Economics 2010b Problem Set 4

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Due: Friday, 2/23 by 11:59pm

Exercise 18 (Adapted from MWG 14.B.4). Suppose there are three possible effort levels, $\mathcal{E} = \{e_1, e_2, e_3\}$, and two possible output levels, $\mathcal{Y} = \{0, 10\}$, and the output price is $p = 1$. The probability that $y = 10$ conditional on each of the effort levels is given by the probability mass function $f(10|\, e_1) = 2/3$, $f(10|\, e_2) = 1/2$, and $f(10|\, e_3) = 1/3$. The Agent's effort cost function satisfies $c(e_1) = 5/3$, $c(e_2) = 8/5$, and $c(e_3) = 4/3$. Finally, the Agent's utility over income is given by $u(w) = \sqrt{w}$, so that

$$U(w, e) = E\left[\sqrt{w}\middle|\, e\right] - c(e),$$

and his outside option yields utility $\bar{u} = 0$.

(a) What is the optimal contract for the Principal when effort is contractible?

(b) Show that if effort is noncontractible, and $\mathcal{W} = \{w : \mathcal{Y} \to \mathbb{R}\}$, then there is no contract w for which the Agent will choose e_2. For what levels of $c(e_2)$ would there exist a contract w under which the Agent would choose e_2?

(c) What is the optimal contract when effort is noncontractible, and $\mathcal{W} = \{w : \mathcal{Y} \to \mathbb{R}\}$?

(d) Suppose instead that $c(e_1) = \sqrt{8}$, and let $f(10|\, e_1) = x \in (0, 1)$. If effort is noncontractible, and $\mathcal{W} = \{w : \mathcal{Y} \to \mathbb{R}\}$, what is the optimal contract for the Principal as x approaches 1? Is the level of effort implemented higher or lower than when effort is contractible?

Exercise 19. Suppose the Agent can allocate time to two different tasks. Let e_i be the amount of time spent on task $i \in \{1, 2\}$. The Principal cares only about task 1 and obtains payoff $y = e_1 + \varepsilon$, where $\varepsilon \sim N(0, \sigma^2)$. The Agent, however, derives a benefit $v(e_2)$ from spending time on task 2. The Agent has CARA preferences with utility function

$$u(w, e_1, e_2) = -\exp\{-r[w - c(e_1 + e_2) + v(e_2)]\},$$

where $c(e_1 + e_2)$ is the cost of time, with $c'(\cdot) > 0$, $c''(\cdot) > 0$, and $c(0) = 0$. Assume also that $v'(\cdot) > 0$, $v''(\cdot) < 0$, and $v(0) = 0$, and that optimization with respect to (e_1, e_2) results in an interior solution. Let $\bar{w}$ denote the wage the Agent receives from his outside option, so $\bar{u} = -\exp\{-r\bar{w}\}$.

(a) What is the first-best outcome in this setting?

(b) Suppose effort e_1 is noncontractible, and the Principal can write a contract that is an affine function of output and can also allow the Agent to engage in task 2 or not. Under these assumptions, what is the contracting space?

(c) Suppose the Principal must pay the Agent $b = 1$ (i.e., she must pay him one dollar for each unit of output). Will the Principal allow the Agent to engage in task 2? Compare this to your answer in part (a). What if $b < 1$ is set exogenously? Find the difference in the Principal's utility under the two policies, as a function of b.

Exercise 20. This exercise goes through a two-period version of Holmström and Milgrom's (1987) linear contracts argument. In each of two periods, $t \in \{1,2\}$, the Agent chooses whether to "work" or to "shirk": $e_t \in \{0,1\}$ at cost ce_t with $c > 0$. Output is binary, so that $y_t \in \{0,1\}$, and the price of output is normalized to 1. Effort increases the probability that $y_t = 1$:

$$1 > \Pr\left[y_t = 1 | \, e_t = 1\right] = p_H > p_L = \Pr\left[y_t = 1 | \, e_t = 0\right] > 0.$$

The Agent's Bernoulli utility function is

$$u\left(w, e_1, e_2\right) = -\exp\left\{-r\left(w - ce_1 - ce_2\right)\right\},$$

and his outside option yields utility $-\exp\{-r \cdot 0\}$. The Agent can observe the realization of y_1 before choosing y_2.

The Principal's payoff is $y_1 + y_2 - w$, and the payment w can depend on each period's output and is paid at the end of period 2 (i.e., after both realizations of output). Assume it is optimal to induce the Agent to work hard in both periods. Show that a least-cost (optimal) contract that implements $e_1 = e_2 = 1$ has the form

$$w\left(y_1, y_2\right) = s + b\left(y_1 + y_2\right).$$

Guide:

(a) Define w_{y_1,y_2} to be the wage conditional on y_1 in period 1 and y_2 in period 2. Then, using the (IC) constraints for period 2, show that

$$e^{rc}\left[1 + p_H\left\{\frac{\exp\{-rw_{0,1}\}}{\exp\{-rw_{0,0}\}} - 1\right\}\right] = 1 + p_L\left\{\frac{\exp\{-rw_{0,1}\}}{\exp\{-rw_{0,0}\}} - 1\right\}$$

and

$$e^{rc}\left[1 + p_H\left\{\frac{\exp\{-rw_{1,1}\}}{\exp\{-rw_{1,0}\}} - 1\right\}\right] = 1 + p_L\left\{\frac{\exp\{-rw_{1,1}\}}{\exp\{-rw_{1,0}\}} - 1\right\}.$$

This implies there exists

$$b = w_{0,1} - w_{0,0} = w_{1,1} - w_{1,0}.$$

Why did CARA utility matter for this argument?

(b) Now, using the (IC) constraint for period 1, show that we have

$$e^{rc}\left[1 + p_H\left\{\frac{u_1}{u_0} - 1\right\}\right] = 1 + p_L\left\{\frac{u_1}{u_0} - 1\right\},$$

where u_i is the expected utility conditional on success in the first period $(i = 1)$ or failure $(i = 0)$.

(c) Note that

$$\exp\left\{r\left(c - w_{y,1}\right)\right\} = \exp\{-rb\}\exp\left\{r\left(c - w_{y,0}\right)\right\}$$

for each $y \in \{0, 1\}$. Now show that

$$\frac{u_1}{u_0} = \exp\{-r(w_{1,0} - w_{0,0})\} = \exp\{-r(w_{1,1} - w_{0,1})\}.$$

Therefore, we must have $b = w_{0,1} - w_{0,0} = w_{1,0} - w_{0,0} = w_{1,1} - w_{0,1}$.

Exercise 21. This exercise goes through a version of Diamond's (1998) and Barron, Georgiadis, and Swinkels's (2018) argument for why linear contracts are optimal when the Agent is able to "take on risk." Suppose the Principal and the Agent are both risk neutral, and let $\mathcal{Y} = [0, \bar{y}]$ and $\mathcal{E} = \mathbb{R}_+$. There is a limited-liability constraint, and the contracting space is $\mathcal{W} = \{w : \mathcal{Y} \to \mathbb{R}_+\}$. After the Agent chooses an effort level e, he can then choose any distribution function $F(y)$ over output that satisfies $e = \int_0^{\bar{y}} y dF(y)$. In other words, his effort level determines his *average* output, but he can then add mean-preserving noise to his output. Given a contract w, effort e, and distribution F, the Agent's expected utility is

$$\int_0^{\bar{y}} w(y) \, dF(y) - c(e),$$

where c is strictly increasing and strictly convex. The Principal's expected profits are $\int_0^{\bar{y}} (y - w(y)) \, dF(y)$. The Agent's outside option gives both parties a payoff of zero.

(*a*) Show that a linear contract of the form $w(y) = by$ maximizes the Principal's expected profits. To do so, you will want to argue that given any contract $w(y)$ that implements effort level e, there is a linear contract that also implements effort level e but at a weakly lower cost to the Principal. [Hint: instead of thinking about all the possible distribution functions the Agent can choose among, it may be useful to just look at distributions that put weight on two levels of output, $0 \leq y_L < y_H \leq \bar{y}$ satisfying $e = (1 - q) y_L + q y_H$.] [Hint^2: a graphical argument will suffice.]

(*b*) Are there other contracts that maximize the Principal's expected profits? If so, how are they related to the optimal linear contract? If not, provide an intuition for why linear contracts are uniquely optimal.