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Frictional stick-slip oscillation as a first-passage problem

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Abstract – We study complex frictional stick-slip oscillations by formulating a novel first-passage time problem whose solutions are used to compute distributions of stick-slip oscillation periods, displacements, and slip durations for a generic family of stochastic friction models. Approximate solutions are developed using a level-crossing expansion due to Rice and Stratonovich. Sample results for a minimal Langevin sliding friction model agree with simulations over a range of system parameters, and reproduce qualitative features of prior experimental studies of stick-slip motion. The analysis also reveals a complex oscillatory regime near the transition from stick-slip to continuous sliding, in which additional transient modes are excited through boundary-noise interactions.

Stick-slip oscillations, an important source of vibration and acoustic emissions in natural and man-made mechanical systems, are observed in many frictional settings: during macroscopic sliding between unlubricated solid surfaces, such as vehicle brakes [1]; in the bowing of a violin string [2]; in nanoscale friction on an atomic substrate [3]; or in the shear response of a granular material [4,5].

Minimal sliding friction models (fig. 1), are able to capture many salient phenomena, but dry frictional sliding at macroscopic scales also involves localized contact at multiple asperities, which are brought into rapid collision during sliding, contributing a source of noise. In granular friction, fluctuating forces can be highly pronounced [4,6–8], and in both solid and granular sliding, multiple dynamical regimes are observed (fig. 2), including irregular and periodic stick-slip oscillation, and continuous sliding [4,6,9,10], with slip distances, force fluctuations, and periods found to be broadly distributed. Although phenomenological models have been proposed for granular [4,5], solid friction [1,9,11–13], and related seismological processes [14,15], no theory is able to fully explain the complex oscillatory behavior that is observed.

In a stochastic model of sliding friction, random force components due to friction noise lend the velocity $v(t)$ of a sliding body a random character, which can generically be described by a stochastic dynamical process. Transitions to static friction occur when this velocity decreases sufficiently, due to elastic restorative forces or frictional damping (fig. 1B), that the contacting surfaces become stuck in a local energetic minimum. This condition associates the stick-slip motion with a first-passage problem for the random variable $v(t)$. If the body is externally driven at constant velocity v_b (fig. 1), a maximum of the static friction force will, after some time, be reached, sliding motion will recommence, and the process will repeat.

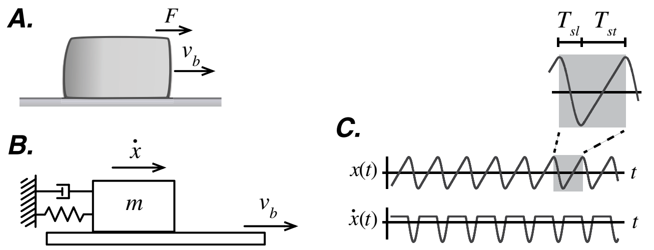

Fig. 1: (A) A sliding body exhibits frictional stick-slip oscillations, modeled as (B). A block, spring, damper, and surface, which (C) oscillate with period $\tau = T_{\rm sl} + T_{\rm st}$.

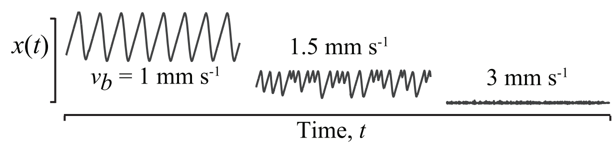

Fig. 2: Illustration of complex stick-slip oscillatory behavior, from high-speed free sliding to low-speed periodic motion.

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The present work develops such a first-passage problem for a generic family of stochastic sliding friction models, and solves it, obtaining analytical expressions for the distributions of periods, slip durations and displacements, which statistically characterize the motion as a function of the dynamical parameters and noise characteristics of the model. Lima *et al.* previously studied the distribution of excursions of a block sliding down an incline before coming to rest [11], modeling experiments of Brito and Gomes [16] via a stochastic friction law that could be expressed as a biased diffusion. The block sliding distribution was effectively reproduced by solutions of a first-passage problem, obtained via a Fokker-Planck equation [11]. Such an analysis is also suitable for modeling first-passage processes in overdamped settings, where the relaxation time t_R of the dynamics is shorter than the typical first-passage time [17]. In such cases, similar to Kramers' problem for the noise-induced escape of a particle from a metastable potential, a constant escape rate is soon achieved, and the first-passage time density decays exponentially for times exceeding t_R, behavior that is also consistent with experimentally observed waiting time densities in mechanical stick-slip motion at high damping ratios [9].

However, in the quasiperiodic frictional regimes studied here, the presence of underdamped oscillation modes render the dynamics essentially non-Markovian [18], with both position and velocity values required to determine the evolution at future times. Further, transitions to static friction result in a reset to sharp initial conditions on each cycle, ensuring the system remains far away from equilibrium, thus making it impossible to solve the Fokker-Planck equation using known techniques. Instead, the present work develops solutions based on an expansion in level crossing statistics, that originates with work of Rice and Stratonovich [19,20] that has recently been used to model distributions of firing intervals in resonant neurons [17,21]. There are similarities between membrane potential oscillations of a spiking neuron and stick-slip frictional motion, with the distribution of stick-slip periods paralleling the neuronal interspike interval density. This paper draws on the aforementioned methods, and extends them so as to address the unique velocity-type boundary relevant to stick-slip oscillation. Additionally, it extracts multivariate first-passage densities required to compute distributions of slip displacements and oscillation periods. Finally, it applies this theoretical treatment to a minimal physically plausible stochastic friction model.

First-passage problem and solutions. – In a minimal deterministic model of stick-slip motion (fig. 3A), the oscillation period, a function of the system parameters, is equal to the sum of time spent during stick and slip phases, $T = T_{st} + T_{sl}$. In the coordinate frame moving with the driving velocity v_b (fig. 1B), the static friction grows until the threshold is reached, at relative displacement x_0. Sliding follows until the two surfaces lock into a local energetic minimum, matching velocities, $v(t) = v_b$, at $t = T_{sl}$. The cycle then repeats. Additional factors (*e.g.*, contact aging, granular dynamics, lubrication) can yield more complex or multiperiodic oscillation [10,22]. Ignoring such factors, the motion is that of a non-smooth dynamical system having two dynamical phases: stick and slip [23].

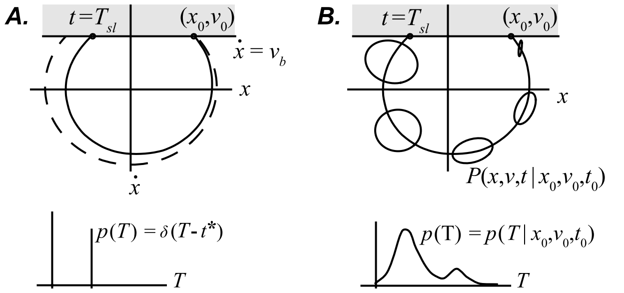

Fig. 3: (A) Periodic phase-space orbit produced by a simple deterministic friction law. (B) Dynamics in a stochastic friction model with a distribution $p(T)$ of periods. The state follows a transition probability density $P(x, v, t|x_0, v_0, t_0)$. The ellipsoids represent level sets (isolines) of this density at successive times.

For the remainder, to account for random forces during sliding, consider a frictional process in which sliding is governed by a continuous time stochastic dynamical system:

$$\dot{\mathbf{x}} = f(\mathbf{x}(t); \eta(t)). \tag{1}$$

Here, η is a noise source, $f(\cdot)$ describes the (possibly nonlinear) sliding dynamics, and the vector $\mathbf{x}(t) = (x, v = \dot{x}, \ldots)$ accounts for x, v, and any other required state variables. The presence of damped oscillations is assumed. At the onset of sliding (fig. 3B), trajectories of the system are localized near $(x, v) = (x_0, v_b)$, and spread out over time, as represented via a transition probability density $P(x, t|x_0, t_0 \equiv 0)$. A transition to static friction is assumed to occur when the velocity $v(t)$ first reaches v_b. During static friction, the state evolves deterministically, with the surfaces in stationary contact, until the static friction force limit is reached. The dynamics of transitions between static and kinetic friction are ignored here. Variations in the static friction threshold (due to asperity size, contact aging, infinitesimal creep [24], etc.) are neglected, but could, to a first approximation, be incorporated in the present analysis by allowing the static friction threshold to be randomly distributed. Thus, we study the distribution $p(T|x_0, 0)$ of periods T, assuming a reset to sharp initial conditions x_0 upon transition to sliding.

Let $\mathcal{F}(x, t)$ be the joint probability density for a transition to static friction (characterized by $v(t) = v_b$) to occur at displacement x and time $t = T_{sl}$. T_{sl} is thus a random variable, while sticking time $T_{st} = (x_0 - x_{sl})/v_b$ is a deterministic function of the (random) displacement x_{sl} at stick onset. The period $T = T_{st} + T_{sl}$ is distributed as

$$p(T|\mathbf{x}_0, 0) = \int_0^T \mathrm{d}\tau \mathcal{F}(\hat{x}, \tau), \;\; \hat{x} \equiv x_0 - v_b(T - \tau). \tag{2}$$

The slip duration T_{sl} and length $d = x_0 - x(T_{sl})$ are distributed according to the marginal densities

$$p(T_{sl}) = \mathcal{F}(T_{sl}) = \int_{-\infty}^{\infty} \mathrm{d}x\ \mathcal{F}(x, T_{sl}), \tag{3}$$

$$p(d) = \mathcal{F}(x = x_0 - d) = \int_0^{\infty} \mathrm{d}t\ \mathcal{F}(x, t). \tag{4}$$

Computing the probability densities $\mathcal{F}(x,t)$, $\mathcal{F}(x)$, and $\mathcal{F}(t)$, which characterize the random stick-slip transitions in the system, requires the solution of a first-passage problem in the velocity $v(t)$ across the value v_b.

The most common approach to solving for the transition state density $P(x,t|x_0,t_0)$ proceeds via a Fokker-Planck partial differential equation. For example, for frictional dynamics modeled on a generalized Langevin equation, it has the form [25]

$$\frac{\partial P}{\partial t} = -v\frac{\partial P}{\partial x} + \frac{U'(x)}{m}\frac{\partial P}{\partial v} + \frac{\gamma}{m}\frac{\partial}{\partial v}vP + \frac{D}{m^2}\frac{\partial^2 P}{\partial v^2}, \tag{5}$$

where $U(x)$ is the elastic potential energy and D is the diffusion coefficient. The required densities are related to solutions of this equation in the presence of an absorbing boundary condition at $v = v_b$, *i.e.*$P(x, v_b, t) = 0$ with $\partial P(x, v_b, t)/\partial v > 0$ [25]. However, analytic solution methods are known only for limiting cases, in which the motion is overdamped [26], the relaxation time is shorter than the mean escape time, or the dynamics can otherwise be reduced to a Brownian motion in a single variable (Smoluchowsi approximation [18]), and are inapplicable for non-Markovian processes with sustained oscillations, which is the case of present interest.

Instead, solutions may be developed based on a level crossing series expansion for the random variable $v(t)$, as first formulated by Rice and Stratonovich [17,19,20]. It has the form

$$\mathcal{F}(t) = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \int_0^t \mathrm{d}t_p \cdots \int_0^t \mathrm{d}t_1 n_{p+1}(t, t_p, \ldots, t_1). \tag{6}$$

The first term, $n_1(t)$, is the density of velocity *upcrossings* at time t, or passages of $v(t)$ across the level $v = v_b$ from below. $n_k(t_k, t_{k-1}, \ldots, t_1)$ are the joint densities for $v(t)$ to perform upcrossings at k successive time instants. $n_1(t)$ overestimates $\mathcal{F}(t)$ by including all trajectories that crossed at time t and at previous times. The second term corrects for trajectories that performed an upcrossing prior to t, and subsequent terms provide further corrections. Thus, the partial sums of order k, here denoted $\mathcal{F}_n^{(k)}(t)$, can be regarded as successive approximations to $\mathcal{F}(t)$ [21]. The k-th estimate is valid for times at which fewer than $k+1$ upcrossings are likely to occur —*i.e.*, for $T < k/(2\pi\omega)$, where ω is the oscillation frequency. Over long time intervals it is not normalized, due to contributions of trajectories that perform multiple upcrossings, but a normalized density can be defined via $\hat{\mathcal{F}}_n^{(k)}(t) = \mathcal{F}_n^{(k)}(t)\,\Theta(t - t_n)$, where $\Theta(\cdot)$ is the Heaviside step function and t_n is chosen so that $\int_0^{t_n} \mathcal{F}_n^{(k)}(t)\mathrm{d}t = 1$.

$\mathcal{F}(t)$ can also be expressed via a cumulant expansion [18,20], by writing $\mathcal{F}(t) = \dot{S}(t)\exp(-S(t))$, where

$$S(T) = -\sum_{p=1}^{\infty} \frac{(-1)^p}{p!} \int_0^T \cdots \int_0^T g_p(t_1, \ldots, t_p)\mathrm{d}t_1 \cdots \mathrm{d}t_p; \tag{7}$$

$\dot{S}(t) = \mathrm{d}S/\mathrm{d}t$ can be interpreted as a time-dependent escape rate. The densities g_k may be expressed through n_k of equal and lower order by comparing terms in the summations. The first two are given by $g_1(t) = n_1(t)$ and $g_2(t_1, t_2) = n_2(t_1, t_2) - n_1(t_1)n_1(t_2)$. If the crossing times t_k occur almost independently, the g_k decrease rapidly in magnitude [18]. This facilitates approximations for $\mathcal{F}(T)$ in which $S(T)$ is replaced by partial sums of order k. The first-order "decoupling" estimate, $\mathcal{F}_g^{(1)}(T)$, replaces the n_k at each order by products of one-crossing distributions, n_1, which yields $\mathcal{F}_g^{(1)}(T) = n_1(T)\exp(-\int_0^T n_1(t)\mathrm{d}t)$. Higher-order approximations $\mathcal{F}_g^{(k)}(T)$ can be expressed in terms of the upcrossing densities n_k [17,20], and are normalized for all positive times if a positive escape rate is ensured.

Similar approximations can be obtained for the joint density $\mathcal{F}(x,T)$, based on a generalization of eq. (6):

$$\mathcal{F}(x,T) = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \int_0^T \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}x_1 \times \cdots \times \int_0^T \mathrm{d}t_p \int_{-\infty}^{\infty} \mathrm{d}x_p\, n_{p+1}(x_1, t_1; x_2, t_2; \ldots; x, T);$$

$n_k(x_k, t_k; \ldots; x_1, t_1)$ is the probability of $v(t)$ performing upcrossings at points (x_i, t_i), $i = 1, 2, \ldots, k$. Truncation and cumulant approximations, $\mathcal{F}_n^{(k)}(x,T)$ and $\mathcal{F}_g^{(k)}(x,T)$, can be derived just as for the univariate densities. A decoupling approximation can likewise be obtained, yielding $\mathcal{F}_g^{(1)}(x,T) = n_1(x,T)\exp(-\int_0^T n_1(t)\mathrm{d}t)$.

Example: Langevin model. – While the forgoing treatment is applicable to a broad class of stochastic friction models, as a concrete example, consider a minimal friction model with sliding dynamics based on the Langevin equation

$$\dot{x} < v_b\colon \quad a = \dot{v} = -\omega_0^2 x - \zeta v + v_b \xi(t), \quad \dot{x} = v, \tag{8}$$

$$\dot{x} = v_b\colon \quad a = \dot{v} = 0, \quad \dot{x} = v(t) = v_b. \tag{9}$$

Equation (8) describes sliding motion (which holds for $\dot{x} < v_b$) in a coordinate frame moving at the driving velocity v_b. In this model, noise generated by the collision of thousands of μm-scale asperities per second is captured by the random force $m v_b \xi(t)$, which vanishes as v_b tends to zero. This is assumed to constitute Gaussian white noise, although the analysis can be extended to correlated noise [17]. This minimal Langevin dynamics has been used to model experimental observations of sliding friction on solid and granular surfaces [5,9]. Here, we assume

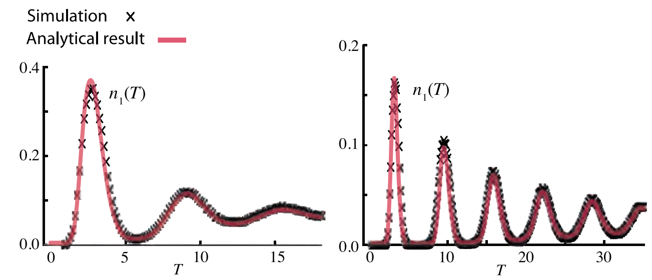

Fig. 4: (Colour on-line) First upcrossing density $n_1(t)$. The exact analytic result (red line) is compared with the Monte Carlo estimate (×). Left: $\omega_0 = 1$, $D = 0.4$, $v_b = 1$, $\zeta = 0.1$. Right: $\omega_0 = 1$, $D = 0.04$, $v_b = 1.3$, $\zeta = 0.11$.

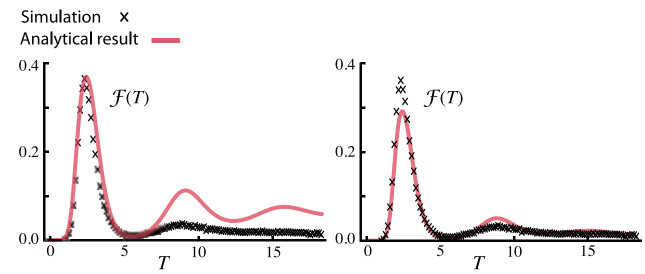

Fig. 5: (Colour on-line) Comparison of analytic approximations for slip time density $\mathcal{F}(T)$ with numerical estimates (marked ×). Left: $\mathcal{F}_n^{(1)}(T)$. Right: $\mathcal{F}_g^{(1)}(T)$. Parameters: $\omega_0 = 1$, $D = 0.4$, $v_b = 1.0$, $\zeta = 0.1$.

the driving velocity v_b to be constant, so define a velocity-scaled force term to be $\eta(t) \equiv v_b\xi(t)$, with $\langle\eta(\tau)\eta(\tau+t)\rangle = 2D\delta(t)$. This noise is not thermal, but due to inelastic collisions of surface asperities, and standard thermodynamic concepts such as temperature or fluctuation-dissipation relations are not straightforward to apply. The line $v = v_b$ is an absorbing boundary condition; first passage to v_b yields a transition to static friction (eq. (9)). The latter persists until the threshold $x = x_0$ is reached, when sliding resumes from a localized state $(x_0, v_0 = v_b)$. Although this example involves a simple piecewise-linear, non-smooth stochastic dynamical system, it nonetheless qualitatively captures a rich variety of oscillatory phenomena.

Absent the boundary at $v = v_b$, the sliding dynamics is linear with Gaussian noise, so the transition state density on the whole phase plane (*i.e.*, without boundary), here denoted $P(x, v, t|x_0, v_0, t_0) \equiv P_{10}$, is a bivariate Gaussian process; its solution is well known [27,28]. The components $\mu_x(t)$ and $\mu_v(t)$ of the mean are trajectories of the damped harmonic oscillator. The variances $\Sigma_{xx}(t)$, $\Sigma_{xv}(t)$ and $\Sigma_{vv}(t)$ grow over time, as qualitatively illustrated in fig. 3B. The free-sliding transition density P_{10} may be used to compute upcrossing and first-passage densities in the presence of the transition boundary $v = v_b$ as follows.

For a velocity upcrossing to occur at time t, it must be that $v_b - a\mathrm{d}t < v(t) < v_b$. The probability for this to occur is

$$\int_{v_b - a\mathrm{d}t}^{v_b} \mathrm{d}v P(x,v,a,t|x_0,v_0,t_0) = P(x,v_b,a,t|x_0,v_0,t_0)|a|\mathrm{d}t = P_{10}\, P(a|x,v_b)\, |a|\mathrm{d}t. \quad (10)$$

Here, $P(a|x, v_b) \equiv P(a|x, v = v_b)$ is, due to (8), a normal density with mean $\mu_a = -\omega_0^2 x - \zeta v_b$ and variance $\sigma^2 = 2D$.

Since $a > 0$ is required for an upcrossing, $n_1(x, v_b, t)$ is obtained by integrating eq. (10) over positive values of a,

$$n_1(x, v_b, t) = P_{10}\left(\int_0^\infty \mathrm{d}a\, aP(a|x,v_b)\right) \equiv P_{10} I_a(x, v_b). \quad (11)$$

The integral $I_a(x, v_b)$ in parentheses yields

$$I_a(x, v) = \sqrt{\frac{\sigma^2}{2\pi}}\exp\left(-\frac{\mu_a^2}{2\sigma^2}\right) - \frac{\mu_a}{2}\mathrm{erfc}\left(\frac{\mu_a}{\sqrt{2}\sigma}\right). \quad (12)$$

The density of upcrossings across $v = v_b$ is thus

$$n_1(t) = \int_{-\infty}^{\infty} \mathrm{d}x P_{10}(x, v_b, t) I_a(x, v_b). \quad (13)$$

As shown in the appendix, this integral over x, which involves factors of Gaussian or Error Function type in x and v_b, yields an expression for $n_1(t)$ with similar terms in v_b alone, and a complicated time evolution, due to its dependence on components of the mean $\boldsymbol{\mu}(t)$ and variance $\boldsymbol{\Sigma}(t)$ of P_{10}. From it, we directly compute the approximations $\mathcal{F}_n^{(1)}(T)$ and $\mathcal{F}_g^{(1)}(T)$ to $\mathcal{F}(T)$. The joint densities $\mathcal{F}_n^{(1)}(x, T)$ and $\mathcal{F}_g^{(1)}(x, T)$ depend on $n_1(x, T)$ in (11). They yield the oscillation period and slip length densities via eqs. (3) and (4). Higher-order approximations, based on the upcrossing densities n_k, involve multiple integrals with similar factors to those arising above. Although too complex to be performed analytically, they may be evaluated via repeated numerical integration.

Numerical estimates for stick-slip densities in the Langevin model were obtained from Monte Carlo simulations. Each estimate was based on 10^5 to 10^6 instances of (8) integrated in time, using the Euler-Maruyama method, from the configuration (x_0, v_0) at slip onset $(t_0 \equiv 0)$ until the time of first passage. The integration time step was $h = 0.002$, with other parameters as noted in captions of figs. 4–6. The results broadly agree with the analytical predictions over a wide range of system parameters. They evidence stick-slip motion with finite-width distributions of periods, slip distances, and slip times.

Density of upcrossings: Figure 4 compares analytic results for $n_1(T)$, which are exact, with corresponding numerical estimates, demonstrating nearly perfect agreement. The location of the maximum of the upcrossing density lies close to the value of the deterministic period $\tau = 2\pi/\omega$, with subsequent peaks lying at multiples of this value.

Slip duration: The density $\mathcal{F}(T)$ of slip durations is shown in fig. 5 (left panel). The truncation approximation $\mathcal{F}_n^{(1)}(T) = n_1(T)$ to $\mathcal{F}(T)$ provides good agreement near the first maximum of $\mathcal{F}(T)$, but overestimates later peaks, as it also counts all upcrossings after the first. As $T \to \infty$, $n_1(T)$ overestimates $\mathcal{F}(T)$ by the asymptotic upcrossing

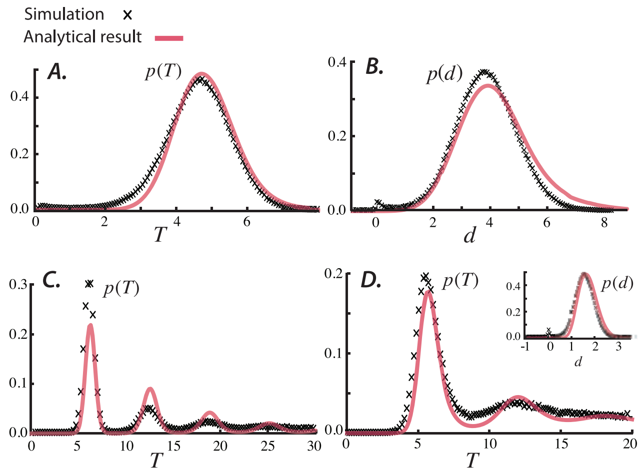

Fig. 6: (Colour on-line) Numerical and analytical oscillation densities. Quasiperiodic regime: (A) period density $p(T)$ computed from the normalized truncation estimate $\hat{\mathcal{F}}_n^{(1)}$. Parameters: $\omega_0 = 2$, $D = 3$, $v_b = 1.25$, $\zeta = 0.1$. (B) Slip size density $p(d)$. Other details as in (A). Multiperiodic regime: (C) period density $p(T)$, in decoupling approx. $\mathcal{F}_g^{(1)}$, $\omega_0 = 1$, $\zeta = 0.4$, $D = 0.1$, $v_b = 1.0$. (D) period $p(T)$ and slip distance $p(d)$ densities from $\mathcal{F}_g^{(1)}$, $\omega_0 = 1$, $\zeta = 0.11$, $D = 0.04$, $v_b = 1.0$.

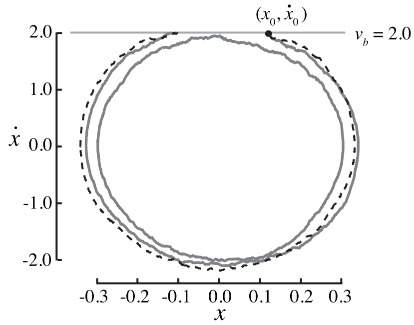

Fig. 7: Two phase-space trajectories of the Langevin system, near the critical speed for stick-slip motion, integrated from (x_0, v_0) until first passage. Dashed line: fundamental mode (mean period τ). Solid line: A double orbit ($\tau' \approx 2\tau$). $D = 3$, $v_b = 2$, $\omega_0 = 1$, $\zeta = 0.02$.

rate, known as the Rice frequency [19], and here given by $n_0 = (\omega_0/2\pi)\exp(-v_b\zeta/2D)$. The decoupling approximation, $\mathcal{F}_g^{(1)}(T)$, displays no such error (fig. 5, right panel), but underestimates the first and overestimates later peaks.

Quasiperiodic motion: Figure 6 (top row) presents results for the densities of oscillation periods $p(T)$ and slip distances $p(d)$ (eq. (4)) obtained using the first-order truncation approximation in comparison with numerical results. In this regime, the two densities are each broadly distributed about a single maximum, evidencing quasiperiodic motion. In this regime, there is good qualitative agreement with densities obtained from experimental observations of solid-on-solid stick-slip motion (cf., Johansen *et al.* [9], figs. 18, 20), also consistent with several studies that have found Gaussian-shaped distributions of force fluctuations in solid and granular stick-slip motion [7,9,29].

Multiperiod oscillation: As the platform velocity v_b approaches the fundamental critical speed, which is the highest speed at which stick-slip motion is expected to occur deterministically [1], some trajectories are observed to "miss" a slip-to-stick transition on one oscillation cycle, but, due to noise perturbations, to attain it on a subsequent pass (fig. 7). Trajectories that pass N cycles of the motion before crossing contribute to peaks of the first-passage density with mean periods of about $\tau = NT$. These transient oscillations occur near subharmonics of the fundamental period, but are randomly spaced in time, so do not correspond to any stationary oscillation mode. Similar near-threshold subharmonic oscillations are seen in real frictional systems [2], are evidenced by anticorrelations in return maps found in experiments on granular sliding [4], and are predicted by deterministic friction models that account for multiple, coupled oscillation modes [1,22,30,31].

Multimodal period and slip densities: The period density, $p(T)$, was computed using the decoupling approximation $\mathcal{F}_g^{(1)}(x, T)$ (figs. 6C, D). At low damping, a decaying sequence of maxima is observed, revealing quasiperiodic oscillations with transient slip periods spaced at integer multiples of the free-sliding period. As noted above, they do not correspond to any stationary subharmonic oscillation mode, as they are randomly distributed in time. The corresponding density $p(d)$ of slip displacements $p(d)$ is approximately Gaussian-shaped, despite the multimodal nature of $p(T)$. Simulations yield a small probability of sticking immediately after slip onset ($d \approx 0$), consistent with experiments (*e.g.*, [9], fig. 18). However, effects due to static friction variations and pre-sliding would affect such transitions [2,3], and are not accounted for here.

Summary. – A novel formulation of frictional stick-slip motion as a first-passage time problem was presented, and used to investigate statistical distributions of stick-slip oscillatory motion. Approximations were obtained for distributions of periods, slip times, and slip lengths for a generic class of stochastic friction models, via a level-crossing statistics expansion. Applied to a Langevin model of sliding friction, analytic solutions yielded good agreement with simulations and prior experiments. While this model ignored the dynamics of static to kinetic friction transitions (which can depend on asperity strengths, contact aging, etc.), such effects could, for example, be accommodated by allowing for random variations in the static threshold, or by applying this analysis to a friction model (*e.g.*, the Burridge-Knopoff model [15]) that accounts for multiple interfacial degrees of freedom. The presence of noise-stimulated transient oscillation modes suggests an intriguing connection with the literature on stochastic resonance and excitable systems [32]. While the case example examined here involved a piecewise-linear non-smooth dynamics, qualitative arguments summarized above suggest similar near-threshold behavior may be expected in disordered frictional systems with non-linearities or additional dynamical states.

Appendix: Langevin upcrossing density. – The upcrossing density is $n_1(t) = \int_{-\infty}^{\infty} n_1(x, v_b, t)\,\mathrm{d}x$, where

$$n_1(x,v,t) = P_{10}(x,v,t)\,I_a(x,v) \equiv P_{10}(I_{a1} + I_{a2}), \quad \text{(A.1)}$$

$$I_{a1} = \sqrt{\frac{\sigma^2}{2\pi}}\exp\left(\frac{-\mu_a^2}{2\sigma^2}\right), \quad I_{a2} = -\frac{\mu_a}{2}\operatorname{erfc}\left(\frac{\mu_a}{\sqrt{2}\sigma}\right), \quad \text{(A.2)}$$

P_{10} is a Gaussian process in $\mathbf{x} = (x, v)$, so the integral in $n_1(t)$ involves factors of Gaussian or Error Function type in x and v_b. One can write

$$P_{10}(x,v,t) = N(\mathbf{x};\boldsymbol{\mu},\boldsymbol{\Sigma}), \quad I_{a1}(x,v) = \sigma\sqrt{2\pi}\,N(\mathbf{x};0,\boldsymbol{S}),$$

where N is the bivariate normal density and

$$\boldsymbol{S} = \begin{pmatrix} \omega_0^4 & \omega_0^2\zeta \\ \omega_0^2\zeta & \zeta^2 \end{pmatrix}\sigma^2 \equiv \boldsymbol{C}^{-1}. \quad \text{(A.3)}$$

The integrand $P_{10}I_{a1}$ is a product of bivariate Gaussian functions, and can be written $P_{10}I_{a1} = \tilde{Z}N(\mathbf{x};\tilde{\boldsymbol{\mu}},\tilde{\boldsymbol{\Sigma}})$,

$$\tilde{Z} = |\boldsymbol{S}|\sigma\sqrt{2\pi}\;N(\boldsymbol{\mu};0,\boldsymbol{S}\tilde{\boldsymbol{\Sigma}}\boldsymbol{\Sigma}^{-1}), \quad \text{(A.4)}$$

$$\tilde{\boldsymbol{\Sigma}} = (\boldsymbol{\Sigma}^{-1} + \boldsymbol{S})^{-1}, \quad \tilde{\boldsymbol{\mu}} = \tilde{\boldsymbol{\Sigma}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}. \quad \text{(A.5)}$$

The integral over x then gives

$$\int_{-\infty}^{\infty} \mathrm{d}x\,P_{10}I_{a,1} \;=\; \tilde{Z}N(v;\tilde{\mu}_v,\tilde{\Sigma}_{vv}). \quad \text{(A.6)}$$

The integral of $P_{10}I_{a2}$ yields

$$\int_{-\infty}^{\infty} \mathrm{d}x\,P_{10}I_{a,2} = Z'\left(n\sqrt{\pi}\operatorname{erfc}(nq) + (q^{-2}-1)e^{-n^2q^2}\right)$$

with

$$n = \frac{\sqrt{2}}{\sigma}\left(v\left(\zeta - \frac{C_{xv}}{C_{xx}}\omega_0^2\right) + \mu_x\omega_0^2\right), \quad \text{(A.7)}$$

$$m = 2\omega_0^2\sigma^{-1}C_{xx}^{-\frac{1}{2}}, \qquad q = \left(1 + \frac{4\omega_0^4}{\sigma^2 C_{xx}}\right)^{-\frac{1}{2}}, \quad \text{(A.8)}$$

$$Z' = \sigma(C_{xx}C_{vv}|4\pi\boldsymbol{\Sigma}|)^{-\frac{1}{2}}\;N(v;\mu_v,C_{vv}). \quad \text{(A.9)}$$

With these definitions, $n_1(t)$ can be written as

$$n_1(t) \;=\; \tilde{Z}N(v;\tilde{\mu}_v,\tilde{\Sigma}_{vv}) + Z'(n\sqrt{\pi}\;\operatorname{erfc}(nq) \;+\; (q^{-1}-q)e^{-n^2q^2}). \quad \text{(A.10)}$$

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