Magnetic Weyl Calculus

Quantizing Classical Systems with Magnetic Fields

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Main Goal

Ascribe meaning to the **quantization**

$$\mathrm{Op}^A : f \mapsto f\big(\hat{x}, -\mathrm{i}\varepsilon\nabla - A(\hat{x})\big)$$

which systematically maps functions on phase space $T^*\mathbb{R}^d \cong \mathbb{R}^{2d}$ to operators on the Hilbert space $L^2(\mathbb{R}^d)$.

- $A : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a vector potential to the magnetic field $B = \mathrm{d}A$
- Op^A is **gauge-covariant**, i. e. $\mathrm{Op}^{A+\mathrm{d}\chi}(f) = \mathrm{e}^{+\mathrm{i}\chi}\,\mathrm{Op}^A(f)\,\mathrm{e}^{-\mathrm{i}\chi}$

"Making Phase Space Non-Commutative"

Consider an algebra of functions $\mathcal{A}_{\mathrm{cl}}$ on phase space, the *classical observables*, where the product is the *pointwise product* of functions

$$(f \cdot g)(q,p) = f(q,p)\; g(q,p)$$

Idea
Replace pointwise product $f \cdot g$ by a **non-commutative** product

$$f \star g \asymp \sum_{n=0}^{\infty} \varepsilon^n \, (f \star g)_{(n)}$$

so that

$$\left[q^\mu , q^\nu \right]_\star = q^\mu \star q^\nu - q^\nu \star q^\mu = B^{\mu\nu}$$

and similarly for the other commutators.

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where $f \star g$ is given as an *asymptotic series* with

$$(f \star g)_{(n)} = \sum_{\alpha_q,\alpha_p,\beta_q,\beta_p} B_{\alpha_q,\alpha_p,\beta_q,\beta_p} \, \partial_q^{\alpha_q} \partial_p^{\alpha_p} f \, \partial_q^{\beta_q} \partial_p^{\beta_p} g$$

"Making Phase Space Non-Commutative"

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and we obtain the algebra of "quantum" observables $\mathcal{A}_{\mathrm{qm}}$.

Relation to Magnetic Weyl Quantization

This is the special case where phase space $T^*\mathbb{R}^d$ is deformed by the **magnetic Weyl product** $\sharp^B$ characterized by the commutation relations

$$\tfrac{\mathrm{i}}{\varepsilon}[q_j, q_k]_{\sharp^B} = 0, \quad \tfrac{\mathrm{i}}{\varepsilon}[p_j, p_k]_{\sharp^B} = B_{jk}(q), \quad \tfrac{\mathrm{i}}{\varepsilon}[q_j, q_k]_{\sharp^B} = \delta_{jk}$$

Corollary (Belmonte, L. & Mantoiu (20213))

$\Big\{\mathcal{S}\big(\mathbb{R}^d, \mathcal{C}_0^\infty(\mathbb{R}^d)\big)_{\mathbb{R}} \hookrightarrow \mathfrak{B}^B_{\varepsilon,\mathbb{R}}\Big\}_{\varepsilon\in[0,1]}$ *defines a* **strict deformation quantization in the sense of Rieffel** *of the Poisson algebra* $\mathcal{S}\big(\mathbb{R}^d, \mathcal{C}_0^\infty(\mathbb{R}^d)\big)_{\mathbb{R}}$ *with magnetic Poisson bracket* $\{\,\cdot\,,\,\cdot\,\}_B$.

Other Applications

- Obtain (indeed, *compute*) **semiclassical dynamics** which approximate quantum dynamics.
- Obtain perturbation expansions which allow for the computation of **effective dynamics**.
- An algebraic point of view which generalizes the notion of **non-commutative torus**.

Depending on the time: More on that at the end.

Important Works on the Subject

Historical References

- Notation $f(\hat{x}, -\mathrm{i}\varepsilon\nabla)$ used by the **founding fathers of quantum mechanics** already (e. g. in books by Dirac and Heisenberg and the papers by Schrödinger on the hydrogen atom)

  $\rightsquigarrow$ Problem of **operator ordering**
- **Weyl (1927):** Definition of Weyl quantization of observables
- **Wigner (1932):** Introduction of Wigner transform
- **Moyal (1949):** Full Weyl calculus including explicit expressions for Moyal commutator

Important Works on the Subject

Mathematics: Pseudodifferential Theory

- **Berezin & Shubin (1970)**
- **Caldéron & Vaillancourt (1971)**
- **Hörmander (1979 & 1983)**
- **Folland (1989)**

Important Works on the Subject

Magnetic Weyl Calculus

- First non-rigorous work: **Müller (1999)**
- First rigorous work: **Purice and Mantoiu (2004)**
- Algebraic approach: **Mantoiu, Purice & Richard (2005)**, **L., Mantoiu & Richard (2010)**
- Further functional analytic works: **L. (2011)**, **Iftimie, Mantoiu & Purice (2007 & 2010)**, etc.

Building Blocks of Physical Theories

1. **States** describe the **configuration** of the system.
2. **Observables** represent experimentally **measurable** quantities.
3. **Dynamics** explain how states or observables **evolve over time**.

Classical Mechanics for $B = 0$

States

- **Phase space** $T^*\mathbb{R}^d$ (space of position and momenta)
- **“Building block” observables:** position q and momentum p
- $T^*\mathbb{R}^d$ endowed with **canonical symplectic form** $\omega = \mathrm{d}q_j \wedge \mathrm{d}p_j = (\mathrm{d}q, \mathrm{d}p) \cdot J(\mathrm{d}q, \mathrm{d}p)$

$$J = \begin{pmatrix} 0 & -\mathbb{1}_{\mathbb{R}^d} \\ +\mathbb{1}_{\mathbb{R}^d} & 0 \end{pmatrix}$$

- **Pure states:** points in phase space (q_0, p_0)
- **Mixed states:** probability measures on phase space

Classical Mechanics for $B = 0$

Observables

Functions on phase space $f : T^* \mathbb{R}^d \longrightarrow \mathbb{R}$

Commutation relations of **Building block observables**

$$\{q_j, q_k\} = 0, \qquad \{p_j, p_k\} = 0, \qquad \{p_j, q_k\} = \delta_{jk}$$

Classical Mechanics for $B = 0$

Dynamical Equation for Pure States

Hamiltonian equations of motion

$$\begin{pmatrix} 0 & -\mathbb{1}_{\mathbb{R}^d} \\ +\mathbb{1}_{\mathbb{R}^d} & 0 \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} \nabla_q \\ \nabla_p \end{pmatrix} h(q(t), p(t))$$

gives rise to the **classical flow** $\Phi_t : T^*\mathbb{R}^d \longrightarrow T^*\mathbb{R}^d$, where $h : T^*\mathbb{R}^d \longrightarrow \mathbb{R}$ is the energy observable (aka Hamilton function).

Symplectic form

Classical Mechanics for $B = 0$

Dynamical Equation for Observables

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t) = \{h, f(t)\}$$

where $\{f, g\} = \nabla_p f \cdot \nabla_q g - \nabla_q f \cdot \nabla_p g$ is the Poisson bracket and $f(t) = f \circ \Phi_t$.

Quantum Mechanics

States

- **Hilbert space** $\mathcal{H}$
- Here: $\mathcal{H} = L^2(\mathbb{R}^d)$ endowed with the usual scalar product

$$\langle \psi, \phi \rangle = \int_{\mathbb{R}^d} \mathrm{d}x \, \overline{\psi(x)} \, \phi(x)$$

- **Pure states:** Normalized wave functions $\psi \in \mathcal{H}$, $\|\psi\|^2 = 1$ up to a global phase $\Longleftrightarrow$ projection $P = |\psi\rangle\langle\psi|$
- **Mixed states:** Density operators $\rho = \rho^*$ with $\rho \geq 0$, $\mathrm{Tr}\rho = 1$

Quantum Mechanics

Observables

Represented by **selfadjoint operators** $F^* = F$

- *Spectrum* $\sigma(F)$ outcomes of measurements
- Projection-valued measure of F contains statistics, i. e. *how frequent* a measurement outcome is
- Associated observable quantities: expectation values, etc.

Commutation relations of **building block observables** $P = -\mathrm{i}\varepsilon\nabla$ and $Q = \hat{x}$

$$\frac{\mathrm{i}}{\varepsilon}[Q_j, Q_k] = 0, \qquad \frac{\mathrm{i}}{\varepsilon}[P_j, P_k] = 0, \qquad \frac{\mathrm{i}}{\varepsilon}[P_j, Q_k] = \delta_{jk}\,\mathbb{1}$$

Quantum Mechanics

Dynamical Equation for Pure States

Schrödinger equation

$$\mathrm{i}\varepsilon\partial_t\psi(t) = H\psi(t), \qquad \psi(0) = \phi \in \mathcal{H}$$

where $H = H^*$ is the **Hamiltonian (operator)**, ε the semiclassical parameter and $\psi(t) = \mathrm{e}^{-\mathrm{i}\frac{t}{\varepsilon}H}\phi$.

Quantum Mechanics

Dynamical Equation for Observables

Heisenberg equation

$$\frac{\mathrm{d}}{\mathrm{d}t} F(t) = \frac{\mathrm{i}}{\varepsilon} [H, F(t)]$$

where $F(t) = \mathrm{e}^{+\mathrm{i}\frac{t}{\varepsilon} H} \, F \, \mathrm{e}^{-\mathrm{i}\frac{t}{\varepsilon} H}$.

Commutation Relations of Building Block Operators

$$\left.\begin{array}{c}\textbf{Classical}\\ \{q_j, q_k\} = 0 \\ \{p_j, p_k\} = 0 \\ \{p_j, q_k\} = \delta_{jk}\, 1\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textbf{Quantum}\\ \frac{\mathrm{i}}{\varepsilon}[Q_j, Q_k] = 0 \\ \frac{\mathrm{i}}{\varepsilon}[P_j, P_k] = 0 \\ \frac{\mathrm{i}}{\varepsilon}[P_j, Q_k] = \delta_{jk}\, \mathbb{1}\end{array}\right.$$

What is Weyl Quantization?

It allows us to systematically relate

1. **States** via the Wigner transform (i. e. a dequantization)
2. **Observables** via Weyl quantization
3. **Dynamics** via a semiclassical limit (e. g. Egorov-type theorem)

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What is Weyl Quantization?

Requirements

- Maps classical onto quantum building block observables

$$q_j \mapsto \mathsf{Op}(q_j) = Q_j = x_j$$
$$p_j \mapsto \mathsf{Op}(p_j) = P_j = -\mathrm{i}\varepsilon\partial_j$$

- **Linearity:** $\mathsf{Op}(\alpha f + g) = \alpha\,\mathsf{Op}(f) + \mathsf{Op}(g)$
- **Compatibility with the involution:** $\mathsf{Op}(\overline{f}) = \mathsf{Op}(f)^*$
- Pointwise product of functions cannot be equivalent:

$$\mathsf{Op}(f\,g) = \mathsf{Op}(g\,f) \neq \mathsf{Op}(f)\,Op(g)$$

- Non-commutative **Weyl product**: $\mathsf{Op}\big(f \sharp g\big) := \mathsf{Op}(f)\,\mathsf{Op}(g)$
- **Poisson bracket approximates commutator**

$$\mathsf{Op}\big(\{f,g\}\big) = \frac{\mathrm{i}}{\varepsilon}\big[\mathsf{Op}(f), \mathsf{Op}(g)\big] + \mathcal{O}(\varepsilon)$$

$B = 0$ and for simplicity $\varepsilon = 1$

What is the Problem to Solve?

How to define $f(Q,P) = f\big(\hat{x}, -\mathrm{i}\nabla\big)$?

In case f depends only on either position or momentum:

- $f(q,p) = V(q)$: Multiplication operator
  $(V(Q)\psi)(x) = V(x)\,\psi(x)$
- $f(q,p) = K(p)$: Use Fourier transform
  $K(P)\psi = \mathcal{F}\,K(\hat{\mathrm{p}})\,\mathcal{F}^{-1}\psi$ where $\big(K(\hat{\mathrm{p}})\widehat{\psi}\big)(p) = K(p)\,\psi(p)$

But what if f depends on *both* variables?
$\rightsquigarrow$ **Non-trivial** because P and Q do not commute!

$f(Q,P)$ already found in earliest books on quantum mechanics, e. g. by Dirac.

The Weyl System

Definition (Weyl System)

$$W(X) = W(x,\xi) := \mathrm{e}^{-\mathrm{i}\sigma(X,(Q,P))} = \mathrm{e}^{-\mathrm{i}(\xi\cdot x - x\cdot P)}$$

where $\sigma(X,Y) = \sigma\big((x,\xi),(y,\eta)\big) = \xi\cdot y - x\cdot\eta$ and $X = (x,\xi), Y = (y,\eta) \in T^*\mathbb{R}^d$.

$X \mapsto W(X)$ is a projective representation of $T^*\mathbb{R}^d$ on $L^2(\mathbb{R}^d)$

$$W(X)\,W(Y) = \mathrm{e}^{\frac{\mathrm{i}}{2}\sigma(X,Y)}\,W(X+Y)$$

⇝ Encodes commutation relations of Q and P!

The Weyl System

Definition (Weyl System)

$$(W(X)\psi)(y) = \mathrm{e}^{-\mathrm{i}\xi\cdot(y+\frac{x}{2})}\,\psi(x+y)$$

where $\sigma(X,Y) = \sigma\big((x,\xi),(y,\eta)\big) = \xi\cdot y - x\cdot\eta$ and $X=(x,\xi), Y=(y,\eta)\in T^*\mathbb{R}^d$.

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$$W(X) = W(x, \xi) := \mathrm{e}^{-\mathrm{i}\sigma(X,(Q,P))} = \mathrm{e}^{-\mathrm{i}(\xi \cdot x - x \cdot P)}$$

defines bounded (indeed unitary) operator on $L^2(\mathbb{R}^d)$.

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$$W(X)\, W(Y) = \mathrm{e}^{\frac{\mathrm{i}}{2}\sigma(X,Y)}\, W(X+Y)$$

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The Symplectic Fourier Transform

$$(\mathcal{F}_\sigma f)(X) = \frac{1}{(2\pi)^d} \int_{T^*\mathbb{R}^d} \mathrm{d}Y \, \mathrm{e}^{+\mathrm{i}\sigma(X,Y)} \, f(Y)$$

is the usual Fourier transform combined with a sign flip and an exchange of position and momentum.

$$\Longrightarrow \mathcal{F}_\sigma^2 = \mathbb{1} \text{ (} \mathcal{F}_\sigma \text{ is its own inverse!)}$$

$$\sigma(X,Y) = \sigma\big((x,\xi),(y,\eta)\big) = \xi \cdot y - x \cdot \eta$$

Weyl Quantization

Definition (Weyl Quantization)

$$\mathrm{Op}(f) := \frac{1}{(2\pi)^d} \int_{T^*\mathbb{R}^d} \mathrm{d}X \, (\mathcal{F}_\sigma f)(X) \, W(X)$$

for suitable f (e. g. $\mathcal{S}(T^*\mathbb{R}^d)$ or $\mathcal{S}'(T^*\mathbb{R}^d)$)

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Heuristic justification: Pretend Q and P are numbers

$$\text{“}\mathsf{Op}(f) = \frac{1}{(2\pi)^{2d}} \int_{T^*\mathbb{R}^d} \mathrm{d}Y \underbrace{\int_{T^*\mathbb{R}^d} \mathrm{d}X \, \mathrm{e}^{\mathrm{i}\sigma(X, Y-(Q,P))}}_{=(2\pi)^{2d} \, \delta(Y-(Q,P))} f(Y)$$

$$= f(Q,P)\text{”}$$

Weyl Quantization

Definition (Weyl Quantization)

$$(\mathrm{Op}(f)\psi)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{d}y \int_{\mathbb{R}^d} \mathrm{d}\eta \, \mathrm{e}^{-\mathrm{i}(y-x)\cdot\eta} \, f\big(\tfrac{1}{2}(x+y), \eta\big) \, \psi(y)$$

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Symmetric operator ordering

Weyl Quantization

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for suitable f (e. g. $\mathcal{S}(T^*\mathbb{R}^d)$ or $\mathcal{S}'(T^*\mathbb{R}^d)$)

Operator kernel of $\mathsf{Op}(f)$:

$$K_f(x,y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{d}\eta \, \mathrm{e}^{-\mathrm{i}(y-x)\cdot\eta} \, f\big(\tfrac{1}{2}(x+y), \eta\big)$$

Dequantization via the Wigner Transform

Apply Wigner transform

$$(\mathcal{W}K_f)(x,\xi) := \int_{\mathbb{R}^d} \mathrm{d}y \, \mathrm{e}^{-\mathrm{i} y \cdot \xi} \, K_f\big(x - \tfrac{y}{2}, x + \tfrac{y}{2}\big) = f(x,\xi)$$

to operator kernel of $\mathrm{Op}(f)$, and we obtain f.

Definition (Weyl Dequantization)

Let T be an operator on $\mathcal{H}$ whose (distributional) operator kernel is denoted with K_T. Then the inverse of Weyl quantization is given by

$$\mathrm{Op}^{-1}(T) := \mathcal{W}K_T$$

Dequantization via the Wigner Transform

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The Weyl Product

$$f \sharp g := \mathsf{Op}^{-1}\Big(\mathsf{Op}(f)\,\mathsf{Op}(g)\Big)$$

The Weyl Product

$$(f \sharp g)(x) = \frac{1}{(2\pi)^{2d}} \int_{T^*\mathbb{R}^d} \mathrm{d}Y \int_{T^*\mathbb{R}^d} \mathrm{d}Z \, \mathrm{e}^{\mathrm{i}\sigma(X, Y+Z)} \, \mathrm{e}^{\frac{\mathrm{i}}{2}\sigma(Y,Z)} \cdot (\mathcal{F}_\sigma f)(Y) \, (\mathcal{F}_\sigma g)(Z)$$

Fourier transform & Non-commutativity of Q and P

The Weyl Product

$$\begin{aligned} f \sharp g &:= \mathsf{Op}^{-1}\Big(\mathsf{Op}(f)\,\mathsf{Op}(g)\Big) \\ &\asymp \sum_{n=0}^{\infty} \varepsilon^n \, (f \sharp g)_{(n)} = f\,g - \varepsilon\,\tfrac{\mathrm{i}}{2}\{f,g\} + \mathcal{O}(\varepsilon^2) \end{aligned}$$

Admits expansion in semiclassical parameter $\varepsilon \ll 1$
$\Longrightarrow$ Expansion of operator product in powers of ε, e. g.

$$\begin{aligned} \big[\mathsf{Op}(f)\,,\mathsf{Op}(g)\big] &= \mathsf{Op}\big([f,g]_\sharp\big) = \mathsf{Op}\big(f\sharp g - g\sharp f\big) \\ &= -\varepsilon\,\mathsf{Op}\big(\mathrm{i}\{f,g\}\big) + \mathcal{O}(\varepsilon^3) \end{aligned}$$

$$B \neq 0$$

Quantum Building Block Observables

$$Q_j = \hat{x}_j$$
$$P_j^A = -\mathrm{i}\,\partial_j - A_j(\hat{x})$$

Highly non-unique vector potential A represents $B = \mathrm{d}A$

Commutation Relations

$$\tfrac{\mathrm{i}}{\varepsilon}\,[Q_j, Q_k] = 0, \quad \tfrac{\mathrm{i}}{\varepsilon}\,[P_j^A, P_k^A] = -B_{jk}(Q), \quad \tfrac{\mathrm{i}}{\varepsilon}\,[P_j^A, Q_k] = \delta_{jk}\,\mathbb{1}$$

Adapt Weyl Calculus to $B \neq 0$

Two slots where magnetic field can be incorporated into quantization rule

$$\mathsf{Op}(f) := \frac{1}{(2\pi)^d} \int_{T^*\mathbb{R}^d} \mathrm{d}X \, (\mathcal{F}_\sigma f)(X) \, W(X)$$

where the vector potential A represents $B = \mathrm{d}A$.

1. Observable via **minimal substitution**:

$$f(x, \xi) \rightsquigarrow f_A(x, \xi) := f(x, \xi - A(x))$$

2. Weyl system: $P \rightsquigarrow P^A = -\mathrm{i}\nabla - A(Q)$

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Gauge Covariance as a Criterion

If A and $A' = A + \mathrm{d}\chi$ are equivalent gauges, then we want that $\mathrm{Op}^{A}(f)$ and $\mathrm{Op}^{A+\mathrm{d}\chi}(f)$ should be **unitarily equivalent,**

$$\mathrm{Op}^{A+\mathrm{d}\chi}(f) = \mathrm{e}^{+\mathrm{i}\chi}\,\mathrm{Op}^{A}(f)\,\mathrm{e}^{-\mathrm{i}\chi}.$$

Physics should not depend on the choice of gauge!

Approach 1: Minimal Substitution

$$\mathsf{Op}_A(f) := \mathsf{Op}(f_A)$$

Approach 1: Minimal Substitution

$$\mathsf{Op}_A(f) := \mathsf{Op}(f_A)$$

Good News
Quantization of building block operators *is* covariant:

$$q_j \mapsto \mathsf{Op}_{A+\mathrm{d}\chi}(q_j) = Q_j$$
$$p_j \mapsto \mathsf{Op}_{A+\mathrm{d}\chi}(p_j) = P_j^{A+\mathrm{d}\chi} = \mathrm{e}^{+\mathrm{i}\chi} \, P_j^A \, \mathrm{e}^{-\mathrm{i}\chi}$$

Approach 1: Minimal Substitution

$$\mathsf{Op}_A(f) := \mathsf{Op}(f_A)$$

Check gauge covariance: $A' = A + \nabla\chi$

$$\Bigl(\Bigl(\mathsf{Op}_{A+\mathsf{d}\chi}(f) - \mathsf{e}^{+\mathsf{i}\chi} \, \mathsf{Op}_A(f) \, \mathsf{e}^{-\mathsf{i}\chi} \Bigr) \psi \Bigr)(x) =$$

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathsf{d}y \int_{\mathbb{R}^d} \mathsf{d}\eta \, \mathsf{e}^{-\mathsf{i}(y-x)\cdot\eta} \Bigl(f\Bigl(\tfrac{1}{2}(x+y) \, , \, \eta - A'\bigl(\tfrac{1}{2}(x+y)\bigr) \Bigr) +$$

$$- \, \mathsf{e}^{+\mathsf{i}\chi(x)} \, f\Bigl(\tfrac{1}{2}(x+y) \, , \, \eta - A\bigl(\tfrac{1}{2}(x+y)\bigr) \Bigr) \, \mathsf{e}^{-\mathsf{i}\chi(y)} \Bigr) \, \psi(y)$$

$$\overset{!}{=} 0$$

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$$\Big(\big(\mathrm{Op}_{A+\mathrm{d}\chi}(f) - \mathrm{e}^{+\mathrm{i}\chi}\,\mathrm{Op}_A(f)\,\mathrm{e}^{-\mathrm{i}\chi}\big)\psi\Big)(x) =$$
$$= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mathrm{d}y\,\left(\int_{\mathbb{R}^d}\mathrm{d}\eta\,\mathrm{e}^{-\mathrm{i}(y-x)\cdot\eta}\,f\big(\tfrac{1}{2}(x+y),\eta\big)\right)\cdot$$
$$\cdot\;\mathrm{e}^{-\mathrm{i}(x-y)\cdot A(\frac{1}{2}(x+y))}\,\cdot$$
$$\cdot\;\left(\mathrm{e}^{-\mathrm{i}(y-x)\cdot\nabla\chi(\frac{1}{2}(x+y))} - \mathrm{e}^{-\mathrm{i}(\chi(y)-\chi(x))}\right)\psi(y)$$
$$\overset{!}{=} 0$$

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Check gauge covariance: $A' = A + \nabla\chi$

$$\Big(\big(\mathrm{Op}_{A+\mathrm{d}\chi}(f) - \mathrm{e}^{+\mathrm{i}\chi}\,\mathrm{Op}_A(f)\,\mathrm{e}^{-\mathrm{i}\chi}\big)\psi\Big)(x) \neq 0$$

Vanishes for linear and quadratic polynomials in *momentum* ξ, but $\neq 0$ **in general**!

Approach 1: Minimal Substitution

$$\mathrm{Op}_A(f) := \mathrm{Op}(f_A)$$

Check gauge covariance for $h = \sqrt{m^2 + \xi^2}$: $A' = A + \nabla\chi$

$$\Big(\big(\mathrm{Op}_{A+\mathrm{d}\chi}(h) - \mathrm{e}^{+\mathrm{i}\chi}\,\mathrm{Op}_A(h)\,\mathrm{e}^{-\mathrm{i}\chi}\big)\psi\Big)(x) \neq 0$$

Vanishes for linear and quadratic polynomials in *momentum* ξ, but $\neq 0$ **in general**!

Approach 2: Put Vector Potential into Weyl System

Definition (Magnetic Weyl System)

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Definition (Magnetic Weyl System)

$$\left(W^A(X)\psi\right)(y) = \mathrm{e}^{-\mathrm{i}(x+\frac{y}{2})\cdot\xi}\,\mathrm{e}^{-\mathrm{i}\int_{[x,y]} A}\,\psi(x+y)$$

- Magnetic circulation along line segment $[x, y]$
- Manifestly gauge covariant

Approach 2: Put Vector Potential into Weyl System

Definition (Magnetic Weyl System)

$$W^{A+\mathrm{d}\chi}(X) = \mathrm{e}^{+\mathrm{i}\chi} \, W^{A}(X) \, \mathrm{e}^{-\mathrm{i}\chi}$$

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- Manifestly gauge covariant

Approach 2: Put Vector Potential into Weyl System

Definition (Magnetic Weyl System)

$$W^A(X) := \mathrm{e}^{-\mathrm{i}\sigma(X,(Q,P^A))}$$

defines a bounded (indeed unitary) operator on $L^2(\mathbb{R}^d)$

$X \mapsto W^A(X)$ **is a generalized projective representation of** $T^*\mathbb{R}^d$

$$W^A(X)\,W^A(Y) = \mathrm{e}^{\frac{\mathrm{i}}{2}\sigma(Y,Z)}\,\mathrm{e}^{-\mathrm{i}\int_{\Delta(Q,Q+x,Q+x+y)} B}\,W^A(Y+Z)$$

where $\int_{\Delta(Q,Q+x,Q+x+y)} B$ is the **magnetic flux through the triangle** with corners Q, $Q+x$ and $Q+x+y$.

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depends only on B!

Magnetic Weyl Quantization

$$\mathsf{Op}^A(f) := \frac{1}{(2\pi)^d} \int_{T^*\mathbb{R}^d} \mathrm{d}X \, (\mathcal{F}_\sigma f)(X) \, W^A(X) \equiv f(Q, P^A)$$

- Inherits gauge covariance from $W^A(X)$
- Magnetic Wigner transform $\mathcal{W}^A$ exists
- Magnetic Weyl product

$$\begin{aligned} f \sharp^B g &= \mathsf{Op}^{A^{-1}}\Big(\mathsf{Op}^A(f) \, \mathsf{Op}^A(g)\Big) \\ &= f\,g - \varepsilon\,\tfrac{i}{2}\{f, g\}_B + \mathcal{O}(\varepsilon^2) \end{aligned}$$

exists and has a $\varepsilon \ll 1$ expansion where

$$\{f, g\}_B = \nabla_p f \cdot \nabla_q g - \nabla_q f \cdot \nabla_p g - \sum_{j,k=1}^{d} B_{jk} \, \partial_{p_j} f \, \partial_{p_k} g$$

is the **magnetic Poisson bracket**

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Overview: Why is Magnetic Weyl Calculus Natural?

$$
\begin{array}{ccc}
\big(x, \xi - A(x)\big) \text{ on } (T^*\mathbb{R}^d, \omega) & \longleftrightarrow & (x, \xi) \text{ on } (T^*\mathbb{R}^d, \omega^B) \\
\Big\downarrow \, \mathrm{Op} & & \Big\downarrow \, \mathrm{Op}^A \\
\big(\hat{x}, -i\varepsilon\nabla_x - A(x)\big) & \overset{X}{\longleftrightarrow} & \big(\hat{x}, -i\varepsilon\nabla_x - A(x)\big)
\end{array}
$$

$\omega = \mathrm{d}x_j \wedge \mathrm{d}\xi_j$ is the **canonical symplectic form**

$\omega^B = \mathrm{d}x_j \wedge \mathrm{d}\xi_j + B_{jk} \, \mathrm{d}x_j \wedge \mathrm{d}x_k$ is the **magnetic symplectic form**

Overview: Why is Magnetic Weyl Calculus Natural?

Hamiltonian Equations when $B \neq 0$

Hamiltonian equations of motion

$$\begin{pmatrix} B & -\mathbb{1}_{\mathbb{R}^d} \\ +\mathbb{1}_{\mathbb{R}^d} & 0 \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} \nabla_q \\ \nabla_p \end{pmatrix} h(q(t), p(t))$$

gives rise to the *classical flow* $\Phi_t : T^*\mathbb{R}^d \longrightarrow T^*\mathbb{R}^d$, where $h : T^*\mathbb{R}^d \longrightarrow \mathbb{R}$ is the energy observable (aka Hamilton function)

Overview: Why is Magnetic Weyl Calculus Natural?

Hamiltonian Equations when $B \neq 0$

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t) = \{h, f(t)\}_B$$

where

$$\{f, g\}_B = \nabla_p f \cdot \nabla_q g - \nabla_q f \cdot \nabla_p g - \sum_{j,k=1}^{d} B_{jk} \, \partial_{p_j} f \, \partial_{p_k} g$$

is the **magnetic** Poisson bracket and $f(t) = f \circ \Phi_t$

Overview: Why is Magnetic Weyl Calculus Natural?

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Weyl Calculus as a Computational Tool

- Semiclassical limit
- Perturbation expansions
- Application to other wave equations (e. g. to obtain ray optics)
- Algebraic formulation available!

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Semiclassical Limit of Magnetic Systems

$$\left.\begin{array}{c}\textbf{Quantum Mechanics}\\ \mathrm{i}\varepsilon\partial_t\psi(t) = H^A\psi(t)\\ H = \mathsf{Op}^A(h)\end{array}\right\} \xrightarrow{\varepsilon\ll 1} \left\{\begin{array}{c}\textbf{Classical Mechanics}\\ \dot{q} = +\nabla_p h + \mathcal{O}(\varepsilon)\\ \dot{p} = -\nabla_q h - B\,\dot{q} + \mathcal{O}(\varepsilon)\end{array}\right.$$

Setting

- Semiclassical parameter $\varepsilon \ll 1$ quantifies **separation of spatial scales**
- Semiclassical equations could contain $\mathcal{O}(\varepsilon)$ **quantum corrections**

Semiclassical Limit of Magnetic Systems

Idea
Work in the **Heisenberg picture** and compare evolution of observable $F^A = \mathrm{Op}^A(f)$

Quantum mechanically evolved observable

$$\frac{\mathrm{d}}{\mathrm{d}t} F^A_{\mathrm{qm}}(t) = \frac{\mathrm{i}}{\varepsilon} \left[\mathrm{Op}^A(h) \, , \, F^A_{\mathrm{qm}}(t) \right]$$

$$F^A_{\mathrm{qm}}(t) = \mathrm{e}^{+\mathrm{i} \frac{t}{\varepsilon} \mathrm{Op}^A(h)} \, \mathrm{Op}^A(f) \, \mathrm{e}^{-\mathrm{i} \frac{t}{\varepsilon} \mathrm{Op}^A(h)}$$

Classically evolved observable

$$\frac{\mathrm{d}}{\mathrm{d}t} f_{\mathrm{cl}}(t) = \left\{ h \, , \, f_{\mathrm{qm}}(t) \right\}_B$$

$$F^A_{\mathrm{cl}} := \mathrm{Op}^A\big(f_{\mathrm{cl}}(t)\big) = \mathrm{Op}^A\big(f \circ \Phi_t\big)$$

Semiclassical Limit of Magnetic Systems

Quantum mechanically evolved observable

$$F^A_{\mathrm{qm}}(t) = \mathrm{e}^{+\mathrm{i}\frac{t}{\varepsilon}\mathrm{Op}^A(h)} \, \mathrm{Op}^A(f) \, \mathrm{e}^{-\mathrm{i}\frac{t}{\varepsilon}\mathrm{Op}^A(h)}$$

Classically evolved observable

$$F^A_{\mathrm{cl}} := \mathrm{Op}^A(f \circ \Phi_t)$$

Theorem (L. (2011))

Under suitable technical conditions on f and h, we have

$$F^A_{\mathrm{qm}}(t) - F^A_{\mathrm{cl}}(t) = \mathcal{O}(\varepsilon^2).$$

Put another way, f is the semiclassical limit of $F^A = \mathrm{Op}^A(f)$.

Semiclassical Limit of Magnetic Systems

Proof.

Ingredients

$\varepsilon \ll 1$ expansion of the magnetic Weyl product

$$\tfrac{\mathrm{i}}{\varepsilon} [f, g]_{\sharp^B} = \{f, g\}_B + \mathcal{O}(\varepsilon^2)$$

Duhamel argument:

$$F_{\mathrm{qm}}(t) - F_{\mathrm{cl}}(t) = \\ = \int_0^t \mathrm{d}s \, \frac{\mathrm{d}}{\mathrm{d}s} \Big(\mathrm{e}^{+\mathrm{i} \frac{t}{\varepsilon} \mathrm{Op}^A(h)} \, \mathrm{Op}^A (f \circ \Phi_{t-s}) \, \mathrm{e}^{-\mathrm{i} \frac{t}{\varepsilon} \mathrm{Op}^A(h)} \Big)$$

□

Semiclassical Limit of Magnetic Systems

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$$\left. - \mathrm{Op}^A(\{h, f \circ \Phi_{t-s}\}_B) \right) \mathrm{e}^{-\mathrm{i}\frac{t}{\varepsilon}\mathrm{Op}^A(h)}$$

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Semiclassical Limit of Magnetic Systems

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Obtaining Recursion Relations from $\varepsilon \ll 1$ Expansion

Setting

- Hamiltonian $H^A_\varepsilon = \mathsf{Op}^A(h)$ where $h = \sum_{n=0}^\infty \varepsilon^n \, h_n$
- $P^A_0 = \mathsf{Op}^A(p_0) = \mathsf{Op}^A(p_0)^*$ satisfies

$$\begin{aligned} {P^A_0}^2 - P^A_0 &= \mathcal{O}(\varepsilon) \\ [H^A_\varepsilon \, , \, P^A_0] &= \mathcal{O}(\varepsilon) \end{aligned}$$

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$$\begin{aligned} p_0 \sharp^B p_0 - p_0 &= \varepsilon \, A_1 + \mathcal{O}(\varepsilon^2) \\ [h, p_0]_{\sharp^B} &= \varepsilon \, B_1 + \mathcal{O}(\varepsilon^2) \end{aligned}$$

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$$[h, p_0]_{\sharp^B} = \varepsilon \, B_1 + \mathcal{O}(\varepsilon^2)$$

Construct function on phase space p_1 from “projection defect” A_1 and “commutation defect” B_1 so that

$$(p_0 + \varepsilon \, p_1) \sharp^B (p_0 + \varepsilon \, p_1) - (p_0 + \varepsilon \, p_1) = \mathcal{O}(\varepsilon^2)$$
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Obtaining Recursion Relations from $\varepsilon \ll 1$ Expansion

Setting

- Hamiltonian $H_\varepsilon^A = \mathrm{Op}^A(h)$ where $h = \sum_{n=0}^\infty \varepsilon^n h_n$
- $P_0^A = \mathrm{Op}^A(p_0) = \mathrm{Op}^A(p_0)^*$ satisfies

$$p^{(n)} \sharp^B p^{(n)} - p^{(n)} = \varepsilon^{n+1} A_{n+1} + \mathcal{O}(\varepsilon^{n+2})$$

$$\left[h \, , \, p^{(n)} \right]_{\sharp^B} = \varepsilon^{n+1} B_{n+1} + \mathcal{O}(\varepsilon^{n+2})$$

Proceed by recursion: given $p^{(n)} = \sum_{j=0}^n \varepsilon^j p_j$, compute p_{n+1} from projection and commutation defects.

Wrap-Up: Magnetic Weyl calculus

- Gauge covariant quantization procedure which maps functions on phase space $(T^*\mathbb{R}^d, \omega^B)$ onto magnetic operators on $L^2(\mathbb{R}^d)$
- Computational tool to obtain semiclassical limits and perturbation expansions
- Admits an algebraic point of view: magnetic pseudodifferential operators affiliated to twisted crossed product C^*-algebras
- Gives rise to "non-commutative phase space"
  $\rightsquigarrow$ Algebras of functions and distributions with non-commutative product $\sharp^B$

Thank you!

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