Spectral Analysis of ΨDOs by Combining Analytic and Algebraic Techniques

Max Lein
in collaboration with Marius Măntoiu and Serge Richard

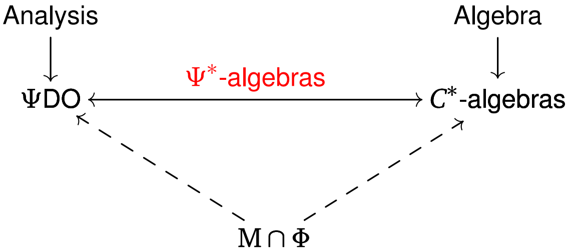
Kyushu University

2013.01.25@RIMS

Talk based on

1. *Magnetic pseudodifferential operators with coefficients in C^*-algebras*, Publ. RIMS Kyoto Univ., Volume 46 (2010), 755–788
2. *Semiclassical Dynamics and Magnetic Weyl Calculus*, PhD thesis

Key idea


Analysis
Algebra
ΨDO
Ψ^*-algebras
C^*-algebras
$\mathrm{M} \cap \Phi$

The notion of Ψ^*-algebra

Definition (Ψ^*-algebra Gramsch (1984))

Let $\Psi \subset \mathfrak{C}$ be a unital $*$-subalgebra. Then Ψ is called a Ψ^*-algebra if and only if

(i) if Ψ can be endowed with a Fréchet topology τ_Ψ such that the embedding $\Psi \hookrightarrow \mathfrak{C}$ is continuous and

(ii) $\Psi \cap \mathfrak{C}^{-1} = \Psi^{-1}$ holds.

My motivation

1. Analysis of *magnetic* ΨDOs using algebraic techniques.
2. *Application to problems in solid state physics:* conduction properties in crystalline solids and photonic crystals (space-adiabatic perturbation theory, existence of an exponentially localized Wannier basis)
3. *Ultimate goal: »Adiabatic perturbation theory« for random media:* Combination of non-commutative geometry with magnetic pseudodifferential calculus

My motivation

1. Analysis of *magnetic* ΨDOs using algebraic techniques.
2. *Application to problems in solid state physics:* conduction properties in crystalline solids and photonic crystals (space-adiabatic perturbation theory, existence of an exponentially localized Wannier basis)
3. *Ultimate goal: »Adiabatic perturbation theory« for random media:* Combination of non-commutative geometry with magnetic pseudodifferential calculus

Simplified setup: $B = 0$

Pseudodifferential operators on $\mathbb{R}^d$

Building block operators

Weyl quantization is a functional calculus for

$$Q = \hat{x}$$
$$P = -i\nabla_x$$

commutation relations

$$i[Q_l, Q_j] = 0 \qquad i[P_l, Q_j] = \delta_{lj} \qquad i[P_l, P_j] = 0$$

Weyl quantization

For suitable functions $h : \Xi \longrightarrow \mathbb{C}$ on phase space $\Xi = \mathbb{R}^d \times \mathbb{R}^{d^*}$, $u \in \mathcal{S}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$:

$$\begin{aligned}\big(\mathfrak{Op}(h)u\big)(x) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{d}y \int_{\mathbb{R}^{d^*}} \mathrm{d}\eta \, e^{-i(y-x)\cdot\eta} \cdot \\ &\qquad\qquad \cdot h\Big(\tfrac{1}{2}(x+y), \eta\Big) \, u(y) \\ &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{d}y \, \mathcal{F}h\Big(\tfrac{1}{2}(x+y), y-x\Big) \, u(y)\end{aligned}$$

The Moyal product

$$\mathfrak{Op}\big(f\sharp g\big) := \mathfrak{Op}(f)\,\mathfrak{Op}(g)$$

$$(f\sharp g)(x,\xi) = \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^d\times\mathbb{R}^{d*}} \mathrm{d}y\,\mathrm{d}\eta \int_{\mathbb{R}^d\times\mathbb{R}^{d*}} \mathrm{d}z\,\mathrm{d}\zeta\, \mathrm{e}^{-i(\eta\cdot y - y\cdot\zeta)} \,. \\ \cdot f\Big(x - \tfrac{1}{2}y, \xi - \eta\Big)\, g\Big(x - \tfrac{1}{2}z, \xi - \zeta\Big)$$

The Moyal product

$$\mathfrak{Op}\big(f\sharp g\big) := \mathfrak{Op}(f)\,\mathfrak{Op}(g)$$

$$\big(f\sharp g\big)(x,\xi) = \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^d\times\mathbb{R}^{d*}} \mathrm{d}y\,\mathrm{d}\eta \int_{\mathbb{R}^d\times\mathbb{R}^{d*}} \mathrm{d}z\,\mathrm{d}\zeta\; \mathrm{e}^{-i(\eta\cdot y - y\cdot\zeta)} \,. \\ \cdot f\Big(x-\tfrac{1}{2}y,\xi-\eta\Big)\, g\Big(x-\tfrac{1}{2}z,\xi-\zeta\Big)$$

The Moyal product

$$\mathfrak{Op}\big(f \sharp g\big) := \mathfrak{Op}(f)\, \mathfrak{Op}(g)$$

$$\begin{aligned}(f \sharp g)(x,\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} & \mathrm{d}y \, \mathrm{e}^{+i\xi \cdot y} \cdot \\ & \cdot \int_{\mathbb{R}^d} \mathrm{d}z \, \mathcal{F}f\Big(x - \tfrac{1}{2}(y-z), y\Big) \, \mathcal{F}g\Big(x - \tfrac{1}{2}z, y - z\Big)\end{aligned}$$

Hörmander symbol classes

Definition

The Fréchet space of Hörmander symbols of order $m \in \mathbb{R}$ and type $\rho \in [0,1]$ are defined by

$$S^m_\rho := \Big\{ f \in \mathcal{C}^\infty(\Xi) \;\Big|\; \forall a, \alpha \in \mathbb{N}_0^d \, \exists C_{a\alpha} > 0 : \\ \big| \partial_x^a \partial_\xi^\alpha f(x,\xi) \big| \leq C_{a\alpha} \, \langle \xi \rangle^m \;\; \forall (x,\xi) \in \Xi \Big\}.$$

Hörmander symbol classes

Theorem

$S_\rho^{m_1} \sharp S_\rho^{m_2} \subseteq S_\rho^{m_1+m_2}$

Hörmander symbol classes

Theorem (Bony)

Let $h = h^* \in S^m_\rho$ *for* $m \geq 0$*. If* $m > 0$*, we assume that in addition* h *is elliptic. Then for all* $z \in \mathbb{C} \setminus \mathbb{R}$

$$(h - z)^{(-1)_\sharp} \in S^{-m}_\rho,$$

where the Moyal resolvent $(h - z)^{(-1)_\sharp}$ *is the inverse of* $h - z$ *with respect to* $\sharp$*.*

Ellipticity $\Rightarrow \mathfrak{Op}(h)^* = \mathfrak{Op}(h)$ with domain $H^m(\mathbb{R}^d)$

Hörmander symbol classes

Theorem (Bony)

Let $h = h^* \in S^m_\rho$ *for* $m \geq 0$. *If* $m > 0$, *we assume that in addition* h *is elliptic. Then for all* $z \in \mathbb{C} \setminus \mathbb{R}$

$$(h - z)^{(-1)_\sharp} \in S^{-m}_\rho,$$

where the Moyal resolvent $(h - z)^{(-1)_\sharp}$ *is the inverse of* $h - z$ *with respect to* $\sharp$.

Ellipticity $\Rightarrow \mathfrak{Op}(h)^* = \mathfrak{Op}(h)$ with domain $H^m(\mathbb{R}^d)$

Simplest relevant C^*-algebra

Definition

$\mathfrak{B} := \mathfrak{Op}^{-1}\left(\mathcal{B}\big(L^2(\mathbb{R}^d)\big)\right)$ mit

1. C^*-norm $\|f\|_{\mathfrak{B}} := \big\|\mathfrak{Op}(f)\big\|_{\mathcal{B}(L^2(\mathbb{R}^d))}$
2. Product $f \sharp g := \mathfrak{Op}^{-1}\big(\mathfrak{Op}(f)\,\mathfrak{Op}(g)\big)$ und
3. Involution $f^* := \mathfrak{Op}^{-1}\big(\mathfrak{Op}(f)^*\big)$

f, g suitable: $f \sharp g$ agrees with Moyal product and f^* is the function which is the pointwise complex conjugate of f.

Simplest relevant C^*-algebra

Definition

$\mathfrak{B} := \mathfrak{Op}^{-1}\big(\mathcal{B}(L^2(\mathbb{R}^d))\big)$ mit

1. C^*-norm $\|f\|_{\mathfrak{B}} := \big\|\mathfrak{Op}(f)\big\|_{\mathcal{B}(L^2(\mathbb{R}^d))}$
2. Product $f \sharp g := \mathfrak{Op}^{-1}\big(\mathfrak{Op}(f)\,\mathfrak{Op}(g)\big)$ und
3. Involution $f^* := \mathfrak{Op}^{-1}\big(\mathfrak{Op}(f)^*\big)$

f, g suitable: $f \sharp g$ agrees with Moyal product and f^* is the function which is the pointwise complex conjugate of f.

Simplest relevant C^*-algebra

Problem

$\mathfrak{B}$ is often »too big.« $\rightsquigarrow$ *twisted product* C^*-algebras

Simplest relevant C^*-algebra

Problem

$\mathfrak{B}$ is often »too big.« $\rightsquigarrow$ *twisted product* C^*-algebras

Anisotropy algebra

Definition

Let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{bu}}(\mathbb{R}^d)$ be a C^*-subalgebra, which is stable under translations.

Encodes the behavior of the ΨDO $\mathfrak{Op}(h)$ in the x variable of $h : \mathbb{R}^d \times \mathbb{R}^{d^*} \longrightarrow \mathbb{R}$.

Anisotropy algebra

Definition

Let $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{bu}}(\mathbb{R}^d)$ be a C^*-subalgebra, which is stable under translations.

Encodes the behavior of the ΨDO $\mathfrak{Op}(h)$ in the x variable of $h : \mathbb{R}^d \times \mathbb{R}^{d^*} \longrightarrow \mathbb{R}$.

Anisotropy algebra

The triple $(\mathcal{A}, \mathbb{R}^d, \theta)$ forms a C^*-dynamical system, where $\theta_x[\varphi] := \varphi(\cdot + x)$ is the group action by translation.

crossed product C^*-algebra

Proposition

The triple $\big(L^1(\mathbb{R}^d, \mathcal{A}), \star, {}^{\star}\big)$ *is a Banach-$*$-algebra, where* $L^1(\mathbb{R}^d, \mathcal{A})$ *is the Banach space of Bochner-integrable functions with*

1. *product*

$$(F \star G)(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{d}y \; \theta_{-\frac{1}{2}(x-y)}\big[F(y)\big] \; \theta_{\frac{1}{2}y}\big[G(x-y)\big] \in \mathcal{A}$$

2. *and involution* $F^{\star}(x) := F^*(-x)$.

crossed product C^*-algebra

Proposition

The triple $\big(L^1(\mathbb{R}^d, \mathcal{A}), \star, {}^\star\big)$ *is a Banach-$*$-algebra, where* $L^1(\mathbb{R}^d, \mathcal{A})$ *is the Banach space of Bochner-integrable functions with*

1. *product*

$$(F \star G)(x,z) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{d}y\, F\Big(y, z - \tfrac{1}{2}(x-y)\Big)\, G\Big(x-y, z + \tfrac{1}{2}y\Big)$$

2. *and involution* $F^\star(x) := F^*(-x)$*.*

crossed product C^*-algebra

Definition (*crossed product* C^*-algebra)

$\mathcal{A} \rtimes \mathbb{R}^d$ is defined as the completion of the Banach-$*$ algebra $\left(L^1(\mathbb{R}^d, \mathcal{A}), \star, ^\star\right)$ with respect to the C^*-norm

$$\|F\|_{\rtimes} := \sup\left\{ \left\| \pi(F) \right\|_{\mathcal{B}(\mathcal{H})} \mid \pi \text{ non-degenerate representation on } \mathcal{H} \right\}.$$

Fourier-transformed *crossed product* C^*-algebra

Definition (Fourier transformation $\mathcal{F}$)

$\mathcal{F}^{-1} : L^1(\mathbb{R}^d, \mathcal{A}) \longrightarrow \mathcal{C}_0\big(\mathbb{R}^{d^*}, \mathcal{A}\big)$

$$\big(\mathcal{F}^{-1}F\big)(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{d}x \, \mathrm{e}^{+i\xi\cdot x} F(x) \in \mathcal{A}$$

Fourier-transformed *crossed product* C^*-algebra

Definition

$\mathfrak{C}_{\mathcal{A}} := \mathcal{F}^{-1}\big(\mathcal{A} \rtimes \mathbb{R}^d\big)$ mit

1. Norm $\|f\|_{\mathfrak{C}_{\mathcal{A}}} := \|\mathcal{F}f\|_{\rtimes}$,
2. Produkt $f \sharp g := \mathcal{F}^{-1}\big(\mathcal{F}f \star \mathcal{F}g\big)$,

$$(f \sharp g)(x,\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{d}y \, \mathrm{e}^{+i\xi\cdot y} \cdot \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{d}z \, \mathcal{F}f\Big(x - \tfrac{1}{2}(y-z), y\Big) \, \mathcal{F}g\Big(x - \tfrac{1}{2}z, y - z\Big)$$

and

3. Involution $f^{\sharp} := \mathcal{F}^{-1}\big((\mathcal{F}f)^{\star}\big) = f^*$

f, g suitable: $f \sharp g$ coincides with Moyal produkt

Fourier-transformed *crossed product* C^*-algebra

Definition

$\mathfrak{C}_{\mathcal{A}} := \mathcal{F}^{-1}(\mathcal{A} \rtimes \mathbb{R}^d)$ mit

1. Norm $\|f\|_{\mathfrak{C}_{\mathcal{A}}} := \|\mathcal{F}f\|_{\rtimes}$,
2. Produkt $f \sharp g := \mathcal{F}^{-1}(\mathcal{F}f \star \mathcal{F}g)$,

$$(f \sharp g)(x,\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{d}y \, \mathrm{e}^{+i\xi\cdot y} \cdot \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{d}z \, \mathcal{F}f\Big(x - \tfrac{1}{2}(y-z), y\Big) \, \mathcal{F}g\Big(x - \tfrac{1}{2}z, y-z\Big)$$

and

3. Involution $f^{\sharp} := \mathcal{F}^{-1}\big((\mathcal{F}f)^{\star}\big) = f^*$

f, g suitable: $f \sharp g$ coincides with Moyal produkt

Representations of $\mathfrak{C}_\mathcal{A}$

Theorem

1. $\mathfrak{Op}$ *is the position representation of* $\mathfrak{C}_\mathcal{A}$ *on* $L^2(\mathbb{R}^d)$.
2. $\mathcal{F}\,\mathfrak{Op}\,\mathcal{F}^{-1}$ *is the momentum representation of* $\mathfrak{C}_\mathcal{A}$ *on* $L^2(\mathbb{R}^{d^*})$.

$\Rightarrow \mathfrak{C}_\mathcal{A} \hookrightarrow \mathfrak{B}$

Representations of $\mathfrak{C}_{\mathcal{A}}$

Theorem

1. $\mathfrak{Op}$ *is the position representation of* $\mathfrak{C}_{\mathcal{A}}$ *on* $L^2(\mathbb{R}^d)$.
2. $\mathcal{F}\,\mathfrak{Op}\,\mathcal{F}^{-1}$ *is the momentum representation of* $\mathfrak{C}_{\mathcal{A}}$ *on* $L^2(\mathbb{R}^{d^*})$.

$\Rightarrow \mathfrak{C}_{\mathcal{A}} \hookrightarrow \mathfrak{B}$

Representations of $\mathfrak{C}_{\mathcal{A}}$

Theorem

1. *$\mathfrak{Op}$ is the position representation of $\mathfrak{C}_{\mathcal{A}}$ on $L^2(\mathbb{R}^d)$.*
2. *$\mathcal{F}\,\mathfrak{Op}\,\mathcal{F}^{-1}$ is the momentum representation of $\mathfrak{C}_{\mathcal{A}}$ on $L^2(\mathbb{R}^{d^*})$.*

$\Rightarrow \mathfrak{C}_{\mathcal{A}} \hookrightarrow \mathfrak{B}$

Ψ^*-property of S^0_ρ

Proposition

$S^0_\rho \hookrightarrow \mathfrak{B}$ is a Ψ^*-algebra.

Ψ^*-property of S^0_ρ

Proof.

1. $\mathfrak{Op} : S^0_\rho \hookrightarrow \mathcal{B}\big(L^2(\mathbb{R}^d)\big)$ continuous (Caldéron-Vaillancourt)
2. $S^0_\rho \sharp S^0_\rho \subseteq S^0_\rho$ (closedness under $\sharp$)
3. $S^0_\rho \cap \mathfrak{B}^{(-1)_\sharp} = \big(S^0_\rho\big)^{(-1)_\sharp}$ (corollary of Bony criterion)

□

Ψ^*-property of S^0_ρ

Proof.

1. $\mathfrak{Op} : S^0_\rho \hookrightarrow \mathcal{B}\big(L^2(\mathbb{R}^d)\big)$ continuous (Caldéron-Vaillancourt)
2. $S^0_\rho \sharp S^0_\rho \subseteq S^0_\rho$ (closedness under $\sharp$)
3. $S^0_\rho \cap \mathfrak{B}^{(-1)_\sharp} = \big(S^0_\rho\big)^{(-1)_\sharp}$ (corollary of Bony criterion)

□

Ψ^*-property of S^0_ρ

Proof.

1. $\mathfrak{Op} : S^0_\rho \hookrightarrow \mathcal{B}\big(L^2(\mathbb{R}^d)\big)$ continuous (Caldéron-Vaillancourt)
2. $S^0_\rho \sharp S^0_\rho \subseteq S^0_\rho$ (closedness under $\sharp$)
3. $S^0_\rho \cap \mathfrak{B}^{(-1)_\sharp} = \big(S^0_\rho\big)^{(-1)_\sharp}$ (corollary of Bony criterion)

□

Anistropic Hörmander symbol classes

Definition ($S^m_\rho(\mathcal{A})$)

$$S^m_\rho(\mathcal{A}) := \left\{ f \in S^m_\rho \;\middle|\; \forall a, \alpha \in \mathbb{N}_0^d, \xi \in \mathbb{R}^{d^*} \, x \mapsto \partial_x^a \partial_\xi^\alpha f(x, \xi) \in \mathcal{A} \right\}$$

Ψ^*-property of $S^0_\rho(\mathcal{A})$

Theorem (L.-Măntoiu-Richard (2010))

$S^0_\rho(\mathcal{A}) \hookrightarrow \mathfrak{B}$ *is a* Ψ^**-algebra.*

Ψ^*-property of $S^0_\rho(\mathcal{A})$

Proof.

1. $\mathfrak{Op} : S^0_\rho(\mathcal{A}) \hookrightarrow \mathcal{B}\big(L^2(\mathbb{R}^d)\big)$ continuous ($S^m_\rho(\mathcal{A}) \subseteq S^m_\rho$)
2. $S^0_\rho(\mathcal{A}) \sharp S^0_\rho(\mathcal{A}) \subseteq S^0_\rho(\mathcal{A})$ (L.-Măntoiu-Richard (2010), easy)
3. $S^0_\rho(\mathcal{A}) \cap \mathfrak{B}^{(-1)_\sharp} = \big(S^0_\rho(\mathcal{A})\big)^{(-1)_\sharp}$???

□

Ψ^*-property of $S^0_\rho(\mathcal{A})$

Proof.

1. $\mathfrak{Op} : S^0_\rho(\mathcal{A}) \hookrightarrow \mathcal{B}\big(L^2(\mathbb{R}^d)\big)$ continuous ($S^m_\rho(\mathcal{A}) \subseteq S^m_\rho$)
2. $S^0_\rho(\mathcal{A}) \sharp S^0_\rho(\mathcal{A}) \subseteq S^0_\rho(\mathcal{A})$ (L.-Măntoiu-Richard (2010), easy)
3. $S^0_\rho(\mathcal{A}) \cap \mathfrak{B}^{(-1)_\sharp} = \big(S^0_\rho(\mathcal{A})\big)^{(-1)_\sharp}$???

□

Ψ^*-property of $S^0_\rho(\mathcal{A})$

Proof.

1. $\mathfrak{Op} : S^0_\rho(\mathcal{A}) \hookrightarrow \mathcal{B}\big(L^2(\mathbb{R}^d)\big)$ continuous ($S^m_\rho(\mathcal{A}) \subseteq S^m_\rho$)
2. $S^0_\rho(\mathcal{A}) \sharp S^0_\rho(\mathcal{A}) \subseteq S^0_\rho(\mathcal{A})$ (L.-Măntoiu-Richard (2010), easy)
3. $S^0_\rho(\mathcal{A}) \cap \mathfrak{B}^{(-1)_\sharp} = \big(S^0_\rho(\mathcal{A})\big)^{(-1)_\sharp}$???

$\square$

Ψ^*-property of $S^0_\rho(\mathcal{A})$

Proving spectral invariance is not necessary:

Theorem (Lauter (1998))

Let $\Psi \hookrightarrow \mathfrak{B}$ be a Ψ^*-algebra and $\Psi' \subset \Psi$ a closed unital $*$-subalgebra. Then $\Psi' \hookrightarrow \mathfrak{B}$ endowed with the restricted topology $\tau_\Psi|_{\Psi'}$ is also a Ψ^*-algebra.

Ψ^*-property of $S^0_\rho(\mathcal{A})$

Proof.

1. $\mathfrak{Op} : S^0_\rho(\mathcal{A}) \hookrightarrow \mathcal{B}\big(L^2(\mathbb{R}^d)\big)$ continuous ($S^m_\rho(\mathcal{A}) \subseteq S^m_\rho$)
2. $S^0_\rho(\mathcal{A}) \sharp S^0_\rho(\mathcal{A}) \subseteq S^0_\rho(\mathcal{A})$ (L.-Măntoiu-Richard (2010), easy)
3. follows immediately from Ψ^*-property of S^0_ρ and Theorem by Lauter

□

What have we gained?

Determining the essential spectrum of a ΨDO

ΨDO $H = \mathfrak{Op}(h)$ associated to the function h with certain properties

Analysis	Algebra
↓	↓
ΨDO	C^*-algebras

Determining the essential spectrum of a ΨDO

Essential spectrum of an operator $H = H^* \in \mathcal{B}(\mathcal{H}) \rightsquigarrow$ *Calkin-Algebra* $\mathcal{C}(\mathcal{H}) := \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$

Analysis	Algebra
↓	↓
ΨDO	C^*-algebras

Determining the essential spectrum of a ΨDO

What is $[\mathfrak{Op}(h)]_{\mathcal{K}(\mathcal{H})} \in \mathcal{C}(\mathcal{H}) = \mathcal{B}(\mathcal{H}) / \mathcal{K}(\mathcal{H})$?

Analysis $\longrightarrow$ ΨDO

Algebra $\longrightarrow$ C^*-algebras

How do you combine both points of view?

Determining the essential spectrum of a ΨDO

1. Let h be an anisotropic Hörmander symbol, e. g. $S^0_\rho(\mathcal{A})$.

Analysis	Algebra
↓	↓
ΨDO	C^*-algebras

Determining the essential spectrum of a ΨDO

2. Identification of $S^0_\rho(\mathcal{A})$ with a subalgebra of $\mathfrak{C}_\mathcal{A} \subset \mathfrak{B}$.

Determining the essential spectrum of a ΨDO

3. Analysis of the quotient algebra $\mathcal{A}/\mathcal{C}_0(\mathbb{R}^d) \cong \mathcal{C}_0(\mathcal{S}_\mathcal{A} \setminus \mathbb{R}^d) \cong \bigcup_{j \in \mathcal{I}} \mathcal{C}(\mathcal{Q}_{\infty,j})$

Analysis	Algebra
↓	↓
ΨDO	C^*-algebras

Determining the essential spectrum of a ΨDO

4. $\mathcal{A} \rtimes \mathbb{R}^d / \mathcal{C}_0(\mathbb{R}^d) \rtimes \mathbb{R}^d \cong (\mathcal{A}/\mathcal{C}_0(\mathbb{R}^d)) \rtimes \mathbb{R}^d \cong \bigcup_{j \in \mathcal{I}} \mathcal{C}(\mathcal{Q}_{\infty,j}) \rtimes \mathbb{R}^d$

Analysis → ΨDO

Algebra → C^*-algebras

Determining the essential spectrum of a ΨDO

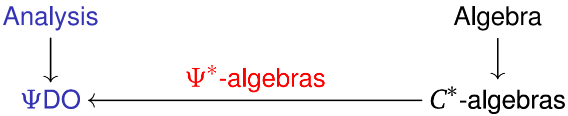
5. Morphism $\mathfrak{C}_{\mathcal{A}} \ni h \mapsto h_{\infty,j} \in \mathfrak{C}_{\mathcal{C}(\mathcal{Q}_{\infty,j})}$

Analysis $\longrightarrow$ ΨDO

Algebra $\longrightarrow$ C^*-algebras

Determining the essential spectrum of a ΨDO

6. The $h_{\infty,j}$ are again functions on $\mathbb{R}^d \times \mathbb{R}^{d^*}$ $\rightsquigarrow$ analysis of the spectra of the $\mathfrak{Op}(h_{\infty,j})$.



Concrete result

Theorem (L., Măntoiu, Richard (2010))

Suppose the components of B *are of class* $\mathcal{A}^\infty$, $m > 0$, $\rho \in [0,1]$ *and* $h \in S^m_\rho(\mathcal{A})$ *be elliptic. Then*

$$\sigma_{\text{ess}}\big(\mathfrak{Op}^A(h)\big) = \overline{\bigcup_{j \in \mathcal{I}} \sigma\big(\mathfrak{Op}^{A_{\infty,j}}(h_{\infty,j})\big)}$$

where $\bigcup_{j \in \mathcal{I}} \mathcal{Q}_{\infty,j}$ *is a covering of the points at infinity* $\Omega_\mathcal{A} \setminus \iota(\mathbb{R}^d)$.

Further results

1. Inclusion of magnetic fields B with components in $\mathcal{A}^\infty$
2. Asymtptotic expansions of $\sharp$ are compatible with anisotropic Hörmander classes.
3. Extension of more general anisotropy algebras $\mathcal{A} = \mathcal{C}(\Omega)$ are available.

Further results

1. Inclusion of magnetic fields B with components in $\mathcal{A}^\infty$
2. Asymtptotic expansions of $\sharp$ are compatible with anisotropic Hörmander classes.
3. Extension of more general anisotropy algebras $\mathcal{A} = \mathcal{C}(\Omega)$ are available.

Further results

1. Inclusion of magnetic fields B with components in $\mathcal{A}^\infty$
2. Asymtptotic expansions of $\sharp$ are compatible with anisotropic Hörmander classes.
3. Extension of more general anisotropy algebras $\mathcal{A} = \mathcal{C}(\Omega)$ are available.

Take-away message

1. The notion of Ψ^*-algebras mediates between analytic and algebraic point of view.
2. Allows applications of algebraic tools to problems from $\mathrm{M} \cap \Phi$.
3. Algebraic point of view sometimes simplifies arguments involving pseudodifferential calculus.

Take-away message

1. The notion of Ψ^*-algebras mediates between analytic and algebraic point of view.
2. Allows applications of algebraic tools to problems from $\mathrm{M} \cap \Phi$.
3. Algebraic point of view sometimes simplifies arguments involving pseudodifferential calculus.

Take-away message

1. The notion of Ψ^*-algebras mediates between analytic and algebraic point of view.
2. Allows applications of algebraic tools to problems from $\mathrm{M} \cap \Phi$.
3. Algebraic point of view sometimes simplifies arguments involving pseudodifferential calculus.

Referenzen

- *Spectral and Propagation Results for Magnetic Schroedinger Operators; a C*-Algebraic Framework*, M.Măntoiu, R. Purice, S. Richard, Journal of Functional Analysis, Vol. 250, Issue 1 (2007)
- *Commutator Criteria for Magnetic Pseudodifferential Operators*, V. Iftimie, M. Măntoiu, R. Purice, Communications in Partial Differential Equations, Vol. 35, No. 6 (2010), 1058–1094
- *Twisted Crossed Products and Magnetic Pseudodifferential Operators*, M. Măntoiu, R. Purice, S. Richard, arxiv:math-ph/0403016, Journal of Functional Analysis, Vol. 250, No. 1 (2007), 42–67

Referenzen

- *An Operator Theoretical Approach to Enveloping Ψ^*- and C^*-Algebras of Melrose Algebras of Totally Characteristic Pseudodifferential Operators*, R. Lauter, Mathematische Nachrichten, Vol. 196, pp. 141-166 (1998)
- *Magnetic pseudodifferential operators with coefficients in C^*-algebras*, Publ. RIMS Kyoto Univ., Volume 46 (2010), 755–788
- *Semiclassical Dynamics and Magnetic Weyl Calculus*, PhD thesis