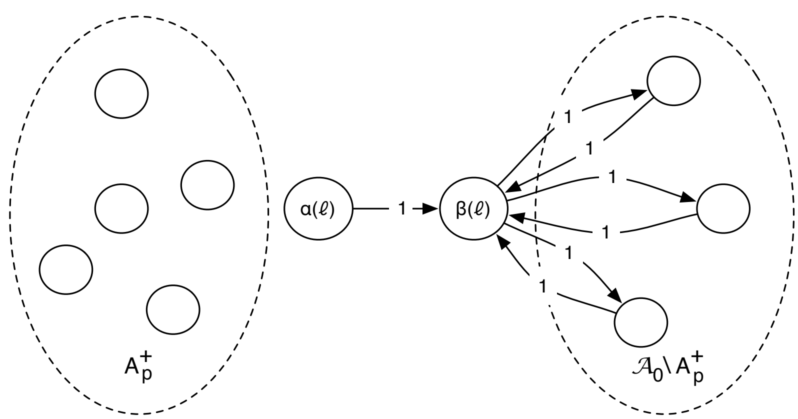

Figure 4: Towards the definition of the influence function in the canonical social network.

For any $\ell = (A_p^+, p)$, we assume that upon adopting the product agent $\alpha(\ell)$ puts peer pressure on agent $\beta(\ell)$, agent $\beta(\ell)$ puts peer pressure on each agent in set $\mathcal{A}_0 \setminus A_p^+$ and each agent in set $\mathcal{A}_0 \setminus A_p^+$, in turn, puts peer pressure on agent $\beta(\ell)$. The peer pressure structure is illustrated in Figure 4. Note that the same agent $a \in \mathcal{A}_0$ can belong to set $\mathcal{A}_0 \setminus A_p^+$ for several different values of p. Such agent a could experience (or put) peer pressure from (on) several different agents $\beta(\ell)$. The structure is formally specified in Definition 17.

Definition 17. *For any $a, b \in \mathcal{A}$,*

$$w(a,b) = \begin{cases} 1, & \text{if } a = \alpha(\ell) \text{ and } b = \beta(\ell) \text{ for some } \ell \in \mathbb{L}, \\ 1, & \text{if } a \in \mathcal{A}_0 \setminus A_p^+,\ b = \beta(A_p^+, p), \text{ and } (A_p^+, p) \in \mathbb{L}, \\ 1, & \text{if } a = \beta(A_p^+, p),\ (A_p^+, p) \in \mathbb{L}, \text{ and } b \in \mathcal{A}_0 \setminus A_p^+, \\ 0, & \text{otherwise.} \end{cases}$$

We assume that only agents $\{\alpha(\ell) \mid \ell \in \mathbb{L}\}$ are responsive to preventive marketing. This is formally captured in the definition of the receptivity function below.

Definition 18. *For any $a \in \mathcal{A}$,*

$$\lambda(a) = \begin{cases} -1, & \text{if } a = \alpha(\ell) \text{ for some } \ell \in \mathbb{L}, \\ 0, & \text{otherwise.} \end{cases}$$

To finish the definition of canonical social network $N = (\mathcal{A}, w, \lambda, \theta)$, we only need to define threshold function $\theta(a)$ for each $a \in \mathcal{A}$. There are three different cases to consider: $a = \alpha(\ell)$ for some $\ell \in \mathbb{L}$, $a = \beta(\ell)$ for some $\ell \in \mathbb{L}$, and $a \in \mathcal{A}_0$.

Recall that by Definition 17 and Definition 18, agent $\alpha(\ell)$ is not responsive to peer pressure of any other agent. It is only responsive to the marketing pressure with receptivity -1. We set the threshold value of this agent to $\varepsilon - p$, where $\ell = (A_p^+, p)$. Thus, if an amount at least p is spent on the preventive marketing to this agent, it will not adopt the product.

We set threshold value of agent $\beta(\ell)$ to 1. Thus, for each $\ell = (A_p^+, p)$, if either agent $\alpha(\ell)$ or any of the agents in the set $\mathcal{A}_0 \setminus A_p^+$ adopts the product, then agent $\beta(\ell)$ will also adopt the product.

Finally, recall from Definition 17 that agent $a \in \mathcal{A}_0$ can experience peer pressure from any agent $\beta(A_p^+, p)$ such that $a \in \mathcal{A}_0 \setminus A_p^+$. There are exactly $|\{(A_p^+, p) \in \mathbb{L} \mid a \in \mathcal{A}_0 \setminus A_p^+\}|$ such β-agents. We set the threshold value $\theta(a)$

high enough so that it adopts the product only if *all* of these β-agents adopt the product.

The next definition captures the three cases discussed above.

Definition 19. *For any $a \in \mathcal{A}$,*

$$\theta(a) = \begin{cases} \varepsilon - p, & \text{if } a = \alpha(A_p^+, p), \\ 1, & \text{if } a = \beta(A_p^+, p), \\ |\{(A_p^+, p) \in \mathbb{L} \mid a \in \mathcal{A}_0 \setminus A_p^+\}|, & \text{if } a \in \mathcal{A}_0. \end{cases}$$

For any $c \in \mathcal{A}_0$, we have chosen $\theta(c)$ to be equal to the number of $\beta(A_p^+, p)$ such that $c \in \mathcal{A}_0 \setminus A_p^+$. Thus, if all such β-agents adopt the product, then the total peer pressure on agent c would reach $\theta(c)$ and agent c also would adopt the product. This observation is formalized by the next lemma.

Lemma 31. *Let c be an agent in $\mathcal{A}_0$, set B be a subset of $\mathcal{A}_0$, and s be an arbitrary spending function for the social network N. If for each $(A_p^+, p) \in \mathbb{L}$ at least one of the following is true: (i) $c \in A_p^+$, (ii) $\beta(A_p^+, p) \in B_s^*$, then $c \in B_s^*$.*

Proof. By Corollary 3, $B_s^* = B_s^n$ for some $n \geq 0$. Thus, by the assumption of this lemma, for each $(A_p^+, p) \in \mathbb{L}$ at least one of the following is true: (i) $c \in A_p^+$, (ii) $\beta(A_p^+, p) \in B_s^n$. In other words, $\{\beta(A_p^+, p) \mid c \in \mathcal{A}_0 \setminus A_p^+\} \subseteq B_s^n$. Hence,

$$\sum_{b \in B_s^n} w(b, c) \geq \sum_{\ell \in \{(A_p^+, p) \in \mathbb{L} \mid c \in \mathcal{A}_0 \setminus A_p^+\}} w(\beta(\ell), c).$$

Thus, by Definition 17,

$$\sum_{b \in B_s^n} w(b, c) \geq \sum_{\ell \in \{(A_p^+, p) \in \mathbb{L} \mid c \in \mathcal{A}_0 \setminus A_p^+\}} 1 = |\{(A_p^+, p) \in \mathbb{L} \mid c \in \mathcal{A}_0 \setminus A_p^+\}|.$$

At the same time $\lambda(c) = 0$ by Definition 18. Hence,

$$\lambda(c) \cdot s(c) + \sum_{b \in B_s^n} w(b, c) \geq |\{(A_p^+, p) \in \mathbb{L} \mid c \in \mathcal{A}_0 \setminus A_p^+\}|.$$

Then, by Definition 19,

$$\lambda(c) \cdot s(c) + \sum_{b \in B_s^n} w(b, c) \geq \theta(c).$$

Thus, $c \in B_s^{n+1}$ by Definition 4. Therefore, $c \in B_s^*$ by Definition 5. $\boxtimes$

Referring back to Figure 4, note that if an agent in set $\mathcal{A}_0 \setminus A_p^+$ adopts the product, then it will put enough pressure on $\beta(A_p^+, p)$ so that agent $\beta(A_p^+, p)$ also adopts the product. We formally state this observation as the lemma below.

Lemma 32. *If there is $b_0 \in B_s^*$ such that $b_0 \in \mathcal{A}_0 \setminus A_p^+$, then $\beta(A_p^+, p) \in B_s^*$, where $(A_p^+, p) \in \mathbb{L}$, set B is a subset of $\mathcal{A}_0$, and s is an arbitrary spending function for the social network N.*

Proof. By Corollary 3, $B_s^* = B_s^n$ for some $n \geq 0$. At the same time, by Definition 17, assumption $b_0 \in \mathcal{A}_0 \setminus A_p^+$ implies that $w(b_0, \beta(A_p^+, p)) = 1$. Thus,

$$\sum_{b \in B_s^n} w(b, \beta(A_p^+, p)) \geq w(b_0, \beta(A_p^+, p)) = 1,$$

since $b_0 \in B_s^* = B_s^n$. Note that $\lambda(\beta(A_p^+, p)) = 0$ by Definition 18. Hence,

$$\lambda(\beta(A_p^+, p)) \cdot s(\beta(A_p^+, p)) + \sum_{b \in B_s^n} w(b, \beta(A_p^+, p)) \geq w(b_0, \beta(A_p^+, p)) = 1.$$

Thus, by Definition 19,

$$\lambda(\beta(A_p^+, p)) \cdot s(\beta(A_p^+, p)) + \sum_{b \in B_s^n} w(b, \beta(A_p^+, p)) \geq \theta(\beta(A_p^+, p)).$$

Hence, $\beta(A_p^+, p) \in B_s^{n+1}$ by Definition 4. Therefore, $\beta(A_p^+, p) \in B_s^*$ by Definition 5. $\boxtimes$

Recall that we have set the threshold value of agent $\alpha(A_p^+, p)$ to be $\varepsilon - p$, so that by spending at least p on preventive marketing one would prevent an adoption of the product by agent $\alpha(A_p^+, p)$. At the same time, spending no greater than $p - \varepsilon$ will result in $\alpha(A_p^+, p)$ adopting the product. Once agent $\alpha(A_p^+, p)$ adopts the product, it will put enough pressure on agent $\beta(A_p^+, p)$ to adopt the product as well. This observation is formally stated below.

Lemma 33. *If $s(\alpha(A_p^+, p)) \leq p - \varepsilon$, then $\beta(A_p^+, p) \in B_s^*$, where $(A_p^+, p) \in \mathbb{L}$, set B is a subset of $\mathcal{A}_0$, and s is an arbitrary spending function for the social network N.*

Proof. Suppose that $s(\alpha(A_p^+, p)) \leq p - \varepsilon$. Note that $\lambda(\alpha(A_p^+, p)) = -1$ by Definition 18 and $w(b, \alpha(A_p^+, p)) = 0$ for each $b \in \mathcal{A}$ by Definition 17. Thus,

$$\begin{aligned}
&\lambda(\alpha(A_p^+, p)) \cdot s(\alpha(A_p^+, p)) + \sum_{b \in B_s^0} w(b, \alpha(A_p^+, p)) \\
&= -1 \cdot s(\alpha(A_p^+, p)) + 0 = -s(\alpha(A_p^+, p)) \geq \varepsilon - p.
\end{aligned}$$

Thus, by Definition 19,

$$\lambda(\alpha(A_p^+, p)) \cdot s(\alpha(A_p^+, p)) + \sum_{b \in B_s^0} w(b, \alpha(A_p^+, p)) \geq \theta(\alpha(A_p^+, p)).$$

Hence, $\alpha(A_p^+, p) \in B_s^1$ by Definition 4. Since w is a non-negative function, by Definition 17,

$$\begin{aligned}
\sum_{b \in B_s^1} w(b, \beta(A_p^+, p)) &= w(\alpha(A_p^+, p), \beta(A_p^+, p)) + \sum_{b \in B_s^1 \setminus \{\alpha(A_p^+, p)\}} w(b, \beta(A_p^+, p)) \\
&\geq w(\alpha(A_p^+, p), \beta(A_p^+, p)) = 1.
\end{aligned}$$

Note that $\lambda(\beta(A_p^+, p)) = 0$ by Definition 18. Thus,

$$\lambda(\beta(A_p^+, p)) \cdot s(\beta(A_p^+, p)) + \sum_{b \in B_s^1} w(b, \beta(A_p^+, p)) \geq 1.$$

Hence, by Definition 19,

$$\lambda(\beta(A_p^+, p)) \cdot s(\beta(A_p^+, p)) + \sum_{b \in B_s^1} w(b, \beta(A_p^+, p)) \geq \theta(\beta(A_p^+, p)).$$

Thus, $\beta(A_p^+, p) \in B_s^2$ by Definition 4. Therefore, $\beta(A_p^+, p) \in B_s^*$ by Definition 5. ⊠

The next lemma states that if we do spend at least p on preventive marketing to agent $\alpha(A_p^+, p)$, then this agent will never adopt the product.

Lemma 34. *For any $(A_p^+, p) \in \mathbb{L}$ and any spending function s, if $s(\alpha(A_p^+, p)) \geq p$, then $\alpha(A_p^+, p) \notin A_s^*$.*

Proof. By Definition 5, it suffices to show that $\alpha(A^+_p, p) \notin A^k_s$ for each $k \geq 0$. We prove this statement by induction on k.

Base Case: By Definition 4, we have $A^0_s = A$. At the same time, by Definition 15, $(A^+_p, p) \in \mathbb{L}$ implies that $A \subseteq \mathcal{A}_0$. Hence, $A^0_s \subseteq \mathcal{A}_0$. Therefore, $\alpha(A^+_p, p) \notin A^0_s$ by Definition 16.

Induction Step: Suppose that $\alpha(A^+_p, p) \notin A^k_s$ and $\alpha(A^+_p, p) \in A^{k+1}_s$. Thus, by Definition 4,

$$\lambda(\alpha(A^+_p, p)) \cdot s(\alpha(A^+_p, p)) + \sum_{b \in A^k_s} w(b, \alpha(A^+_p, p)) \geq \theta(\alpha(A^+_p, p)).$$

Hence, by Definition 18, Definition 17, and Definition 19,

$$1 \cdot s(\alpha(A^+_p, p)) + 0 \geq \varepsilon - p.$$

Thus, $s(\alpha(A^+_p, p)) \leq p - \varepsilon$. Therefore, $s(\alpha(A^+_p, p)) < p$ since $\varepsilon > 0$. This contradicts the assumption $s(\alpha(A^+_p, p)) \geq p$ of the lemma. ⊠

As we have seen in the previous lemma, spending at least p on preventive marketing to agent $\alpha(A^+_p, p)$ prevents it from adopting the product. We now show that spending at least p on agent $\alpha(A^+_p, p)$ prevents all agents in set $\mathcal{A}_0 \setminus A^+_p$ from adopting the product. See Figure 4.

Lemma 35. *$A^*_s \cap \mathcal{A}_0 \subseteq A^+_p$, where $(A^+_p, p) \in \mathbb{L}$ and s is an arbitrary spending function such that $s(\alpha(A^+_p, p)) \geq p$.*

Proof. Let $(A^+_p, p) \in \mathbb{L}$ and s be an arbitrary spending function such that $s(\alpha(A^+_p, p)) \geq p$. By Definition 5, it suffices to show that $A^k_s \cap \mathcal{A}_0 \subseteq A^+_p$ for each $k \geq 0$. Instead, we prove the following two statements simultaneously by induction on k:

$$\begin{cases} A^k_s \cap \mathcal{A}_0 \subseteq A^+_p, \\ \beta(A^+_p, p) \notin A^k_s. \end{cases}$$

Base Case: Suppose that $a \in A^0_s$. Thus, $a \in A$ by Definition 4. Hence, $\vdash A \rhd_p a$ by Reflexivity axiom. Therefore, $a \in A^+_p$ by Definition 14. Assume now that $\beta(A^+_p, p) \in A^0_s$. Thus, $\beta(A^+_p, p) \in A \subseteq \mathcal{A}_0$ by Definition 4 and Definition 15, which is a contradiction with $\beta(A^+_p, p) \notin \mathcal{A}_0$ by the choice of $\alpha(\ell)$ and $\beta(\ell)$.

Induction Step: Assume that

$$\begin{cases} A_s^k \cap \mathcal{A}_0 \subseteq A_p^+, \\ \beta(A_p^+, p) \notin A_s^k. \end{cases}$$

We need to show that

$$\begin{cases} A_s^{k+1} \cap \mathcal{A}_0 \subseteq A_p^+, \\ \beta(A_p^+, p) \notin A_s^{k+1}. \end{cases} \tag{14}$$

To prove the first statement, suppose that there is $a \in \mathcal{A}_0$ such that $a \in A_s^{k+1} \setminus A_p^+$. Note that $a \notin A_s^k$ by the induction hypothesis. Thus, by Definition 4,

$$\lambda(a) \cdot s(a) + \sum_{b \in A_s^k} w(b, a) \geq \theta(a).$$

Hence, by Definition 18 and Definition 19,

$$0 \cdot s(a) + \sum_{b \in A_s^k} w(b, a) \geq |\{(A_q^+, q) \in \mathbb{L} \mid a \in \mathcal{A}_0 \setminus A_q^+\}|.$$

Therefore, by Definition 17,

$$\sum_{\ell \in \{(A_q^+, q) \in \mathbb{L} \mid a \in \mathcal{A}_0 \setminus A_q^+, \beta(\ell) \in A_s^k\}} w(\beta(\ell), a) \geq |\{(A_q^+, q) \in \mathbb{L} \mid a \in \mathcal{A}_0 \setminus A_q^+\}|.$$

Hence, again by Definition 17,

$$\sum_{\ell \in \{(A_q^+, q) \in \mathbb{L} \mid a \in \mathcal{A}_0 \setminus A_q^+, \beta(\ell) \in A_s^k\}} 1 \geq |\{(A_q^+, q) \in \mathbb{L} \mid a \in \mathcal{A}_0 \setminus A_q^+\}|.$$

Thus,

$$|\{(A_q^+, q) \in \mathbb{L} \mid a \in \mathcal{A}_0 \setminus A_q^+, \beta(\ell) \in A_s^k\}| \geq |\{(A_q^+, q) \in \mathbb{L} \mid a \in \mathcal{A}_0 \setminus A_q^+\}|.$$

At the same time,

$$\{(A_q^+, q) \in \mathbb{L} \mid a \in \mathcal{A}_0 \setminus A_q^+, \beta(\ell) \in A_s^k\} \subseteq \{(A_q^+, q) \in \mathbb{L} \mid a \in \mathcal{A}_0 \setminus A_q^+\}.$$

Then it must be the case that

$$\{(A_q^+, q) \in \mathbb{L} \mid a \in \mathcal{A}_0 \setminus A_q^+, \beta(\ell) \in A_s^k\} = \{(A_q^+, q) \in \mathbb{L} \mid a \in \mathcal{A}_0 \setminus A_q^+\}.$$

Hence, $\beta(\ell) \in A_s^k$ for all $\ell \in \{(A_q^+, q) \in \mathbb{L} \mid a \in \mathcal{A}_0 \setminus A_q^+\}$. In particular, $\beta(A_p^+, p) \in A_s^k$, which is a contradiction to the induction hypothesis.

To prove the second statement from (14), suppose that $\beta(A_p^+, p) \in A_s^{k+1}$. Note that $\beta(A_p^+, p) \notin A_s^k$ due to the induction hypothesis. Thus, by Definition 4,

$$\lambda(\beta(A_p^+, p)) \cdot s(\beta(A_p^+, p)) + \sum_{b \in A_s^k} w(b, \beta(A_p^+, p)) \geq \theta(\beta(A_p^+, p)).$$

Hence, by Definition 18 and Definition 19,

$$0 \cdot s(\beta(A_p^+, p)) + \sum_{b \in A_s^k} w(b, \beta(A_p^+, p)) \geq 1.$$

Thus, there must exist at least one $b \in A_s^k$ such that $w(b, \beta(A_p^+, p)) > 0$. By Definition 17 and Lemma 34, this implies that $b \in \mathcal{A}_0 \setminus A_p^+$, which is a contradiction to the first part of the induction hypothesis, i.e. $A_s^k \cap \mathcal{A}_0 \subseteq A_p^+$. ⊠

Lemma 36. *For each $B, C \subseteq \mathcal{A}_0$ and each $q \in P$, if $B \rhd_q C \in X$, then $N \vDash B \rhd_q C$.*

Proof. Consider any spending function s such that $\|s\| \leq q$. By Definition 13, it suffices to show that $C \subseteq B_s^*$. Suppose that there is $c_0 \in C$ such that $c_0 \notin B_s^*$.

Thus, by Lemma 31, there exists $(A_p^+, p) \in \mathbb{L}$ such that $c_0 \notin A_p^+$ and $\beta(A_p^+, p) \notin B_s^*$. The latter, by Lemma 32, implies that $B_s^* \cap (\mathcal{A}_0 \setminus A_p^+) = \varnothing$. Hence, $(B_s^* \cap \mathcal{A}_0) \setminus A_p^+ = \varnothing$. Then, $B_s^* \cap \mathcal{A}_0 \subseteq A_p^+$. Thus, $B \subseteq B_s^* \cap \mathcal{A}_0 \subseteq A_p^+$ by Definition 4 and Definition 5. We next consider the following two cases:
Case I: $p \leq q$. In this case, assumption $B \rhd_q C \in X$, by Monotonicity axiom, implies that $X \vdash B \rhd_p C$. At the same time, by Reflexivity axiom, $B \subseteq A_p^+$ implies that $\vdash A_p^+ \rhd_p B$. Thus, $X \vdash A_p^+ \rhd_p C$ by Transitivity axiom. Again by Reflexivity axiom, we have $\vdash C \rhd_p c_0$. Hence, $X \vdash A_p^+ \rhd_p c_0$ by Transitivity axiom. Thus, $X \vdash A \rhd_p c_0$ by Lemma 30 and Transitivity axiom. Therefore, $c_0 \in A_p^+$, which is a contradiction with the choice of set A.
Case II: $p > q$. Then, $p - \varepsilon > q$ by the choice of ε. Hence,

$$s(\alpha(A_p^+, p)) \leq \|s\| \leq q < p - \varepsilon.$$

Therefore, $\beta(A_p^+, p) \in B_s^*$ by Lemma 33, which is a contradiction with the choice of set A. $\boxtimes$

Lemma 37. *For each $B, C \subseteq \mathcal{A}_0$ and each $q \in P$, if $N \vDash B \rhd_q C$, then $B \rhd_q C \in X$.*

Proof. Suppose that $B \rhd_q C \notin X$. Thus, by Lemma 25 and the maximality of set X, there is $c_0 \in C$ such that $X \nvdash B \rhd_q c_0$. Hence, $c_0 \notin B_q^+$ by Definition 14. Consider spending function s such that

$$s(a) = \begin{cases} q, & \text{if } a = \alpha(B_q^+, q), \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\|s\| = q$. Thus, $C \subseteq B_s^*$ by the assumption $N \vDash B \rhd_q C$ of the lemma. Hence, $c_0 \in B_s^*$. This together with $c_0 \notin B_q^+$ contradicts with Lemma 35 and $c_0 \in C \subseteq \mathcal{A}_0$. $\boxtimes$

Lemma 38. *$\psi \in X$ iff $N \vDash \psi$, for each $\psi \in \Phi(\mathcal{A}_0)$.*

Proof. We prove the lemma by induction on the structural complexity of formula ψ. The base case follows from Lemma 36 and Lemma 37. The induction step follows from Definition 13 and maximality and consistency of set X in the standard way. $\boxtimes$

To finish the proof of Theorem 4 note that $\neg\phi \in X$ due to the choice of the set X. Thus, $\phi \notin X$ due to consistency of set X. Therefore, $N \nvDash \phi$ by Lemma 38.

5. Conclusion

In this article we have suggested a way of adding marketing to the standard threshold model of diffusion in social networks. The model is general enough to simulate both promotional and preventive marketing. We have also defined formal logical systems for reasoning about influence relation in social networks with marketing of these two types. Both systems are based on Armstrong's axioms from the database theory. The main technical results of the article are the completeness theorems for these two systems.

As a possible extension of this work, marketing could be added to other logical systems for social networks mentioned in the introduction. Another possible extension of this work is an analysis of the computational complexity of both the proposed model and the logical system. However, this would probably require switching from the real numbers to the rational numbers since the real numbers, generally speaking, can not be given as an input to a Turing machine.

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