Rigorous Analogies Between Quantum Systems and Classical Waves

Using The Quantum Hall Effect for Light as a Lens

Max Lein
Advanced Institute of Materials Research, Tohoku University
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Idea

Realizing Quantum Effects with Classical Waves

Today

Focus on Quantum Hall Effect for Light

Quantum-Wave Analogies: Electromagnetism

Quantum Mechanics

$$\mathsf{i}\,\partial_t \Psi = H \Psi$$
$$H = (-\mathsf{i}\nabla - A)^2 + V$$

(Schrödinger equation)

$$\longleftarrow\!\!\longrightarrow$$

?

Classical Electromagnetism

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix}$$

(dynamical equations)

$$\begin{pmatrix} \nabla \cdot \\ \nabla \cdot \end{pmatrix} \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(constraint equation)

The Quantum Hall Effect: the Prototypical System

$$B \neq 0 \implies \textbf{time-reversal symmetry broken}$$

$$\sigma^{xy}_{\text{bulk}}(t) \approx \frac{e^2}{h}\,\mathrm{Ch}_{\text{bulk}} = \frac{e^2}{h}\,\mathrm{Ch}_{\text{edge}} \approx \sigma^{xy}_{\text{edge}}(t)$$

transverse conductivity = Chern #

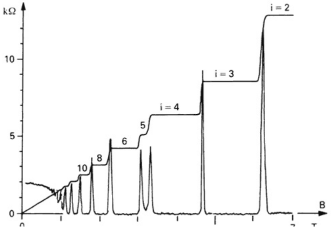
$$\mathrm{Ch}_{\text{bulk/edge}} = \frac{1}{2\pi}\int_{\mathcal{B}} \mathrm{d}k\,\Omega_{\text{bulk/edge}}(k) \in \mathbb{Z}$$

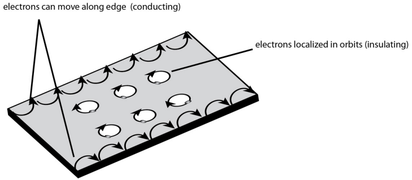
- Edge modes in spectral gaps
- Signed # edge channels $= \mathrm{Ch}(P_{\text{Fermi}})$
- Edge modes unidirectional
- Robust against disorder

***Two* Nobel Prizes**

1985 for experiment: von Klitzing

2016 for theory: Thouless



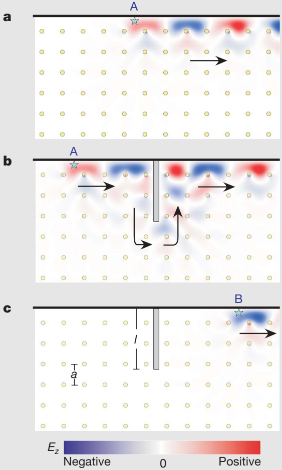


von Klitzing et al (1980)

The Quantum Hall Effect for Light

Predicted theoretically by Raghu & Haldane (2005) ...

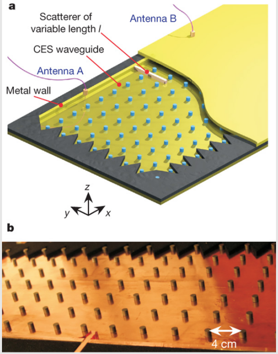
$$\left.\begin{matrix}\begin{pmatrix}\bar{\varepsilon} & 0 \\ 0 & \bar{\mu}\end{pmatrix} \neq \begin{pmatrix}\varepsilon & 0 \\ 0 & \mu\end{pmatrix} \\ \text{symmetry breaking}\end{matrix}\right\} \Longrightarrow$$



Joannopoulos, Soljačić et al (2009)

The Quantum Hall Effect for Light

… and realized experimentally by Joannopoulos et al (2009)



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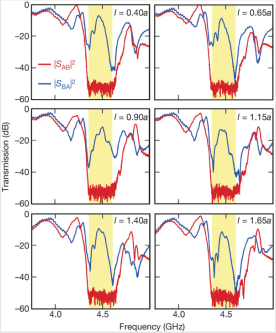


Joannopoulos, Soljačić et al (2009)

Haldane's Argument: "Derivation by Analogy"

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix}$$

(dynamical equation)

$$\begin{pmatrix} \nabla \cdot \\ \nabla \cdot \end{pmatrix} \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(constraint equation)

$$\xrightarrow{\lambda \ll 1} \begin{cases} \dot{r} = +\nabla_k \varpi + \lambda \, \dot{k} \times \Omega \\ \dot{k} = -\nabla_r \varpi \end{cases}$$

(ray optics equations)

Setting

- ϖ dispersion relation, Ω Berry curvature
- Perturbation parameter $\lambda \ll 1$
- Slowly varying *electric permittivity* $\varepsilon = \varepsilon(\lambda)$ and *magnetic permeability* $\mu = \mu(\lambda)$ are 3×3-matrix-valued
- ε and μ: periodic to "leading order"

Goal of Today's Talk

The *Quantum Hall Effect for Light* as a Lens for Other Quantum-Wave Analogies

How and to what extent can Haldane's argument be made rigorous?

1. Find out how semiclassical techniques can be applied to Maxwell's equations.
2. Identify similarities and differences between quantum and classical systems.

Key: Physics Is Not Just a Bunch of Equations ...

... but also how to interpret them, and provides additional information on typical circumstances.

Fundamental Constituents of Physical Theories

1. States
2. Dynamical equation
3. Observables

1. The Schrödinger Formalism for EM: States and Dynamics
2. Observables in Electromagnetism
3. Input from Physics

Recap: States and Dynamics in Quantum Mechanics

States and Dynamics

1. A selfadjoint Hamilton operator, e. g.

$$H = \tfrac{1}{2m}\big(-\mathrm{i}\nabla - A\big)^2 + V$$
$$H = m\,\beta + \big(-\mathrm{i}\nabla - A\big)\cdot\alpha + V$$

2. A Hilbert space $\mathcal{H}$ and states are represented by its elements, e. g. $L^2(\mathbb{R}^d, \mathbb{C}^n)$ with $\langle\phi,\psi\rangle = \int_{\mathbb{R}^d}\mathrm{d}x\,\phi(x)\cdot\psi(x)$.
3. Dynamics given by the Schrödinger equation

$$\mathrm{i}\,\partial_t\psi(t) = H\psi(t), \qquad \psi(0) = \phi$$

Recap: States and Dynamics in Quantum Mechanics

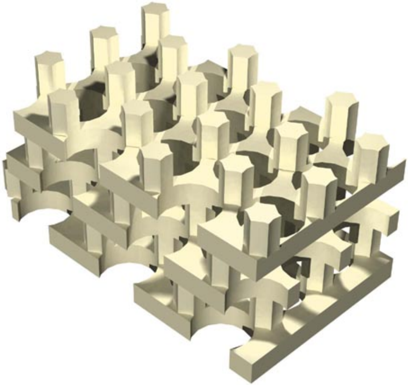
States and Dynamics

1. A selfadjoint Hamilton operator
2. A Hilbert space $\mathcal{H}$ and states are represented by its elements.
3. The Schrödinger equation

Properties

- $H = H^*$
- $\psi(t) = \mathrm{e}^{-\mathrm{i}tH}\phi$
- $\left\|\psi(t)\right\|^2 = \|\phi\|^2$ (conservation of propability)

Maxwell's Equations for Non-Gyrotropic Dielectrics



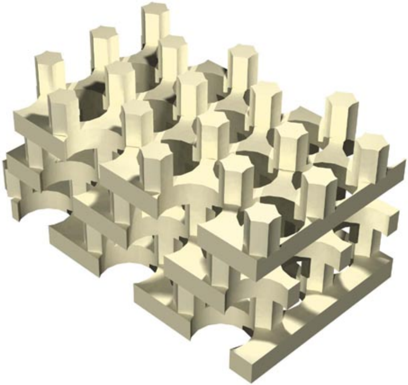
Johnson & Joannopoulos (2004)

Assumption (Material weights)

$$W(x) = \begin{pmatrix} \varepsilon(x) & 0 \\ 0 & \mu(x) \end{pmatrix}^{-1}$$

1. $W = \overline{W}$ real (non-gyrotropic)
2. $W^* = W$ (lossless)
3. $0 < c\,\mathbb{1} \leq W \leq C\,\mathbb{1}$ (excludes metamaterials)

Maxwell's Equations for Non-Gyrotropic Dielectrics



Johnson & Joannopoulos (2004)

Maxwell equations

Dynamical equations

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix}$$

Absence of sources

$$\begin{pmatrix} \mathrm{div} & 0 \\ 0 & \mathrm{div} \end{pmatrix} \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = 0$$

Schrödinger Formalism of Electromagnetism

$$\left.\begin{matrix}\begin{pmatrix}\varepsilon & 0 \\ 0 & \mu\end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix}\mathbf{E} \\ \mathbf{H}\end{pmatrix} = \begin{pmatrix}-\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E}\end{pmatrix} \\ \text{dynamical Maxwell equations}\end{matrix}\right\} \iff \left\{\begin{matrix}\mathrm{i}\,\partial_t \Psi = M \Psi \\ \text{“Schrödinger-type equation”}\end{matrix}\right.$$

$$\Psi(t) = (\mathbf{E}(t), \mathbf{H}(t)) \in \mathcal{H} = \left\{\Psi \in L^2_W(\mathbb{R}^3, \mathbb{C}^6) \;\middle|\; \Psi \text{ transversal}\right\}$$

$$M = \underbrace{\begin{pmatrix}\varepsilon & 0 \\ 0 & \mu\end{pmatrix}^{-1}}_{=W} \underbrace{\begin{pmatrix}0 & +(-\mathrm{i}\nabla)^{\times} \\ -(-\mathrm{i}\nabla)^{\times} & 0\end{pmatrix}}_{=D} = M^{*_W}$$

$$\left.\begin{matrix}\text{Maxwell equations} \\ \iff \\ \text{Maxwell operator } M = M^{*_W}\end{matrix}\right\} \Longrightarrow$$

Adaptation of **techniques from quantum mechanics** to electromagnetism

Schrödinger Formalism for Classical Waves

States and Dynamics

1. *"Hamilton" operator* $M = W\, D$ where
   - $W = W^*$, $0 < c\,\mathbb{1} \leq W \leq C\,\mathbb{1}$
     (positive, bounded, bounded inverse)
   - $D = D^*$ (potentially unbounded)
2. **Complex (!)** weighted *Hilbert space* $\mathcal{H} \subseteq L^2_W(\mathbb{R}^d, \mathbb{C}^n)$ where

$$\langle \Phi, \Psi \rangle_W = \langle \Phi, W^{-1} \Psi \rangle = \int_{\mathbb{R}^d} \mathrm{d}x \, \Phi(x) \cdot W^{-1} \Psi(x)$$

3. Dynamics given by *Schrödinger equation*

$$\mathrm{i}\, \partial_t \Psi(t) = M \Psi(t), \qquad \Psi(0) = \Phi$$

4. **Real-valuedness** of physical solutions

Schrödinger Formalism for Classical Waves

States and Dynamics

1. *"Hamilton" operator* $M = W\, D$ with **product structure**
2. **Complex (!)** weighted *Hilbert space* $\mathcal{H} \subseteq L^2_W(\mathbb{R}^d, \mathbb{C}^n)$
3. Dynamics given by *Schrödinger equation*
4. **Real-valuedness** of physical solutions

Properties

- $M^{*_W} = M$
- $\Psi(t) = \mathrm{e}^{-\mathrm{i} t M} \Phi$
- $\left\lVert \Psi(t) \right\rVert_W^2 = \lVert \Phi \rVert_W^2$ (conserved quantity, here field energy)
- $\mathrm{Re}\, \mathrm{e}^{-\mathrm{i} t M} = \mathrm{e}^{-\mathrm{i} t M}\, \mathrm{Re}$ where $\mathrm{Re} = \frac{1}{2}(\mathbb{1} + C)$
  (existence of real solutions)

Doubling of Degrees of Freedom

One of the **tenets of electromagnetism**:

E and **H** are **real** vector fields.

$\Longrightarrow$ Replacing $L^2_{W,\perp}(\mathbb{R}^3, \mathbb{R}^6) \rightsquigarrow L^2_{W,\perp}(\mathbb{R}^3, \mathbb{C}^6)$ **doubles the degrees of freedom**!

On the other hand, if we want to apply the theory of selfadjoint operators we need to work with **complex** Hilbert spaces!

Restriction to Complex Fields with $\omega > 0$

$C\,M\,C = -M \implies C\,\mathrm{e}^{-\mathrm{i}t\,M} = \mathrm{e}^{-\mathrm{i}t\,M}\,C$ implies

$$\mathrm{e}^{-\mathrm{i}t\,M}\,(\mathbf{E}_0, \mathbf{H}_0) = \mathrm{e}^{-\mathrm{i}t\,M}\,(\mathrm{Re}\,\Psi_{\pm}) = \mathrm{Re}\left(\mathrm{e}^{-\mathrm{i}t\,M}\,\Psi_{\pm}\right)$$

where Re $= \frac{1}{2}(\mathbb{1} + C)$ is the real part operator and

$$\Psi_{+} = 1_{\{\omega > 0\}}(M)\,(\mathbf{E}_0, \mathbf{H}_0) = P_{+}(\mathbf{E}_0, \mathbf{H}_0)$$
$$\Psi_{-} = 1_{\{\omega < 0\}}(M)\,(\mathbf{E}_0, \mathbf{H}_0) = P_{-}(\mathbf{E}_0, \mathbf{H}_0) = C\Psi_{+}$$

are the positive and negative frequency contributions

Restriction to Complex Fields with $\omega > 0$

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$\mathrm{Re} = P_+^{-1} \implies$ Study $M_+ := M|_{\mathrm{ran}\,P_+}$

Real transversal states

$$(\mathbf{E}, \mathbf{H}) = \mathrm{Re}\,\Psi_+$$

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix}$$

$\longleftrightarrow$

Complex states with $\omega > 0$

$$\Psi_+ = P_+(\mathbf{E}, \mathbf{H})$$

$$\mathrm{i}\,\partial_t \Psi_+ = M_+ \Psi_+$$

Fundamental Constituents

Reduced Description

1. *"Hamilton" operator* $M_+ = W\, D\big\vert_{\mathrm{ran}\, P_+}$
2. *Hilbert space* $\mathcal{H}_+ = \mathrm{ran}\, P_+ \subset L^2_W(\mathbb{R}^3, \mathbb{C}^6)$
3. Dynamics given by *Schrödinger equation*

$$\mathrm{i}\, \partial_t \Psi_+(t) = M_+ \Psi_+(t), \qquad \Psi_+(0) = P_+(\mathbf{E}, \mathbf{H}) \in \mathrm{ran}\, P_+$$

4. *Real-valuedness* of physical solutions:

$$\big(\mathbf{E}(t), \mathbf{H}(t)\big) = \mathrm{Re}\, \Psi_+(t)$$

Note

This also applies to **gyrotropic** materials where $W \neq \overline{W}$.

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Which Symmetries *Are* Broken in QHE for Light?

Non-Gyrotropic Materials

$$W = \overline{W}$$

1 Relevant Symmetry of Complexified Equation

$T : (\psi^E, \psi^H) \mapsto \big(\overline{\psi^E}, -\overline{\psi^H}\big)$ with $T \, M_+ \, T = +M_+$ (+TR)

reverses arrow of time: $T \, \mathrm{e}^{-\mathrm{i}t \, M_+} = \mathrm{e}^{-\mathrm{i}(-t) \, M_+} \, T$

$\Longrightarrow$ *Needs to be broken to have unidirectional edge modes!*

Which Symmetries *Are* Broken in QHE for Light?

Gyrotropic Materials

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Ray Optics in Topological Photonic Crystals

Vague physical question ⇝ Concrete mathematical problem

- Reformulate Maxwell's equations in Schrödinger form
- Adapt your **semiclassical technique** of choice, e. g.
  - wave packet methods à la Niu (implemented by Onoda, Murakami & Nagaosa for photonic crystals (2006)) or
  - by establishing an Egorov-type theorem (cf. e. g. Panati, Spohn & Teufel (2002) or Teufel & Stiepan (2011))

$\Longrightarrow$ Theorem relating Maxwell's equations to ray optics equations

But what is the **physical content of** the resulting **Theorem**?

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Semiclassics via an Egorov Theorem

- Relies on **pseudodifferential** techniques
- "Quantization" Op $: f \mapsto F$ associates **operator** F **on a Hilbert space** to a suitable **function** f **on phase space**

Theorem (Prototypical form)

$$\langle \psi(t) \, , \, \mathsf{Op}(f) \, \psi(t) \rangle = \langle \psi(0) \, , \, \mathsf{Op}(f \circ \Phi_t) \, \psi(0) \rangle + \mathcal{O}(\varepsilon^2)$$

for bounded times where

- ε *is the **semiclassical parameter,***
- $\psi(t)$ *the **solution to the Schrödinger equation** and*
- Φ_t *the **flow** associated to the **semiclassical equations of motion**.*

Semiclassics via an Egorov Theorem

Theorem (Prototypical form)

$$\big\langle \psi(t) \, , \, \mathsf{Op}(f) \, \psi(t) \big\rangle = \big\langle \psi(0) \, , \, \mathsf{Op}\big(f \circ \Phi_t\big) \, \psi(0) \big\rangle + \mathcal{O}(\varepsilon^2)$$

Upshot

- Quantifies difference of two expectation values.
- Needs a quantum observable.

Observables in QM and EM

Quantum Mechanics

- Selfadjoint operators on $\mathcal{H}$
- Wave $\psi(t,x)$ **not** an observable
- *Typical examples:* position $Q = \hat{x}$, momentum $P = -\mathrm{i}\varepsilon\nabla$

Electromagnetism

- Functionals of the fields
- $E_j(t,x)$ and $H_j(t,x)$ **are** observable!
- Not all of them can be written as "quantum expectation values"

$\Longrightarrow$ Egorov Theorem gives ray optics limit only for a certain class of **quadratic** electromagnetic observables!

Quadratic EM Observables Covered by Egorov Theorem

$$\mathcal{F}(\mathbf{E},\mathbf{H}) = \mathrm{Re}\;\Big\langle P_+(\mathbf{E},\mathbf{H})\,,\,\mathrm{Op}(f)\;P_+(\mathbf{E},\mathbf{H})\Big\rangle_W$$
$$= \mathrm{Re}\;\int_{\mathbb{R}^3}\mathrm{d}x\,\big(P_+(\mathbf{E},\mathbf{H})\big)(x)\cdot\Big(W^{-1}\,\mathrm{Op}(f)\,P_+(\mathbf{E},\mathbf{H})\Big)(x)$$

- Covers local averages of *field energy*, the *Poynting vector* and components of the *Maxwell stress tensor*.
- Quadratic functionals which evaluate fields at a point are not covered!

Ray Optics in Adiabatically Perturbed Photonic Crystals

$$\left.\begin{array}{c} \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix} \\ \text{(dynamical equation)} \\ \begin{pmatrix} \nabla \cdot \\ \nabla \cdot \end{pmatrix} \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \text{(constraint equation)} \end{array} \right\} \xrightarrow{\lambda \ll 1} \begin{cases} \dot{r} = +\nabla_k \varpi + \mathcal{O}(\lambda) \\ \dot{k} = -\nabla_r \varpi + \mathcal{O}(\lambda) \\ \text{(ray optics equations)} \end{cases}$$

Setting

- Perturbation parameter $\lambda \ll 1$
- Slowly varying *electric permittivity* $\varepsilon = \varepsilon(\lambda)$ and *magnetic permeability* $\mu = \mu(\lambda)$ are 3×3-matrix-valued
- ε and μ: periodic to “leading order”

Ray Optics in Adiabatically Perturbed Photonic Crystals

Theorem (De Nittis & L. (2016))

$$\mathcal{F}\big(\mathbf{E}(t),\mathbf{H}(t)\big) = \mathrm{Re}\,\Big\langle P_+(\mathbf{E},\mathbf{H})\,,\,\mathrm{Op}\big(f_{\mathrm{ro}}\circ\Phi_t\big)\,P_+(\mathbf{E},\mathbf{H})\Big\rangle_W + \mathcal{O}(\lambda^2)$$

- $\lambda \ll 1$ *perturbation parameter*
- $\big(\mathbf{E}(t),\mathbf{H}(t)\big)$ *solution to Maxwell's equations*
- Φ_t *ray optics flow*
- $f \rightsquigarrow f_{\mathrm{ro}}$ **modified** *ray optics observable*

End of Story?

Not yet!

1. The Schrödinger Formalism for EM: States and Dynamics
2. Observables in Electromagnetism
3. Input from Physics

Semiclassical Argument for Quantum Hall Effect

Apply in-plane constant electric field to drive current

$$H = \big(-\mathrm{i}\nabla - A(\varepsilon x)\big)^2 + V_{\mathrm{per}}(x) - \mathbf{E} \cdot \varepsilon x$$

Egorov theorem yields

$$\mathrm{i}\,\partial_t \Psi = H \Psi \quad \longrightarrow \quad \begin{cases} \dot{r} = + \nabla_k H_{\mathrm{sc}} + \varepsilon\, \dot{k} \times \Omega \\ \dot{k} = \mathbf{E} + \mathcal{O}(\varepsilon) \end{cases}$$

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Semiclassical Argument for Quantum Hall Effect

- All bands up to Fermi energy completely filled.
- H_{sc} is k-periodic
- Ch vector composed of Chern numbers
- Conductivity coefficients = Chern numbers

$$\begin{aligned}\langle J\rangle &= \left\langle \tfrac{\mathrm{i}}{\varepsilon}[H,\varepsilon x]\right\rangle \\ &= \frac{1}{\varepsilon}\int_{\mathcal{B}} \mathrm{d}k\ \dot{r}(t) + \mathcal{O}(\varepsilon) = \frac{1}{\varepsilon}\int_{\mathcal{B}} \mathrm{d}k\,\big(\nabla_k H_{\text{sc}} + \varepsilon\,\mathbf{E}\times\Omega\big) + \mathcal{O}(\varepsilon)\end{aligned}$$

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Semiclassical Argument for Quantum Hall Effect

- All bands up to Fermi energy completely filled.
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$$
\begin{aligned}
\langle J \rangle &= \left\langle \tfrac{\mathrm{i}}{\varepsilon} [H, \varepsilon x] \right\rangle \\
&= \frac{1}{\varepsilon} \int_{\mathcal{B}} \mathrm{d}k \; \dot{r}(t) + \mathcal{O}(\varepsilon) = \mathbf{E} \times \mathrm{Ch} + \mathcal{O}(\varepsilon)
\end{aligned}
$$

Attempting to Apply these QM Arguments to EM

$$\mathcal{P}_{\mathsf{avg}}\big(\mathbf{E}(t),\mathbf{H}(t)\big) = \mathrm{Re} \int_{\mathbb{R}^3} \mathrm{d}x \, \chi_{\mathsf{avg}}(x) \, \Big(\Psi_+^E(t,x) \times \overline{\Psi_+^H}(t,x) \Big) =$$

$$= 2 \, \mathrm{Re} \, \Big\langle P_+(\mathbf{E},\mathbf{H}) \, , \, \chi_{\mathsf{avg}} \, \tfrac{\mathrm{i}}{\lambda} [M_+, \lambda x] \; P_+(\mathbf{E},\mathbf{H}) \Big\rangle_W$$

$$= 2 \, \mathrm{Re} \, \Big\langle P_+(\mathbf{E},\mathbf{H}) \, , \, \mathrm{Op}(f_{\mathsf{ro}} \circ \Phi_t) \, P_+(\mathbf{E},\mathbf{H}) \Big\rangle_W + \mathcal{O}(\lambda^2)$$

Good News

- Local average of **Poynting vector** is a quadratic observable
- $\mathcal{P}_{\mathsf{avg}}$ can be expressed as "quantum expectation value" of the current operator
- Invoke Egorov theorm of De Nittis & L.

Attempting to Apply these QM Arguments to EM

$$\mathcal{P}_{\mathrm{avg}}\big(\mathbf{E}(t),\mathbf{H}(t)\big) = \mathrm{Re} \int_{\mathbb{R}^3} \mathrm{d}x \, \chi_{\mathrm{avg}}(x) \, \Big(\Psi_+^E(t,x) \times \overline{\Psi_+^H}(t,x) \Big) =$$

$$= 2 \, \mathrm{Re} \Big\langle P_+(\mathbf{E},\mathbf{H}) \, , \, \chi_{\mathrm{avg}} \, \tfrac{\mathrm{i}}{\lambda} [M_+, \lambda x] \; P_+(\mathbf{E},\mathbf{H}) \Big\rangle_W$$

$$= 2 \, \mathrm{Re} \Big\langle P_+(\mathbf{E},\mathbf{H}) \, , \, \mathrm{Op}(f_{\mathrm{ro}} \circ \Phi_t) \; P_+(\mathbf{E},\mathbf{H}) \Big\rangle_W + \mathcal{O}(\lambda^2)$$

Good News

- Local average of **Poynting vector** is a quadratic observable
- $\mathcal{P}_{\mathrm{avg}}$ can be expressed as "quantum expectation value" of the current operator
- Invoke Egorov theorm of De Nittis & L.

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Bad News

- Assumption of completely filled bands in photonics **unphysical** (there is *no Fermi projection*)
- Linear response theory for QM and EM completely different: "$M_+ + \mathbf{E} \cdot \varepsilon r$" makes **no physical sense** (use either antennas or perturbations of ε and μ)
- Ray optics equations for $f_{\mathrm{ro}} = f_{\mathrm{ro},0} + \lambda\, f_{\mathrm{ro},1}$ do not include Berry curvature (**mathematics** also different)
  $\rightsquigarrow$ instead Berry geometric terms contained in $f_{\mathrm{ro},1}$

Attempting to Apply these QM Arguments to EM

$$\mathcal{P}_{\mathsf{avg}}\big(\mathbf{E}(t), \mathbf{H}(t)\big) = 2\,\mathrm{Re}\,\Big\langle P_+(\mathbf{E}, \mathbf{H})\,,\,\mathrm{Op}\big(f_{\mathsf{ro}} \circ \Phi_t\big)\,P_+(\mathbf{E}, \mathbf{H})\Big\rangle_W + \mathcal{O}(\lambda^2)$$

Bad News

$\Longrightarrow$ **"Semiclassical" arguments do not explain quantization of conductivity!**

Conclusion

Quantum Hall Effect for Light

- The "semiclassical" ray optics limit does not furnish an explanation.
- Haldane's heuristic arguments never went as far as claiming that ray optics equations make quantitative predictions.
- A first-principles explanation is still an open problem. (Work in progress.)

Conclusion

Broader Implications

- Arguments generalize to many other classical wave equations (e. g. transverse acoustic waves, spin waves, etc.).
- Quantum-wave analogies are more subtle than just bringing the dynamical equation in Schrödinger form.
- Careful physical interpretation of mathematical results necessary.
- Schrödinger formalism allows us to adapt and apply techniques initially developed for quantum mechanics.
- Plenty of open mathematical and physical problems (e. g. scattering theory, topological classification, bulk-boundary correspondences, Krein-Schrödinger formalism, …)

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Thank you very much &

Happy (70+ε)th Birthday, Herbert!