Lighthouse Principle for Diffusion in Social Networks

Sanaz Azimipour

University of Tehran, Tehran, Iran

Pavel Naumov

Vassar College, Poughkeepsie, New York, USA

Abstract

The article investigates an influence relation between two sets of agents in a social network. It proposes a logical system that captures propositional properties of this relation valid in all threshold models of social networks with the same structure. The logical system consists of Armstrong axioms for functional dependence and an additional Lighthouse axiom. The main results are soundness, completeness, and decidability theorems for this logical system.

1. Introduction

1.1. Social Networks

In this article we study influence in social networks. When a new product is introduced to the market, it is usually first adopted by a few users that are called "early adopters". These users might adopt the product because they are fans of the company introducing the product, as a result of the marketing campaign conducted by the company, or because they have a genuine need for this type of product. Once the early adopters start using the product, they put peer pressure on their friends and acquaintances in the social network, who might eventually follow them in adopting the product. The friends of the early adopters might eventually influence their own friends and so on, until the product is potentially adopted by a significant part of the network.

Email addresses: `Sanaz.a234@gmail.com` (Sanaz Azimipour), `pnaumov@vassar.edu` (Pavel Naumov)

A similar phenomenon could be observed with the diffusion of certain behaviours, like smoking, the adoption of new words and technical innovations, and the propagation of beliefs.

There are two most widely used models that formally capture diffusion process in social networks. One of them is the *stochastic* model [20, 12]. This model distinguishes active and inactive vertices of the network. Once a vertex v becomes active, it gets a single chance to activate each neighbour u with a given probability $p_{v,u}$. This process continues until no more activations can happen.

In this article we focus on the second model, called *threshold* model [26, 14, 11, 1], originally introduced by Granovetter [8] and Schelling [21]. In this model each agent has a non-negative threshold value representing the agent's resistance to adoption of a given product. If the pressure from those peers of the agent who already adopted the product reaches the threshold value, then the agent also adopts the product. We assume that each of the other agents has a non-negative, but possibly zero, influence on the given agent. The peer pressure on an agent to adopt a product is the sum of influences on the agent of all agents who have already adopted the product. It is assumed in this model that, once the product is adopted, the agent keeps using the product and putting pressure on her peers indefinitely.

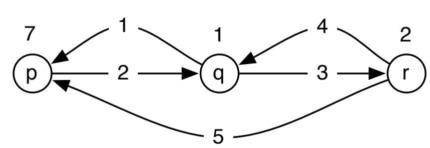

Figure 1: Social Network N_1

Consider, for example, social network N_1 depicted in Figure 1. This network consists of three agents: p, q, and r that have threshold values 7, 1, and 2 respectively. The threshold value of a node is shown on the diagram above the node representing the agent. The influence of one agent on another is shown in this figure by the label on the directed edge connecting the two agents. For instance, the influence of agent r on agent p is 5. If an agent has zero influence on another agent, no edge is shown. Thus, influence of agent p on agent r is zero.

Suppose that a marketing company gives agent p a free sample of the product and the agent starts using it. Since agent p has influence 2 on agent

q and threshold value of agent q is only 1, she will eventually also adopt the product. In turn, the adoption of the product by agent q will eventually lead to an adoption by agent r because threshold value of agent r is only 2 and the influence of agent q on agent r is 3. Thus, the adoption by agent p eventually leads to an adoption of this product by agent r. We denote this fact by $N_1 \vDash p \rhd r$.

In this article we study relation $A \rhd B$ between group of agents A and B that could be informally described[1] as "if all agents in set A use the product, then all agents in set B will eventually adopt the product". For example, for the above discussed social network N_1, we have $N_1 \vDash \{p\} \rhd \{q, r\}$, which we usually write as just $N_1 \vDash p \rhd q, r$.

At the same time, if a free sample of the product is given to agent r, then agent q will eventually adopt it because her threshold value is 1 and the influence of agent r on her is 4. Once agent q adopts the product, however, the product diffusion stops and the product will never be adopted by agent p because her threshold value is 7 and the total peer pressure from agents q and r on p will be only $1 + 5 = 6$. Therefore, for example, $N_1 \vDash \neg(r \rhd p)$.

The properties of relation $A \rhd B$ that we have discussed so far were specific to social network N_1. Let us now consider social network N_2 depicted in Figure 2. If a free sample of the product is given in network N_2 to agent r and

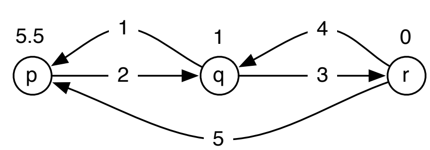

Figure 2: Social Network N_2

she starts using it, then, like it was for the network N_1, agent q will eventually adopt the product because her threshold value is only 1 and influence of agent r on agent q is 4. Unlike network N_1, however, the product diffusion does not stop at this point because now the total peer pressure of agents q and r on agent p is still $1 + 5 = 6$, but the threshold value of agent p in this network is only 5.5. Thus, agent p eventually will adopt the product. In other words, $N_2 \vDash r \rhd p$.

[1]We formally specify this relation in Definition 7.

An interesting property of network N_2 is that agent r has threshold value 0. Thus, she will eventually adopt the product even if no free product samples are given to any of the agents: $N_2 \vDash \varnothing \rhd r$.

1.2. Sociograms

So far, we have discussed properties of specific social networks. In this article we study properties common to a class of networks. The classes of networks can be defined on different levels of abstraction. Perhaps the most natural approach is to study common properties of social networks that have the same topological structure. In other words, to study properties that do not depend on a specific choice of influence and threshold values, but only on the (unlabeled) graph of the network. Although such an approach appears to be the most natural, it unexpectedly results in a very complicated principles that seems to capture more properties of real numbers than properties of the influence relation.

We adopt a different level of abstraction in which we assume that the graph and the distribution of influences is fixed. We study all properties that are universal no matter what the threshold values are. This level of abstraction results in a simple set of properties that can be captured by the complete logic system presented in this paper. In the conclusion we discuss examples of properties of influence that are true for all graphs without fixing distribution of influences *and* distribution of the thresholds. To distinguish graphs labeled with influences and thresholds from those labeled with influences only, we call the former *social networks* and the latter *sociograms*. To some degree, the threshold values characterize the relation that exists between the product and the individual agents and the sociogram describes the influence relation between the agents. The term sociogram has been first introduced by psychosociologist Jacob Levy Moreno [16]. The sociograms, as defined in this article, are directed labeled graphs. The original Moreno's sociograms were neither directed nor labeled.

For example, the above discussed social networks N_1 and N_2 are different only by the threshold values that the agents have. Thus, we say that social networks N_1 and N_2 have the same sociogram. This common sociogram S_1 for networks N_1 and N_2 is depicted in Figure 3.

We write $S \vDash \varphi$ if property φ is true for all social networks with sociogram S. For example, as we show in Proposition 1,

$$S_1 \vDash p \rhd r \to q \rhd r. \tag{1}$$

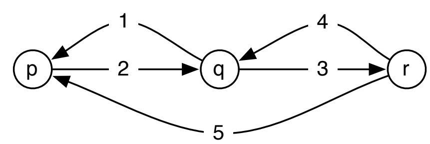

Figure 3: Sociogram S_1

In other words, under any assignment of threshold values on sociogram S_1, if giving a free sample of the product to agent p will eventually lead to agent r adopting the product, then giving a free sample of the product to agent q would have the same effect.

1.3. Lighthouse Axiom

The main result of this article is a complete axiomatization of the propositional properties of relation $A \rhd B$ for any given sociogram. Such an axiomatization consists of three axioms common to all sociograms and a sociogram-specific fourth axiom. The first three axioms are

1. Reflexivity: $A \rhd B$ if $B \subseteq A$,
2. Transitivity: $A \rhd B \to (B \rhd C \to A \rhd C)$,
3. Augmentation: $A \rhd B \to (A, C \rhd B, C)$,

where A, B denotes the union of sets A and B. These axioms were originally proposed by Armstrong [2] to describe functional dependence relation in database theory. They became known in database literature as Armstrong's axioms [7, p. 81]. Väänänen proposed a first order version of these principles [24] and their generalization for reasoning about approximate dependency [25]. Beeri, Fagin, and Howard [4] suggested a variation of Armstrong's axioms that describes properties of multi-valued dependence. Naumov and Nicholls [17] proposed another variation of these axioms that describes rationally functional dependence. The influence semantics of these axioms that we introduce in this article does not appear to be connected to the functional dependency semantics.

The sociogram-dependent fourth axiom captures the fact that in every group of agents in which at least one agent eventually adopts the product there is always an agent (or a nonempty subgroup of agents) who adopts the product first. In marketing such agents are sometimes called *lighthouse customers*. In any given group of agents, the distinctive property of lighthouse customers is that they adopt the product without any peer pressure coming from other agents in this group. The lighthouse customers adopt the product as a result of the peer pressure from the outside of the group. Our fourth axiom postulates the existence of lighthouse customers in any group of agents in which at least one agent eventually will adopt the product. Thus, we call this postulate *Lighthouse axiom*.

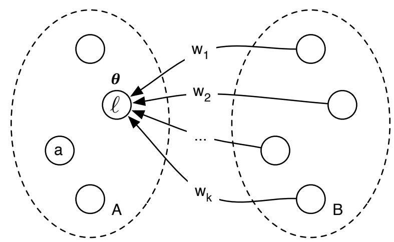

Figure 4: Lighthouse Axiom

One possible way to state Lighthouse axiom is to say that if all agents in network N are partitioned into disjoint sets A and B, see Figure 4, and there is an agent $a \in A$ such that $N \vDash B \rhd a$, then there must exist a "lighthouse" agent $\ell \in A$ such that the total peer pressure of all agents in set B on agent ℓ is no less than the threshold value of agent ℓ:

$$\theta \leq w_1 + w_2 + \cdots + w_k.$$

Unfortunately, when stated this way, Lighthouse axiom refers to threshold value θ of agent ℓ. Thus, in this form, it is a property of the social network, rather than the corresponding sociogram.

It turns out, however, that there is a way to re-word the axiom so that it does not refer to threshold values. Namely, let us assume that for every agent $a \in A$ we choose a set of agents $C_a \subseteq A \cup B$ such that peer pressure of set C_a on agent a is no less than peer pressure of set B on agent a. The new form of Lighthouse axiom states that, under the above condition, if $N \vDash B \rhd a$, then there exists a "lighthouse" agent $\ell \in A$ such that $N \vDash C_\ell \rhd \ell$. The main result of this article is the completeness theorem for the logical system consisting of this form of Lighthouse axiom and the three Armstrong axioms.

1.4. Related Literature

Several logical frameworks for reasoning about diffusion in social networks have been studied before. Seligman, Liu, and Girard [22] proposed Facebook Logic for capturing properties of epistemic social networks in modal language, but did not give any axiomatization for this logic. They further developed this approach in papers [23, 13] where they introduced dynamic friendship relations. Christoff and Hansen [5] simplified Seligman, Liu, and Girard setting and gave a complete axiomatization of the logical system for this new setting. Christoff and Rendsvig proposed Minimal Threshold Influence Logic [6] that uses modal language to capture dynamic of diffusion in a threshold model and gave a complete axiomatization of this logic. Baltag, Christoff, Rendsvig, and Smets [3] discussed logics for informed update and prediction update. Informally, the languages of the described above systems feel significantly richer than the more succinct language of our system. However, neither of these systems capture principles similar to our Lighthouse axiom. Naumov and Tao [19, 18] used Armstrong's axioms to describe influence in social networks. They considered relation $A \rhd_b B$ that stands for "given marketing budget b, group of agents A can influence group of agents B". They gave modified versions of Armstrong axioms that capture properties of this relation for preventive and promotional marketing. Since they do not assume a fixed sociogram of the network, their approach does not capture any properties similar to our Lighthouse principle.

Diffusion in social networks is a special case of information flow on graphs. Logical systems for reasoning about various types of graph information flow has been studied before. Lighthouse axiom has certain resemblance with Gateway axiom for functional dependence on hypergraphs of secrets [15], Contiguity axiom [9] for graphical games, and Shield Wall axiom for fault tolerance in belief formation networks [10].

1.5. Outline

This article is organized as following. In Section 2 we introduce formal syntax and semantics of our logical system. Section 3 list the four axioms of the system. In Section 4, we give several examples of formal proofs in our system. In Section 5 we show some auxiliary results that are used later. Section 6 and Section 7 prove soundness and completeness theorems respectively. Section 9 concludes with a discussion of logical properties of unlabeled sociograms.

2. Syntax and Semantics

In this section we formally define a social network, a sociogram, and the influence relation.

Definition 1. *For any finite set $\mathcal{A}$, let $\Phi(\mathcal{A})$ be the minimal set of formulas such that*

1. $\bot \in \Phi(\mathcal{A})$,
2. $A \rhd B \in \Phi(\mathcal{A})$, *for each subsets* $A, B \subseteq \mathcal{A}$,
3. $\varphi \to \psi \in \Phi(\mathcal{A})$ *for each* $\varphi, \psi \in \Phi(\mathcal{A})$.

We assume that disjunction $\vee$ is defined through implication $\to$ and false constant $\bot$ in the standard way.

Definition 2. *A sociogram is pair* $(\mathcal{A}, w)$, *where*

1. $\mathcal{A}$ *is an arbitrary finite set (of agents),*
2. w *is a function that maps* $\mathcal{A}^2$ *into non-negative real numbers. Value* $w(a, b)$ *represents influence of agent* a *on agent* b.

Definition 3. *A social network is triple* $(\mathcal{A}, w, \theta)$, *where*

1. $(\mathcal{A}, w)$ *is a sociogram,*
2. θ *is a function that maps* $\mathcal{A}$ *into non-negative real numbers. Value* $\theta(a)$ *represents threshold value of agent* $a \in \mathcal{A}$.

We say that social network $(\mathcal{A}, w, \theta)$ is based on sociogram $(\mathcal{A}, w)$. We now proceed to define peer pressure on an agent by a group of agents in a given sociogram.

Definition 4. *For any sociogram* $(\mathcal{A}, w)$ *and any subset of agents* $A \subseteq \mathcal{A}$, *let* $\|A\|_b = \sum_{a \in A} w(a, b)$.

In the introduction we said that if, at some moment in time, an agent experiences peer pressure higher than her threshold value, then at some point in the future she will adopt the product. For the sake of simplicity, in our formal model we assume that time is discrete and that if at moment k an agent experiences sufficient peer pressure, then she adopts the product at moment $k + 1$. Although this assumption, generally speaking, affects the "time dynamics" of product diffusion, it does not affect the final outcome of diffusion. Thus, this assumption, while simplifying the formal setting, does not change the properties of influence relation $A \rhd B$. Given this assumption, if free samples of the product are given to all agents in set A at moment 0, then by A^k we mean the set of all agents who will adopt the product by moment k. The formal definition of A^k is below.

Definition 5. *For any $A \subseteq \mathcal{A}$ and any $k \in \mathbb{N}$, let subset $A^k \subseteq \mathcal{A}$ be defined recursively as follows:*

1. $A^0 = A$,
2. $A^{k+1} = A^k \cup \{x \in \mathcal{A} \mid \|A^k\|_x \geq \theta(x)\}$.

Corollary 1. $(A^n)^k = A^{n+k}$.

If free samples of the product are given to all agents in set A, then by A^* we mean the set of all agents who will eventually adopt the product. The formal definition of A^* is below.

Definition 6.

$$A^* = \bigcup_{k \geq 0} A^k.$$

The next definition specifies the formal semantics of our logical system. In particular, item 2 in this definition specifies the formal meaning of the influence relation.

Definition 7. *For any social network $N = (\mathcal{A}, w, \theta)$ and any $\varphi \in \Phi(\mathcal{A})$, let satisfiability relation $N \vDash \varphi$ be defined as follows*

1. $N \nvDash \bot$,
2. $N \vDash A \rhd B$ *if* $B \subseteq A^*$,
3. $N \vDash \psi \to \chi$ *if* $N \nvDash \psi$ *or* $N \vDash \chi$.

3. Axioms

Our logical system for an arbitrary sociogram $S = (\mathcal{A}, w)$ consists of propositional tautologies in language $\Phi(\mathcal{A})$ and the following additional axioms:

1. Reflexivity: $A \rhd B$ if $B \subseteq A$,

2. Transitivity: $A \rhd B \to (B \rhd C \to A \rhd C)$,
3. Augmentation: $A \rhd B \to (A, C \rhd B, C)$,
4. Lighthouse: if $A \sqcup B$ is a partition of the set of all agents $\mathcal{A}$ and $\{C_a\}_{a \in A}$ is a family of sets of agents such that $\|B\|_a \leq \|C_a\|_a$ for each $a \in A$, then

$$\bigvee_{a \in A} B \rhd a \to \bigvee_{a \in A} C_a \rhd a.$$

We write $\vdash_S \varphi$ if formula φ can be derived in our system using Modus Ponens inference rule. We sometimes write just $\vdash \varphi$ if the value of subscript S is clear from the context. We also write $X \vdash_S \varphi$ if formula φ could be derived in our system extended by a set of additional axioms X.

4. Examples

In this section we give three examples of formal proofs in our logical system to illustrate how the system works. Soundness of the system is shown in Section 6. We start by proving statement (1) from the introduction.

Proposition 1. *$\vdash_{S_1} p \rhd r \to q \rhd r$, where S_1 is the sociogram depicted in Figure 3.*

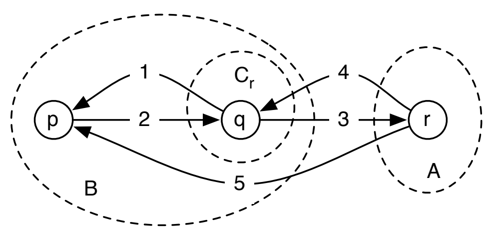

Figure 5: Towards Proof of Proposition 1

Proof. Let $A = \{r\}$, $B = \{p, q\}$, and $C_r = \{q\}$, see Figure 5. Note that

$$\|B\|_r = w(p, r) + w(q, r) = 0 + 3 = 3 = w(q, r) = \|C_r\|_r.$$

Hence, by Lighthouse axiom,

$$\vdash p, q \rhd r \to q \rhd r. \tag{2}$$

At the same time, by Transitivity axiom,

$$\vdash p, q \rhd p \to (p \rhd r \to p, q \rhd r).$$

By Reflexivity axiom, $\vdash p, q \rhd p$. Thus, by Modus Ponens inference rule,

$$\vdash p \rhd r \to p, q \rhd r.$$

Therefore, $\vdash p \rhd r \to q \rhd r$ using statement (2) and propositional logic reasoning. ⊠

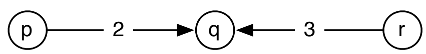

Figure 6: Sociogram S_2

Let us now consider sociogram S_2 depicted in Figure 6. Since in this sociogram agent r has higher influence on agent q than agent p, one might expect the following statement to be true for all social networks over sociogram S_2:

$$p \rhd q \to r \rhd q. \tag{3}$$

Surprisingly, this is false. Namely, this statement is false for the social network depicted in Figure 7. This happens because agent r in this social

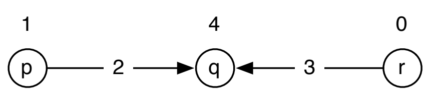

Figure 7: Social Network

network has threshold value 0. In other words, agent r is an "early adopter" who does not need any external peer pressure in order to buy the product. As a result, see Figure 8, we have $\{p\}^1 = \{p, r\}$. Once agent r adopts the product, the total peer pressure on agent q becomes $2 + 3 = 5$ and she will adopt the product as well. On the other hand, if the free sample is given to agent r, then neither agent p nor agent q ever adopt the product.

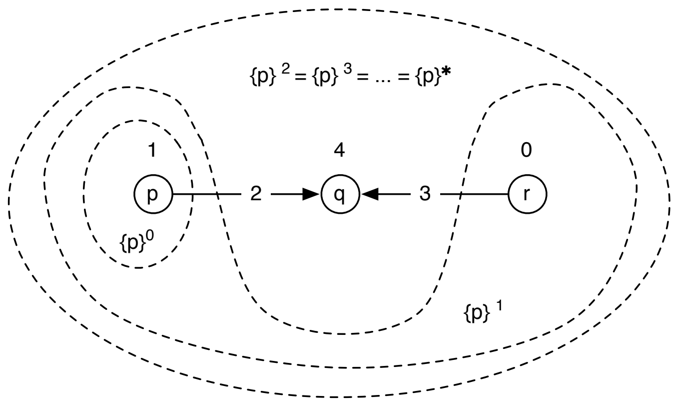

Figure 8: Social Network

Although statement (3) does not hold for some social networks over sociogram S_2, in the next proposition we show that a slightly modified version of this statement does hold for all such networks.

Proposition 2. $\vdash_{S_2} p \rhd q \to (r \rhd q \vee \varnothing \rhd r)$*, where* S_2 *is the sociogram depicted in Figure 6.*

Proof. Let $A = \{q, r\}$, $B = \{p\}$, $C_q = \{r\}$, and $C_r = \varnothing$, see Figure 9. Note

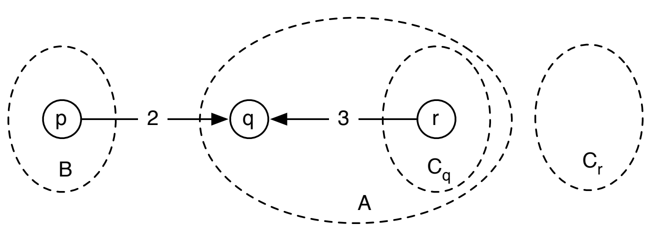

Figure 9: Towards Proof of Proposition 2

that

$$\|B\|_q = w(p, q) = 2 < 3 = w(r, q) = \|C_q\|_q$$

and

$$\|B\|_r = w(p, r) = 0 = \|\varnothing\|_r = \|C_r\|_r.$$

Thus, by Lighthouse axiom,

$$\vdash p \rhd q \vee p \rhd r \to r \rhd q \vee \varnothing \rhd r.$$

Therefore, $\vdash p \rhd r \to r \rhd q \vee \varnothing \rhd r$. ⊠

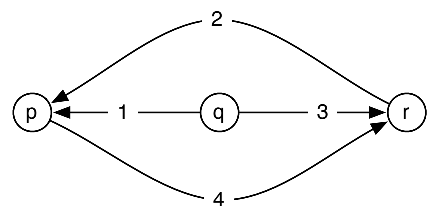

Figure 10: Sociogram S_3

Proposition 3. *$\vdash_{S_3} q \rhd p \vee q \rhd r \to p \rhd r \vee r \rhd p$, where S_3 is the sociogram depicted in Figure 10.*

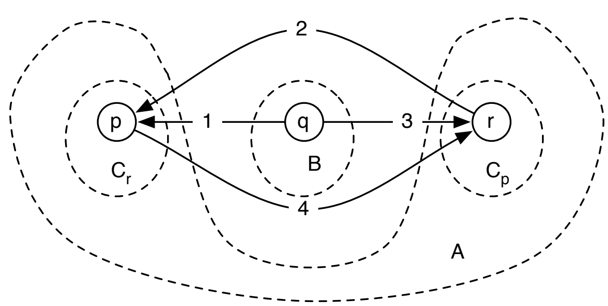

Figure 11: Towards Proof of Proposition 3

Proof. Let $A = \{p, r\}$, $B = \{q\}$, $C_p = \{r\}$, and $C_r = \{p\}$, see Figure 11. Note that

$$\|B\|_p = w(q, p) = 1 < 2 = w(r, p) = \|C_p\|_p$$

and

$$\|B\|_r = w(q, r) = 3 < 4 = w(p, r) = \|C_r\|_r.$$

Therefore, by Lighthouse axiom, $\vdash q \rhd p \vee q \rhd r \to p \rhd r \vee r \rhd p$. ⊠

5. Properties of Star Closure

In this section we prove several technical properties of A^* that are used later in the proofs of soundness and completeness.

Lemma 1. *If $A^1 = A$, then $A^k = A$ for each $k \geq 0$.*

Proof. We prove this lemma by induction on k. If $k = 0$, then $A^0 = A$ by Definition 5. If $k > 0$, then by Corollary 1, assumption $A^1 = A$, and the induction hypothesis, $A^k = (A^1)^{k-1} = A^{k-1} = A$. ⊠

Lemma 2. *$A^* = A^k$ for some $k \geq 0$.*

Proof. The statement of the lemma follows from the assumption in Definition 3 that set $\mathcal{A}$ is finite. ⊠

Lemma 3. *If $x \notin A^*$, then $\theta(x) > \|A^*\|_x$, for each subset $A \subseteq \mathcal{A}$ and each agent $x \in \mathcal{A}$.*

Proof. By Lemma 2, there is $k \geq 0$ such that $A^* = A^k$. Suppose that $\|A^*\|_x \geq \theta(x)$. Thus, $\|A^k\|_x \geq \theta(x)$. Hence, $x \in A^{k+1}$, by Definition 5. Thus, $x \in A^*$ by Definition 6, which is a contradiction to the assumption of the lemma. ⊠

Lemma 4. *$A \subseteq A^*$.*

Proof. By Definition 5 and Definition 6, $A = A^0 \subseteq \bigcup_{k \geq 0} A^k = A^*$. ⊠

Lemma 5. *$(A^*)^* \subseteq A^*$.*

Proof. By Lemma 2, there are $n, k \geq 0$ such that $A^* = A^n$ and $(A^*)^* = (A^*)^k$. Thus, by Corollary 1 and Definition 6,

$$(A^*)^* = (A^*)^k = (A^n)^k = A^{n+k} \subseteq \bigcup_{m \geq 0} A^m = A^*.$$

⊠

Lemma 6. *If $A \subseteq B$, then $A^k \subseteq B^k$, for each $k \geq 0$.*

Proof. We prove the statement of the lemma by induction on k. If $k = 0$, then $A^0 = A \subseteq B = B^0$ by Definition 5.

Suppose that $A^k \subseteq B^k$. Let $x \in A^{k+1}$. It suffices to show that $x \in B^{k+1}$. Indeed, by Definition 5, assumption $x \in A^{k+1}$ implies that either $x \in A^k$ or $\|A^k\|_x \geq \theta(x)$. In the first case, by the induction hypothesis, $x \in A^k \subseteq B^k$. Thus, $x \in B^k$. Therefore, $x \in B^{k+1}$ by Definition 5.

In the second case, by Definition 4 and assumption $A^k \subseteq B^k$,

$$\|B^k\|_x = \sum_{b \in B^k} w(b,x) \geq \sum_{a \in A^k} w(a,x) = \|A^k\|_x \geq \theta(x).$$

Therefore, $x \in B^{k+1}$ by Definition 5. ⊠

Corollary 2. *If $A \subseteq B$, then $A^* \subseteq B^*$.*

Lemma 7. *$A^* \cup B^* \subseteq (A \cup B)^*$.*

Proof. Note that $A \subseteq A \cup B$ and $B \subseteq A \cup B$. Thus, $A^* \subseteq (A \cup B)^*$ and $B^* \subseteq (A \cup B)^*$ by Corollary 2. Therefore, $A^* \cup B^* \subseteq (A \cup B)^*$. ⊠

6. Soundness

In this section we prove the soundness of our logical system with respect to the semantics given in Definition 7. The soundness of propositional tautologies and Modus Ponens inference rule is straightforward. Below we show the soundness of each of the remaining four axioms as separate lemmas. In the lemmas that follow we assume that $S = (\mathcal{A}, w, \theta)$ is a social network and A, B, and C are subsets of $\mathcal{A}$.

Lemma 8. *If $B \subseteq A$, then $S \vDash A \rhd B$.*

Proof. By Lemma 4, $A \subseteq A^*$. Thus, $B \subseteq A^*$ by the assumption of the lemma. Therefore, $S \vDash A \rhd B$, by Definition 7. ⊠

Lemma 9. *If $S \vDash A \rhd B$ and $S \vDash B \rhd C$, then $S \vDash A \rhd C$.*

Proof. By Definition 7, assumption $S \vDash A \rhd B$ implies that $B \subseteq A^*$. Hence, $B^* \subseteq (A^*)^*$ by Corollary 2. Thus, $B^* \subseteq A^*$ by Lemma 5. At the same time, $C \subseteq B^*$ by assumption $S \vDash B \rhd C$ and Definition 7. Thus, $C \subseteq A^*$. Therefore, $S \vDash A \rhd C$ by Definition 7. $\boxtimes$

Lemma 10. *If $S \vDash A \rhd B$, then $S \vDash A, C \rhd B, C$.*

Proof. Suppose that $S \vDash A \rhd B$. Thus, $B \subseteq A^*$ by Definition 7. Note that $C \subseteq C^*$ by Lemma 4. Thus, $B \cup C \subseteq A^* \cup C^* \subseteq (A \cup C)^*$, by Lemma 7. Therefore, $S \vDash A, C \rhd B, C$, by Definition 7. $\boxtimes$

Lemma 11. *If $S \vDash B \rhd a_0$ for some $a_0 \in A$, then there is $\ell \in A$ such that $S \vDash C_\ell \rhd \ell$, where $A \sqcup B$ is a partition of the set of all agents $\mathcal{A}$ and $\{C_a\}_{a \in A}$ is a family of sets of agents such that $\|B\|_a \le \|C_a\|_a$ for each $a \in A$.*

Proof. Note that assumption $S \vDash B \rhd a_0$ by Definition 7 implies that $a_0 \in B^*$. On the other hand, assumption $a_0 \in A$ implies that $a_0 \notin B$ because $A \sqcup B$ is a partition of set $\mathcal{A}$. Thus, $B^* \neq B$. Hence, by Definition 6, there must exist k such that $B^k \neq B$. Then, $B^1 \neq B$ by Lemma 1. Thus, there must exist $\ell \in B^1 \setminus B$. Hence, $\|B\|_\ell \ge \theta(\ell)$ by Definition 5. Then, by the assumption of the lemma, $\|C_\ell\|_\ell \ge \|B\|_\ell \ge \theta(\ell)$. Thus, $\ell \in C^1_\ell$, by Definition 5. Hence, $\ell \in C^*_\ell$ by Definition 6. Therefore, $S \vDash C_\ell \rhd \ell$ by Definition 7. Finally, note that $\ell \in A$ because $\ell \in B^1 \setminus B$ and $A \sqcup B$ is a partition of the set $\mathcal{A}$. $\boxtimes$
This concludes the proof of the soundness of our logical system.

7. Completeness

In this section we prove the completeness of our logical system with respect to the semantics given in Definition 7. This result is formally stated as Theorem 1 in the end of this section. The proof of completeness consists in the construction of a "canonical" social network. We start, however, we a few technical lemmas and definitions.

7.1. Preliminaries

Let us first prove a useful property of real numbers.

Lemma 12. *If $\varepsilon > 0$ is a real number and x and y are any real numbers such that either $x = y$ or $|x - y| > \varepsilon$. Then, $x + \varepsilon > y$ implies $x \ge y$.*

Proof. Suppose $y > x$. Hence, $x \neq y$. Thus, $|x - y| > \varepsilon$, by the assumption of the lemma. Then, $y - x > \varepsilon$, because $y > x$. Therefore, $x + \varepsilon < y$. ☒

We now assume a fixed sociogram $(\mathcal{A}, w)$ and a fixed maximal consistent subset X of $\Phi(\mathcal{A})$.

Definition 8. $\widehat{A} = \{a \in \mathcal{A} \mid X \vdash A \rhd a\}$ *for each subset* $A \subseteq \mathcal{A}$.

Choose ε to be any positive real number such that $\varepsilon < \|A\|_a - \|B\|_a$ for each agent $a \in \mathcal{A}$ and each subsets $A, B \subseteq \mathcal{A}$, such that $\|A\|_a > \|B\|_a$. This could be achieved because set $\mathcal{A}$ is finite.

Lemma 13. *For any subsets* $A, B \subseteq \mathcal{A}$ *and any agent* $a \in \mathcal{A}$ *if* $\|A\|_a + \varepsilon > \|B\|_a$, *then* $\|A\|_a \geq \|B\|_a$.

Proof. By the choice of ε, we have either $\|A\|_a = \|B\|_a$ or $|(\|A\|_a - \|B\|_a)| > \varepsilon$. Thus, $\|A\|_a \geq \|B\|_a$ by Lemma 12. ☒

Lemma 14. $A \subseteq \widehat{A}$ *for each subset* $A \subseteq \mathcal{A}$.

Proof. Suppose that $a \in A$. Thus, $\vdash A \rhd a$ by Reflexivity axiom. Therefore, $a \in \widehat{A}$ by Definition 8. ☒

Lemma 15. $X \vdash A \rhd \widehat{A}$, *for each subset* $A \subseteq \mathcal{A}$.

Proof. Let $\widehat{A} = \{a_1, \ldots, a_n\}$. By the definition of $\widehat{A}$, $X \vdash A \rhd a_i$, for any $i \leq n$. We prove, by induction on k, that $X \vdash A \rhd a_1, \ldots, a_k$ for each $0 \leq k \leq n$.
Base Case: $X \vdash A \rhd \varnothing$ by Reflexivity axiom.
Induction Step: Assume that $X \vdash A \rhd a_1, \ldots, a_k$. By Augmentation axiom,

$$X \vdash A, a_{k+1} \rhd a_1, \ldots, a_k, a_{k+1}. \tag{4}$$

Recall that $X \vdash A \rhd a_{k+1}$. Again by Augmentation axiom, $X \vdash A \rhd A, a_{k+1}$. Hence, $X \vdash A \rhd a_1, \ldots, a_k, a_{k+1}$, by (4) and Transitivity axiom. ☒

7.2. Canonical Social Network

Next, based on the sociogram $(\mathcal{A}, w)$ and the maximal consistent set X, we define the "canonical" social network $N_X = (\mathcal{A}, w, \theta)$. We then proceed to prove the core properties of this network.

Definition 9.

$$\theta(a) = \begin{cases} 0, & \textit{if } X \vdash \varnothing \rhd a, \\ \max_{a \notin \widehat{B}} \|\widehat{B}\|_a + \varepsilon, & \textit{otherwise.} \end{cases}$$

The maximum in the above definition is taken over all subsets B of $\mathcal{A}$ such that $\widehat{B}$ does not contain agent a.

Lemma 16. *Function $\theta(a)$ is well-defined for each $a \in \mathcal{A}$.*

Proof. We need to show that if $X \nvdash \varnothing \rhd a$, then there is at least one subset $B \subseteq \mathcal{A}$ such that $a \notin \widehat{B}$. It suffices to show that $a \notin \widehat{\varnothing}$, which is true due to assumption $X \nvdash \varnothing \rhd a$ and Definition 8. ⊠

Lemma 17. *For any subset $B \subseteq \mathcal{A}$, if $a \in \mathcal{A} \setminus B^*$, then there is $C \subseteq \mathcal{A}$ such that $a \notin \widehat{C}$ and $\theta(a) = \|\widehat{C}\|_a + \varepsilon$.*

Proof. If $\theta(a) = 0$, then $a \in B^1$ due to Definition 5. Thus, $a \in B^*$ by Definition 6, which is a contradiction to the assumption $a \in \mathcal{A} \setminus B^*$. Suppose now that $\theta(a) > 0$, thus, by Definition 9, there is at least one $C \subseteq \mathcal{A}$ such that $a \notin \widehat{C}$ and $\theta(a) = \|\widehat{C}\|_a + \varepsilon$. ⊠

Lemma 18. *If $B \subseteq \mathcal{A}$ and $a \in \mathcal{A} \setminus \widehat{B}$, then $\theta(a) > \|\widehat{B}\|_a$.*

Proof. *Case I: $X \vdash \varnothing \rhd a$.* Note that $X \vdash B \rhd \varnothing$ by Reflexivity axiom. Thus, $X \vdash B \rhd a$ by Transitivity axiom. Hence, $a \in \widehat{B}$ by Definition 8, which is a contradiction to the assumption of the lemma.

Case II: $X \nvdash \varnothing \rhd a$. Thus, $\theta(a) > \|\widehat{B}\|_a$ by Definition 9. ⊠

Lemma 19. $(\widehat{B})^k = \widehat{B}$ *for each $B \subseteq \mathcal{A}$ and each $k \geq 0$.*

Proof. We prove this statement by induction on k. If $k = 0$, then $(\widehat{B})^k = \widehat{B}$, by Definition 5. Note next that by Definition 5, the induction hypothesis, and Lemma 18,

$$\begin{aligned}(\widehat{B})^{k+1} &= (\widehat{B})^k \cup \{a \in \mathcal{A} \mid \|(\widehat{B})^k\|_a \geq \theta(a)\} \\ &= \widehat{B} \cup \{a \in \mathcal{A} \mid \|\widehat{B}\|_a \geq \theta(a)\} \\ &= \widehat{B} \cup \{a \in \mathcal{A} \setminus \widehat{B} \mid \|\widehat{B}\|_a \geq \theta(a)\} = \widehat{B} \cup \varnothing = \widehat{B}.\end{aligned}$$

⊠

Lemma 20. $(\widehat{B})^* = \widehat{B}$ *for each* $B \subseteq \mathcal{A}$.

Proof. By Definition 6 and Lemma 19, $(\widehat{B})^* = \bigcup_{k\geq 0}(\widehat{B})^k = \bigcup_{k\geq 0} \widehat{B} = \widehat{B}$. ⊠

Lemma 21. *For each* $B \subseteq \mathcal{A}$*, if* $a \in B^*$*, then* $X \vdash B \rhd a$.

Proof. Suppose $a \in B^*$. By Lemma 14, $B \subseteq \widehat{B}$. Then, $B^* \subseteq (\widehat{B})^*$ by Corollary 2. Thus, $a \in (\widehat{B})^*$. Hence, $a \in \widehat{B}$ by Lemma 20. Therefore, $X \vdash B \rhd a$ by Definition 8. ⊠

Lemma 22. *For each* $B \subseteq \mathcal{A}$ *and each* $a \in \mathcal{A}$*, if* $X \vdash B \rhd a$*, then* $a \in B^*$.

Proof. By Lemma 3, $\theta(x) > \|B^*\|_x$ for each $x \in \mathcal{A} \setminus B^*$. At the same time, by Lemma 17, for each $x \in \mathcal{A} \setminus B^*$ there is C_x such that $x \notin \widehat{C_x}$ and $\theta(x) = \|\widehat{C_x}\|_x + \varepsilon$. Hence, $\|\widehat{C_x}\|_x + \varepsilon > \|B^*\|_x$ for each $x \in \mathcal{A} \setminus B^*$. Thus, by Lemma 13, $\|\widehat{C_x}\|_x \geq \|B^*\|_x$ for each $x \in \mathcal{A} \setminus B^*$.

Consider partition $(\mathcal{A} \setminus B^*) \sqcup B^*$ of $\mathcal{A}$. By Lighthouse axiom,

$$\vdash \bigvee_{x\in\mathcal{A}\setminus B^*} B^* \rhd x \to \bigvee_{x\in\mathcal{A}\setminus B^*} \widehat{C_x} \rhd x. \tag{5}$$

Suppose that $a \notin B^*$, Lemma 4 and Reflexivity axiom imply that $\vdash B^* \rhd B$. Thus, by assumption $X \vdash B \rhd a$ and Transitivity axiom, $X \vdash B^* \rhd a$. Hence, statement (5) implies that

$$X \vdash \bigvee_{x\in\mathcal{A}\setminus B^*} \widehat{C_x} \rhd x.$$

Then, due to the maximality of set X, there must exist $x_0 \in \mathcal{A} \setminus B^*$ such that $X \vdash \widehat{C_{x_0}} \rhd x_0$. Thus, $X \vdash C_{x_0} \rhd x_0$, due to Lemma 15 and Transitivity axiom: $\vdash C_{x_0} \rhd \widehat{C_{x_0}} \to (\widehat{C_{x_0}} \rhd x_0 \to C_{x_0} \rhd x_0)$. Hence, $x_0 \in \widehat{C_{x_0}}$ by Definition 8, which is a contradiction with the choice of set C_x. ⊠

Lemma 23. *$N_X \vDash \varphi$ if and only if $\varphi \in X$, for each formula $\varphi \in \Phi(\mathcal{A})$.*

Proof. We prove this lemma by induction on structural complexity of formula φ. Cases when formula φ is $\bot$ or has form $\psi_1 \to \psi_2$ follow in the standard way from Definition 7 and the assumptions of maximality and consistency of set X. Suppose that φ has form $A \rhd B$.

($\Rightarrow$) : Suppose that $N_X \vDash A \rhd B$. Then $B \subseteq A^*$ by Definition 7. Hence, $b \in A^*$ for each $b \in B$. Thus, $X \vdash A \rhd b$ for each $b \in B$ by Lemma 21. Hence, $b \in \widehat{A}$ for each $b \in B$ by Definition 8. In other words, $B \subseteq \widehat{A}$. Thus, by Reflexivity axiom, $\vdash \widehat{A} \rhd B$. On the other hand, $X \vdash A \rhd \widehat{A}$ by Lemma 15. Therefore, $X \vdash A \rhd B$ by Transitivity axiom.

($\Leftarrow$) : Assume $X \vdash A \rhd B$. By Reflexivity axiom, $\vdash B \rhd b$ for every $b \in B$. Hence, $X \vdash A \rhd b$ for each $b \in B$ by Transitivity axiom. Thus, $b \in A^*$ for each $b \in B$, by Lemma 22. In other words, $B \subseteq A^*$. Therefore, $N_X \vDash A \rhd B$ by Definition 7. ⊠

7.3. Main Result

We are now ready to state and prove the completeness theorem for our logical system with respect to the semantics given in Definition 7.

Theorem 1. *For any sociogram $(\mathcal{A}, w)$ and any formula $\varphi \in \Phi(\mathcal{A})$, if $N \vDash \varphi$ for each social network N based on sociogram $(\mathcal{A}, w)$, then $\vdash \varphi$.*

Proof. Suppose that $\nvdash \varphi$. Let X be a maximal consistent subset of $\Phi(\mathcal{A})$ such that $\varphi \notin X$. By Lemma 23, $N_X \nvDash \varphi$. ⊠

8. Decidability

In this section we discuss decidability of our logical system for any fixed sociogram $(\mathcal{A}, w)$. Note that we allow arbitrary real numbers as subscripts in formula $A \rhd_c B$. Thus, the set of all formulas $\Phi(\mathcal{A})$ is uncountable and its elements can not be used as inputs of a Turing machine. In order to avoid this issue, in this section we modify Definition 1, Definition 3, and Definition 2 by assuming that only rational numbers could be used as subscripts in our atomic formulas $A \rhd_c B$, as influence values, and as threshold values. It is easy to see that the above proof of completeness is still valid. From this change point of view, the only non-trivial place is the choice of ε for the given sociogram $(\mathcal{A}, w)$ that we have made right after Definition 8. Note, however, that the required ε could always be choose to be a rational number because 0 is a limit point of the set of positive rational numbers.

Theorem 2. *For any given sociogram $S = (\mathcal{A}, w)$, set $\{\varphi \in \Phi(\mathcal{A}) \mid \vdash_S \varphi\}$ is decidable.*

Proof. According to Theorem 1, $\vdash_S \varphi$ if and only if formula φ is true for each social network $(\mathcal{A}, w, \theta)$ based on sociogram S. This, of course, does not imply the decidability because there are infinitely many social networks based on sociogram S. However, it turns out that the proof of Theorem 1 that we gave above actually shows a stronger result: $\vdash_S \varphi$ if and only if formula φ is true for each social network from a specific finite class $C(S)$ of networks based on sociogram S.

Once existence of such *finite* class of social networks $C(S)$ is establish, we should be able to claim the decidability result because one can always verify if a formula φ is true for each out of *finitely* many given networks.

We are now ready to describe the finite class of social networks $C(S)$. The social network over sociogram S is completely defined by specifying threshold function θ. In the proof of Theorem 1, this is done in Definition 9. This definition depends on ε and maximal consistent set of formulas X. Note however that the choice of ε does not depend on X and could be made based on sociogram S alone. Once ε is fixed, the set of all values of function θ, as specified in Definition 9, belongs to *finite* set

$$\{0\} \cup \{\|A\|_a + \varepsilon \mid a \in \mathcal{A}, A \subseteq \mathcal{A}\}.$$

The set of all social networks over sociogram S whose threshold functions use only values from the above set is the desired finite class of social networks $C(S)$. ⊠

9. Conclusion

In this article we have studied properties of influence common to all social networks with the same weighted sociogram. We introduced a logical system for reasoning about these properties and proved soundness and completeness of this system. We have established that the logical system is decidable if its syntax and semantics are restricted to rational numbers.

As has been mentioned in that introduction, perhaps more natural question to consider is axiomatization of all common influence properties of social networks with the same graph, without fixing distribution of either weights or thresholds. Surprisingly, such setting yields a much more complicated set of properties. We discuss some of these properties below.

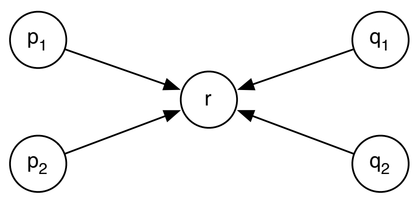

Figure 12: Unweighted Sociogram U_1

Consider, for example, unweighted sociogram U_1 depicted in Figure 12. Let $N = (\mathcal{A}, w, \theta)$ be a social network based on U_1. Furthermore, assume that in social network N (i) neither of the agents p_1, p_2, q_1, q_2 is an early adopter, (ii) $N \vDash p_1, p_2 \rhd r$, and (iii) $N \vDash q_1, q_2 \rhd r$. Thus, $w(p_1, r) + w(p_2, r) \geq \theta(r)$ and $w(q_1, r) + w(q_2, r) \geq \theta(r)$. The first inequality implies that at least one out of $w(p_1, r)$ and $w(p_2, r)$ is greater or equal than $\theta(r)/2$. In other words, there is $i \in \{1, 2\}$ such that $w(p_i, r) \geq \theta/2$. Similarly, the second inequality implies that there is $j \in \{1, 2\}$ such that $w(q_j, r) \geq \theta/2$. Thus,

$$\|\{p_i, q_j\}\|_r = w(p_i, r) + w(q_j,) \geq \theta/2 + \theta/2 = \theta.$$

Hence, $r \in \{p_i, q_j\}^1 \subseteq \{p_i, q_j\}^*$. Then, $N \vDash p_i, q_j \rhd r$. So, we have shown that for any social network N based on unweighted sociogram U_1 and satisfying the conditions (i), (ii), (iii), there are $i, j \in \{1, 2\}$ such that $N \vDash p_i, q_j \rhd r$.

This could be formally stated as

$$U_2 \vDash \quad p_1, p_2 \rhd r \wedge q_1, q_2 \rhd r \to \bigvee_{i=1}^{2} \bigvee_{j=1}^{2} p_i, q_j \rhd r \vee \bigvee_{x \in \{p_1, p_2, q_1, q_2\}} \varnothing \rhd x,$$

where disjunction $\bigvee_{x \in \{p_1, p_2, q_1, q_2\}} \varnothing \rhd x$ captures the statement that one of agents p_1, p_2, q_1, q_2 is an early adopter. The above principle is just an example of a non-trivial property of diffusion common to all social networks with the same unweighted sociogram. This example can be stated in a more general form as

$$\begin{aligned} U_2 \vDash \quad & \bigwedge_{i=1}^{n} p_{i1}, p_{i2}, \dots, p_{in} \rhd q \\ & \to \bigvee_{j_1=1}^{n} \bigvee_{j_2=1}^{n} \cdots \bigvee_{j_n=1}^{n} p_{1j_1}, p_{2j_2}, \dots, p_{nj_n} \rhd q \vee \bigvee_{i=1}^{n} \bigvee_{j=1}^{n} \varnothing \rhd p_{ij}, \end{aligned}$$

where U_2 is unweighted sociogram depicted in Figure 13. Complete axiomatization of properties of influence common to all social networks with a given graph remains an open problem.

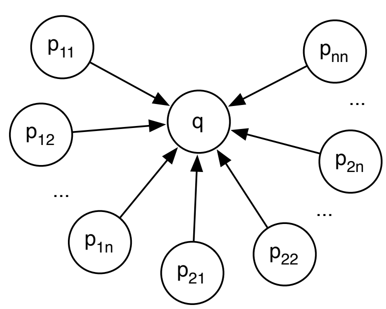

Figure 13: Unweighted Sociogram U_2

Another possible extension of our work, suggested by an anonymous reviewer, is to consider common logical principles of all social networks in which all agents have the same threshold values. Such more narrow class of models would results in a larger set of universally true principles, some of which will not be provable from the axioms of our logical system. Formula $p \rhd q \to p \rhd r$ is an example of such principle for the sociogram depicted in Figure 14. To

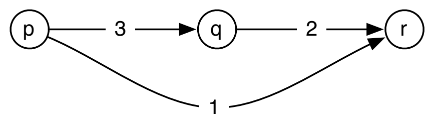

Figure 14: Sociogram S_4

see how much different this new setting is from the one discussed earlier in the article, note that in all models from this class, either all agents are early adopters or none is.

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