MECS 570-1: Organizational Economics I

Michael Powell

Incentives in Organizations, Part II: Limited Liability, Misaligned Incentives

Limited Liability (Updated: Jan 10 2017)

We saw in the previous model that the optimal contract sometimes involved upfront payments from the Agent to the Principal. To the extent that the Agent is unable to afford such payments (or legal restrictions prohibit such payments), the Principal will not be able to extract all the surplus that the Agent creates. Further, in order to extract surplus from the Agent, the Principal may have to put in place contracts that reduce the total surplus created. In equilibrium, the Principal may therefore offer a contract that induces effort below the first-best.

Description Again, there is a risk-neutral Principal (P). There is also a **risk-neutral** Agent (A). The Agent chooses an effort level $e \in \mathbb{R}_+$ at a private cost of $c(e)$, with $c'', c' > 0$, and this effort level affects the distribution over outputs $y \in Y$, with y distributed according to cdf $F(\cdot | e)$. These outputs can be sold on the product market for price p. The Principal can write a contract $w \in W \subset \{w : Y \to \mathbb{R}, w(y) \geq \underline{w} \text{ for all } y\}$ that determines a transfer $w(y)$ that she is compelled to pay the Agent if output y is realized. The Agent has an outside option that provides utility $\bar{u}$ to the Agent and $\bar{\pi}$ to the Principal. If the outside option is not exercised, the Principal's and Agent's preferences are, respectively,

$$\begin{aligned} \Pi(w, e) &= \int_{y \in Y} (py - w(y))\, dF(y | e) = E_y[py - w | e] \\ U(w, e) &= \int_{y \in Y} (w(y) - c(e))\, dF(y | e) = E_y[w - c(e) | e]. \end{aligned}$$

There are two differences between this model and the model in the previous subsection. The first difference is that the Agent is risk-neutral (so that absent any other changes, the equilibrium contract would induce first-best effort). The second difference is that the wage payment from the Principal to the Agent has to exceed, for each realization of output, a value $\underline{w}$. Depending on the setting, this constraint is described as a liquidity constraint or a limited-liability constraint. In repeated settings, it is more naturally thought of as the latter—due to legal restrictions, the Agent cannot be legally compelled to make a transfer (larger than $-\underline{w}$) to the Principal. In static settings, either interpretation may be sensible depending on the particular application—if the Agent is a fruit picker, for instance, he may not have much liquid wealth that he can use to pay the Principal.

Timing The timing of the game is exactly the same as before.

1. P offers A a contract $w(y)$, which is commonly observed.
2. A accepts the contract $(d = 1)$ or rejects it $(d = 0)$ and receives $\bar{u}$, and the game ends. This decision is commonly observed.
3. If A accepts the contract, A chooses effort level e and incurs cost $c(e)$. e is only observed by A.
4. Output y is drawn from distribution with cdf $F(\cdot|\, e)$. y is commonly observed.
5. P pays A an amount $w(y)$. This payment is commonly observed.

Equilibrium The solution concept is the same as before. A **pure-strategy subgame-perfect equilibrium** is a contract $w^* \in W$, an acceptance decision $d^* : W \to \{0, 1\}$, and an effort choice $e^* : W \times \{0, 1\} \to \mathbb{R}_+$ such that given the contract w^*, the Agent optimally chooses d^* and e^*, and given d^* and e^*, the Principal optimally offers contract w^*. We will say that the optimal contract induces effort e^*.

The Program The principal offers a contract $w \in W$ and proposes an effort level e in order to solve

$$\max_{w \in W, e} \int_{y \in Y} (py - w(y))\, dF(y|\, e)$$

subject to three constraints: the incentive-compatibility constraint

$$e \in \operatorname*{argmax}_{\hat{e} \in \mathbb{R}_+} \int_{y \in Y} (w(y) - c(\hat{e}))\, dF(y|\, \hat{e}),$$

the individual-rationality constraint

$$\int_{y \in Y} (w(y) - c(e))\, dF(y|\, e) \geq \bar{u},$$

and the limited-liability constraint

$$w(y) \geq \underline{w} \text{ for all } y.$$

Binary-Output Case Jewitt, Kadan, and Swinkels (2008) solves for the optimal contract in the general environment above (and even allows for agent risk aversion). Here, I will instead focus on an elementary case that highlights the main trade-off.

Assumption 1. Output is $y \in \{0, 1\}$, and given effort e, its distribution satisfies $\Pr[y = 1|\, e] = e$.

Assumption 2. The agent's costs have a non-negative third derivative: $c''' \geq 0$, and they satisfy conditions that ensure an interior solution: $c'(0) = 0$ and $c'(1) = +\infty$. Or for comparison across models in this module, $c(e) = \frac{c}{2}e^2$, where $p \leq c$ to ensure that $e^{FB} < 1$.

Finally, we can restrict attention to affine, nondecreasing contracts

$$\begin{aligned} W &= \{w(y) = (1 - y)\, w_0 + y w_1, w_0, w_1 \geq 0\} \\ &= \{w(y) = s + by, s \geq \underline{w}, b \geq 0\}. \end{aligned}$$

When output is binary, this restriction to affine contracts is without loss of generality. Also, the restriction to nondecreasing contracts is not restrictive (i.e., any optimal contract of a relaxed problem in which we do not impose that contracts are nondecreasing will also be the solution to the full problem). This result is something that needs to be shown and is not in general true, but in this case, it is straightforward.

In principal-agent models, it is often useful to break the problem down into two steps. The first step takes a target effort level, e, as given and solves for the set of cost-minimizing contracts implementing effort level e. Any cost-minimizing contract implementing effort level e results in an expected cost of $C(e)$ to the principal. The second step takes the function $C(\cdot)$ as given and solves for the optimal effort choice.

In general, the cost-minimization problem tends to be a well-behaved convex-optimization problem, since (even if the agent is risk-averse) the objective function is weakly concave, and the constraint set is a convex set (since given an effort level e, the individual-rationality constraint and the limited-liability constraint define convex sets, and each incentive constraint ruling out effort level $\hat{e} \neq e$ also defines a convex set, and the intersection of convex sets is itself a convex set). The resulting cost function $C(\cdot)$ need not have nice properties, however, so the second step of the optimization problem is only well-behaved under restrictive assumptions. In the present case, assumptions 1 and 2 ensure that the second step of the optimization problem is well-behaved.

Cost-Minimization Problem Given an effort level e, the cost-minimization problem is given by

$$C(e, \bar{u}, \underline{w}) = \min_{s,b} s + be$$

subject to the agent's incentive-compatibility constraint

$$e \in \underset{\hat{e}}{\operatorname{argmax}} \left\{ s + b\hat{e} - c(\hat{e}) \right\},$$

his individual-rationality constraint

$$s + be - c(e) \geq \bar{u},$$

and the limited-liability constraint

$$s \geq \underline{w}.$$

I will denote a **cost-minimizing contract implementing effort level** e by (s_e^*, b_e^*).

The first step in solving this problem is to notice that the agent's incentive-compatibility constraint implies that any cost-minimizing contract implementing effort level e must have $b_e^* = c'(e)$.

If there were no limited-liability constraint, the principal would choose s_e^* to extract the agent's surplus. That is, given $b = b_e^*$, s would solve

$$s + b_e^* e = \bar{u} + c(e).$$

That is, s would ensure that the agent's expected compensation exactly equals his expected effort costs plus his opportunity cost. The resulting s, however, may not satisfy the limited-liability constraint. The question then is: given $\bar{u}$ and $\underline{w}$, for what effort levels e is the principal able to extract all the agent's surplus (i.e., for what effort levels does the limited-liability constraint not bind?), and for what effort levels is she unable to do so? Figure 1 below shows cost-minimizing contracts for effort levels e_1 and e_2. Any contract can be represented as a line in this figure, where the line represents the expected pay the agent will receive given an effort level e. The cost-minimizing contract for effort level e_1 is tangent to the $\bar{u} + c(e)$ curve at e_1 and its intercept is $s_{e_1}^*$. Similarly for e_2. Both $s_{e_1}^*$ and $s_{e_2}^*$ are greater than $\underline{w}$, which implies that for such effort levels, the limited-liability constraint is not binding.

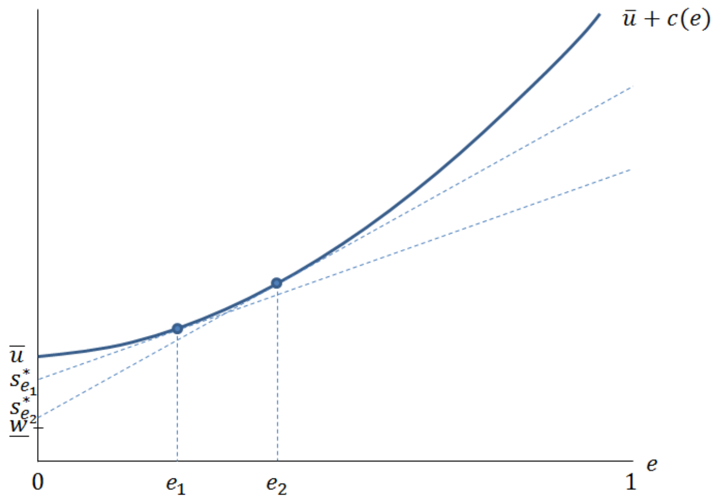


Figure 1

For effort sufficiently high, the limited-liability constraint will be binding in a cost-minimizing contract, and it will be binding for all higher effort levels. Define the threshold $\bar{e}(\bar{u}, \underline{w})$ to be the effort level such that for all $e \geq \bar{e}(\bar{u}, \underline{w})$, $s_e^* = \underline{w}$. Figure 2 illustrates that $\bar{e}(\bar{u}, \underline{w})$ is the effort level at which the contract tangent to the $\bar{u} + c(e)$ curve at $\bar{e}(\bar{u}, \underline{w})$ intersects the vertical axis at exactly $\underline{w}$. That is, $\bar{e}(\bar{u}, \underline{w})$ solves

$$c'(\bar{e}(\bar{u}, \underline{w})) = \frac{\bar{u} + c(\bar{e}(\bar{u}, \underline{w})) - \underline{w}}{\bar{e}(\bar{u}, \underline{w})}.$$

Figure 2 also illustrates that for all effort levels $e > \bar{e}(\bar{u}, \underline{w})$, the cost-minimizing contract involves giving the agent strictly positive surplus. That is, the cost to the principal of getting the agent to choose effort $e > \bar{e}(\bar{u}, \underline{w})$ is equal to the agent's opportunity costs $\bar{u}$ plus his effort costs $c(e)$ plus **incentive costs** $IC(e, \bar{u}, \underline{w})$.

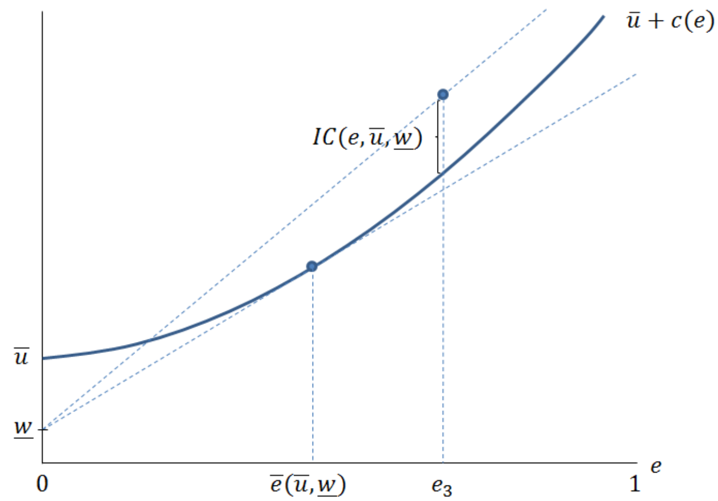


Figure 2

The incentive costs $IC(e, \bar{u}, \underline{w})$ are equal to the agent's expected compensation given effort choice e and cost-minimizing contract (s_e^*, b_e^*) minus his costs:

$$\begin{aligned} IC(e, \bar{u}, \underline{w}) &= \begin{cases} 0 & e \leq \bar{e}(\bar{u}, \underline{w}) \\ \underline{w} + c'(e)\, e - c(e) - \bar{u} & e \geq \bar{e}(\bar{u}, \underline{w}) \end{cases} \\ &= \max\{0, \underline{w} + c'(e)\, e - c(e) - \bar{u}\} \end{aligned}$$

where I used the fact that for $e \geq \bar{e}(\bar{u}, \underline{w})$, $s_e^* = \underline{w}$ and $b_e^* = c'(e)$. This incentive-cost function $IC(\cdot, \bar{u}, \underline{w})$ is the key object that captures the main contracting friction in this model. I will sometimes refer to $IC(e, \bar{u}, \underline{w})$ as the **incentive rents** required to get the agent to choose effort level e. Putting these results together, we see that

$$C(e, \bar{u}, \underline{w}) = \bar{u} + c(e) + IC(e, \bar{u}, \underline{w}).$$

That is, the principal's total costs of implementing effort level e are the sum of the agent's costs plus the incentive rents required to get the agent to choose effort level e.

Since $IC(e,\bar{u},\underline{w})$ is the main object of interest in this model, I will describe some of its properties. First, it is continuous in e (including, in particular, at $e=\bar{e}(\bar{u},\underline{w})$). Next, $\bar{e}(\bar{u},\underline{w})$ and $IC(e,\bar{u},\underline{w})$ depend on $(\bar{u},\underline{w})$ only inasmuch as $(\bar{u},\underline{w})$ determines $\bar{u}-\underline{w}$, so I will abuse notation and write these expressions as $\bar{e}(\bar{u}-\underline{w})$ and $IC(e,\bar{u}-\underline{w})$. Also, given that $c''>0$, IC is increasing in e (since $\underline{w}+c'(e)e-c(e)-\underline{u}$ is strictly increasing in e, and IC is just the max of this expression and zero). Further, given that $c'''\geq 0$, IC is convex in e. For $e\geq\bar{e}(\bar{u}-\underline{w})$, this property follows, because

$$\frac{\partial^2}{\partial e^2}IC=c''(e)+c'''(e)\,e\geq 0.$$

And again, since IC is the max of two convex functions, it is also a convex function. Finally, since $IC(\cdot,\bar{u}-\underline{w})$ is flat when $e\leq\bar{e}(\bar{u}-\underline{w})$ and it is strictly increasing (with slope independent of $\bar{u}-\underline{w}$) when $e\geq\bar{e}(\bar{u}-\underline{w})$, the slope of IC with respect to e is (weakly) decreasing in $\bar{u}-\underline{w}$, since $\bar{e}(\bar{u}-\underline{w})$ is increasing in $\bar{u}-\underline{w}$. That is, $IC(e,\bar{u}-\underline{w})$ satisfies decreasing differences in $(e,\bar{u}-\underline{w})$.

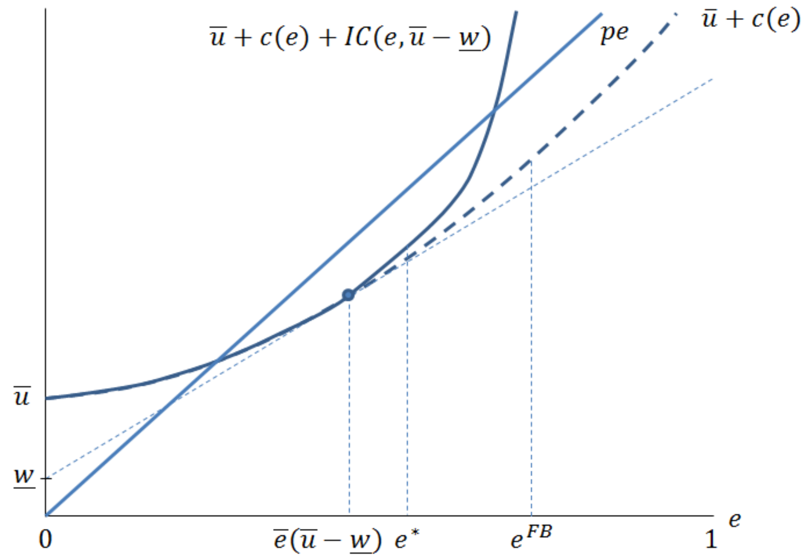
Motivation-Rent Extraction Trade-off The second step of the optimization problem takes as given the function

$$C(e,\bar{u}-\underline{w})=\bar{u}+c(e)+IC(e,\bar{u}-\underline{w})$$

and solves for the optimal effort choice by the principal:

$$\begin{aligned}&\max_e pe-C(e,\bar{u}-\underline{w})\\ =\;&\max_e pe-\bar{u}-c(e)-IC(e,\bar{u}-\underline{w}).\end{aligned}$$

Note that total surplus is given by $pe - \bar{u} - c(e)$, which is therefore maximized at $e = e^{FB}$ (which, if $c(e) = ce^2/2$, then $e^{FB} = p/c$). Figure 3 below depicts the principal's expected benefit line pe, and her expected costs of implementing effort e at minimum cost, $C(e, \bar{u} - \underline{w})$. The first-best effort level, e^{FB} maximizes the difference between pe and $\bar{u} + c(e)$, while the equilibrium effort level e^* maximizes the difference between pe and $C(e, \bar{u} - \underline{w})$.



If $c(e) = ce^2/2$, we can solve explicitly for $\bar{e}(\bar{u} - \underline{w})$ and for $IC(e, \bar{u} - \underline{w})$ when $e > \bar{e}(\bar{u} - \underline{w})$. In particular,

$$\bar{e}(\bar{u} - \underline{w}) = \left(\frac{2(\bar{u} - \underline{w})}{c} \right)^{1/2}$$

and when $e > \bar{e}(\bar{u} - \underline{w})$,

$$IC(e, \bar{u} - \underline{w}) = \underline{w} + \frac{1}{2} ce^2 - \bar{u}.$$

If $\underline{w} < 0$ and p is sufficiently small, we can have $e^* = e^{FB}$ (i.e., these are the conditions required to ensure that the limited-liability constraint is not binding for the cost-minimizing contract implementing $e = e^{FB}$). If p is sufficiently large relative to $\bar{u} - \underline{w}$, we will have $e^* = \frac{1}{2}\frac{p}{c} = \frac{1}{2} e^{FB}$. For p somewhere in between, we will have $e^* = \bar{e}(\bar{u} - \underline{w}) < e^{FB}$. In particular, $C(e, \bar{u} - \underline{w})$ is kinked at this point.

As in the risk–incentives model, we can illustrate through a partial characterization why (and when) effort is less-than first-best. Since we know that e^{FB} maximizes $pe - \bar{u} - c(e)$, we therefore have that

$$\frac{d}{de}\left[pe - \bar{u} - c(e) - IC(e, \bar{u} - \underline{w})\right]_{e=e^{FB}} = -\frac{\partial}{\partial e} IC\left(e^{FB}, \bar{u} - \underline{w}\right) \leq 0,$$

with strict inequality if the limited-liability constraint binds at the cost-minimizing contract implementing e^{FB}. This means that, even though e^{FB} maximizes total surplus, if the principal has to provide the agent with rents at the margin, she may choose to implement a lower effort level. Reducing the effort level away from e^{FB} leads to second-order losses in terms of total surplus, but it leads to first-order gains in profits for the principal. In this model, there is a tension between total-surplus creation and rent extraction, which yields less-than first-best effort in equilibrium.

In my view, liquidity constraints are extremely important and are probably one of the main reasons for why many jobs do not involve first-best incentives. The Vickrey-Clarke-Groves logic that first-best outcomes can be obtained if the firm transfers the entire profit stream to each of its members in exchange for a large up-front payment seems simultaneously compelling, trivial, and obviously impracticable. In for-profit firms, in order to make it worthwhile to transfer a large enough share of the profit stream to an individual worker to significantly affect his incentives, the firm would require a large up-front transfer that most workers cannot afford to pay. It is therefore not surprising that we do not see most workers' compensation tied directly to the firm's overall profits in a meaningful way. One implication of this logic is that firms have to find alternative instruments to use as performance measures, which we will turn to next. In principle, models in which firms do not motivate their workers by writing contracts directly on profits should include assumptions under which the firm optimally chooses not to write contracts directly on profits, but they almost never do.

Exercise. *Let $\Delta \equiv (2c(\bar{u} - \underline{w}))^{1/2}$. Part 1: Show that when $p \leq \Delta$, the contract-augmented possibilities set is $\tilde{Y}^f = \left\{(y, -C) : y \leq \left(\frac{2C}{c}\right)^{1/2}\left(\frac{C-\underline{w}}{2C}\right)^{1/2}\right\}$. Part 2: Show that when $p \geq 2\Delta$,* *the contract-augmented possibilities set is* $\tilde{Y}^f = \left\{(y, -C) : y \leq \left(\frac{2C}{c}\right)^{1/2}\right\}$. *Part 3: Solve for* $\tilde{Y}^f$ *for* $\Delta < p < 2\Delta$. *(This part is somewhat more complicated.) Part 4: In this model, the contract-augmented possibilities depend on the equilibrium price level, which implies that in a competitive-equilibrium framework, the firm's production possibilities are endogenous to the equilibrium. This was not the case for the risk-incentives trade-off model. If we define* $\tilde{Y}^f(p)$ *as the contract-augmented possibilities set given price level* p*, how does* $\tilde{Y}^f(p)$ *vary in* p*? (Note that since* $\tilde{Y}^f(p)$ *is a set, you will have to think about what it means for a set to vary in a parameter.)*

Exercise. *Holmstrom (1979) shows that in the risk-incentives model in the previous subsection, if there is a costless additional performance measure* m *that is informative about* e*, then an optimal formal contract should always put some weight on* m *unless* y *is a sufficient statistic for* y *and* m*. This is known as Holmstrom's "informativeness principle" and suggests that optimal contracts should always be extremely sensitive to the details of the environment the contract is written in. Suppose instead that the agent is risk-neutral but liquidity-constrained, and suppose there is a performance measure* $m \in \{0, 1\}$ *such that* $\Pr[m = 1 | e] = e$ *and conditional on* e*,* m *and* y *are independent. Suppose contracts of the form* $w(y, m) = s + b_y y + b_m m + b_{ym} ym$ *can be written but must satisfy* $w(y, m) \geq \bar{w}$ *for each realization of* (y, m)*. Is it again always the case that* $b_m \neq 0$ *and/or* $b_{ym} \neq 0$*?*

Further Reading Jewitt, Kadan, and Swinkels (2008) derive optimal contracts in a broad class of environments with risk-averse agents and bounded payments (in either direction). Chaigneau, Edmans, and Gottlieb (2015) provide necessary and sufficient conditions for additional informative signals to have strictly positive value to the Principal. Wu (2015) shows that firms' contract-augmented possibilities sets are endogenous to the competitive environment they face when their workers are subject to limited-liability constraints.

Multiple Tasks and Misaligned Performance Measures (Updated: Jan 10 2017)

In the previous two models, what the Principal cared about was output, and output, though a noisy measure of effort, was perfectly measurable. This assumption seems sensible when we think about overall firm profits (ignoring basically everything that accountants think about every day), but as we alluded to in the discussion above, overall firm profits are generally too blunt of an instrument to use to motivate individual workers within the firm if they are liquidity-constrained. As a result, firms often try to motivate workers using more specific performance measures, but while these performance measures are informative about what actions workers are taking, they may be less useful as a description of how the workers' actions affect the objectives the firm cares about. And paying workers for what is measured may not get them to take actions that the firm cares about. This observation underpins the title of the famous 1975 paper by Steve Kerr called "On the Folly of Rewarding A, while Hoping for B."

Description Again, there is a risk-neutral Principal (P) and a risk-neutral Agent (A). The Agent chooses an effort vector $e = (e_1, e_2) \in \mathbb{R}_+^2$ at a private cost of $\frac{c}{2}\left(e_1^2 + e_2^2\right)$. This effort vector affects the distribution of output $y \in Y = \{0, 1\}$ and a performance measure $m \in M = \{0, 1\}$ as follows:

$$
\begin{aligned}
\Pr\left[\left. y = 1 \right| e\right] &= f_1 e_1 + f_2 e_2 \\
\Pr\left[\left. m = 1 \right| e\right] &= g_1 e_1 + g_2 e_2,
\end{aligned}
$$

where it may be the case that $f = (f_1, f_2) \neq (g_1, g_2) = g$. Assume that $f_1^2 + f_2^2 = g_1^2 + g_2^2 = 1$ (i.e., the norms of the f and g vectors are unity). The output can be sold on the product market for price p. The Principal can write a contract $w \in W \subset \{w : M \to \mathbb{R}\}$ that determines a transfer $w(m)$ that she is compelled to pay the Agent if performance measure m is realized. Since the performance measure is binary, contracts take the form $w = s + bm$.

The Agent has an outside option that provides utility $\bar{u}$ to the Agent and $\bar{\pi}$ to the Principal. If the outside option is not exercised, the Principal's and Agent's preferences are, respectively,

$$\begin{aligned} \Pi(w,e) &= f_1 e_1 + f_2 e_2 - E\left[w(m)\mid e\right] \\ U(w,e) &= s + b\left(g_1 e_1 + g_2 e_2\right) - \frac{c}{2}\left(e_1^2 + e_2^2\right). \end{aligned}$$

Timing The timing of the game is exactly the same as before.

1. P offers A a contract $w(m)$, which is commonly observed.
2. A accepts the contract $(d = 1)$ or rejects it $(d = 0)$ and receives $\bar{u}$ and the game ends. This decision is commonly observed.
3. If A accepts the contract, A chooses effort vector e and incurs cost $c(e)$. e is only observed by A.
4. Performance measure m is drawn from distribution with pdf $f(\cdot\mid e)$ and output y is drawn from distribution with pdf $g(\cdot\mid e)$. m is commonly observed.
5. P pays A an amount $w(m)$. This payment is commonly observed.

Equilibrium The solution concept is the same as before. A **pure-strategy subgame-perfect equilibrium** is a contract $w^* \in W$, an acceptance decision $d^* : W \to \{0,1\}$, and an effort choice $e^* : W \times \{0,1\} \to \mathbb{R}_+^2$ such that given the contract w^*, the Agent optimally chooses d^* and e^*, and given d^* and e^*, the Principal optimally offers contract w^*. We will say that the optimal contract induces effort e^*.

The Program The principal offers a contract $w = s + bm$ and proposes an effort level e in order to solve

$$\max_{s,b,e} p\left(f_1 e_1 + f_2 e_2\right) - \left(s + b\left(g_1 e_1 + g_2 e_2\right)\right)$$

subject to the incentive-compatibility constraint

$$e \in \underset{\hat{e} \in \mathbb{R}_+}{\operatorname{argmax}}\, s + b\left(g_1 \hat{e}_1 + g_2 \hat{e}_2\right) - \frac{c}{2}\left(\hat{e}_1^2 + \hat{e}_2^2\right)$$

and the individual-rationality constraint

$$s + b\left(g_1 e_1 + g_2 e_2\right) - \frac{c}{2}\left(e_1^2 + e_2^2\right) \geq \bar{u}.$$

Equilibrium Contracts and Effort Given a contract $s + bm$, the Agent will choose efforts

$$\begin{aligned} e_1^*(b) &= \frac{b}{c} g_1 \\ e_2^*(b) &= \frac{b}{c} g_2. \end{aligned}$$

The Principal will choose s so that the individual-rationality constraint holds with equality

$$s + b\left(g_1 e_1^*(b) + g_2 e_2^*(b)\right) = \bar{u} + \frac{c}{2}\left(e_1^*(b)^2 + e_2^*(b)^2\right).$$

Since contracts send the Agent off in the "wrong direction" relative to what maximizes total surplus, providing the Agent with higher-powered incentives by increasing b sends the agent farther of in the wrong direction. This is costly for the Principal, because in order to get the Agent to accept the contract, she has to compensate him for his effort costs, even if they are in the wrong direction.

The Principal's unconstrained problem is therefore

$$\max_b p\left(f_1 e_1^*(b) + f_2 e_2^*(b)\right) - \frac{c}{2}\left(e_1^*(b)^2 + e_2^*(b)^2\right) - \bar{u}.$$

Taking first-order conditions,

$$pf_1\frac{\partial e_1^*}{\partial b}+pf_2\frac{\partial e_2^*}{\partial b}=ce_1^*\left(b^*\right)\frac{\partial e_1^*}{\partial b}+ce_2^*\left(b^*\right)\frac{\partial e_2^*}{\partial b},$$

or

$$\begin{aligned}pf_1g_1+pf_2g_2 &= b^*g_1g_1+b^*g_2g_2\\ b^* &= p\frac{f_1g_1+f_2g_2}{g_1^2+g_2^2}=p\frac{f\cdot g}{g\cdot g}=p\frac{||f||}{||g||}\cos\theta=p\cos\theta,\end{aligned}$$

where $\cos\theta$ is the angle between the vectors f and g. That is, the optimal incentive slope depends on the relative magnitudes of the f and g vectors (which in this model were assumed to be the same, but in a richer model this need not be the case) as well as how well-aligned they are. If m is a perfect measure of what the firm cares about, then g is a linear transformation of f and therefore the angle between f and g would be zero, so that $\cos\theta=1$. If m is completely uninformative about what the firm cares about, then f and g are orthogonal, and therefore $\cos\theta=0$. As a result, this model is often referred to as the **"cosine of theta model."**

Another way to view this model is as follows. Since formal contracts allow for unrestricted lump-sum transfers between the Principal and the Agent, the Principal would optimally like efforts to be chosen in such a way that they maximize total surplus:

$$\max_e p\left(f_1e_1+f_2e_2\right)-\frac{c}{2}\left(e_1^2+e_2^2\right),$$

or $e_1^*=\frac{p}{c}f_1$ and $e_2^*=\frac{p}{c}f_2$. That is, the Principal would like to choose a vector of efforts that is collinear with the vector f :

$$\left(e_1^*,e_2^*\right)=\frac{p}{c}\cdot\left(f_1,f_2\right).$$

Since contracts can only depend on m and not directly on y, the Principal has only limited control over the actions that the Agent chooses. That is, given a contract specifying incentive slope b, the Agent chooses $e_1^*(b) = \frac{b}{c} g_1$ and $e_2^*(b) = \frac{b}{c} g_2$. Therefore, the Principal can only (indirectly) choose a vector of efforts that is collinear with the vector g:

$$\left(e_1^*(b), e_2^*(b)\right) = \frac{b}{c} \cdot (g_1, g_2).$$

The question is then: which such vector maximizes total surplus (which the Principal will extract with an ex-ante lump-sum transfer)? That is, which point along the $k \cdot (g_1, g_2)$ ray minimizes the mean-squared error distance to $\frac{p}{c} \cdot (f_1, f_2)$?

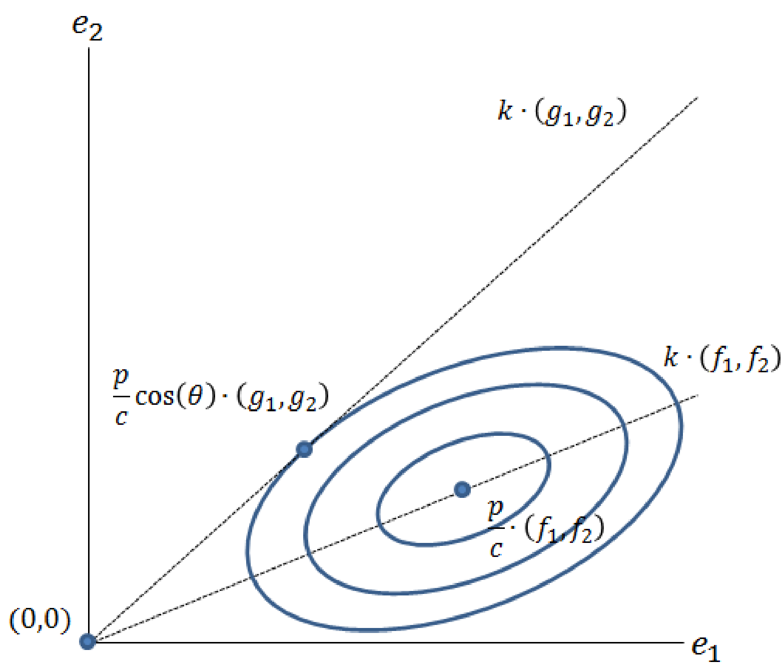


Figure 4

This is a more explicit "incomplete contracts" model of motivation. That is, we are explicitly restricting the set of contracts that the Principal can offer the Agent in a way that directly determines a subset of the effort space that the Principal can induce the Agent to choose among. And it is founded not on the idea that certain measures (in particular, y) are unobservable, but rather that they cannot be contracted upon.

Exercise. *Suppose there are N tasks rather than 2 (i.e., $e = (e_1, \ldots, e_N)$ and $c(e) = \frac{1}{2}(e_1^2 + \cdots + e_N^2)$) and $M < N$ linearly independent performance measures rather than 1 (i.e., $m_j = g_{1j}a_1 + \cdots + g_{Nj}a_N$ for $j = 1, \ldots, M$). Show that the optimal incentive slope vector is equal to the regression coefficient that would be obtained if one ran the regression $f_i = \alpha + \beta_1 g_{i1} + \cdots + \beta_M g_{iM} + \varepsilon_i$.*

Finally, we can derive the contract-augmented possibilities set and compare it to the technological possibilities set in this setting. First, let us derive the technological possibilities set. We can write

$$y(C) = \max_{e_1, e_2} f_1 e_1 + f_2 e_2$$

subject to

$$\frac{c}{2}\left(e_1^2 + e_2^2\right) \leq C.$$

The Lagrangian for this problem is

$$\mathcal{L} = p\left(f_1 e_1 + f_2 e_2\right) + \lambda\left(C - \frac{c}{2}\left(e_1^2 + e_2^2\right)\right),$$

and its first-order conditions are

$$\begin{aligned} p f_1 &= \lambda c e_1 \\ p f_2 &= \lambda c e_2, \end{aligned}$$

which implies that the optimum must always satisfy

$$\frac{e_1^*}{e_2^*} = \frac{f_1}{f_2}.$$

If we plug this condition into the constraint, which will hold with equality, we get

$$C = \frac{c}{2}\left(\left(\frac{f_1}{f_2} e_2\right)^2 + e_2^2\right) = \frac{c}{2}\frac{f_1^2 + f_2^2}{f_2^2} e_2^2 = \frac{c}{2}\frac{||f||^2}{f_2^2} e_2^2,$$

so that

$$e_2 = \left(\frac{2C}{c}\right)^{1/2} \frac{f_2}{||f||} \text{ and } e_1 = \left(\frac{2C}{c}\right)^{1/2} \frac{f_1}{||f||}.$$

The frontier of the technological possibilities set is therefore

$$y(C) = \left(\frac{2C}{c}\right)^{1/2} \frac{f_1^2 + f_2^2}{||f||} = \left(\frac{2C}{c}\right)^{1/2},$$

and the technological possibilities set is

$$Y^f = \{(y, -C) : y \leq y(C)\}.$$

To solve for the contract-augmented possibilities set, note that given b, the Agent chooses

$$(e_1(b), e_2(b)) = \frac{b}{c}(g_1, g_2).$$

At cost C, maximum production solves

$$\hat{y}(C) = \max_b f_1 e_1(b) + f_2 e_2(b)$$

subject to

$$\frac{c}{2}\left(e_1(b)^2 + e_2(b)^2\right) \leq C.$$

This cost constraint will hold with equality, which gives us

$$\frac{c}{2}\left(\frac{b}{c}\right)^2 \left(g_1^2 + g_2^2\right) = C$$

or

$$b(C) = (2Cc)^{1/2} \frac{1}{||g||}.$$

The frontier of the contract-augmented possibilities set is therefore

$$\begin{aligned}\hat{y}(C) &= f_1 \frac{b(C)}{c} g_1 + f_2 \frac{b(C)}{c} g_2 = \frac{b(C)}{c} f \cdot g = \left(\frac{2C}{c}\right)^{1/2} \frac{1}{||g||} ||f|| \, ||g|| \cos\theta \\ &= \left(\frac{2C}{c}\right)^{1/2} \cos\theta = \cos\theta \cdot y(C),\end{aligned}$$

and therefore the contract-augmented possibilities set is given by

$$\hat{Y}^f = \{(y, -C) : y \leq \hat{y}(C)\}.$$

Further Reading Holmstrom and Milgrom (1991, 1994) explore many interesting organizational implications of misaligned performance measures in multi-task settings. In particular, they show that when performance measures are misaligned, it may be optimal to put in place rules that restrict the actions an agent is allowed to perform, it may be optimal to split up activities across agents (job design), and it may be optimal to adjust the boundaries of the firm. Job restrictions, job design, boundaries of the firm, and incentives should be designed to be an internally consistent system. The model described in this section is formally equivalent to Baker's (1992) model in which the agent receives noncontractible private information about the effectiveness of his (single) task before making his effort decision, since his contingent plan of effort choices can be viewed as a vector of effort choices that differentially affect his expected pay. This particular specification was spelled out in Baker's (2002) article, and it is related to Feltham and Xie's (1994) model.