Examples as Tools for Constructing Justifications

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Every odd number greater than 3 can be written as the sum of a prime and a power of 2.

If asked if this conjecture is true, a student would probably test it using several examples or perhaps find a counterexample. Examples are powerful tools for understanding mathematics (Watson and Mason 2005), which may be why students often generate examples when trying to understand a mathematical idea. However, finding several examples that work does not prove that a statement is true. For instance, a student may think that the conjecture above is true because it works for the numbers 5, 7, 9, and 11. It is not. One counterexample is 127.

Reasoning and sense making should start from the early elementary grades on (CCSSI 2010; NCTM 2000). However, more formal work with mathematical proof typically occurs in high school. To prepare for this work, middle school students should have opportunities to bridge their informal, intuitive ways of reasoning about mathematics to develop arguments that treat the general case.

Focus in High School Mathematics: Reasoning and Sense Making describes a progression of students' mathematical reasoning that occurs in three phases: *empirical*, *preformal*, and *formal* (NCTM 2009, pp. 10–11). *Empirical reasoning* involves generating confirming examples to support a conjecture. Research shows that students at all grade levels rely too much on empirical reasoning when proving. *Formal reasoning* is what we want all students to achieve. In this phase, students logically justify mathematical statements by applying general statements, definitions, and axioms. The middle, or *preformal*, stage is often overlooked in teaching reasoning and proving in math classrooms. This stage is described as "the role of intuitive explanations and partial arguments that lend insight into what is happening" (NCTM 2009, p. 10).

Middle school teachers can help students engage in preformal reasoning by carefully unpacking examples to reveal the underlying mathematical structures. By carefully using examples, teachers can reveal hidden structure and general patterns, an essential

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practice outlined in the Common Core State Standards for Mathematics (CCSSI 2010). We will describe aspects of middle school students' empirical reasoning, in particular, the distinction between the productive and unproductive use of examples. We will then provide recommendations for instruction that can guide students toward productive empirical reasoning and the kind of example use that will lead to preformal reasoning.

WAYS THAT STUDENTS USE EXAMPLES

The Sum of Three Odd Numbers task was given to approximately 200 seventh-grade students to determine if they could provide formal reasoning to justify the odd sum (see **fig. 1**). While working with these seventh-grade students, we found that most were in the empirical stage of reasoning. This finding is not surprising; Knuth, Choppin, and Bieda (2009) reported that middle school students tend to show that a statement is true by providing confirming examples. In our analysis, we found that how students generated examples may have influenced whether they could move toward justification. We classified students' use of examples into two categories: *unproductive* and *productive.*

Fig. 1 The Sum of Three Odd Numbers task provides an opportunity for formal reasoning to justify the odd sum.

Consider the following facts:

- An odd number plus an odd number equals an even number.
- An even number plus an even number equals an even number.
- An odd number plus an even number equals an odd number.

Using those facts, determine if the sum of three odd numbers is odd or even. Write an explanation that would convince someone that you are correct.

We found that how students were able to generate examples may have influenced whether they could move toward justification.

Unproductive Use of Examples

Students overwhelmingly used three examples in their justifications. For example, one student wrote the following response to The Sum of Three Odd Numbers problem:

$$3 + 5 + 7 = 15$$
$$5 + 5 + 5 = 15$$
$$1 + 5 + 11 = 17$$

The types of examples that students generated show that their computations result in an odd number. When using this method, however, students do not develop a conceptual *lever* that can later help them formulate a deductive argument to justify the truth of the statement. By finding the sum of odd numbers in one step, students are unable to see the emerging patterns between intermediate sums that correspond to the given statements (odd + odd = even; even + odd = odd).

Instead of providing justification, students tend to use examples to merely provide answers. Balacheff (1991) noticed that students' use of examples when proving is tied to their desire to produce an answer rather than communicate knowledge. By approaching the task in this way, students are only able to verify that statements are sometimes true.

Productive Use of Examples

Finding mathematical structure by *recognizing* general patterns when generating examples is an important mathematical practice. The mathematical structure of a problem is revealed when "we carry out several concrete examples of a process that we don't quite have . . . to find regularity and build a generic algorithm that describes every instance" (Graham, Cuoco, and Zimmermann 2010, p. 29). That is, as students pay attention to repeated calculations in various examples, they can step back and begin to see general properties of the problem emerge. Although students might not be conceptually ready to write out a formal proof of their ideas, reasoning about general patterns noted in examples indicates a preformal stage of reasoning (NCTM 2009). We discuss two tips for helping students use examples in powerful ways.

TEACHING TIPS

1. Scaffold Students' Use of Examples

As previously mentioned, the majority of students tend to use examples to *test* a conjecture, which leads them to confirm the validity of a statement instead of understanding the underlying mathematical relationships. While planning a lesson, it is critical for the teacher to consider ways to scaffold students' work using examples.

To illustrate, we return to the Sum of Three Odd Numbers task. To promote students' preformal reasoning about this statement, a teacher could discuss how the mathematical operations unfold when odd numbers are added. To begin, students could be asked to conjecture whether the sum of two odd numbers is odd or even. A visual representation can reinforce

Fig. 2 Using manipulatives to add two odd numbers can provide a visual representation of why their sum is always even.

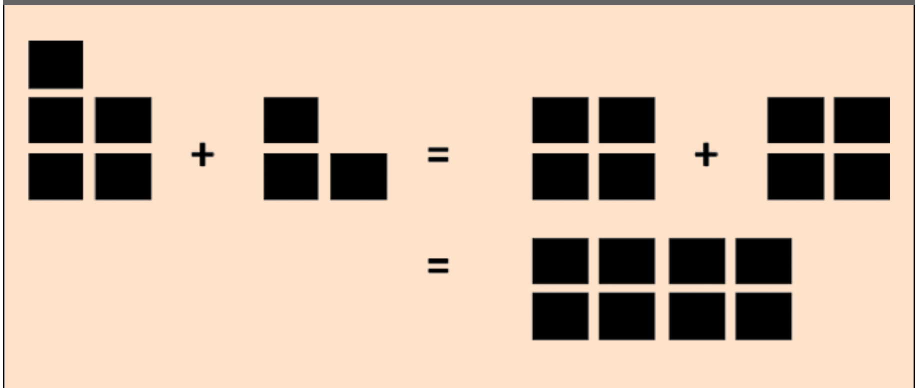
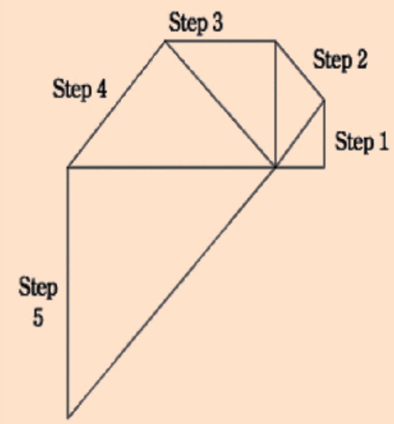


Fig. 3 The Spiraling Triangles problem promotes an understanding of special triangles and the Pythagorean theorem by encouraging students to articulate a pattern.



Each step in a science museum's spiral staircase is an isosceles right triangle whose leg matches the hypotenuse of the previous step, as shown in the overhead view of the staircase. If the first step has an area of 0.5 square feet, what is the area of the *n*th step?

students' understanding of the definitions of odd and even numbers. Students can visualize how an even number result includes a factor of two (see **fig. 2**).

Once students have informally justified why the sum of any two odd numbers is even, they can then explore what happens when the third odd number is added. Students may already have learned the "odd plus odd is even" fact from elementary school, but using visual manipulatives or drawings in middle school can help them reason about relationships between odd and even numbers. Decomposing a relatively complicated operation into smaller chunks and using examples to model intermediate steps can reveal the hidden structure of the mathematical idea and promote preformal reasoning.

Students can benefit greatly from a teacher's guidance in reasoning from examples in all areas of mathematics. **Figure 3** illustrates a problem that can be used when students are learning about properties of special triangles and the Pythagorean theorem. A typical student approach to the Spiraling Triangles problem is to determine the side length of the triangle in one step and then use that result to find the area of the triangle in the next step. Asking students to record their results in a table can help them recognize the geometric pattern.

Table 1 shows how the area increases by a factor of 2 for every step. This helps students understand that the expression 2^{n-1}, where n is the step number, gives the area of the triangle in each step. However, to guide students to develop a deeper understanding of the mathematics, teachers can pose this question: "What is the relationship between the side length and the area of each triangle?"

Students can reference the examples in the table to investigate this relationship. Asking "What do we know about the bases and heights of isosceles right triangles?" will help students see that they can modify the general formula for the area of a triangle for the special case, the area of the isosceles right triangle, to be $1/2 \times (\text{side length})^2$. Students can now be prompted to consider what happens to the value for the area of the isosceles right triangle when the value for the side length is squared. As the equal side lengths increase by a factor of

$$\sqrt{2},$$

the area increases by a factor of 2. As students recognize this relationship, they will be able to not only justify the expression that gives the side length for step n, namely,

$$\left(\sqrt{2}\right)^{n-2},$$

but also generate the following expressions using the law of exponents:

$$\frac{1}{2}\left(\sqrt{2}^{\,n-1}\right)^2 \text{ or } \frac{1}{2}\left(\sqrt{2}^{\,2}\right)^{n-1}$$

Since

$$\left(\sqrt{2}\right)^2$$

is equivalent to 2, the expression above can be simplified to

$$\frac{1}{2}(2)^{n-1}$$

to find the area of step n. By having students attend to not only the patterns of growth in the area of the spiraling isosceles triangles but also the patterns of growth in the side

Table 1 The area increases by a factor of 2 for every step.

Step	Side Length (ft.)	Area (sq. ft.)
1	1	0.5
2	$\sqrt{2}$	1
3	2	2
4	$2\sqrt{2}$	4

lengths, they can use their findings to generalize mathematical relationships that reveal much richer connections between the geometric quantities. In so doing, students will understand *why* such patterns hold.

2. Ask Something General about Something Specific

Sometimes the simplest questions can yield the richest answers. Asking "How do you know this is true?" while pointing to a part of a worked example can prompt students to provide more general reasoning. Students must learn that examples, although useful for noticing patterns, are insufficient to prove that something is true in math. One of the best ways to teach this expectation is to encourage students to question their own assumptions. These questions are useful in developing students' preformal reasoning:

- How do you know this is true?
- Is this true for all cases?
- What general statements could you use to show that this is true for *every* case?
- Is there a relationship between these quantities?
- What patterns do you see with these examples?
- Can we break this into a smaller problem?

Students often make this false conjecture when learning to order fractions:

> The greater the denominator, the smaller the fraction.

Students typically make this conclusion after comparing fractions with the same numerator, such as 1/2 and 1/4 or 3/7 and 3/16. Asking specific questions about these confirming examples can push students to think about alternative cases. For instance,

> What is similar about these examples: 1/2, 1/4 and 3/7, 3/16?

The most obvious similarity is that the numerators are the same for both sets. One might then ask, "Is the conjecture true if the numerators are different? Why, or why not?" This may generate a counterexample and a revised original conjecture: "If two or more fractions have the same numerator, then the smaller fraction is the one with the greater denominator."

PUTTING IT INTO PRACTICE

The important message is that not just any set of examples or cases will provide students with those aha! moments when they begin to understand the relationships between quantities in a problem. Guiding students' thinking about the examples they generate and their use of representations, such as tables, to organize their work can draw their attention away from finding an answer to noticing important patterns within the cases they test.

Asking questions to press students to reason about specific examples in general ways is a necessary aspect of instruction that moves students from the empirical to the preformal stages of reasoning. More important, these strategies are not specific to only some mathematical ideas. Considering these strategies when planning and teaching almost any lesson will generate opportunities to push your students' reasoning and deepen their understanding.

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