Economics 2010b Problem Set 2

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Due: Friday, 2/9 by 11:59pm

Exercise 6 (Adapted from MWG 16.C.3). In this exercise, you are asked to establish the first welfare theorem under a set of assumptions compatible with satiation. First, we will define the appropriate notion of equilibrium. Given an economy $\mathcal{E}$, an allocation $(x_i^*)_{i\in\mathcal{I}}$ and a price vector $p = (p_1, \ldots, p_L)$ constitutes a **price equilibrium with transfers** if there is an assignment of wealth levels $(w_1, \ldots, w_I)$ with $\sum_{i\in\mathcal{I}} w_i = p \cdot \left(\sum_{i\in\mathcal{I}} \omega_i\right)$ such that: (i) consumers optimize: $x_i^*(p, w_i) = x_i^*$ and (ii) markets clear: $\sum_{i\in\mathcal{I}} x_i^* = \sum_{i\in\mathcal{I}} \omega_i$. Suppose that every $\mathcal{X}_i$ is nonempty and convex and that every u_i is strictly concave. Prove the following:

(a) For every consumer i, there is at most one consumption bundle at which she is locally satiated. Such a bundle, if it exists, uniquely maximizes u_i on $\mathcal{X}_i$.

(b) Any price equilibrium with transfers is a Pareto optimum.

Exercise 7. This exercise illustrates that the importance of the assumption that there are a finite number of commodities for the first welfare theorem. Consider an economy in which there is one physical good, available at infinitely many dates: $t = 1, 2, \ldots,$ so there are effectively an infinite number of commodities: the physical good at date 1, the physical good at date 2, and so on. One consumer (or "generation") is born at each date $t = 1, 2, \ldots,$ and lives and consumes at dates t and $t + 1$ ("young" and "old"). We will refer to the consumer born in date t as consumer t. There is also one old consumer alive at date $t = 1$ (call her consumer 0). Each consumer is endowed with one unit of the good when she is young, and no storage is possible. Consumption in each period is non-negative, and each consumer t's preferences over consumption is given by $u_t(x_{t,t}, x_{t+1,t}) = u(x_{t,t}) + u(x_{t+1,t})$, where u is smooth, increasing, and strictly concave, with $u'(0) < \infty$.

(a) Show that there is a Walrasian equilibrium in which each consumer consumes her endowment and gets utility $u(1) + u(0)$.

(b) Show that the above Walrasian equilibrium is unique.

(c) Show that the above Walrasian equilibrium allocation is not Pareto optimal. In other words, construct a feasible allocation that is strictly better for each consumer.

[Optional] Exercise 8. This question is intended to guide you through a proof of the separating hyperplane theorem. This is more of an exercise in math than in economics, so feel free to skip to the next step if you get stuck.

(a) Prove that if $y \in \mathbb{R}^N$ and $\mathcal{C} \subseteq \mathbb{R}^N$ is closed, then there exists a point $z \in \mathcal{C}$ such that $||z - y|| \leq ||x - y||$ for all $x \in \mathcal{C}$. That is, there exists a point in $\mathcal{C}$ that is closest to y. (You may assume that $||\cdot||$ is the Euclidean norm.) Hint: use the Weierstrass extreme value theorem—if f is a real-valued and continuous function on domain $\mathcal{S}$, and $\mathcal{S}$ is compact and non-empty, then there exists x such that $f(x) \geq f(y)$ for all $y \in \mathcal{S}$.

(b) Suppose further that $\mathcal{C} \subseteq \mathbb{R}^N$ is convex, and note from above that if $y \notin \mathcal{C}$, then there exists $z \in \mathcal{C}$ that is closest to y. Let $x \in \mathcal{C}$ with $x \neq z$.

(i) Show that $(y - z) \cdot z \geq (y - z) \cdot x$. Hint: consider $||y - (z + t(x - z))||$ for $t \in [0, 1]$, the distance between y and a convex combination of x and z.

(ii) Use the above result to show that for all $x \in \mathcal{C}$, $(y - z) \cdot y > (y - z) \cdot x$.

(iii) Explain how this is a special case of the separating hyperplane theorem, which states that for any disjoint convex sets $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^N$, there exists nonzero $p \in \mathbb{R}^N$ such that $p \cdot u \geq p \cdot v$ for any $u \in \mathcal{A}$ and $v \in \mathcal{B}$.

(iv) Use the result of (ii) to deduce the separating hyperplane theorem. Hint: consider $y = 0$ and $\mathcal{C} = \mathcal{A} - \mathcal{B} = \{u - v : u \in \mathcal{A}, v \in \mathcal{B}\}$.

Exercise 9 (Adapted from MWG 16.C.4). Suppose that for each consumer, there is a "happiness function" depending on her own consumption only, given by $u(x_i)$. Every consumer's utility depends positively on her own and everyone else's "happiness" according to the utility function

$$U_i(x_1, \ldots, x_l) = U_i(u_1(x_1), \ldots, u_l(x_l)).$$

Show that if $x = (x_1, \ldots, x_l)$ is Pareto optimal relative to the $U_i(\cdot)$'s, then $x = (x_1, \ldots, x_l)$ is also a Pareto optimum relative to the u_i's. Does this mean a community of altruists can use competitive markets to attain Pareto optima? Does your argument depend on the concavity of the u_i's or the U_i's?

Exercise 10 (Adapted from MWG 17.C.4). Consider a pure exchange economy. The only novelty is that a progressive tax system is instituted according to the following rule: individual wealth is no longer $p \cdot \omega_i$; instead, anyone with wealth above the mean of the population must contribute half of the excess over the mean into a fund, and those below the mean receive a contribution from the fund in proportion to their deficiency below the mean.

(a) For a two-consumer society with endowments $\omega_1 = (1, 2)$ and $\omega_2 = (2, 1)$, write the after-tax wealths of the two consumers as a function of prices.

(b) If the consumer preferences are continuous, strictly convex, and strongly monotone, will the excess demand functions satisfy the conditions required for existence stated in Lemma 3?

Exercise 11 (Adapted from MWG 17.D.1). Consider an exchange economy with two commodities and two consumers. Both consumers have homothetic preferences of the constant elasticity variety. Moreover, the elasticity of substitution is the same for both consumers and is small (i.e., commodities are close to perfect complements). Specifically,

$$u_1(x_{1,1}, x_{2,1}) = \left(2x_{1,1}^{\rho} + x_{2,1}^{\rho}\right)^{1/\rho} \text{ and } u_2(x_{1,2}, x_{2,2}) = \left(x_{1,2}^{\rho} + 2x_{2,2}^{\rho}\right)^{1/\rho},$$

and $\rho = -4$. The endowments are $\omega_1 = (1, 0)$ and $\omega_2 = (0, 1)$. Compute the excess demand function of this economy and find the set of competitive equilibria.