Ray Optics in Topologically Non-trivial Photonic Crystals

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in collarboration with Giuseppe De Nittis

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2015.04.21@Newton Institute

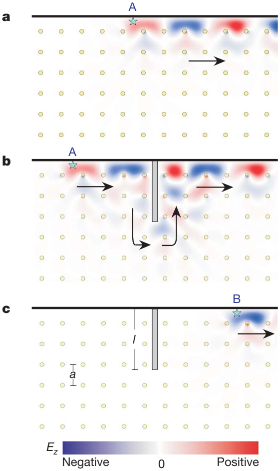
Talk Based on

Collaboration with **Giuseppe De Nittis**

- *On the Role of Symmetries in the Theory of Photonic Crystals*
  Annals of Physics **350**, pp. 568--587, 2014
- *Effective Light Dynamics in Perturbed Photonic Crystals*
  Comm. Math. Phys. **332**, issue 1, pp. 221--260, 2014
- *Derivation of Ray Optics Equations in Photonic Crystals Via a Semiclassical Limit*
  arxiv:1502.07235, submitted for publication, 2015

Motivation

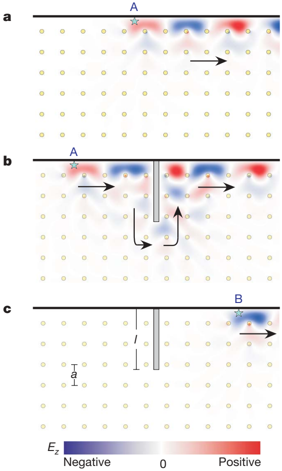
$$\begin{pmatrix} \overline{\varepsilon} & 0 \\ 0 & \overline{\mu} \end{pmatrix} \neq \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \qquad \Longrightarrow$$



Wang et al (2009)

Motivation

Chern numbers $\neq 0$ (?) $\Longleftrightarrow$



Wang et al (2009)

Motivation

Understand ***how topological effects emerge from electrodynamics****,* *starting from Maxwells equations.*

Motivation

... many natural connected problems, e. g.

- Role of symmetries $\longleftrightarrow$ **photonic topological insulators**
- Topological invariants and their physical interpretation
  $\rightsquigarrow$ **bulk-edge correspondences**
- Persistence of topological effects in random photonic crystals
  $\rightsquigarrow$ Anderson localization

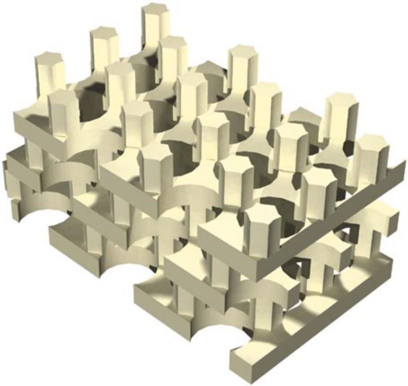
Part 1
Photonic Crystals

Part 2
Single-Band Ray Optics

Part 3
Ray Optics for Real Fields

Part 1
Photonic Crystals

Photonic Cyrstals



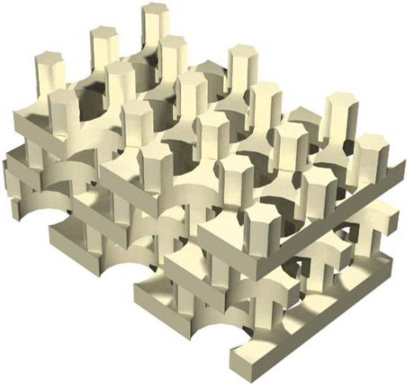
Johnson & Joannopoulos (2004)

Assumption (Material weights)

$$W(x) = \begin{pmatrix} \varepsilon(x) & 0 \\ 0 & \mu(x) \end{pmatrix}$$

1. $0 < c\,\mathbf{1} \leq W \leq C\,\mathbf{1}$ *(excludes metamaterials)*
2. $W^* = W$ *(lossless)*
3. *W frequency-independent (response instantaneous)*
4. *W periodic wrt lattice* $\Gamma \simeq \mathbb{Z}^3$

Photonic Cyrstals



Johnson & Joannopoulos (2004)

Maxwell equations

Dynamical equations

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla_x \times \mathbf{H} \\ +\nabla_x \times \mathbf{E} \end{pmatrix}$$

Absence of sources

$$\begin{pmatrix} \operatorname{div}(\varepsilon\, \mathbf{E}) \\ \operatorname{div}(\mu\, \mathbf{H}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Idea of Ray Optics

Find **simpler, effective dynamics** for states
from a **narrow range of frequencies**
where Chern numbers $\neq 0$.

Option 1
Effective tight-binding operators
$\rightsquigarrow$ CMP article, but up to now with limitation Chern numbers $= 0$

Option 2
Ray optics limit
$\rightsquigarrow$ Preprint, *considered here*

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Idea of Ray Optics

An incarnation of a **rigorous quantum-light analogy**.

Semiclassics $\longleftrightarrow$ Ray Optics

Idea of Ray Optics

Maxwell Equations $\xrightarrow{\lambda \ll 1}$ **Ray Optics Equations**

$$\left.\begin{matrix} \begin{pmatrix} \varepsilon_\lambda & 0 \\ 0 & \mu_\lambda \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla_x \times \mathbf{H} \\ +\nabla_x \times \mathbf{E} \end{pmatrix} \\ \begin{pmatrix} \mathrm{div}(\varepsilon_\lambda \mathbf{E}) \\ \mathrm{div}(\mu_\lambda \mathbf{H}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{matrix} \right\} \xrightarrow{\lambda \ll 1} \begin{cases} \dot{r} = +\nabla_k \Omega + \mathcal{O}(\lambda) \\ \dot{k} = -\nabla_r \Omega + \mathcal{O}(\lambda) \end{cases}$$

$\lambda \ll 1$ perturbation parameter

Ω dispersion relation

Idea of Ray Optics

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Ω dispersion relation

Schrödinger Formalism of the Maxwell Equations

1. *Field energy*

$$\mathcal{E}(\mathbf{E}, \mathbf{H}) = \frac{1}{2} \int_{\mathbb{R}^3} \mathrm{d}x \begin{pmatrix} \mathbf{E}(x) \\ \mathbf{H}(x) \end{pmatrix} \cdot \begin{pmatrix} \varepsilon(x) & 0 \\ 0 & \mu(x) \end{pmatrix} \begin{pmatrix} \mathbf{E}(x) \\ \mathbf{H}(x) \end{pmatrix}$$

2. *Dynamical equations*

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla_x \times \mathbf{H} \\ +\nabla_x \times \mathbf{E} \end{pmatrix}$$

3. *No sources*

$$\begin{pmatrix} \mathrm{div}(\varepsilon \, \mathbf{E}) \\ \mathrm{div}(\mu \, \mathbf{H}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Schrödinger Formalism of the Maxwell Equations

1. *Field energy*

$$\mathcal{E}\big(\mathbf{E},\mathbf{H}\big) = \mathcal{E}\big(\mathbf{E}(t),\mathbf{H}(t)\big)$$

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Schrödinger Formalism of the Maxwell Equations

1. *Field energy* $(\mathbf{E}, \mathbf{H}) \in L^2_w(\mathbb{R}^3, \mathbb{C}^6)$ with energy norm

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2. *Dynamical equations* $\rightsquigarrow$ »Schrödinger equation«

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$$J_w := \left\{ \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \in L^2_w(\mathbb{R}^3, \mathbb{C}^6) \;\middle|\; \begin{pmatrix} \mathrm{div} & 0 \\ 0 & \mathrm{div} \end{pmatrix} \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = 0 \right\}$$

Schrödinger Formalism of the Maxwell Equations

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$$\left\lVert (\mathbf{E}, \mathbf{H}) \right\rVert^2_{L^2_w} = 2\,\mathcal{E}(\mathbf{E}, \mathbf{H})$$

2. *Dynamical equations* $\rightsquigarrow$ »Schrödinger equation«

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$$\mathrm{i} \frac{\partial}{\partial t} \underbrace{\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}}_{=\Psi} = \underbrace{\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & +\mathrm{i}\nabla^{\times} \\ -\mathrm{i}\nabla^{\times} & 0 \end{pmatrix}}_{=M} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}$$

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The Frequency Band Picture

$$M \cong M^{\mathcal{F}} = \int_{\mathbb{B}}^{\oplus} \mathrm{d}k \, M(k)$$
$$= \int_{\mathbb{B}}^{\oplus} \mathrm{d}k \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & +(-\mathrm{i}\nabla_y + k)^{\times} \\ -(-\mathrm{i}\nabla_y + k)^{\times} & 0 \end{pmatrix}$$

$$\mathfrak{D}\big(M(k)\big) = \underbrace{\big(H^1(\mathbb{T}^3, \mathbb{C}^6) \cap J_w(k)\big)}_{\text{physical states}} \oplus G(k) \subset L^2_w(\mathbb{T}^3, \mathbb{C}^6)$$

$M(k)|_{G(k)} = 0 \Rightarrow$ focus on $M(k)|_{J_w(k)}$

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The Frequency Band Picture

Physical bands

$$M(k)\varphi_n(k) = \omega_n(k)\,\varphi_n(k)$$

- Frequency band functions $k \mapsto \omega_n(k)$
- Bloch functions $k \mapsto \varphi_n(k)$
- both locally continuous everywhere
- both locally analytic *away from band crossings*

The Frequency Band Picture

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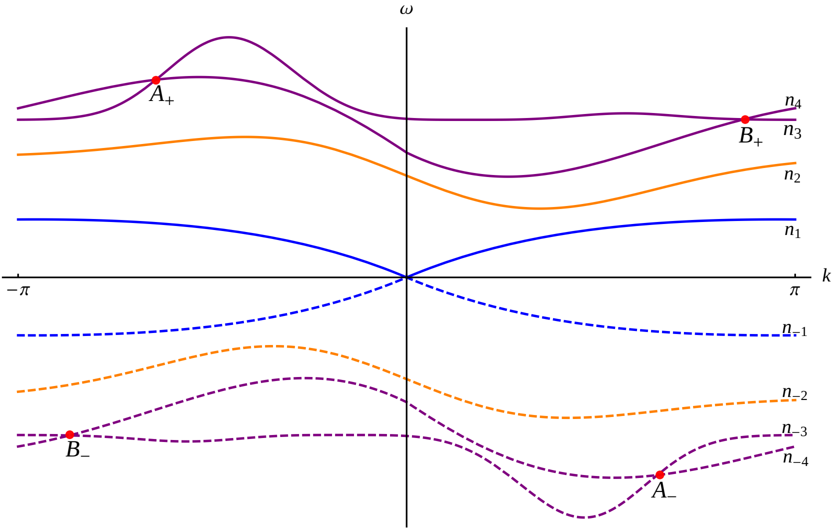
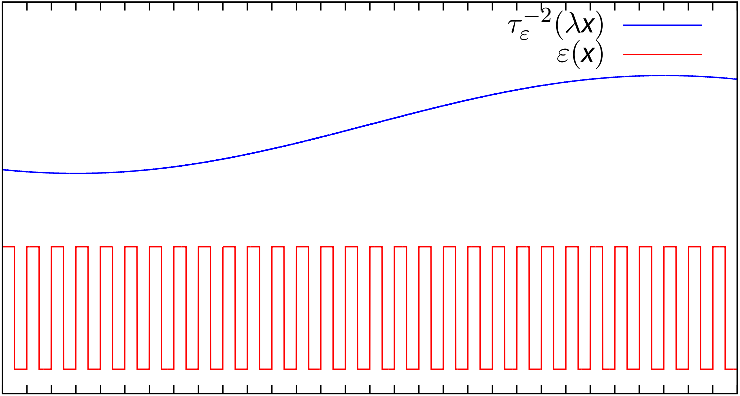
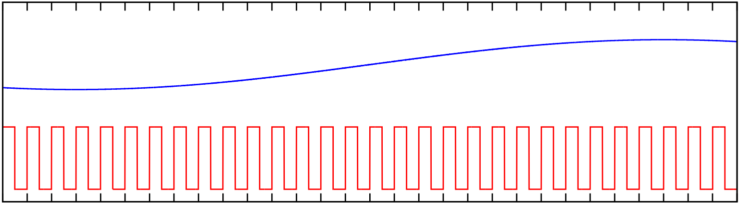


Perturbed Photonic Crystals



x [lattice constants]

Perturbed Photonic Crystals

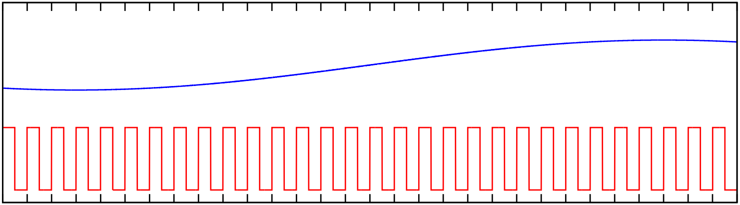


Perturbation of material constants

$$\lambda = \frac{\text{[lattice spacing]}}{\text{[length scale of modulation]}} \ll 1$$

$$\varepsilon(x) \rightsquigarrow \varepsilon_\lambda(x) := \tau_\varepsilon^{-2}(\lambda x)\, \varepsilon(x), \quad \mu(x) \rightsquigarrow \mu_\lambda(x) := \tau_\mu^{-2}(\lambda x)\, \mu(x)$$

Perturbed Photonic Crystals



Assumption (Slow modulation)

$\tau_\varepsilon, \tau_\mu \in \mathcal{C}^\infty_{\mathsf{b}}(\mathbb{R}^3)$, $\tau_\varepsilon, \tau_\mu \geq c > 0$

Adiabatically Perturbed Maxwell Operator

$$M_\lambda = S_\lambda^{-2} \, M$$
$$= \begin{pmatrix} \tau_\varepsilon^2(\lambda x) & 0 \\ 0 & \tau_\mu^2(\lambda x) \end{pmatrix} \begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & \mu^{-1} \end{pmatrix} \begin{pmatrix} 0 & +(-\mathrm{i}\nabla_x)^\times \\ -(-\mathrm{i}\nabla_x)^\times & 0 \end{pmatrix}$$

Slow modulation & periodic Maxwell operator

$\rightsquigarrow$ Perturbations are multiplicative!

Defined on λ-dependent Hilbert space $\mathfrak{H}_\lambda := L^2_{w_\lambda}(\mathbb{R}^3, \mathbb{C}^6)$

Adiabatically Perturbed Maxwell Operator

$$M_\lambda = S_\lambda^{-2}\, M$$
$$= \begin{pmatrix} \tau_\varepsilon^2(\lambda x) & 0 \\ 0 & \tau_\mu^2(\lambda x) \end{pmatrix} \begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & \mu^{-1} \end{pmatrix} \begin{pmatrix} 0 & +(-\mathrm{i}\nabla_x)^\times \\ -(-\mathrm{i}\nabla_x)^\times & 0 \end{pmatrix}$$

Slow modulation & periodic Maxwell operator
$\rightsquigarrow$ Perturbations are multiplicative!
Defined on λ-dependent Hilbert space $\mathfrak{H}_\lambda := L^2_{w_\lambda}(\mathbb{R}^3, \mathbb{C}^6)$

Part 2
Single-Band Ray Optics

Ray Optics Limit via Semiclassical Techniques

Idea

For $(\mathbf{E}, \mathbf{H}) \in \mathfrak{H}_{\mathrm{rel}}$ from a closed subspace of relevant states:

$$\begin{aligned} F\big(\mathbf{E}(t), \mathbf{H}(t)\big) &= \Big\langle \big(\mathbf{E}(t), \mathbf{H}(t)\big), \mathrm{Op}_\lambda(f) \, \big(\mathbf{E}(t), \mathbf{H}(t)\big) \Big\rangle_\lambda \\ &= \Big\langle (\mathbf{E}, \mathbf{H}), \mathrm{Op}_\lambda\big(f_{\mathrm{ro}} \circ \Phi^\lambda_t\big) \, (\mathbf{E}, \mathbf{H}) \Big\rangle_\lambda + \mathcal{O}(\lambda^2) \end{aligned}$$

where

1. F is an observable associated to a ΨDO,
2. $\big(\mathbf{E}(t), \mathbf{H}(t)\big) \in \mathfrak{H}_{\mathrm{rel}}$ solves Maxwell's equations, and
3. Φ^λ is the ray optics flow.

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Class of
Observables

- Functionals $F : L^2_w(\mathbb{R}^3, \mathbb{C}^6) \longrightarrow \mathbb{C}$,

$$F(\Psi) = \big\langle \Psi, \mathrm{Op}_\lambda(f)\Psi \big\rangle_\lambda := \big\langle \Psi \, , \, S_\lambda^{-1} \, \mathcal{F}^{-1} \, f\big(\mathrm{i}\lambda \nabla_k, \hat{k}\big) \, \mathcal{F} \, S_\lambda \Psi \big\rangle_\lambda$$

- $\neq$ Quantum mechanics
- Here: quadratic observables, defined in terms of ΨDO $\mathrm{Op}_\lambda(f)$
- Result holds for all observables of this class
- Includes local averages of energy density, Poynting vector, field intensities

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Definition (Quadratic observables)

Suppose the electromagnetic observable

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is defined in terms of a ΨDO associated to a function *f*.

1. We call *F* **scalar** if $f \equiv f \otimes \mathrm{id}_{\mathfrak{h}}$ and $f \in \mathcal{C}_{\mathrm{b}}^\infty(\mathbb{R}^6, \mathbb{C})$ are periodic in *k*.
2. We call *F* **non-scalar** if $f \in \mathcal{C}_{\mathrm{b}}^\infty\big(\mathbb{R}^6, \mathcal{B}\big(L^2_w(\mathbb{T}^3, \mathbb{C}^6)\big)\big)$ is an operator-valued function satisfying the equivariance condition

$$f(r, k - \gamma^*) = \mathrm{e}^{+\mathrm{i}\gamma^* \cdot \hat{y}} \, f(r, k) \, \mathrm{e}^{-\mathrm{i}\gamma^* \cdot \hat{y}}.$$

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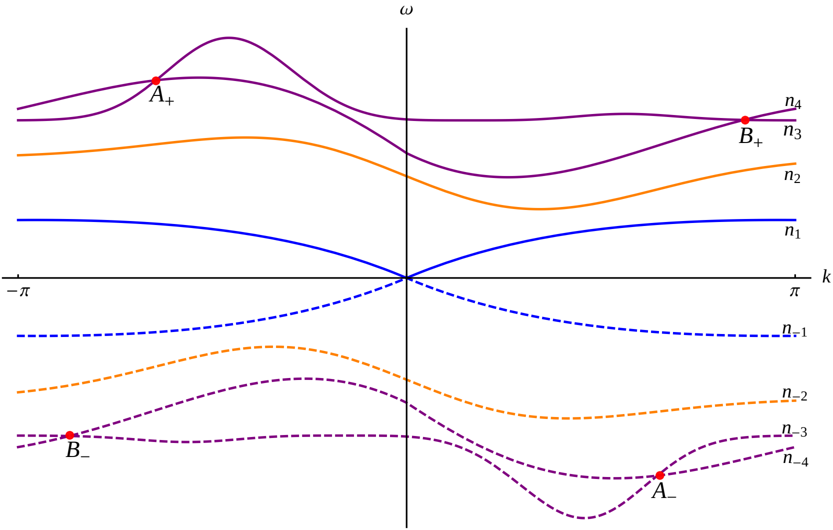
States From a Narrow Range of Frequencies

Relevant states

Projection onto $\mathfrak{H}_{\mathrm{rel}} = \mathrm{ran}\, \Pi_\lambda$

- Closedness $\Longleftrightarrow \exists$ projection Π_λ
- Invariant under dynamics up to $\mathcal{O}(\lambda^2) \Longleftrightarrow \big[M_\lambda, \Pi_\lambda \big] = \mathcal{O}(\lambda^2)$

States From a Narrow Range of Frequencies



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States From a Narrow Range of Frequencies

Definition (Relevant states: unperturbed)

$$\Pi_0 \cong \int_{\mathbb{B}}^{\oplus} \mathrm{d}k \, 1_{\{\omega_*(k)\}}\big(M(k)\big) \text{ so that}$$

1. $\sigma_{\mathrm{rel}}(k) = \big\{\omega_*(k)\big\}$ **isolated** band
2. **Source free**: $\mathrm{ran}\, \Pi_0 \subset J_0$
3. Invariant under dynamics:

$$\mathrm{e}^{-\mathrm{i}tM} \, \Pi_0 = \Pi_0 \, \mathrm{e}^{-\mathrm{i}tM} \, \Pi_0$$

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$$\Pi_\lambda \cong \int_{\mathbb{B}}^{\oplus} \mathrm{d}k \, 1_{\{\omega_*(k)\}}\big(M(k)\big) + \mathcal{O}(\lambda) \text{ so that}$$

1. $\sigma_{\mathrm{rel}}(k) = \big\{\omega_*(k)\big\}$ **isolated** band
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States From a Narrow Range of Frequencies

Theorem (De Nittis-L. 2014 (CMP))

Suppose band $\sigma_{\mathrm{rel}}(k) = \bigl\{\omega_*(k)\bigr\}$ *is an* ***isolated*** *band and* $0 \notin \sigma_{\mathrm{rel}}(0)$. *Then there exists an orthogonal projection*

$$\Pi_\lambda \cong \int_{\mathbb{B}}^{\oplus} \mathrm{d}k \, 1_{\{\omega_*(k)\}}\bigl(M(k)\bigr) + \mathcal{O}(\lambda)$$

which can be computed explicitly order-by-order so that

$$\bigl[M_\lambda \, , \, \Pi_\lambda\bigr] = \mathcal{O}(\lambda^\infty)$$

and whose ***range*** *supports* ***physical states***.

Modified dispersion relation

$$\Omega = \Omega_0 + \lambda\,\Omega_1 := \tau^2\,\omega - \lambda\,\tau^2\,\mathcal{P} \cdot \nabla_r \ln \tfrac{\tau_\varepsilon}{\tau_\mu}$$

where

$$\mathcal{P}(k) := \mathrm{Im} \int_{\mathbb{T}^3} \mathrm{d}y\, \overline{\varphi^E(k,y)} \times \varphi^H(k,y)$$

The Ray Optics Limit: The Scalar Case

Theorem (De Nittis-L. 2015)

*For **scalar** observables where f* $\in \mathcal{C}^{\infty}_{\mathrm{b}}(\mathbb{R}^6, \mathbb{C})$, *the **ray optics flow*** Φ^{λ} *associated to the hamiltonian equations*

$$\begin{pmatrix} \dot{r} \\ \dot{k} \end{pmatrix} = \begin{pmatrix} -\lambda \Xi & +\mathrm{id} \\ -\mathrm{id} & 0 \end{pmatrix} \begin{pmatrix} \nabla_r \Omega \\ \nabla_k \Omega \end{pmatrix},$$

which include the Berry curvature $\Xi := \big(\nabla_k \times \mathrm{i} \langle \varphi, \nabla_k \varphi \rangle_{L^2_w(\mathbb{T}^3, \mathbb{C}^6)} \big)^{\times}$ *as part of the sympletic form, approximates the full light dynamics for* $\Psi \in \mathrm{ran}\, \Pi_\lambda$ *and bounded times in the sense*

$$F\big(\Psi(t)\big) = F\big(\mathrm{e}^{-\mathrm{i}\frac{t}{\lambda} M_\lambda} \Psi\big) = \Big\langle \Psi \, , \, \mathrm{Op}_\lambda\big(f \circ \Phi^\lambda_t\big) \Psi \Big\rangle_\lambda + \mathcal{O}(\lambda^2).$$

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The Ray Optics Limit: The Scalar Case

Remark

*The result also **holds** in case ω is a Bloch band with **non-zero Chern charge**.*

The Ray Optics Limit: The Non-Scalar Case

Theorem (De Nittis-L. 2015)

*For **non-scalar** observables where* $f \in \mathcal{C}^\infty_{\mathrm{b}}\big(\mathbb{R}^6, \mathcal{B}\big(L^2_w(\mathbb{T}^3, \mathbb{C}^6)\big)\big)$*, the ray optics flow* Φ^λ *associated to the hamiltonian equations*

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approximates the full light dynamics for $\Psi \in \mathrm{ran}\, \Pi_\lambda$ *and bounded times in the sense*

$$F\big(\Psi(t)\big) = F\big(\mathrm{e}^{-\mathrm{i}\frac{t}{\lambda} M_\lambda} \Psi\big) = \Big\langle \Psi \, , \, \mathrm{Op}_\lambda\big(f_{\mathrm{ro}} \circ \Phi^\lambda_t\big) \Psi \Big\rangle_\lambda + \mathcal{O}(\lambda^2)$$

where we have transported the modified non-scalar observable $f_{\mathrm{ro}} := \pi_\lambda \sharp f \sharp \pi_\lambda + \mathcal{O}(\lambda^2)$ *along the flow* Φ^λ*.*

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Idea of Proof

Proof.

- Crucial technical tool: **pseudodifferential calculus**
- Projection onto almost-invariant subspace:

$$\Pi_\lambda = \pi_\lambda\big(\mathrm{i}\lambda\nabla_k, \hat{k}\big) + \mathcal{O}(\lambda^\infty)$$

- **2 defining relations:**

$$\pi_\lambda \sharp \pi_\lambda = \pi_\lambda + \mathcal{O}(\lambda^\infty) \qquad \big[\mathcal{M}_\lambda \, , \, \pi_\lambda\big]_\sharp = \mathcal{O}(\lambda^\infty)$$

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Proof.

Construction of almost-invariant projection:

$$\pi_\lambda(r,k) \asymp \frac{\mathrm{i}}{2\pi} \int_{\Gamma(r,k)} \mathrm{d}z \, \big(\mathcal{M}_\lambda - z\big)^{(-1)_\sharp}(r,k)$$

- Here: $\big(\mathcal{M}_\lambda - z\big)^{(-1)_\sharp}$ is the **Moyal resolvent**

$$\big(\mathcal{M}_\lambda - z\big)^{(-1)_\sharp} \sharp \big(\mathcal{M}_\lambda - z\big) = 1 + \mathcal{O}(\lambda^\infty)$$

- Local spectral gap
  $\Rightarrow \Gamma(r,k)$ can be chosen to be locally constant

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Crucial ingredient: Work of Stiepan & Teufel (2013) who prove Egorov-type theorem:

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- Replacement of M_λ with Ω: $\pi_\lambda \sharp \big(\mathcal{M}_\lambda - \Omega\big) \sharp \pi_\lambda = \mathcal{O}(\lambda^2)$
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Part 3
Ray Optics for Real Fields

Real Fields: A Multiband Problem

Complex conjugation in Bloch-Floquet represenation:

$$\big(C\Psi\big)(x) = \overline{\Psi(x)} \iff \big(C^{\mathcal{F}}\varphi\big)(k,y) = \overline{\varphi(-k,y)}$$

$C\,W\,C = \overline{W} = W$ induces symmetry in band spectrum:

$$M(k)\varphi_n(k) = \omega_n(k)\,\varphi_n(k)$$
$$\iff$$
$$M(k)\,C\varphi_n(-k) = -\omega_n(-k)\,C\varphi_n(-k)$$

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$$M(k)\varphi_n(k) = \omega_n(k)\,\varphi_n(k)$$

$$\Longleftrightarrow$$

$$M(k)\,C\varphi_n(-k) = -\omega_n(-k)\,C\varphi_n(-k)$$

Real Fields: A Multiband Problem

$\varphi_+(k)$ Bloch function associated to $\omega_+(k)$

$$\psi_{\mathrm{Re}}(k) := \frac{1}{\sqrt{2}}\Big(\varphi_+(k) + \overline{\varphi_+(-k)}\Big) = \frac{1}{\sqrt{2}}\Big(\varphi_+(k) + \varphi_-(k)\Big),$$
$$\psi_{\mathrm{Im}}(k) := \frac{1}{\mathrm{i}\sqrt{2}}\Big(\varphi_+(k) - \overline{\varphi_+(-k)}\Big) = \frac{1}{\mathrm{i}\sqrt{2}}\Big(\varphi_+(k) - \varphi_-(k)\Big)$$

Real states $\Rightarrow$ *bona fide* multiband problem

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Real states from a narrow range of frequencies

Definition (Physical states: perturbed)

$$\Pi_\lambda \cong \int_{\mathbb{B}}^{\oplus} \mathrm{d}k \; 1_{\sigma_{\mathrm{rel}}(k)}\big(M(k)\big) + \mathcal{O}(\lambda) \text{ so that}$$

1. $\sigma_{\mathrm{rel}}(k) = \sigma_{\mathrm{rel}}(-k) = \bigcup_{n \in \mathcal{I}} \{\omega_n(k)\}$ **isolated** family of bands
2. **source free**: "ran $\Pi_\lambda \subset J_\lambda$" up to $\mathcal{O}(\lambda^\infty)$
3. **invariant under dynamics** up to $\mathcal{O}(\lambda^\infty)$
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Real states from a narrow range of frequencies

Theorem (De Nittis-L. 2014 (CMP))

There exist orthogonal projections

$$\Pi_\lambda = \Pi_{+,\lambda} + \Pi_{-,\lambda} + \mathcal{O}(\lambda^\infty)$$

so that

$$\big[M_\lambda \, , \, \Pi_{\pm,\lambda} \big] = \mathcal{O}(\lambda^\infty)$$

whose ***range*** ran Π_λ *supports* ***physical states***.

The Ray Optics Limit for Real Fields

Theorem (De Nittis-L. 2015)

If in addition $\overline{f(r,-k)} = f(r,k)$, *then*

$$\mathcal{F}\big(\Psi(t)\big) = 4 \, \Big\langle \Psi \, , \, \mathrm{Re} \; \Pi_{+,\lambda} \, \mathrm{Op}_\lambda\big(f_{\mathrm{ro}} \circ \Phi_t^\lambda\big) \, \Pi_{+,\lambda} \, \mathrm{Re} \, \Psi \Big\rangle_\lambda + \mathcal{O}(\lambda^2)$$

where $f_{\mathrm{ro}} = f$ *(scalar) or* $f_{\mathrm{ro}} = \pi_\lambda \sharp \, f \, \sharp \pi_\lambda$ *(non-scalar).*

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Idea of the Proof

Lemma

$$[F, C] = 0 \implies \Pi_\lambda \, F \, \Pi_\lambda \, \mathrm{Re} \; = 4\mathrm{Re} \; \Pi_{+,\lambda} \, F \, \Pi_{+,\lambda} \, \mathrm{Re} \; + \mathcal{O}_{\|\cdot\|}(\lambda^\infty)$$

Proof.

Step 1

Reduction to single-band case

- $[M_\lambda, C] = 0 \Longrightarrow \left[\mathrm{e}^{-\mathrm{i}tM_\lambda}, C\right] = 0$
- $f(r, -k) = \overline{f(r, k)} \Longrightarrow \left[\mathrm{Op}_\lambda(f), C\right] = 0$
- Set $F = \mathrm{e}^{+\mathrm{i}\frac{t}{\lambda}M_\lambda} \, \mathrm{Op}_\lambda(f) \, \mathrm{e}^{-\mathrm{i}\frac{t}{\lambda}M_\lambda}$ □

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Proof.

Step 2

Invoke result on single-band case

$$\implies \Pi_{+,\lambda} \, e^{+i \frac{t}{\lambda} M_\lambda} \, \mathrm{Op}_\lambda(f) \, e^{-i \frac{t}{\lambda} M_\lambda} \, \Pi_{+,\lambda} = \\ = \Pi_{+,\lambda} \, \mathrm{Op}_\lambda\big(f_{\mathrm{ro}} \circ \Phi^\lambda_t\big) \, \Pi_{+,\lambda} + \mathcal{O}(\lambda^2)$$

□

Idea of the Proof

Lemma

$$[F, C] = 0 \implies \Pi_\lambda \, F \, \Pi_\lambda \, \mathrm{Re} \; = 4\mathrm{Re} \; \Pi_{+,\lambda} \, F \, \Pi_{+,\lambda} \, \mathrm{Re} \; + \mathcal{O}_{\|\cdot\|}(\lambda^\infty)$$

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$$\implies \quad \Pi_{+,\lambda} \, \mathrm{e}^{+\mathrm{i}\frac{t}{\lambda} M_\lambda} \, \mathrm{Op}_\lambda(f) \, \mathrm{e}^{-\mathrm{i}\frac{t}{\lambda} M_\lambda} \, \Pi_{+,\lambda} =$$

$$= \Pi_{+,\lambda} \, \mathrm{Op}_\lambda\big(f_{\mathrm{ro}} \circ \Phi^\lambda_t\big) \, \Pi_{+,\lambda} + \mathcal{O}(\lambda^2)$$

□

Thank you for your attention!

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