Economics 2010b: Microeconomic Theory II

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Foundations, Production

1 Foundations of General Equilibrium Theory

In one of the last topics in general equilibrium theory, we will ask when the GE model we have spent the couple weeks talking about is a reasonable model. When does a Walrasian equilibrium arise in situations in which many individuals in an economy interact with one another? We will consider two alternative foundations for Walrasian equilibrium, and in both, the answer to this question will be: when individuals in the economy are "small."

What is potentially problematic about Walrasian equilibrium as a description of the economy is that prices are endogenous variables, but they are not the explicit choices of anyone in the economy. In reality, individuals set prices—they bid in auctions, they post prices in their stores, they negotiate prices with their suppliers. Setting prices are individual decisions. One of the main premises of GE is that, when individuals are small relative to the economy, "market forces" pin down the prices at which trade occurs, and although it may be possible, it would be unwise for individuals to choose any other prices or any other consumption bundles.

Providing microfoundations for GE theory boils down to providing an answer to the question, "Under what conditions do small individuals lack market power, in the sense that they are forced to trade only at competitive prices?" There are two main approaches we will consider here. The first is the *cooperative game theory approach* in which the primitives of the model remain the agents' endowments and preferences, and the process of price-setting and trade is still specified only implicitly. Under this approach, however, the solution concept we will be using does not directly involve prices. Yet as the economy becomes large, consumers will receive the same allocations they would receive in a Walrasian equilibrium.

The second approach we will consider is the *non-cooperative game theory approach* in which we will explicitly model price-setting and trade and think about the (Nash) equilibria of the resulting trading processes. Consumers will have actions that can directly affect the prices they and other consumers pay for different commodities, and therefore equilibria will generically be inefficient. In the limit as the economy becomes large, however, consumers' actions will have little effect on prices, and equilibrium consumption choices will converge to Walrasian equilibrium allocations.

A benefit of the non-cooperative approach relative to the cooperative approach is that Pareto optimality will arise as a result rather than as a maintained assumption. We will therefore be able to develop some deeper intuition for why, exactly, Walrasian equilibrium allocations are Pareto optimal. We will then conclude this section with a brief discussion of who gets what in equilibrium and how under the notion of *competitive equilibrium* in which consumers' impact on prices is miniscule, consumers receive exactly what they contribute to the economy. A common theme in these approaches is that while Walrasian equilibrium is not necessarily a good description of small-numbers interactions, it may be a reasonable description of large-numbers interactions.

1.1 The Cooperative Approach

Going back to his 1881 classic, *Mathematical Psychics*, Edgeworth proposed that in economies with a small number of individuals, the outcome might be "indeterminate." We saw an example of this in the first week when we looked at Edgeworth boxes—with two consumers, Edgeworth believed that the only prediction we could reasonably make is that the final allocation would lie on the contract curve: the set of Pareto optimal allocations that are preferred by each consumer to her endowment. But he also conjectured that as the number of consumers grows, the scope for contracting among different consumers grows, and the resulting contract curve shrinks until it reaches only the set of Walrasian equilibrium allocations.

In the 20th century, economists formalized a version of this argument in what is known as the *core convergence theorem*. In order to describe what the core convergence theorem is, we will first have to define what the *core* is. The idea of the core is that it is the set of allocations for which no group of consumers can get together and trade with each other and do strictly better. Formally, consider a pure exchange economy $\mathcal{E}$ with I consumers whose preferences are continuous, strictly convex, and strongly monotone. We will define the core by defining what it is not. We will say that a coalition $\mathcal{S} \subseteq \mathcal{I}$ of consumers *blocks* an allocation if its members can all do strictly better by trading among themselves. In the case of $\mathcal{I} = \{1, 2\}$ that we considered in the Edgeworth box, any allocation that is not in the Pareto set is blocked by the coalition $\{1, 2\}$, and any allocation in the Pareto set but not on the contract curve is blocked either by coalition $\{1\}$ or by coalition $\{2\}$.

Definition 5. A coalition $\mathcal{S} \subseteq \mathcal{I}$ **blocks** the allocation $x^* = (x_1^*, \ldots, x_I^*) \in \mathbb{R}_+^{LI}$ if there exists another allocation such that:

1. $u_i(x_i) > u_i(x_i^*)$ for all $i \in \mathcal{S}$, and
2. $\sum_{i \in \mathcal{S}} x_i \leq \sum_{i \in \mathcal{S}} \omega_i$.

The core is the set of feasible unblocked allocations.

Definition 6. A feasible allocation x^* is in the **core** if it is not blocked by any coalition. The core is therefore the set of unblocked feasible allocations.

In terms of the Edgeworth box example, the core corresponds to the contract curve, since all other allocations are blocked by some coalition. The core convergence theorem provides conditions under which, when the economy grows large, the set of *core* allocations coincides with the set of *Walrasian equilibrium* allocations. We will break this claim up into two parts. First, we will show that any Walrasian equilibrium allocation is in the core. Then, we will show that any allocation that remains in the core as the economy grows large is a Walrasian equilibrium allocation.

Proposition 3. Any Walrasian equilibrium allocation is in the core.

Proof of Proposition 3. Let $\left(p^*, (x_i^*)_{i\in\mathcal{I}}\right)$ be a Walrasian equilibrium. Suppose $(x_i^*)_{i\in\mathcal{I}}$ is not in the core. Then there is some coalition $\mathcal{S} \subseteq \mathcal{I}$ that can block x^* with some other feasible allocation $\hat{x}$. Then $p^* \cdot \hat{x}_i > p^* \cdot \omega_i$ for all $i \in \mathcal{S}$ by consumer optimality. Since this holds for all $i \in \mathcal{S}$, we must also have $p^* \cdot \left(\sum_{i\in\mathcal{S}} \hat{x}_i\right) > p^* \cdot \left(\sum_{i\in\mathcal{S}} \omega_i\right)$. Since $p^* \geq 0$, this implies that $\sum_{i\in\mathcal{S}} \hat{x}_{l,i} > \sum_{i\in\mathcal{S}} \omega_{l,i}$ for some commodity l. But this means that $\hat{x}$ was not feasible to begin with.■

In order to establish the other direction of the core convergence theorem, we will have to define formally what we mean when we say that an economy grows large. As we know from the Edgeworth box example, when there are only two consumers, not every core allocation is a Walrasian equilibrium allocation. Whether this result remains true as we add more consumers to the economy depends on *how* we add more consumers to the economy. For example, if consumers 1 and 2 only care about their consumption of pens and pencils, and they are endowed with pens and pencils and nothing else, then if we add a bunch of other consumers who care only about their consumption of paper clips and have an endowment of paper clips and nothing else, then this will not do anything to make the terms of trade between consumers 1 and 2 more competitive.

If instead, we "grow the economy" by adding more consumers like consumer 1 (i.e., consumers who have the same preferences and endowment as consumer 1) and adding more consumers like consumer 2, then core allocations do begin to look more like Walrasian equilibrium allocations. Roughly speaking, the reason why is that if any particular consumer is getting a "good deal" from the rest of the consumers at a particular allocation, then the other consumers would prefer to cut her out of the deal and redistribute her net trade among themselves. This may not work when there are only a couple consumers in the economy because the excluded consumer may be hard to replace.

In order to formalize this argument, suppose there are H **types** of consumers $h \in \mathcal{H} = \{1, \ldots, H\}$. A type-$h$ consumer has preferences u_h and endowment ω_h. For each integer

$N > 0$, we consider the N**-replica economy**, which is a pure exchange economy consisting of $I_N \equiv N \cdot H$ consumers, N of each type. We will refer to an allocation in which consumers of the same type consume the same consumption bundle as an **equal-treatment allocation**. The next lemma establishes that any core allocation of the N-replica economy is an equal-treatment allocation. Denote by $x_{h,n}$ the allocation of the n^{th} consumer of type h.

Lemma 5. Suppose x is in the core of the N-replica economy, for some $N > 0$. Then all consumers of the same type receive the same allocation: $x_{h,n} = x_{h,m}$ for all $n, m \leq N$ and all types $h \in \mathcal{H}$.

Proof of Lemma 5. We will proceed by way of contradiction. Suppose x is in the core of the N-replica economy for some $N > 0$, but for some type of consumer—without loss of generality, say type 1—not all consumers of that type receive the same allocation. We will want to show that in fact, such an allocation is not in the core. In particular, we will show that the coalition consisting of the worst-off consumer of every type can block the allocation x.

To see why this is true, let $\hat{x}_h = \frac{1}{N} \sum_n x_{h,n}$ denote the average allocation of type-h consumers. Without loss of generality, suppose that it is consumer number 1 of each type h who is worst off within type h. By strict convexity of preferences, $u_h(\hat{x}_h) \geq u_h(x_{h,1})$ for all h, and $u_1(\hat{x}_1) > u_1(x_{1,1})$. The coalition $\{(1,1), \ldots, (H,1)\}$ can attain consumption vector $(\hat{x}_1, \ldots, \hat{x}_H)$ for its members, since feasibility implies

$$\sum_{h \in \mathcal{H}} \hat{x}_h = \frac{1}{N} \sum_{h \in \mathcal{H}} \sum_{n=1}^{N} x_{h,n} \leq \frac{1}{N} \sum_{h \in \mathcal{H}} \sum_{n=1}^{N} \omega_h = \sum_{h \in \mathcal{H}} \omega_h.$$

Finally, continuity and strong monotonicity of preferences imply that the consumption vector $(\hat{x}_1, \ldots, \hat{x}_H)$ can be perturbed to satisfy $u_h(\hat{x}_h) > u_h(x_{h,1})$ for all h, so the strict inequalities required to apply the definition of blocking are satisfied.■

This Lemma shows that any core allocation for an N-replica economy takes the form of a **type allocation** $(x_1, \ldots, x_H) \in \mathbb{R}_+^{LH}$, where each consumer of type h receives allocation

x_h. Let $\mathcal{C}_N \subseteq \mathbb{R}_+^{LH}$ be the set of core allocations in the N-replica economy. Note that the set of core allocations shrinks as we replicate the economy: $\mathcal{C}_{N+1} \subseteq \mathcal{C}_N$ for all N. This is because any type allocation that is blocked by some coalition in the N-replica economy will be blocked by exactly the same coalition in the $N+1$-replica economy. At the same time, from Proposition 3, we know that the the set of Walrasian equilibrium allocations is independent of N and is always contained in $\mathcal{C}_N$. Debreu and Scarf (1963) proved that as $N \to \infty$, the set $\mathcal{C}_N$ shrinks to exactly the set of Walrasian equilibrium allocations. The version of the theorem we will prove will rely on two additional assumptions about preferences and endowments, although these assumptions can be relaxed.

Assumption A1' (continuous differentiability). For all consumers of type $h \in \mathcal{H}$, u_h is continuously differentiable.

Assumption A4' (interiority). For each $h \in \mathcal{H}$, ω_h is strictly preferred to any consumption bundle x_h that is not strictly positive.

Theorem 7 (Core Convergence Theorem). Suppose $\mathcal{E}$ satisfies $(A1'), (A2'), (A3), (A4')$. If $x \in \mathcal{C}_N$ for all N, then x is a Walrasian equilibrium allocation.

Proof of Theorem 7. At a high level, the proof of this theorem first argues that if $x \in \mathcal{C}_N$ for all N, then it is Pareto-optimal, which means that marginal rates of substitutions are equal across consumers and proportional to a price vector that will be used to construct a Walrasian equilibrium. It then argues that if at this price vector, some type of consumer is getting a "good deal" in that they consuming a bundle that is more expensive than their endowment, then $N-1$ consumers of this type along with all the other consumers in the economy can form a blocking coalition. This means that no type of consumer can be getting a good deal if $x \in \mathcal{C}_N$ for all N. The proof concludes with an argument that if no consumers are getting a good deal at $x \in \mathcal{C}_N$ for all N, then x can be decentralized as a Walrasian equilibrium allocation.

Step 1. Pareto-optimal allocations equate marginal rates of substitution across consumers and can be used to construct candidate prices.

Take an $x^* \in \mathcal{C}_N$ for all N. Since x^* is in the core, it is a Pareto-optimal allocation. Assumptions $(A1')$ and $(A4')$ ensure that at x^*,

$$\frac{\partial u_h/\partial x_{l,h}}{\partial u_h/\partial x_{l',h}} = \frac{\partial u_{h'}/\partial x_{l,h'}}{\partial u_{h'}/\partial x_{l',h'}} \text{ for all } h, h', l, l'.$$

Construct a price vector p^* for which $p_1^* = 1$, and

$$p_l^* = \frac{\partial u_h/\partial x_{l,h}}{\partial u_h/\partial x_{1,h}} \text{ for any } h,$$

so that relative prices match relative marginal utilities. We will now argue that (p^*, x^*) is a Walrasian equilibrium.

Step 2. No consumer types are getting a "good deal" at p^*.

Suppose that type-1 consumers are getting a "good deal" in the sense that their consumption is worth more than their endowment at prices p^*: $p^* \cdot x_1^* > p^* \cdot \omega_1$. We want to show that if this is the case, then x^* is not, in fact, in $\mathcal{C}_N$ for all N. To see why this is the case, note the marginal utility to any consumer type h of consuming an additional ε amount of consumer 1's net trade, $x_1^* - \omega_1$, is, to first order,

$$\varepsilon \sum_{l \in \mathcal{L}} \frac{\partial u_h}{\partial x_{l,h}} \left(x_{l,1} - \omega_{l,1}\right).$$

Since $p^* \cdot (x_1^* - \omega_1) > 0$, and the vector $(\partial u_h/\partial x_{1,h}, \ldots, \partial u_h/\partial x_{L,h})$ is proportional to p^*, this marginal utility is strictly positive. For $\varepsilon > 0$ sufficiently small, therefore, each consumer type h strictly prefers consuming $x_h^* + \varepsilon \left(x_1^* - \omega_1\right)$ to consuming x_h^*.

Now, consider allocation x^* in the N-replica economy. Suppose the coalition $\mathcal{S}$ consisting of everyone except a single type-1 consumer proposes an allocation that gives each coalition member of type h consumption $\hat{x}_h = x_h^* + \frac{1}{NH-1}\left(x_1^* - \omega_1\right)$. This allocation $\hat{x}$ is feasible for the coalition (you can check MWG, p. 658 for the argument for feasibility). Moreover, by the argument in the previous paragraph, if N is sufficiently large, $\hat{x}$ is strictly preferred to x^* by every coalition member. The coalition $\mathcal{S}$ therefore blocks the allocation x^*, so x^* is not in $\mathcal{C}_N$ for N sufficiently large. This contradicts the hypothesis that $p^* \cdot x_1^* > p^* \cdot \omega_1$, so it must be the case that no consumer types are getting a "good deal."

Step 3: Show that (p^*, x^*) is a Walrasian equilibrium.

From the previous step, we know that x_h^* is affordable for type h at prices p^* for all types h: $p^* \cdot x_h^* \leq p^* \cdot \omega_h$ for all $h \in \mathcal{H}$. The bundle x_h^* also satisfies consumer optimality. This is because under our interiority, differentiability, and convexity assumptions, each consumer type h will choose a consumption bundle that equates $\frac{\partial u_h}{\partial x_{l,h}} / p_l^*$ across commodities and therefore will optimally choose x_h^* at prices p^*.

Finally, since x^* is Pareto-optimal, it must also be feasible: $N \sum_{h \in \mathcal{H}} x_h^* \leq N \sum_{h \in \mathcal{H}} \omega_h$. Since preferences are monotone, this inequality must hold with equality, so the market-clearing condition is also satisfied. The vector (p^*, x^*) is therefore a Walrasian equilibrium.■

The core convergence theorem is an important result that is probably the best-known statement of the idea that large markets are approximately competitive. Note that there are no prices in the notion of the core. Yet what the core convergence theorem is saying is that in a sufficiently large economy, any allocation in the core corresponds to exactly what consumers would consume at equilibrium prices in a Walrasian equilibrium. The theorem itself has a number of shortcomings, however.

First, the notion of a replica economy is extreme. We typically think of each individual as being unique, yet the thought experiment the core convergence theorem carries out requires that there are, in the limit, infinitely many people who have exactly the same preferences and endowments as you. Second, the theorem itself is not an approximation result—it does not say that for any finite N, any allocation in the core is *approximately* a Walrasian equilibrium allocation, since it does not say anything about distance. There is a large literature at the intersection of cooperative game theory and general equilibrium theory that tries to extend this result into something that is more convincing. One branch (following Arrow and Hahn, 1971, and others) relaxes the assumption of exact replication and tries to say something about core allocations in large but finite economies. Another branch (following Aumann, 1964) instead looks directly at economies with a continuum of consumers, for which the core convergence theorem provides an exact equivalence between core allocations and Walrasian equilibrium allocations.

1.2 The Non-Cooperative Approach

The cooperative approach imposes no structure on the underlying trading institutions and as a result, it has little to say about how prices are determined and under what conditions they are likely to correspond to Walrasian equilibrium prices. In contrast, under the non-cooperative approach, individual consumers make decisions that "aggregate up" to determine prices.

Suppose there are I consumers, a set $\mathcal{P} \subseteq \mathbb{R}^L$ of possible price vectors, and a set $\mathcal{A}$ of **market actions**. Each consumer $i \in \mathcal{I}$ has a set $\mathcal{A}_i \subset \mathcal{A}$ and an endowment vector $\omega_i \in \mathbb{R}^L$. For each $a_i \in \mathcal{A}_i$ and $p \in \mathcal{P}$, a **trading rule** assigns a net trade vector $g(a_i; p) \in \mathbb{R}^L$ to consumer i, satisfying $p \cdot g(a_i; p) = 0$. Given a vector of market actions $a = (a_1, \ldots, a_I)$, a market-clearing process generates a price vector $p(a) \in \mathcal{P}$. Throughout, we will assume each i has a utility function of the form $u_i(g(a_i; p) + \omega_i)$. An equilibrium of the resulting game is just a Nash equilibrium.

Definition 7. The profile $a^* = (a_1^*, \ldots, a_I^*)$ of market actions is a **trading equilibrium** if, for every consumer $i \in \mathcal{I}$,

$$u_i(g(a_i^*; p(a^*)) + \omega_i) \geq u_i\left(g\left(a_i; p\left(a_i, a_{-i}^*\right)\right) + \omega_i\right) \text{ for all } a_i \in \mathcal{A}_i.$$

We will consider a particular trading rule referred to as Shapley and Shubik's (1977) **trading posts**. It is not particularly realistic, but it does form a complete general equilibrium model in which all consumers interact strategically. Suppose there are I consumers and

L commodities. Commodity L, which we will call "money," is treated differently from the other commodities, and we normalize its price to 1. For each of the other $L-1$ commodities, there is a *trading post* at which consumers can exchange money for the commodity.

At each trading post $l \leq L-1$, each consumer i places bids $a_{l,i} = \left(a'_{l,i}, a''_{l,i}\right) \in \mathbb{R}^2_+$. The first value, $a'_{l,i}$, is interpreted as the amount of commodity l that consumer i is willing to put up for sale in exchange for money. The second value, $a''_{l,i}$ is the amount of money that she puts up in exchange for commodity l. These bids must therefore satisfy $a'_{l,i} \leq \omega_{l,i}$, and $\sum_{l \leq L-1} a''_{l,i} \leq \omega_{L,i}$. Given the consumers' bids, the price of commodity l is set to be equal to the total amount spent on commodity l divided by the total quantity of commodity l supplied:

$$p_l = \frac{\sum_{i \in \mathcal{I}} a''_{l,i}}{\sum_{i \in \mathcal{I}} a'_{l,i}}.$$

Each consumer i receives allocation $x_{l,i} = g_l\left(a_i; p\right) + \omega_{l,i}$, where

$$g_l\left(a_i; p\right) = \frac{a''_{l,i}}{p_l} - a'_{l,i}$$

for all $l \leq L-1$ and $x_{L,i} = \omega_L - \sum_{l=1}^{L-1} a''_{l,i}$.

If there is a large number of consumers trading each commodity, then each consumer's bids would have a negligible effect on prices, and each consumer's allocation will be arbitrarily close to the solution to their problem

$$\max_{x_i \in \mathbb{R}^L_+} u_i\left(x_i\right) \text{ s.t. } p \cdot x_i \leq p \cdot \omega_i.$$

Thus, even though prices *are* determined as the aggregation of individual consumers' actions, when the economy is sufficiently large, each *individual* consumer's actions have no effect on prices. Under this approach, price-taking behavior is therefore a result rather than an assumption. We will refer to the resulting equilibrium as a *competitive equilibrium.*

One important difference between the cooperative approach and the non-cooperative approach to the foundations of GE theory is that under the cooperative approach, the allocations we considered were always Pareto optimal. In contrast, under the non-cooperative approach, allocations are *not* Pareto efficient for any finite market size. When the size of the economy grows does the set of equilibrium allocations become approximately Pareto optimal. Only the non-cooperative approach can, therefore, really tell us anything about *why* Walrasian equilibrium allocations are Pareto optimal.

1.3 Who Gets What? The No-Surplus Condition

This section concludes our discussion of the competitive foundations of general equilibrium theory. In particular, we will ask whether Walrasian equilibria can be characterized by the idea that consumers get exactly what they contribute to the welfare of society. To answer this question, we will consider a special class of preferences in which the notion of the *welfare of society* is well-defined. In particular, suppose there are H types of consumers, $h \in \mathcal{H} = \{1, \ldots, H\}$, and each type of consumer is endowed with ω_h and has *quasi-linear preferences.*

Assumption QL (quasilinearity). For each type $h \in \mathcal{H}$, there is a concave, differentiable, strictly increasing function $v_h(x_{1,h}, \ldots, x_{L-1,h})$ such that type h preferences are $u_h(x_h) = v_h(x_{1,h}, \ldots, x_{L-1,h}) + x_{L,h}$, where $x_h \in \mathbb{R}_+^{L-1} \times \mathbb{R}$.

When consumers have quasilinear preferences, commodity L is what is referred to as the **money commodity**. It is a commodity for which all consumers have the same marginal utility and which consumers can consume any (positive or negative) amount of. The assumption of quasilinear preferences allows for cardinal measures of individuals' private rewards and their contribution to social welfare.

An economy is defined by a profile $(I_1, \ldots, I_H)$ of consumers of the different types, for a total of $I = \sum_{h \in \mathcal{H}} I_h$ consumers. For any economy, we can define the **social welfare**,

$V(I_1, \ldots, I_H)$, as the solution to the following problem:

$$V(I_1, \ldots, I_H) = \max_{(x_h)_{h \in \mathcal{H}}} \sum_{h \in \mathcal{H}} I_h u_h(x_h)$$

subject to feasibility: $\sum_{h \in \mathcal{H}} I_h x_h \leq \sum_{h \in \mathcal{H}} I_h \omega_h$ and $x_{l,h} \geq 0$ for all $l \in \{1, \ldots, L-1\}$ and for all h. This function is homogeneous of degree one in its arguments, so we can describe the economy in terms of its per-capita social welfare $V(I_1/I, \ldots, I_H/I) = V(I_1, \ldots, I_H)/I$, and therefore if we extend the model to one in which there are a continuum of consumers, with mass $\mu_h \geq 0$ of type $h \in \mathcal{H}$ with $\sum_{h \in \mathcal{H}} \mu_h = 1$, we can write $\mu = (\mu_1, \ldots, \mu_H)$ and

$$V(\mu) = \max_{(x_h)_{h \in \mathcal{H}}} \sum_{h \in \mathcal{H}} \mu_h u_h(x_h) \tag{1}$$

subject to feasibility: $\sum_{h \in \mathcal{H}} \mu_h x_h \leq \sum_{h \in \mathcal{H}} \mu_h \omega_h$ and $x_{l,h} \geq 0$ for all $l \in \{1, \ldots, L-1\}$.

Given a continuum population of consumers, we can define a consumer of type h's **marginal contribution to social welfare** as $\partial V(\mu)/\partial \mu_h$. We will say that a feasible allocation $(x_h^*)_{h \in \mathcal{H}}$ is a **no-surplus allocation** if

$$u_h(x_h^*) = \frac{\partial V(\mu)}{\partial \mu_h} \text{ for all } h \in \mathcal{H}.$$

In other words, at a no-surplus allocation, each consumer is receiving in utility exactly what she contributes to social welfare. With this definition in mind, we can state the no-surplus characterization of Walrasian equilibrium.

Theorem 8 (No-Surplus Characterization). For any continuum population $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_H) >> 0$, a feasible allocation $(x_1^*, \ldots, x_H^*) >> 0$ is a no-surplus allocation if and only if it is a Walrasian equilibrium allocation.

Proof of Theorem 8. The structure of the proof is as follows. We will show that if $(x_h^*)_{h \in \mathcal{H}}$ is a no-surplus allocation, then it solves (1). We will then show that if $(x_h^*)_{h \in \mathcal{H}}$ solves (1), then $(x_h^*)_{h \in \mathcal{H}}$ is a Walrasian equilibrium allocation for a suitable price vector p^*. Finally, we will show that if $(x_h^*)_{h\in\mathcal{H}}$ is a Walrasian equilibrium allocation, then it is a no-surplus allocation.

Step 1: $(x_h^*)_{h\in\mathcal{H}}$ is no-surplus $\Rightarrow$ $(x_h^*)_{h\in\mathcal{H}}$ solves (1).

Suppose $(x_h^*)_{h\in\mathcal{H}}$ is a no-surplus allocation. We know that the function $V(\bar{\mu})$ is homogeneous of degree one in $\bar{\mu}$, so by Euler's formula, we can write

$$V(\bar{\mu}) = \sum_{h\in\mathcal{H}} \bar{\mu}_h \frac{\partial V(\bar{\mu})}{\partial \mu_h} = \sum_{h\in\mathcal{H}} \bar{\mu}_h u_h(x_h^*),$$

where the last equality used the fact that $(x_h^*)_{h\in\mathcal{H}}$ is a no-surplus allocation. This implies that $(x_h^*)_{h\in\mathcal{H}}$ is a solution to the social welfare-maximization problem for $\mu = \bar{\mu}$.

Step 2: $(x_h^*)_{h\in\mathcal{H}}$ solves (1) $\Rightarrow$ $(x_h^*)_{h\in\mathcal{H}}$ is a WE allocation.

Suppose now that $(x_h^*)_{h\in\mathcal{H}}$ is a feasible allocation that yields social welfare $V(\bar{\mu})$. Denote by p_l^*, $l = 1,\ldots,L$, the values of the Lagrange multipliers for commodity-l feasibility constraint, $\sum_{h\in\mathcal{H}} \bar{\mu}_h (x_{l,h} - \omega_{l,h}) \leq 0$, in the social-welfare-maximization problem. Because $u_h(\cdot)$ is quasilinear for all $h \in \mathcal{H}$, we will have $p_L^* = 1$ and $p_l^* = \partial u_h(x_h^*)/\partial x_{l,h}$ for all $l \in \{1,\ldots,L-1\}$ and for all $h \in \mathcal{H}$. It follows then that if we let $p^* = (p_1^*,\ldots,p_L^*)$, then $\left(p^*, (x_h^*)_{h\in\mathcal{H}}\right)$ is a Walrasian equilibrium.

Step 3: $(x_h^*)_{h\in\mathcal{H}}$ is a WE allocation $\Rightarrow$ $(x_h^*)_{h\in\mathcal{H}}$ is no-surplus.

Finally, we can apply the envelope theorem to (1) to get

$$\frac{\partial V(\bar{\mu})}{\partial \mu_h} = u_h(x_h^*) + p^* \cdot (\omega_h - x_h^*).$$

Since $(x_h^*)_{h\in\mathcal{H}}$ is consumer-optimal given prices p^*, by Walras's law, the second term is zero. We therefore have that $\partial V(\bar{\mu})/\partial \mu_h = u_h(x_h^*)$, so $(x_h^*)_{h\in\mathcal{H}}$ is a no-surplus allocation.$\blacksquare$

Viewed in light of the no-surplus characterization of Walrasian equilibrium, we can finally develop some intuition for the first welfare theorem result that Walrasian equilibrium allocations are Pareto optimal. If, at the margin, each consumer is receiving exactly what she contributes to society's welfare, then in some sense, the rest of society is indifferent to her presence. Since each consumer is not affecting the welfare of the rest of society, of course each consumer doing the best she can—which she is, by the consumer optimality condition of Walrasian equilibrium—is going to lead to a result that is best for society.

It is important to realize that when there are a finite number of individuals in society, there generically do not exist any no-surplus allocations. The reason for this is that it is typically impossible to give each consumer the full extent of her marginal contribution while maintaining feasibility. For example, when there are only two consumers in the economy, each consumer's contribution to social welfare is equal to the utility she would get if she consumes her endowment plus the *entire gains from trade*, and we cannot simultaneously give both consumers the entire gains from trade. This means that in smaller economies, Walrasian equilibrium allocations generically are not no-surplus allocations.

2 Firms and Production in General Equilibrium

So far in this class, we have focused on pure exchange economies. In doing so, we have assumed that all the commodities in the economy come essentially from nowhere. In other words, we have completely abstracted away from the supply side of the economy. The GE framework can be readily extended to allow for firms and productions as long as two conditions are satisfied: (1) firms' production technologies do not exhibit increasing returns to scale, and (2) firms are price-takers. In this section, we will describe how to extend the GE framework to allow for production and we will show that versions of the welfare theorems and the existence theorem hold. We will then consider some simple examples and conclude with a result that shows that in this framework, we can think of the entire supply side of the economy as a single firm.

2.1 Extending the Framework

There are I consumers $i \in \mathcal{I}$ with utility functions $(u_i)_{i \in \mathcal{I}}$ defined over the consumption of L commodities $l \in \mathcal{L}$, and there are J firms $j \in \mathcal{J}$. Each firm possesses a production set $\mathcal{Y}_j \in \mathbb{R}^L$. The production set $\mathcal{Y}_j$ describes a set of feasible production plans: if $y_j = (y_{1,j}, \ldots, y_{L,j}) \in \mathcal{Y}_j$, then $y_{l,j} < 0$ means that commodity l is being used as an input, and $y_{l,j} > 0$ means that commodity l is being produced as an output. The firms are owned by the households. **Consumer** i's **ownership share of firm** j is a $\theta_{i,j} \in [0,1]$. A **production economy** is then a collection $\mathcal{E} = \left(\left(u_i, \omega_i, (\theta_{i,j})_{j \in \mathcal{J}} \right)_{i \in \mathcal{I}}, (\mathcal{Y}_j)_{j \in \mathcal{J}} \right)$ of consumer preferences, consumer endowments, ownership shares, and production sets. Firm j takes prices $p \in \mathbb{R}^L$ as given and chooses a production plan $y_j \in \mathcal{Y}_j$ to maximize its profits:

$$\max_{y_j \in \mathcal{Y}_j} p \cdot y_j.$$

Our definition of Walrasian equilibrium extends naturally to production economies.

Definition 8. A **Walrasian equilibrium** for the production economy $\mathcal{E}$ is a vector $\left(p^*, (x_i^*)_{i \in \mathcal{I}}, \left(y_j^*\right)_{j \in \mathcal{J}} \right)$ that satisfies:

1. Firm profit maximization: for all $j \in \mathcal{J}$,

$$y_j^* \in \operatorname*{argmax}_{y_j \in \mathcal{Y}_j} p \cdot y_j,$$

2. Consumer optimization: for all consumers $i \in \mathcal{I}$,

$$x_i^* \in \operatorname*{argmax}_{x_i \in \mathcal{X}_i} u_i(x_i)$$

subject to

$$p \cdot x_i \leq p \cdot \omega_i + \sum_{j \in \mathcal{J}} \theta_{i,j} p \cdot y_j^*,$$

3. Market-clearing: for all commodities $l \in \mathcal{L}$

$$\sum_{i \in \mathcal{I}} x^*_{l,i} = \sum_{i \in \mathcal{I}} \omega_{l,i} + \sum_{j \in \mathcal{J}} y^*_{l,j}.$$

2.2 Assumptions on Production Sets

Just as we made a number of assumptions on consumer preferences and endowments, we will make several assumptions on production sets to ensure that a Walrasian equilibrium exists in a production economy. The simplest such assumption would be that Y_j is a convex and compact set for all firms $j \in \mathcal{J}$, but assuming that a production set is bounded is stronger than we need.

Assumption A5 (closed and convex): For all firms $j \in \mathcal{J}$, $\mathcal{Y}_j$ is closed and convex.

Assumption A6 (no production is feasible and free disposal): For all firms $j \in \mathcal{J}$, $0 \in \mathcal{Y}_j$, and for all $y_j \in \mathcal{Y}_j$, $\{y_j\} + \mathbb{R}^L_{- -} \subset \mathcal{Y}_j$.

These two assumptions rule out increasing returns to scale. To see why, note that if $y \in \mathcal{Y}_j$, then since $0 \in \mathcal{Y}_j$, so is αy_j for any $0 < \alpha < 1$, so it is always possible to scale down production or break it up into arbitrarily small productive units.

We will also need to make one further assumption on aggregate production to ensure that the supply side of the economy as a whole cannot produce something with nothing. We want to rule out, for example, situations where one firm can turn one pound of coffee beans into one cup of coffee, while another firm can turn one cup of coffee into two pounds of coffee beans. Define the **aggregate production set** to be the Minkowski sum of all the firms' production sets:

$$\mathcal{Y} = \sum_{j \in \mathcal{J}} \mathcal{Y}_j = \left\{ y : \text{there exist } y_1 \in \mathcal{Y}_1, \dots, y_J \in \mathcal{Y}_J \text{ such that } y = \sum_{j \in \mathcal{J}} y_j \right\}.$$

The following assumption is sufficient to rule out the implausible situations described above.

Assumption A7 (irreversibility): $\mathcal{Y} \cap -\mathcal{Y} = \{0\}$.

It is worth spending some time thinking about why assumptions $(A6)$ and $(A7)$ rule out the situations I just described.

2.3 Welfare Theorems and Existence of Walrasian Equilibrium

The definitions of feasibility and Pareto efficiency are easily extended to production economies.

Definition 9. An allocation and production plan $\left((x_i)_{i\in\mathcal{I}}, (y_j)_{j\in\mathcal{J}}\right)$ is **feasible** if

$$\sum_{i\in\mathcal{I}} x_{l,i} \leq \sum_{i\in\mathcal{I}} \omega_{l,i} + \sum_{j\in\mathcal{J}} y_{l,j} \text{ for all } l \in \mathcal{L}.$$

A feasible allocation and production plan $\left((x_i)_{i\in\mathcal{I}}, (y_j)_{j\in\mathcal{J}}\right)$ is **Pareto optimal** if there is no other feasible allocation and production plan $\left((\hat{x}_i)_{i\in\mathcal{I}}, (\hat{y}_j)_{j\in\mathcal{J}}\right)$ satisfying $u_i(\hat{x}_i) \geq u_i(x_i)$ for all i, with strict inequality for at least one i'.

We can now state the extensions of the two welfare theorems.

Theorem 9 (First Welfare Theorem). Suppose $\left(p^*, (x_i^*)_{i\in\mathcal{I}}, (y_j^*)_{j\in\mathcal{J}}\right)$ is a Walrasian equilibrium for production economy $\mathcal{E}$. Then if $(A2)$ holds, the allocation and production $\left((x_i^*)_{i\in\mathcal{I}}, (y_j^*)_{j\in\mathcal{J}}\right)$ is Pareto optimal.

The proof of the first welfare theorem for production economies is essentially the same as the proof for pure exchange economies. It is worth trying to extend each of the steps from our previous proof to allow for production. The second welfare theorem can be similarly extended.

Theorem 10 (Second Welfare Theorem). Let $\mathcal{E}$ be a production economy that satisfies $(A1)-(A6)$. Suppose $\left((x_i)_{i\in\mathcal{I}}, (y_j)_{j\in\mathcal{J}}\right)$ is Pareto optimal, and suppose $x_i >> 0$ for all $i \in \mathcal{I}$. Then there is a price vector p, ownership shares $(\theta_{i,j})_{i\in\mathcal{I},j\in\mathcal{J}}$, and endowments $(\omega_i)_{i\in\mathcal{I}}$ such that $\left(p, (x_i)_{i\in\mathcal{I}}, (y_j)_{j\in\mathcal{J}}\right)$ is a Walrasian equilibrium given these endowments and ownership shares.

The proof of the second welfare theorem again relies on the separating hyperplane theorem. Whereas the separating hyperplane in the earlier proof separated the aggregate demand set (i.e., the set of points preferred to the endowment) from the endowment, the proof in production economies requires separation between the aggregate demand set and a suitably constructed aggregate supply set (i.e., the endowment plus the set of feasible aggregate production plans). Convexity of production sets is required in order to invoke the separating hyperplane theorem.

Finally, we can also show that if we impose all the assumptions $(A1) - (A7)$, then a Walrasian equilibrium exists.

Theorem 11 (Existence of Equilibrium). Let $\mathcal{E}$ be a production economy that satisfies $(A1) - (A7)$. Then there exists a Walrasian equilibrium of $\mathcal{E}$.

Exercise 12. Consider an economy with two consumers and two commodities. Consumer 1's endowment vector is $(\lambda, 0)$ and consumer 2's is $(\mu, 0)$. Each consumer's utility is the sum of their consumption of the two commodities. Consumer 1 owns a technology for transforming commodity 1 into commodity 2. The production function is $y_2 = y_1^2$ for $y_1 \leq 0$, where y_1 is the input of commodity 1.

(a) Does this economy have a Walrasian equilibrium?

(b) What allocation would a planner choose to maximize the sum of utilities? [Be careful about second-order conditions.]

(c) What is the core of this economy?

Exercise 13 (Adapted from MWG, 16.F.2-4). In the first week, we discussed the first-order conditions for Pareto optimality in exchange economies. This exercise asks you to extend these conditions to production economies with I consumers and J firms. Define the utility possibility set:

$$\mathcal{U} = \left\{ (u_1, \ldots, u_I) \in \mathbb{R}^I : \exists \text{ feasible } (x_i)_{i \in \mathcal{I}}, (y_j)_{j \in \mathcal{J}} \text{ with } u_i(x_i) \geq u_i \text{ for all } i \right\}.$$

Assume the production set for firm j takes the form $\mathcal{Y}_j = \left\{ y \in \mathbb{R}^L : F_j(y) \leq 0 \right\}$, where $F_j(y) = 0$ defines firm j's **transformation frontier**, and $F_j : \mathbb{R}^L \to \mathbb{R}$ is twice continuously differentiable with $F_j(0) \leq 0$ and $\nabla F_j(y_j) >> 0$ for all $y_j \in \mathbb{R}^L$.

(a) Show that if F_j is a convex function, then $\mathcal{Y}_j$ is a convex set.

(b) [Optional] Show that if, for all $i \in \mathcal{I}$, $\mathcal{X}_i$ is convex and u_i is concave, and for all $j \in \mathcal{J}$, F_j is convex, then $\mathcal{U}$ is a convex set.

(c) Suppose $\lambda \geq 0$ is a non-zero vector of Pareto weights, and consider the Pareto problem

$$\max_{u \in \mathcal{U}} \lambda \cdot u.$$

Show that the optimality conditions for an interior solution (i.e. $x_i >> 0$ for all i) for this problem satisfy

$$\frac{\partial u_i / \partial x_{l,i}}{\partial u_i / \partial x_{l',i}} = \frac{\partial u_{i'} / \partial x_{l,i'}}{\partial u_{i'} / \partial x_{l',i'}} \text{ for all } i, i', l, l' \tag{1}$$

$$\frac{\partial F_j / \partial y_{l,j}}{\partial F_j / \partial y_{l',j}} = \frac{\partial F_{j'} / \partial y_{l,j'}}{\partial F_{j'} / \partial y_{l',j'}} \text{ for all } j, j', l, l' \tag{2}$$

$$\frac{\partial u_i / \partial x_{l,i}}{\partial u_i / \partial x_{l',i}} = \frac{\partial F_j / \partial y_{l,j}}{\partial F_j / \partial y_{l',j}} \text{ for all } i, j, l, l'. \tag{3}$$

(d) Consider the aggregate problem of maximizing the production of commodity 1 subject to minimum production levels $(\bar{y}_2, \ldots, \bar{y}_L)$ for the other commodities.

$$\max_{(y_1, \ldots, y_J)} \sum_{j \in \mathcal{J}} y_{1,j}$$

subject to

$$\sum_{j \in \mathcal{J}} y_{l,j} \geq \bar{y}_l \text{ for all } l = 2, \ldots, L$$

and

$$F_j(y_j) \leq 0 \text{ for all } j = 1, \ldots, J.$$

Show that the optimality conditions for this problem satisfy (2). What does these conditions imply about how production is carried out across firms in a Pareto optimal allocation?

2.4 A Constant Returns-to-Scale Example

For a production economy, we have to specify both consumers' preferences as well as firms' production sets. A simple class of production sets that satisfy assumptions $(A5) - (A7)$ are linear production sets. Such production sets are convex cones spanned by finitely many rays.[1] There is a single firm that has access to M linear activities $a_m \in \mathbb{R}^L$, $a_m \in \mathcal{M} = \{a_1, \ldots, a_M\}$,

[1] Let $\mathcal{X} \subset \mathbb{R}^N$ be a set that contains $\{0\}$. Take two vectors $x, y \in \mathcal{X}$. We will say that a vector $w = \alpha x + \beta y$, where $\alpha \geq 0, \beta \geq 0$ is a **conic combination** of the vectors x and y. If the set $\mathcal{X}$ contains all conic combinations of its elements, we say that $\mathcal{X}$ is a **convex cone**. The way to think about a convex cone is to imagine a convex set $\mathcal{A}$ located some distance from the origin. The convex cone generated by the set $\mathcal{A}$ is the set of all points that lie on a ray from the origin that goes through any point in $\mathcal{A}$. If $\mathcal{A}$ is a disk in $\mathbb{R}^2$, then the convex cone generated by $\mathcal{A}$ is what you would normally think of as a cone.

and it can operate each activity at some level γ_m. Its production set $\mathcal{Y}$ is the convex hull of these activities:

$$\mathcal{Y} = \left\{ y \in \mathbb{R}^L : y = \sum_{m=1}^{M} \gamma_m a_m \text{ for some } \gamma \in \mathbb{R}_+^M \right\}.$$

Assumption $(A5)$ is satisfied, and the free disposal part of assumption $(A6)$ is satisfied if the vectors

$$(-1, 0, \ldots, 0), (0, -1, 0, \ldots, 0), \ldots, (0, \ldots, 0, -1)$$

are all in $\mathcal{M}$.

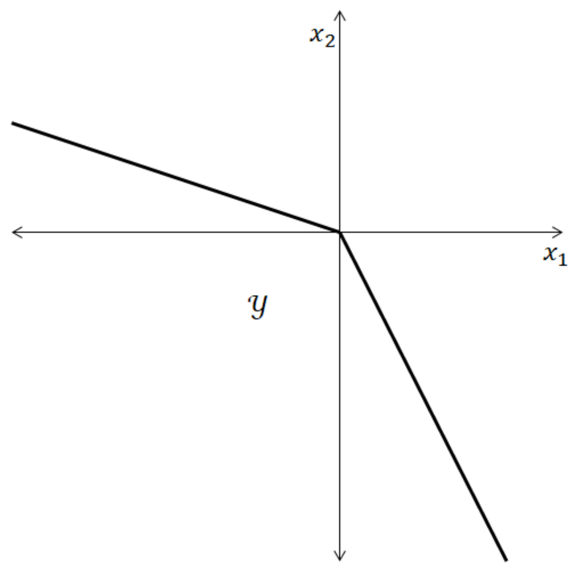

Figure 14: Linear Production Set

Figure 14 illustrates a linear production set in the special case of $M = 4$ and $L = 2$. There are two productive activities: activity 1 allows 2 units of commodity 2 to be converted into 1 unit of commodity 1. Activity 2 allows 3 units of commodity 1 to be converted into 1 unit of commodity 2. Activities 3 and 4 are the activities I described above that ensure that the free disposal assumption is satisfied. In this case,

$$\mathcal{M} = \{(1, -2), (-3, 1), (0, -1), (-1, 0)\}.$$

Note that the production set $\mathcal{Y}$ generated by $\mathcal{M}$ is the same production set that would be generated by activities $\{(1,-2),(-3,1)\}$ because $\frac{3}{5}(1,-2)+\frac{1}{5}(-3,1)=(0,-1)$ and $\frac{1}{5}(1,-2)+\frac{3}{5}(-3,1)=(-1,0)$.

Given a price vector p, a profit-maximizing plan exists if and only if $p\cdot a_m\leq 0$ for all $m=1,\ldots,M$. If this were not the case, the firm's potential profits would be unbounded: if $p\cdot a_m>0$ for some m, the firm could choose a sequence of production vectors $\gamma_m a_m$ with $\gamma_m\to\infty$, and its profits would increase without bound along that sequence. If $p\cdot a_m<0$ for some m, then it is clear that optimal production $y=\sum_{m\in\mathcal{M}}\gamma_m a_m$ satisfies $\gamma_m=0$.

When production sets are convex cones, as in this example, market clearing implies that equilibrium prices are pinned down by zero-profit conditions. This specification of production sets is not as much of a special case as it first appears—it is satisfied by any constant returns to scale production technology, including the Cobb-Douglas production functions you have used in macroeconomics.

2.5 The Representative Firm Theorem

When production sets are convex cones, firms do not really play much of a role in the economy—they earn zero profits in equilibrium, and it is actually irrelevant whether there is a single firm that possesses the entire set of activities $\mathcal{M}$ or a collection of M firms that each possess only a single activity a_m (plus the free disposal activities). The result that it is without loss of generality to focus on a single firm that possesses the sum of firms' production technologies is actually a much more general result than this example illustrates, as the following theorem highlights.

Theorem 12 (Representative Firm Theorem). Let $\mathcal{E}$ be a production economy satisfying $(A5)-(A7)$. Given a price vector $p\in\mathbb{R}_+^L$, denote the set of profit-maximizing net supplies of firm $j\in\mathcal{J}$ by $y_j(p)$. Then there exists a representative firm with production possibilities set $\mathcal{Y}$ and a set of profit-maximizing net supplies $y(p)$ such that $y^*\in y(p)$ if and only if $y^*=\sum_{j\in\mathcal{J}}y_j^*$ for some $y_j^*\in y_j(p)$ for each $j\in\mathcal{J}$.

This theorem shows us that we can "aggregate" the production side of the economy. Exercise 14 asks you to prove this result. We know that when there are no income effects for consumers, we can represent all the consumers in the economy with a single representative consumer. The same idea holds for firms. But for firms, at least for the simplistic way we are currently thinking about firms, there are never income effects—prices have no impact on firms' *feasible* production plans. One implication of the representative firm theorem is that, if we take this simplistic view of firms, we are left with a somewhat simplistic view of the supply side of the economy as a whole. This point will be important for us to remember when we talk about the theory of the firm and why we might care about firm boundaries.

Exercise 14. This exercise asks you to prove the representative firm theorem.

(a) Fix p and construct $y^* = \sum_{j \in \mathcal{J}} y_j^*$ for some $y_j^* \in y_j(p)$ for each $j \in \mathcal{J}$. Prove that we must have $y^* \in y(p)$.

(b) Let $y^* \in y(p)$ be a profit-maximizing choice for the representative firm. Show that if $y^* = \sum_{j \in \mathcal{J}} y_j$ for some $y_j \in \mathcal{Y}_j$ for each $j \in \mathcal{J}$, then $y_j \in y_j(p)$ for each $j \in \mathcal{J}$.