MECS 475: The Economics of Organizations

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Performance Differences: Productivity Measures

Productivity Measures

In this note, I will discuss two commonly used measures of total factor productivity that are used in the literature. Which one is used in a particular application is typically determined by data availability, but we will see that the two measures have significantly different interpretations. A firm chooses capital K and labor L at constant unit costs r and w respectively to produce quantity according to a Cobb-Douglas production function $q(K, L) = AK^{\alpha}L^{1-\alpha}$, which it sells on a product market at price p.

The first measure of productivity we will be concerned with is **quantity total factor productivity** (referred to as $TFPQ$), which is given by:

$$TFPQ = \frac{q(K, L)}{K^{\alpha}L^{1-\alpha}} = A.$$

That is, $TFPQ$ is a ratio of physical output to physical inputs, appropriately weighted according to their production elasticities. (i.e., $\alpha = \frac{d \log q}{d \log K}$) Differences in $TFPQ$ across firms correspond to variations in output across firms that are not explained by variation in inputs. In other words, it is a measure of our ignorance about the firm's underlying production process. One objective of organizational economics is to improve our understanding of firms' production processes.

The second measure of productivity we will discuss is **revenue total factor productivity** (referred to as $TFPR$), which is given by

$$TFPR = \frac{p \cdot q(K, L)}{(rK)^{\alpha}(wL)^{1-\alpha}} = \frac{p}{r^{\alpha}w^{1-\alpha}} \cdot A$$

or sometimes

$$TFPR = \frac{p \cdot q(K,L)}{K^{\alpha}L^{1-\alpha}} = p \cdot A.$$

That is, $TFPR$ is a ratio of revenues to input costs, appropriately weighted according to their production elasticities. Differences in $TFPR$ across firms correspond to variations in revenues across firms that are not explained by variation in measured costs. Since $TFPR$ depends on unit costs and output prices, it may depend on the market conditions that determine them. Under some assumptions, $TFPR$ may depend exclusively on market conditions, and variations in $TFPR$ across firms is indicative of misallocation of productive resources across firms. (Hsieh and Klenow '09) In this note, I will focus primarily on $TFPR$.

TFPR Under Constraints and Market Power

A firm produces according to a Cobb-Douglas production function $q(K,L) = AK^{\alpha}L^{1-\alpha}$ and is subject to constraints in either labor or capital. Suppose a firm is constrained to produce with $L \leq \bar{L}$ and $K \leq \bar{K}$. Let $p(q(K,L))$ denote the firm's residual inverse demand curve, and let $\varepsilon_{qp}(q) = \frac{1}{\frac{dp(q)}{dq}\frac{q}{p(q)}}$. The firm's problem is to

$$\max p(q(K,L))\, q(K,L) - wL - rK$$

subject to $L \leq \bar{L}$ and $K \leq \bar{K}$. The Lagrangian is

$$\mathcal{L} = p(q(K,L))\, q(K,L) - wL - rK + \lambda_K \left(\bar{K} - K\right) + \lambda_L \left(\bar{L} - L\right).$$

Taking first-order conditions, we get

$$\begin{aligned} MRP_K^* &= p'(q^*)\, q_K q^* + p(q^*)\, q_K = r + \lambda_K \\ MRP_L^* &= p'(q^*)\, q_L q^* + p(q^*)\, q_L = w + \lambda_L \end{aligned}$$

We can rearrange these expressions and derive

$$\begin{aligned}\left(p'\left(q^*\right)\frac{q^*}{p\left(q^*\right)}+1\right)p\left(q^*\right)q_K &= r+\lambda_K \\ \left(p'\left(q^*\right)\frac{q^*}{p\left(q^*\right)}+1\right)p\left(q^*\right)q_L &= w+\lambda_L\end{aligned}$$

or

$$\begin{aligned}p\left(q^*\right)q_K &= \frac{\left|\varepsilon_{qp}\left(q^*\right)\right|}{\left|\varepsilon_{qp}\left(q^*\right)\right|-1}\left(r+\lambda_K\right) \\ p\left(q^*\right)q_L &= \frac{\left|\varepsilon_{qp}\left(q^*\right)\right|}{\left|\varepsilon_{qp}\left(q^*\right)\right|-1}\left(w+\lambda_L\right).\end{aligned}$$

Under Cobb-Douglas, we know that $q_K=\alpha\frac{q}{K}$ and $q_L=(1-\alpha)\frac{q}{L}$. We therefore have

$$\begin{aligned}p\left(q^*\right)\frac{q^*}{rK^*} &= \frac{1}{\alpha}\frac{\left|\varepsilon_{qp}\left(q^*\right)\right|}{\left|\varepsilon_{qp}\left(q^*\right)\right|-1}\frac{MRP_K^*}{r} \\ p\left(q^*\right)\frac{q^*}{wL^*} &= \frac{1}{1-\alpha}\frac{\left|\varepsilon_{qp}\left(q^*\right)\right|}{\left|\varepsilon_{qp}\left(q^*\right)\right|-1}\frac{MRP_L^*}{w}\end{aligned}$$

Revenue total factor productivity is defined as revenue divided by a geometric average of capital expenditures and labor expenditures, and is therefore given by

$$\begin{aligned}TFPR^* &= p\left(q^*\right)\frac{q^*}{\left(rK^*\right)^{\alpha}\left(wL^*\right)^{1-\alpha}}=\left[p\left(q^*\right)\frac{q^*}{rK^*}\right]^{\alpha}\left[p\left(q^*\right)\frac{q^*}{wL^*}\right]^{1-\alpha} \\ &= \frac{1}{\alpha^{\alpha}\left(1-\alpha\right)^{1-\alpha}}\frac{\left|\varepsilon_{qp}\left(q^*\right)\right|}{\left|\varepsilon_{qp}\left(q^*\right)\right|-1}\left(\frac{MRP_K^*}{r}\right)^{\alpha}\left(\frac{MRP_L^*}{w}\right)^{1-\alpha}\end{aligned}$$

In this model, heterogeneity in $TFPR$ arises from heterogeneity in α, $\left|\varepsilon_{qp}\left(q^*\right)\right|$, $\frac{MRP_K^*}{r}$, or $\frac{MRP_L^*}{w}$. Heterogeneity in α arises from differences in technology. Heterogeneity in $\left|\varepsilon_{qp}\left(q^*\right)\right|$ can result from idiosyncratic demand shocks. Heterogeneity in $\frac{MRP_K^*}{r}$ or $\frac{MRP_L^*}{w}$ results from either heterogeneity in the capital and labor constraints or heterogeneity in A.

If $TFPR$ is defined as revenue divided by a geometric average of capital and labor inputs (rather than expenditures), it is given by

$$\begin{aligned} TFPR^* &= p(q^*) \frac{q^*}{(K^*)^\alpha (L^*)^{1-\alpha}} = \left[p(q^*) \frac{q^*}{K^*} \right]^\alpha \left[p(q^*) \frac{q^*}{L^*} \right]^{1-\alpha} \\ &= \frac{1}{\alpha^\alpha (1-\alpha)^{1-\alpha}} \frac{|\varepsilon_{qp}(q^*)|}{|\varepsilon_{qp}(q^*)| - 1} (MRP_K^*)^\alpha (MRP_L^*)^{1-\alpha}. \end{aligned}$$

In this case, heterogeneity can arise from differences in the price of inputs.

Why doesn't this equal $p \cdot A$ when $|\varepsilon_{qp}| \to \infty$ and $\lambda_K, \lambda_L \to 0$? In fact it does. The Kuhn-Tucker conditions can only be replaced by the first-order conditions I derived above when p, r, and w are such that this expression equals $p \cdot A$.

There are two special cases of this expression in the literature:

Hsieh and Klenow '09

Hsieh and Klenow (2009) measure $TFPR^*$ as value-added (revenues) divided by a share-weighted geometric average of the net book value of fixed capital of a firm, net of depreciation (rK^*) and labor compensation, which is the sum of wages, bonuses, and benefits. They derive an expression for $TFPR^*$ under the assumption of constant elasticity demand of the form $q(p) = \frac{D}{p^{-\varepsilon}}$, so that $\varepsilon_{qp} = \varepsilon$. In this case,

$$TFPR^* \propto \left(\frac{MRP_K^*}{r} \right)^\alpha \left(\frac{MRP_L^*}{w} \right)^{1-\alpha}.$$

Heterogeneity in $TFPR^*$ is then interpreted as firm-specific wedges (i.e. heterogeneity in MRP_K^*/r or MRP_L^*/w, which should both be equal to 1 at the non-distorted optimum).

Foster, Haltiwanger, Syverson '08

Foster, Haltiwanger, and Syverson measure $TFPR^*$ as plant-level prices times $TFPQ^*$, which uses physical output data, labor measured in hours, capital as plant's book values of equipment and structures deflated to 1987 dollars, and materials expenditures (which I will ignore). They assume firms are not constrained, so $MRP_K^* = r$ and $MRP_L^* = w$ for all firms. Their definition of $TFPR^*$ corresponds to the second measure listed above, and therefore

$$TFPR^* \propto \frac{|\varepsilon_{qp}(q^*)|}{|\varepsilon_{qp}(q^*)| - 1}$$

They interpret differences in $TFPR^*$ as arising from differences in $\frac{|\varepsilon_{qp}(q^*)|}{|\varepsilon_{qp}(q^*)|-1}$.

Alternative Interpretation of TFPR

Let $\tilde{\alpha} = \frac{rK^*}{rK^*+wL^*}$ and $1 - \tilde{\alpha} = \frac{wL^*}{rK^*+wL^*}$ denote the realized cost shares of capital and labor, respectively. These need not be equal to α and $1 - \alpha$, because constraints may tilt the optimal input mix. Note that

$$\begin{aligned}
\frac{(rK^*)^{\alpha}(wL^*)^{1-\alpha}}{rK^* + wL^*} &= (1-\tilde{\alpha})\,\tilde{\alpha}\left(\frac{wL^*}{rK^*}\right)^{1-\alpha} + \tilde{\alpha}\,(1-\tilde{\alpha})\left(\frac{rK^*}{wL^*}\right)^{\alpha} \\
&= \tilde{\alpha}^{\alpha}(1-\tilde{\alpha})^{1-\alpha} \\
rK^* + wL^* &= \frac{(rK^*)^{\alpha}(wL^*)^{1-\alpha}}{\tilde{\alpha}^{\alpha}(1-\tilde{\alpha})^{1-\alpha}}
\end{aligned}$$

Then,

$$\begin{aligned}
TFPR &= \frac{p(q^*)\,q^*}{(rK^*)^{\alpha}(wL^*)^{1-\alpha}} = \frac{1}{\tilde{\alpha}^{\alpha}(1-\tilde{\alpha})^{1-\alpha}}\frac{p(q^*)\,q^*}{rK^* + wL^*} \\
&= \frac{1}{\tilde{\alpha}^{\alpha}(1-\tilde{\alpha})^{1-\alpha}}\frac{REV}{TVC}
\end{aligned}$$

In particular, we see TFPR is proportional to the revenue (REV) to total variable cost (TVC) ratio. This ratio is given by

$$\frac{REV}{TVC} = \frac{|\varepsilon_{qp}(q^*)|}{|\varepsilon_{qp}(q^*)| - 1}\left(\frac{\tilde{\alpha}}{\alpha}\frac{MRP_K^*}{r}\right)^{\alpha}\left(\frac{1-\tilde{\alpha}}{1-\alpha}\frac{MRP_L^*}{w}\right)^{1-\alpha}.$$

The profit/cost ratio is this expression plus 1. Average profits per dollar of inputs are therefore increasing in markups and distortions (corrected by distortions in input mix).

Heterogeneous Returns-to-Scale

Suppose a firm faces a downward-sloping residual demand curve $p(q)$ for its product, and it has a Cobb-Douglas production function $q = AL^{\beta}$, where L is labor inputs, and β is the elasticity of production with respect to labor. Further, suppose the firm faces a labor constraint $L \leq \bar{L}$. The firm's Lagrangian is

$$\mathcal{L} = p\left(AL^{\beta}\right) AL^{\beta} - wL + \lambda_L \left(\bar{L} - L\right)$$

and its first-order conditions are given by:

$$\begin{aligned} w + \lambda_L^* &= MRP_L^* = \left(p'\left(q^*\right) \frac{q^*}{p\left(q^*\right)} + 1\right) p\left(q^*\right) q_L^* \\ MRP_L^* &= \frac{\left|\varepsilon^*\left(q^*\right)\right| - 1}{\left|\varepsilon^*\left(q^*\right)\right|} \beta \frac{pq^*}{L^*}, \end{aligned}$$

where $|\varepsilon|$ is the elasticity of the firm's (strategic) residual demand curve. Average labor productivity is given by

$$ALP = \frac{p^*\left(q^*\right) q^*}{wL^*} = \frac{|\varepsilon|}{|\varepsilon| - 1} \frac{MRP_L^*/w}{\beta}$$

Heterogeneity in average labor productivity is driven by heterogeneity in either MRP_L/w (i.e. labor wedges), heterogeneity in β (i.e. heterogeneous technologies), or heterogeneity in $|\varepsilon|$ (which could be due to idiosyncratic demand shocks). Under perfect competition, $|\varepsilon| = \infty$, so this becomes

$$ALP = \frac{MRP_L^*/w}{\beta}. \tag{1}$$

Here, prices are exogenous to the model, which should eliminate the concerns about the differences between TFP and $TFPR$. However, we see from this expression that heterogeneity in ALP still does not depend on TFP. In fact, all heterogeneity in average labor productivity is driven by heterogeneous returns to scale (and if so-desired, labor constraints).

Mismeasured Scale Effects

Let us maintain the assumption that $|\varepsilon| = \infty$, so that $TFPR$ should just be a constant multiple of TFP (and should therefore reflect A). The real issue is that ALP does not correct for scale effects. If it did, then it would not be driven by β, and it would reflect $TFPR$ (and hence TFP, since prices are exogenous):

$$TFPR^{*} = \frac{pq^{*}}{\left(wL^{*}\right)^{\beta}} = p\frac{A\left(L^{*}\right)^{\beta}}{\left(wL^{*}\right)^{\beta}} = p\frac{A}{w^{\beta}}. \tag{2}$$

More generally, suppose the scale is assumed (by the econometrician) to be $\gamma \geq \beta$. Then

$$\frac{pq}{\left(wL\right)^{\gamma}} = ALP\left(wL\right)^{1-\gamma} \tag{3}$$

We know that for $\gamma = \beta$, $p\frac{A}{w^{\beta}} = ALP\left(wL\right)^{1-\beta}$. Solving this for $\left(wL\right)^{1-\gamma}$, we have

$$\left(wL\right)^{1-\gamma} = \left(p\frac{A}{w^{\beta}}\frac{1}{ALP}\right)^{\frac{1-\gamma}{1-\beta}}$$

Plugging this into (3), we get

$$\frac{pq}{\left(wL\right)^{\gamma}} = \left(p\frac{A}{w^{\beta}}\right)^{1-\frac{\gamma-\beta}{1-\beta}}\left(\frac{MRP_{L}/w}{\beta}\right)^{\frac{\gamma-\beta}{1-\beta}}.$$

Thus, when TFP is calculated using the incorrect returns to scale, the result is a geometric average of (2), which depends on actual TFP (A), and (1), which depends on labor constraints (MRP_L/w) and the actual returns to scale (β).

When is $TFPR$ increasing in TFP?

Let ρ denote the pass-through rate characterized by the demand system. That is, $\rho = \frac{dp}{d(C')}$, where $C' = \frac{c}{TFP}$ is the marginal cost of production. In the no-constraints case, we can derive the following expression:

$$TFPR'(TFP) = \left(\frac{|\varepsilon(p^*(TFP))|}{|\varepsilon(p^*(TFP))| - 1} - \rho \right) \frac{c}{TFP},$$

which is positive whenever $\rho < \frac{|\varepsilon(p^*(TFP))|}{|\varepsilon(p^*(TFP))|-1}$. For the case of linear demand (as in Foster, Haltiwanger, and Syverson), $\rho < 1 < \frac{|\varepsilon(p^*(TFP))|}{|\varepsilon(p^*(TFP))|-1}$, so $TFPR$ is increasing in TFP. For the case of constant elasticity demand, $\rho = \frac{|\varepsilon(p^*(TFP))|}{|\varepsilon(p^*(TFP))|-1}$, so $TFPR'(TFP) = 0$, and therefore $TFPR$ is independent of TFP (which is emphasized in Hsieh and Klenow). If pass-through is sufficiently high (i.e. significantly greater than one-for-one), then it can in fact be the case that $TFPR$ is decreasing in TFP. Intuitively, this would happen if prices fell by more than TFP increased following an increase in TFP.