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Stationary Markovian equilibrium in overlapping generation models with stochastic nonclassical production and Markov shocks

Olivier F. Morand [a,*], Kevin L. Reffett [b]

[a] University of Connecticut, Storrs, CT, United States
[b] Arizona State University, Tempe, AZ, United States

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Abstract

This paper provides new sufficient conditions for the existence and computation of Markovian equilibrium for a large class of OLG models with Markov shocks to production. The economies considered allow for a very large class of "reduced-form" production functions, including those that are nonclassical, encompassing stochastic OLG models with social security, income redistribution policies, taxation, valued fiat money, production nonconvexities, and monopolistic competition. Our approach combines aspects of both topological and order theoretic fixed point theory and provides globally stable successive approximations algorithms for computing extremal Markovian equilibrium objects.

Keywords: Markov equilibrium; Overlapping generations

1. Introduction

This paper develops a monotone iterative approach for studying the existence, characterization, stability, and computation of Markovian equilibrium for a large class of overlapping generation (OLG) models with Markovian aggregate technology shocks to production. We cover stochastic OLG models with very general one-sector production technologies, and thus encompass many economies considered in the existing literature. We also include some economies with public policy (e.g., fiscal spending and taxation, intergenerational transfers, social security), monopolistic

* Corresponding author.
E-mail address: morand@uconn.edu (O.F. Morand).

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competition, and nonconvex ("nonclassical") stochastic production functions.[1] We identify our approach as "monotone" (or, more precisely, "isotone") because it relies primarily on the order preserving structure of particular equilibrium fixed point mappings, as opposed to the existing approaches in the literature which rely exclusively on topological fixed point constructions.

It is well known that stochastic OLG models can exhibit complicated equilibrium dynamics in the form of endogenous fluctuations, sunspots, and local or global indeterminacies. For instance, as shown in Wang (1994), different types of time homogeneous Markovian equilibrium (THME) can exist. A key contribution of our work is to provide sharp sufficient conditions for constructing specific types of Markovian equilibrium decision processes or MEDP (a MEDP is identified as a recursive equilibrium in Stokey et al. (1989), a self-fulfilling equilibria in Wang (1993), and an optimal policy in Hopenhayn and Prescott (1992)). Our main focus is on MEDP that are measurable, isotone, and defined on "minimal" state spaces (i.e., MEDP that can be written only in terms of current equilibrium aggregate state variables).[2] This ability to identify MEDP with a very specific structure is a main advantage of our constructive monotone approach, since it is something that cannot easily be done using existing topological approaches. Although sufficient conditions for the existence of measurable and isotone MEDP have been obtained using "social planning" methods in the context of iid shocks (see, for instance, Hopenhayn and Prescott, 1992 or Nishimura and Stachurski (2005)), none of these methods apply for economies where the second welfare theorem fails or for economies with Markov shocks. As a result, we construct measurable isotone MEDP through a new application of the "Euler equation method" pioneered by Coleman (1991) for infinite horizon economies.[3]

In addition to studying the structure of MEDP, we also provide important characterizations of the long-run properties for equilibrium induced by such MEDP. Specifically, we first prove the existence of a stationary Markov equilibrium for any measurable MEDP when technology shocks follow a quasi-compact first order Markov process (we define a SME to be a non-degenerate invariant probability measure for the capital to labor ratio). Since quasi-compactness is equivalent to Doeblin's condition, and given that we make no assumptions on the MEDP beyond its measurability, this existence result is quite general relative to all existing work on this problem. Next, we produce successive approximation methods for obtaining extremal SME corresponding to any isotone and measurable MEDP. In this sense, we provide a new answer to the stochastic stability question raised in previous work as we give sufficient conditions for the existence of a "pure" stationary Markov equilibrium, (i.e., a *nonsunspot* THME in the terminology of Wang (1994)). The algorithms used to compute extremal SME corresponding to each isotone and measurable MEDP are shown to be stable in a very important sense (i.e., stable when iterating from initial "upper solutions" or "lower solutions").

[1] Our production specification is more general than in any existing work of which we are aware, including the important papers by Zilcha (1990), Wang (1993, 1994), Demange and Laroque (1999, 2000), and Hausenchild (2002). Our production functions are analogous to those nonclassical reduced-form specifications found in the literature on nonoptimal infinite horizon economies with public policy and one-sector nonclassical production studied for instance in Coleman (1991), Boldrin and Rustichini (1994), Benhabib and Farmer (1994), Farmer and Guo (1994), and Greenwood and Huffman (1995).

[2] A key implication of Wang's work (along with the storyline in the paper of Kubler and Polemarchakis (2004)) is that when sunspots and multiplicities are considered, the state space that is used to construct THME can potentially get very large.

[3] The application of the Euler equation method in Erikson et al. (2005) is restricted to OLG models with independently and identically distributed (iid) shocks.

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It is important to note that the existing literature concerning the existence of SME in OLG models with Markov shocks (i.e., Wang, 1994) relies on the nonconstructive topological approach of Duffie et al. (1994) and thus requires fixed point correspondences to be convex valued. Consequently, there are no results concerning the existence and construction of SME associated with *nonsunspot* equilibria. The monotone approach in this paper is thus a constructive alternative to the topological approach of Duffie et al. (1994) which also removes the need for the standard regularity conditions often found in existing work (e.g., requiring shocks to satisfy the Feller property).[4] Finally, we give sufficient conditions for these SME to be nondegenerate (i.e., the aggregate capital stock does not collapse to zero asymptotically from initial strictly positive capital stocks). This result provides some direct answers to questions pertaining to the stochastic stability of Markovian equilibrium for the class of economies under consideration, even if it falls short of the long-run stability results recently obtained for infinite horizon economies with iid shocks in Nishimura and Stachurski (2005).

The remainder of the paper is organized as follows. The next section gives a brief presentation of some notions from the theory of partially ordered sets and lattices. Section 3 develops our results concerning the existence of MEDP using an Euler equation method and the fixed point theory for isotone (i.e., "increasing") operators on complete lattices. Section 4 addresses the issue of existence and computation of SME associated with particular MEDP.

2. Some mathematical prerequisites

In this paper the fixed point theory for order preserving transformations plays a central role in the construction of MEDP and SME. This section presents some definitions related to ordered spaces and order preserving mappings which are used in the sequel.[5]

A *partially ordered set* (or *poset*) is a pair $(P, \leq)$ where the set P is endowed with a partial order $\leq$ (i.e., a binary relation that is reflexive, antisymmetric, and transitive). If $(P, \leq)$ is a poset, then an *upper bound* (respectively, *lower bound*) of any subset $A \subset P$ is any element u (respectively, v) such that $\forall p \in A$, $u \geq p$ (respectively, $v \leq p$). If there is a point p^u (respectively, p^l) such that p^u is the least element in the subset of upper bounds of the subset $A \subset P$ (respectively, the greatest element in the subset of lower bounds of the subset $A \subset P$), we say p^u (respectively, p^l) is the *supremum* or *least upper bound* (respectively, *infimum* or *greatest lower bound*) of the subset A. Clearly both the supremum and infimum if they exist must be unique. A *chain* C is a linearly ordered (or totally ordered) subset of P (i.e., any pair of elements in C are ordered under $\geq$). A *sequence* is a countable subset of P of the form $\{p_n\}_{n\in\mathbb{N}}$. If $p_i \leq p_{i+1}$ (respectively, $p_i \geq p_{i+1}$) for all $i \in N$ the sequence is called *increasing* (respectively, *decreasing*). A *countable chain* is a linearly ordered subset of P that can be written in the form of a "double sequence" $\{p_n\}_{n=-\infty}^{+\infty}$ with $p_i \leq p_{i+1}$ for all $i \in Z$. A poset $(P, \leq)$ is *countable chain complete* if $\vee C$ and $\wedge C$ exist in P for any countable chain C.

A poset $(P, \leq)$ is called a *lattice* if the infimum and supremum, respectively $p \wedge p'$ and $p \vee p'$, of any two elements p and p' in P exist. A poset $(P, \leq)$ is a *complete lattice* if the infimum and

4 Although our state space for shocks is compact, recent work by Stachurski (2003) indicates the possibility of generalizing our results to unbounded shock spaces.

5 For a thorough venture into partially ordered sets and lattice theory, we suggest consulting for instance Davey and Priestley (1990), and Veinott (1992). See, also Topkis (1998) for a discussion of this material with an eye for economic applications.

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supremum of any subset $P' \subseteq P$ exist in P, in which case they are respectively denoted $\wedge P'$ and $\vee P'$. If $(P, \leq_P)$ and $(L, \leq_L)$ are posets, a function $F : (P, \leq_P) \rightarrow (L, \leq_L)$ is said to be *increasing* or *isotone* if it is order-preserving, i.e. if:

$$\forall (p, p') \in P \times P,\ p \geq_P p' \text{ implies } F(p) \geq_L F(p').$$

Finally, given two elements $p \leq p'$ in a poset $(P, \leq)$, the interval order $[p, p']$ is the set $\{x \in P, p \leq x \leq p'\}$. A mapping $F : P \rightarrow P$ is referred to as a *self map* on P (or a transformation of P).

Our existence of equilibrium results rely on a constructive variation of the seminal fixed point theorem of Tarski (1955, Theorem 1). Tarski's theorem states that an isotone self map F on a nonempty complete lattice $(P, \leq)$ has a nonempty complete lattice of fixed points. In this paper, however, the spaces of interest are only countable chain complete, but the increasing map F is shown to have the following property: if F has a "lower solution" x in P (i.e., $x \leq F(x)$), then the increasing sequence $C = \{F^n(x)\}$ is such that $\vee\{F(C)\} = F(\vee C)$ (a symmetric property holds for decreasing sequences whenever x is an upper solution for F, i.e., $x \geq F(x)$), which implies that $\vee C$ is a fixed point of F.[6] Note that the countable chain completeness hypothesis is here to guarantee that $\vee C$ exists. In addition, it is easy to see that $\vee C$ is the smallest fixed point that is greater than x. Thus, recursively applying F to x provide an algorithm "converging" to a particular "extremal" fixed point.[7]

3. Setup, existence and construction of extremal MEDP

We consider a class of stochastic OLG models related to those in Wang (1994), but modified along several important dimensions. First, we assume that the lifetime utility function of each generation is supermodular on the commodity space (in addition to satisfying interiority restrictions that are standard in the literature), an assumption always satisfied for the time additive case often found in applied work. Second, we generalize the production function of Wang by allowing for nonconvexities in production and for various forms of public policy distortions, although the constant returns to scale in private inputs, as in Wang, imply zero profits. This specification is typical of the literature on infinite horizon nonoptimal economies (see for instance, Coleman (1991)), and can be viewed as a "reduced-form" for many cases of economies with fiscal and monetary policy, monopolistic competition, and production nonconvexities (see Greenwood and Huffman, 1995; Datta et al., 2002a). Finally, we put some restrictions on the Markov shock process governing technology shocks that are different that those in Wang (1993, 1994) (in particular, we do no require the Feller property).

3.1. The primitives of the economy

We now describe the preferences, technologies, and stochastic structure. For preferences, our assumptions on lifetime utility functions are standard (e.g., see Wang, 1993, 1994) with the exception of the complementarity conditions between consumption when young (denoted as c_1) and consumption when old (denoted as c_2) in Assumption 1(IV):

[6] This result is similar to Theorem 1.2 in Dugundji and Granas (2003) (Chapter 2) but does not require order continuity of F (which is defined as $\vee\{F(C)\} = F(\vee C)$ for all countable chains C having a supremum).

[7] This method is used extensively in Morand (2006).

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Assumption 1. The utility function $U(c) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$:

I. is twice continuously differentiable;
II. is strictly increasing in each of its arguments and jointly concave;
III. satisfies $\forall c_2 > 0$, $\lim_{c_1 \to 0^+} U_1(c_1, c_2) = +\infty$ and $\forall c_1 > 0$, $\lim_{c_2 \to 0^+} U_2(c_1, c_2) = +\infty$;
IV. has increasing differences in (c_1, c_2) (i.e., $U_{12} \geq 0$).

Production admits in its "reduced-form" a "nonclassical" specification,[8] a case extensively studied in the literature on infinite horizon economies (e.g., see Romer, 1986; Coleman, 1991; Boldrin and Rustichini, 1994; Benhabib and Farmer, 1994; Greenwood and Huffman, 1995), but not in overlapping generations economies. In particular, we assume that the reduced-form production technology, denoted $F(k, n, K, N, z)$, may have social inputs (K, N) but exhibits constant returns to scale in private inputs (k, n). Our assumptions on F, including the monotonic properties of the equilibrium wage rate and rental rate of capital, are adapted from the literature on nonoptimal stochastic growth (e.g., Coleman, 1991; Greenwood and Huffman, 1995), and are completely standard.

Assumption 2. The production function $F(k, n, K, N, z) : \mathbb{R}_+ \times [0, 1] \times \mathbb{R}_+ \times [0, 1] \times Z \to \mathbb{R}_+$ is:

I. twice continuously differentiable in its first two arguments;
II. constant returns to scale in the private inputs (k, n);
III. such that $w(k, z) = F_2(k, 1, k, 1, z)$ is increasing in k and $\lim_{k\to 0} w(k, z) = 0$;
IV. such that $r(k, z) = F_1(k, 1, k, 1, z)$ is strictly decreasing in k and that $\lim_{k\to 0} r(k, z) = +\infty$;
V. such that there exists $k_{\max}$ with $\forall k \geq k_{\max}$ and $\exists z \in Z$ $F(k, 1, k, 1, z) \geq k_{\max}$, and $\forall k \leq k_{\max}, \forall z \in Z, F(k, 1, k, 1, z) \leq k_{\max}$.

Note that in Assumption 2, we anticipate that $n = 1 = N$ in equilibrium (as households do not value leisure), which implies that Assumption 2(III–V) concern properties of the "equilibrium" production function $F(k, 1, k, 1, z)$. Assumption 2(V) implies that the set of feasible capital stock can be restricted to be the compact interval $X = [0, k_{\max}]$ (as long as we place the initial date zero capital stocks $k_0 = K_0$ in X).

To sharpen the characterization of equilibrium, we make an additional assumption on the equilibrium wages and rental rates on capital. This is a complementarity assumption between technology shocks and the inputs of capital and labor, and it is trivially satisfied in many standard cases used in applied work (for instance, in any economy with multiplicative shocks).

Assumption 2′. $w(k, z)$ and $r(k, z)$ are increasing in z.

Finally, we assume that uncertainty enters the production process in the form of an aggregate technology shock following a first-order Markov process with stationary transition function Q. We follow Wang (1994) and assume that shocks take their values in the compact subset $Z = [z_{\min}, z_{\max}]$ of $\mathbb{R}$. The transition function Q is assumed to be increasing (Assumption 3), an assumption also common in the literature (see, for instance Hopenhayn and Prescott, 1992).

[8] See Greenwood and Huffman (1995) for an extensive discussion of "reduced-form" production technologies, and of the type of models with equilibrium distortions that can be mapped into such a specification.

Unlike Wang (1994), we relax the assumption that Q satisfies the Feller property, but require Q to satisfy Doeblin's condition (D) (Assumption $3'$), which is equivalent to the quasi-compactness of the operator Q (see, for instance, Neveu, 1965, V.3.2). Note that all the conditions on the shocks are trivially satisfied in the special case of iid shocks (in particular, they are satisfied for the economies in Wang (1993) and Erikson et al. (2005)).

Assumption 3. Q is increasing, that is, for every increasing $f(z) \in \mathbf{B}(Z, \mathcal{B}(Z))$, the function:

$$Tf(z) = \int f(z')Q(z, \mathrm{d}z') \text{ is increasing in } z.$$

Assumption $3'$**.** Q is a quasi-compact operator; that is, there exists $\gamma \in \Lambda(Z, \mathcal{B}(Z))$ and $\varepsilon > 0$ with:

$$\forall B \in \mathcal{B}(Z), \gamma(B) \leq \varepsilon \text{ implies that } \forall z \in Z, Q(z, B) \leq 1 - \varepsilon.$$

3.2. Existence and computation of MEDP

Having specified the class of economies under consideration, we now apply the "Euler equation" method originally pioneered by Coleman (1991). Coleman's (1991)construction does not directly apply (it is reserved for infinitely-lived agents, as in Greenwood and Huffman (1995), Datta et al. (2002a), Morand and Reffett (2003), and Datta et al. (2005)) and needs to be adapted to OLG models. Specifically, from the Euler equation evaluated along an equilibrium trajectory, we define a nonlinear operator A whose nonzero fixed points are precisely the MEDP. This nonlinear operator is increasing and maps a countable chain complete lattice of candidate equilibrium policies into itself. The construction of extremal nontrivial fixed points is obtained via a simple successive approximation algorithms iterating from upper and lower solutions.

Consider the maximization problem of a typical agent earning the competitive wage w in the first period of his life, and who must decide what part of his earning to consume immediately, and what part to set aside for future consumption. While making these decisions, the agent takes as given a law of motion h for the aggregate per capita capital stock, which he uses to compute the competitive expected return to investing in capital. Returns on labor and capital are taken by the agent to be the competitive prices, namely $w(k, z) = F_2(k, 1, k, 1, z)$ and $r(k, z) = F_1(k, 1, k, 1, z)$ which are obtained from the firm's profit maximization.

More formally, we define W to be the set of functions $h : S \to X$ that are pointwise feasible, i.e. such that $0 \leq h(s) \leq w(s)$ for all $s \in S$. Given a candidate equilibrium law of motion for the capital stock $k' = h(s)$ with $h \in W$, an agent in a Markovian equilibrium governed by h solves:

$$\max_{y \in [0, w(s)]} \int_Z U(w(s) - y, r(h(s), z')y)Q(z, \mathrm{d}z'),$$

Under Assumptions 1 and 2, and the condition that $w(s) > 0$ and $r(s) > 0$ for all $s = (k, z) \in (0, k_{\max}] \times Z$, the necessary and sufficient Euler equation associated with the household maximization problem above is:

$$\int_Z U_1(w(s) - y^*, r(h(s), z')y^*)Q(z, \mathrm{d}z') = \int_Z U_2(w(s) - y^*, r(h(s), z')y^*)r(h(s), z')Q(z, \mathrm{d}z'). \tag{E}$$

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We can now define a Markovian equilibrium decision policy as follows:

Definition 1. A MEDP is a function $h \in W$ distinct from 0 such that, for all $s \in S^*$, $h(s)$ solves (E) for y and $h(0, z) = 0$ for all $z \in Z$.

A MEDP is thus a best response h for household investment to an aggregate environment also parametrized by exactly such an h. We will restrict our search for MEDP in particular subspaces of W which we now present.

3.2.1. Spaces for candidate MEDP

Recalling that $X = [0, k_{\max}] \subset \mathbb{R}$ where $k_{\max} > 0$, $Z = [z_{\min}, z_{\max}] \subset \mathbb{R}$ where $0 < z_{\min} \leq z_{\max}$, we denote the state space by $S = X \times Z$, and $S^* = (0, k_{\max}] \times Z$. All compact subsets of $\mathbb{R}^n$ are endowed with the standard pointwise partial order $\leq$ and the usual topology on $\mathbb{R}^n$. The Borel algebra corresponding to S is denoted $\mathcal{B}(S)$.

Given a function $w : S \longrightarrow X$ that is bounded, increasing[9] and continuous (and thus $\mathcal{B}(S)$-measurable), we endow W with the standard pointwise order $\geq$ defined as follows:

$$w_1 \geq w_2, \quad \text{iff } \forall s \in S, w_1(s) \geq w_2(s).$$

By making successive restrictions on the functions h, we consider several subsets of W, each also endowed with the pointwise order $\geq$. Unlike in the case of iid shocks, we cannot generally expect the MEDP to be isotone, although we present sufficient conditions for this to be the case later in the paper. We first define the subspaces G and H obtained from W by adding the following monotonicity restrictions:

(i) $G = \{h \in W \text{ and } h \text{ increasing on} (S, \leq)\}$,
(ii) $H = \{h \in W \text{ and } h \text{ increasing in } x \text{ for all } z \in Z\}$.

The following three lemma state some important order properties which are easily verified, so we omit their proofs.

Lemma 2. *$(G, \leq)$ and $(H, \leq)$ are complete lattices with minimal and maximal elements* 0 *and* w, *respectively*.

Given the continuity properties in Assumptions 1–3, we can expect the existence of semicontinuity MEDP and thus consider the two subsets U and L of H, each endowed with the pointwise order $\geq$ and defined as follows:

(i) $U = \{h \in H \text{ and upper semicontinuous (“usc”) in } x \text{ for all } z \in Z\}$,
(ii) $L = \{h \in H \text{ and lower semicontinuous (“lsc”) in } x \text{ for all } z \in Z\}$.

Lemma 3. *$(U, \leq)$ and $(L, \leq)$ are complete lattices with minimal and maximal elements* 0 *and* w, *respectively*.

[9] We say that a function $h \in (W, \leq)$ is *increasing* (equivalently *isotone on* $(S, \leq)$) if:

$$\forall (s, s') \in S \times S, s' \geq s \text{ implies that } h(s') \geq h(s).$$

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Finally, we add the algebraic requirement of $\mathcal{B}(S)$-measurability to the sets G, H, U, and L, and correspondingly define the sets G_{m}, H_{m}, U_{m} and L_{m}. For example, $G_{\mathrm{m}} = \{h \in G | h \text{ is } \mathcal{B}(S)\text{-measurable}\}$. The following result is well-known.

Lemma 4. *$(G_{\mathrm{m}}, \leq), (H_{\mathrm{m}}, \leq), (U_{\mathrm{m}}, \leq)$ and $(L_{\mathrm{m}}, \leq)$ are countable chain complete posets with minimal and maximal elements 0 and w, respectively.*

3.3. Computation of measurable MEDP

We now develop an Euler equation method for finding measurable MEDP by first defining the nonlinear operator A as follows: for any $h \in H_{\mathrm{m}}$, and $s = (k, z) \in S^*$, consider the following equation (E′) in y below[10]:

$$\int_Z U_1(w(s) - y, r(h(s), z')y)Q(z, \mathrm{d}z') = \int_Z U_2(w(s) - y, r(h(s), z')y)r(y, z')Q(z, \mathrm{d}z'), \tag{E′}$$

and define $Ah(s)$ depending only on $h(s)$ and defined by:

$$Ah(s) = \left\{ \begin{array}{ll} 0, & \text{if } h(s) = 0 \\ y^*(s, h(s)), & \text{if } h(s) > 0 \end{array} \right\},$$

where y^* is the unique solution to (E′). Note that Assumption 1 is sufficient to establish the uniqueness of the solution y^* since it implies that the left side of (E′) is increasing in y and goes to $+\infty$ as $y \to w$, and that the right side of (E′) is decreasing in y and goes to $+\infty$ as $y \to 0$. Since an element h in H_{m} is a MEDP if and only if it is a nonzero fixed point of A, the search for MEDP in H_{m} is equivalent to finding the (nonzero) fixed points of the operator A.

We first consider the question of existence of MEDP, which we establish as a consequence of the particular order properties of the operator A.

Lemma 5. *A is an isotone self map on $(H_{\mathrm{m}}, \leq)$.*

Proof. For any $h \in H_{\mathrm{m}}$, Assumptions 1 and 2 imply that the left side of (E′) is decreasing in k while the right side is increasing in k, which implies that $Ah : S \to X$ is increasing in k for each $z \in Z$. The unique solution y^* of (E′) can be expressed as a continuous function of h and w, say $y^* = G(h(k, z), w(k, z))$ with G continuous in w and h. The measurability of h and w, and the property that the composition of Borel measurable functions is measurable, imply that

$$s \to G(h(s), w(s)) \text{ is measurable.}$$

Thus, Ah is Borel measurable. Finally, note that the left side of (E′) is decreasing in h while the right side is increasing in h. As a result, $h \leq h'$ implies that $Ah \leq Ah'$, i.e., A is isotone in h. □

Based on this lemma, we now prove a key proposition concerning the existence and computation of MEDP. Let H_A be the set of fixed points of the operator A in $(H_{\mathrm{m}}, \leq)$.

Proposition 6. *Under* Assumptions 1, 2 *and* 3, *the nonempty set of fixed points H_A has a maximal element computed as $\wedge_{H_{\mathrm{m}}}\{A^n w\}_{n\in\mathbb{N}}$ with $\wedge_{H_{\mathrm{m}}}\{A^n w\}_{n\in\mathbb{N}}(s) = \lim_{n\to\infty}\{A^n w\}_{n\in\mathbb{N}}(s)$.*

[10] The operator A differs significantly from that of the infinite horizon case in Coleman (1991). In particular, it is not defined to be the current period "response" to a guess of next period's law of motion.

Proof. Consider the sequence $\{A^n w\}_{n\in\mathbb{N}}$ in $(H_\mathrm{m}, \leq)$. Assumption 1(III) guarantees that for all $s \in S^*$, $Aw(s) < w(s)$, and therefore that the sequence $\{A^n w\}_{n\in\mathbb{N}}$ is decreasing since A is isotone. In particular, for any $s \in S^*$, the sequence $\{A^n w(s)\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ is decreasing and bounded below by 0. This implies that $\lim_{n\to\infty} A^n w(s) = \inf\{A^n w(s)\}_{n\in\mathbb{N}}$. If $\inf\{A^n w(s)\}_{n\in\mathbb{N}} = 0$, then by definition $A(\inf\{A^n w(s)\}_{n\in\mathbb{N}}) = 0$.

Suppose then that $\inf\{A^n w(s)\}_{n\in\mathbb{N}} > 0$. By definition of $A^{n+1}w$, $\forall n \in \mathbb{N}$, and $\forall s = (k, z) \in S^*$:

$$\int_Z U_1(w(s) - \{A^{n+1}w(s)\}_{n\in\mathbb{N}}, r(\{A^n w(s)\}_{n\in\mathbb{N}}, z')\{A^{n+1}w(s)\}_{n\in\mathbb{N}})Q(z, \mathrm{d}z')$$
$$= \int_Z U_2(w(s) - \{A^{n+1}w(s)\}_{n\in\mathbb{N}}, r(\{A^n w(s)\}_{n\in\mathbb{N}}, z')\{A^{n+1}w(s)\}_{n\in\mathbb{N}})$$
$$\times\, r(\{A^{n+1}w(s)\}_{n\in\mathbb{N}}, z')Q(z, \mathrm{d}z'),$$

Since the functions U_1, U_2, and r are each continuous in the relevant arguments, taking limits as n goes to infinity since U_1 and U_2 and r are bounded along the relevant sequences easily shows that $A(\inf\{A^n w(s)\}_{n\in\mathbb{N}}) = \inf\{A^n w(s)\}_{n\in\mathbb{N}}$ by the definition of A. Thus, $\wedge_{H_\mathrm{m}}\{A^n w\}_{n\in\mathbb{N}} = A \wedge_{H_\mathrm{m}} \{A^n w\}_{n\in\mathbb{N}}$. □

Although the minimal element of H_A is clearly zero, we provide in the sequel sufficient conditions for the existence of a strictly positive minimal fixed point (see Proposition 9).

3.4. Computation of monotone measurable MEDP

Establishing the existence of nonzero fixed points of A in the space of monotone functions $(G_\mathrm{m}, \leq)$ is a little more difficult. First, it is to be expected that isotonicity in the exogenous shock (we already know that it is increasing in the endogenous state k) will require additional restrictions on the primitives. Intuitively, an increase in z leads to an immediate increase in current period wealth as well as an expected increase in the rate of return on savings under the assumption of an increasing transition function Q.[11] As a result, young agents do not necessarily respond to an increase in z by increasing their savings. Assumption 4 below presents a set of sufficient conditions for A to be a self map on $(G_\mathrm{m}, \leq)$.[12] Second, recall that by construction the minimal fixed point of A is zero, a trivial object that we exclude from the set of MEDP. Consequently, we give in Assumption 5 a simple sufficient condition for the existence of a nonzero minimal fixed point of A in $(G_\mathrm{m}, \leq)$, which by definition will be the minimal MEDP in $(G_\mathrm{m}, \leq)$. Finally, under Assumptions 1–5 we state our result concerning the construction of extremal increasing and measurable MEDP.

We begin with sufficient conditions for A to map the countable chain complete lattice $(G_\mathrm{m}, \leq)$ into itself. Denote by G_A the set of fixed points of A in $(G_\mathrm{m}, \leq)$.

[11] Naturally, if shocks are independently distributed, an increase in z generates only a contemporaneous positive wealth effect to which an agent respond by increasing savings (see Erikson et al., 2005). No additional assumptions on shocks are then needed.

[12] A careful reading of Assumption 4 reveals that it provides sufficient conditions for the objective in the household's decision problem to have increasing differences between investment and the aggregate technology shock z for each k. Since the objective is trivially supermodular in investment (after substituting binding constraints), Topkis (1998) monotonicity theorem (Theorem 2.8.2) applies.

Assumption 4.

(a) Utility is separable, that is $U(c_1, c_2) = u(c_1) + v(c_2)$; u and v are twice differentiable, and $v''(c_2)c_2/v'(c_2) \geq 0$ for all $c_2 > 0$.
(b) Shocks are multiplicative, that is the production function is such that $f(k, K, z) = zf(k, K)$.

Proposition 7. *Under* Assumptions 1, 2, 2′, 3 and 4, *the nonempty set of fixed points* G_A *has a maximal element computed as* $\wedge_{G_m}\{A^n w\}_{n\in\mathbb{N}}$ with $\wedge_{G_m}\{A^n w\}_{n\in\mathbb{N}}(s) = \lim_{n\to\infty}\{A^n w\}_{n\in\mathbb{N}}(s)$.

Proof. We prove that A is a self map on $(G_m, \leq)$. Under Assumption 4 (E′) becomes:

$$u'(w(s) - y) = \int b(z, y, z')Q(z, dz'), \tag{E''}$$

in which we write $b(z, y, z') = v'(r(h(s), z')y)r(y, z')$ for simplicity. The function b is such that (i) $b(z, y, z')$ is increasing in z' since:

$$\begin{aligned}\frac{\partial b}{\partial z'} &= v''(c_2)yr(y, z')\frac{\partial r(h(s), z')}{\partial z'} + v'(c_2)\frac{\partial r(y, z')}{\partial z'}\\ &= \left[v''(c_2)c_2\left(\frac{r(y, z')}{r(h(s), z')}\right)\left(\frac{\partial r(h(k, z), z')/\partial z'}{\partial r(y, z')/\partial z'}\right) + v'(c_2)\right]\frac{\partial r(y, z')}{\partial z'}\\ &= [v''(c_2)c_2 + v'(c_2)]\frac{\partial r(y, z')}{\partial z'} \geq 0,\end{aligned}$$

by Assumption 4, and (ii) $b(z, y, z')$ is increasing in z whenever $h \in G_m$.

Consider then any $y \in [0, w(k, z_2)]$, and any $z_1 \geq z_2$. We have

$$\int b(z_2, y, z')Q(z_2, dz') \leq \int b(z_1, y, z')Q(z_2, dz') \leq \int b(z_1, y, z')Q(z_1, dz'),$$

where the first inequality results from (i) and the second from (ii) and the property that Q is a stochastically increasing transition function (see for example, Topkis, 1998, Corollary 3.9.1). This establishes that the right side of (E″) is increasing in z, and since the left side is clearly decreasing in z, for all $h \in G_m$, $z_1 \geq z_2$ implies that $Ah(k, z_1) \geq Ah(k, z_2)$; thus A maps G_m into itself. The remaining of the proof follows that of the previous proposition since $\wedge_{G_m}\{A^n w\}_{n\in\mathbb{N}} = \wedge_{H_m}\{A^n w\}_{n\in\mathbb{N}}$. □

Our next assumption concerns the limit behavior of the rental rate of capital when the capital stock goes to 0. We show that it is sufficient to establish the existence of a nontrivial minimal fixed point of A (which is then by definition the minimal MEDP in $(G_m, \leq)$).

Assumption 5. $\lim_{x\to 0^+} xr(x, z_{max}) = 0$.

Lemma 8. *Under* Assumption 1, 2, 2′, 3, 4 and 5, *there exists* $h_0 \in (L_m, \leq) \cap (G_m, \leq)$ *such that*:

$$\forall s \in S^*, Ah_0(s) \geq h_0(s) > 0,$$

and:

$$\forall s \in S^*, 0 < x < h_0(s) \Longrightarrow Ax > x,$$

where Ax solves

$$\int_Z U_1(w(s) - Ax, r(x, z')Ax)Q(z, \mathrm{d}z') = \int_Z U_2(w(s) - Ax, r(x, z')Ax)r(Ax, z')Q(z, \mathrm{d}z').$$

Proof. See Appendix A. □

We can now state an important result concerning the construction of the extremal MEDP in the set G_m (i.e., MEDP that are isotone in (k, z) and $\mathcal{B}(S)$-measurable).

Proposition 9. *Under* Assumptions 1, 2, 2′, 3, 4, *and* 5, *the set of MEDP in* $(G_\mathrm{m}, \leq)$ *is nonempty, with a maximal MEDP given by* $\wedge_{G_\mathrm{m}}\{A^n w\}_{n\in\mathbb{N}}$ *(constructed as* $\wedge_{G_\mathrm{m}}\{A^n w\}_{n\in\mathbb{N}}(s) = \lim_{n\to\infty} A^n w(s)$ *for all* $s \in S$*) and minimal MEDP given by* $\vee_{G_\mathrm{m}}\{A^n h_0\}_{n\in\mathbb{N}}$ *(constructed as* $\vee_{G_\mathrm{m}}\{A^n h_0(s)\}_{n\in\mathbb{N}} = \lim_{n\to\infty} A^n h_0(s)$ *for all* $s \in S$*).*

Proof. By the lemma immediately above, the sequence $\{A^n h_0\}_{n\in N}$ is increasing. Since $h_0 \leq w$, by isotonicity of A, $0 < \vee_{G_\mathrm{m}}\{A^n h_0\}_{n\in N} \leq \wedge_{G_\mathrm{m}}\{A^n w\}_{n\in\mathbb{N}} < w$, and by an argument similar to that of Proposition 6 exploiting the continuity of U_1, U_2, and r, $\vee_{G_\mathrm{m}}\{A^n h_0\}_{n\in N}$ is a fixed point of A. It is easy to see that $\vee_{G_\mathrm{m}}\{A^n h_0\}_{n\in\mathbb{N}}$ is the minimal fixed point of A in the order interval $([h_0, w], \leq) \subset (L_\mathrm{m}, \leq) \subset (G_\mathrm{m}, \leq)$. It is thus strictly positive. By the lemma above $Ax > x$ for all $0 < x < h_0(s)$, and A thus cannot have a fixed point in $[0, h_0]$ other than 0. As a result, $\vee_{G_\mathrm{m}}\{A^n h_0\}_{n\in\mathbb{N}}$ is the minimal nonzero fixed point of A in $(G_\mathrm{m}, \leq)$, and therefore the minimal MEDP. As shown in the previous proposition $\wedge_{G_\mathrm{m}}\{A^n w\}_{n\in\mathbb{N}}$ is the maximal fixed point of A, and therefore the maximal MEDP. □

3.5. Computing semicontinuous and continuous MEDP

We now show that the operator A transforms a countable chain complete poset of semicontinuous (in k) functions $(U_\mathrm{m}, \leq)$ (respectively, $(L_\mathrm{m}, \leq)$). Following an argument similar to that of Proposition 6 permits us to establish the existence of a nonempty set of MEDP in $(U_\mathrm{m}, \leq)$ (respectively, $(L_\mathrm{m}, \leq)$), with maximal and minimal elements obtained by successive approximations from upper and lower solutions w and h_0, respectively.

Lemma 10. *Under* Assumptions 1, 2, 2′, 3, *and* 4, *A is a self map on* $(U_\mathrm{m}, \leq)$ *and on* $(L_\mathrm{m}, \leq)$.

Proof. We prove that A is a self map on $(U_\mathrm{m}, \leq)$ (the case of $(L_\mathrm{m}, \leq)$ is symmetric). Recall that $Ah(k, z)$ is obtained as the unique solution y^* of (E′) and can be expressed as a continuous function of h and w, say $Ah(k, z) = G(h(k, z), w(k, z))$ with G continuous in w and h. Since h is increasing and upper semicontinuous in k, it is right continuous at very $k \in [0, x_{\max}[$. The continuity of G in its argument then implies that $Ah(k, z)$ is right continuous at every k, and therefore also upper semicontinuous in k since increasing. □

Proposition 11. *Under* Assumptions 1, 2, 2′, 3, 4, *and* 5, $\wedge_{G_\mathrm{m}}\{A^n w\}_{n\in\mathbb{N}}$ *is the maximum MEDP in* $(U_\mathrm{m}, \leq)$ *and* $\vee_{G_\mathrm{m}}\{A^n h_0\}_{n\in\mathbb{N}}$ *is the minimum MEDP in* $(L_\mathrm{m}, \leq)$. *As a result, if the MEDP is unique, it is necessarily continuous in k.*

Proof. By the previous lemma A is a self map on $(U_\mathrm{m}, \leq)$; A is increasing on $(H_\mathrm{m}, \leq)$, and therefore increasing on $(U_\mathrm{m}, \leq)$ (respectively, $(L_\mathrm{m}, \leq)$). Further since $w \in (U_\mathrm{m}, \leq)$ and A is a self map on $(U_\mathrm{m}, \leq)$, $A^n w \in (U_\mathrm{m}, \leq)$ for all $n \in \mathbb{N}$. Thus, $\wedge_{G_\mathrm{m}}\{A^n w\}_{n\in\mathbb{N}}$ is the lower envelope of a family of usc functions, and is therefore usc in k on $[0, k_{\max})$. Since $h_0 \in (L_\mathrm{m}, \leq)$ and A is a self map on $(L_\mathrm{m}, \leq)$, $A^n h_0 \in (L_\mathrm{m}, \leq)$ for all $n \in \mathbb{N}$. Thus, $\vee_{G_\mathrm{m}}\{A^n h_0\}_{n\in\mathbb{N}}$ is the upper envelope of a family of lsc functions, and is therefore lsc in k on $]0, k_{\max}]$. □

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Finally, we note that the maximum MEDP in $(L_{\mathrm{m}}, \leq)$ (minimum MEDP in $(U_{\mathrm{m}}, \leq)$) can easily be constructed by altering the maximum MEDP in $(U_{\mathrm{m}}, \leq)$ (respectively, minimum MEDP in $(L_{\mathrm{m}}, \leq)$) at its discontinuity points.

4. Stationary Markov equilibria

In this section, we follow the work of Grandmont and Hildenbrand (1974), Futia (1982) and Hopenhayn and Prescott (1992) and define a stationary Markovian equilibrium (SME) to be a nondegenerate invariant distribution. This, of course, is a weaker requirement than often used in the literature (i.e., SME are required to be nondegenerate ergodic measures.[13]) We initially postulate the existence of a MEDP h in $(W, \leq)$ (i.e., a MEDP that is assumed to be $\mathcal{B}(S)$-measurable),[14] and only require the transition function Q to be quasi-compact. Under these two conditions, we demonstrate that there always exists a SME associated with h. This existence result, however, offers no information concerning how to compute a SME. Most importantly, the result offers no guarantee that a candidate SME is not *trivial* (i.e., all the probability mass is concentrated at stationary capital stock equal to 0 for economies starting with an initial capital stock $k_0 \in (0, k_{\max}]$).

In light of this possibility, we prove two additional results. First, we show that under the combined conditions of an isotone MEDP h (i.e., $h \in (G_{\mathrm{m}}, \leq)$) and a quasi-compact and stochastically increasing transition Q, the minimal and maximal SME associated with h can be constructed through successive approximations from upper and lower solutions. This is a new result, since existing computational results rely on a very different and generally *more restrictive* set of assumptions on h or Q (or both). Second, we present sufficient conditions for the existence of a nontrivial SME associated with the minimal MEDP. In doing this, we also give sufficient conditions under which the minimal nontrivial SME associated with the minimal MEDP can be constructed through successive approximations on the equilibrium Markov operator.

4.1. Spaces for candidate SME

A stationary Markov equilibria will be defined below as a probability measure satisfying certain properties and obtained as a fixed point of a particular operator. Before turning to this discussion, we introduce the following new notations. Denote $\mathbf{B}(S, \mathcal{B}(S))$ the space of bounded and $\mathcal{B}(S)$-measurable real valued functions, and $\mathbf{C}(S, \mathcal{B}(S))$ the space of bounded continuous real valued functions.[15] We use the standard inner product notation:

$$\langle f, \mu \rangle = \int_S f(s)\mu(\mathrm{d}s), \qquad f \in \mathbf{B}(S, \mathcal{B}(S)) \text{ and } \mu \in \Lambda(S, \mathcal{B}(S)).$$

We denote by $\Lambda(S, \mathcal{B}(S))$ the space of probability measures defined on the measurable space $(S, \mathcal{B}(S))$, which we first endow with the stochastic order $\geq_s$ defined as follows: $\mu \geq_s \mu'$ if and only if:

$$\langle f, \mu \rangle \geq \langle f, \mu' \rangle, \qquad \text{for all increasing } f \in \mathbf{B}(S, \mathcal{B}(S)).$$

[13] See Duffie et al. (1994) and both of papers by Wang (1993, 1994) for this definition of SME.

[14] Since we proved in the previous section that there exists a MEDP in the space $(H_{\mathrm{m}}, \leq) \subset (W, \leq)$, this assumption requires no additional conditions beyond the ones used in the previous section.

[15] Given the definition of S, every bounded continuous function $f : S \to \mathcal{R}$ is $\mathcal{B}(S)$-measurable.

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Note that since S is compact, $\mu \geq_s \mu'$ if and only if:

$$\langle f, \mu \rangle \geq \langle f, \mu' \rangle, \quad \text{for all increasing } f \in \mathbf{C}(S, \mathcal{B}(S)).$$

Next, we also endow $\Lambda(S, \mathcal{B}(S))$ with the weak topology for which a sequence of probability measures $\{\mu_n\}_{n\in\mathbb{N}}$ in $\Lambda(S, \mathcal{B}(S))$ is said to *weakly converge* to $\mu \in \Lambda(S, \mathcal{B}(S))$ if for all $f \in \mathbf{C}(S, \mathcal{B}(S))$:

$$\lim_{n\to\infty} \langle f, \mu_n \rangle = \langle f, \mu \rangle, \tag{CV}$$

in which case we write $\mu_n \Longrightarrow \mu$ and call μ the weak limit of the sequence $\{\mu_n\}_{n\in\mathbb{N}}$. Finally, a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ in $\Lambda(S, B(S))$ is said to converge in the total variation norm to $\mu \in \Lambda(S, B(S))$ if condition (CV) above holds for all $f \in B(S, B(S))$, and the convergence is uniform for all functions with $\|f\| \leq 1$. Clearly, convergence in the total variation norm implies weak convergence (for instance, see Stokey et al., 1989 for the various definitions of convergence in this context).

We now demonstrate an important result concerning the order structure [16] of $(\Lambda(S, \mathcal{B}(S)), \geq_s)$ as well as the properties of monotone sequences in $(\Lambda(S, \mathcal{B}(S)), \geq_s)$.[17]

Lemma 12. $(\Lambda(S, \mathcal{B}(S)), \geq_s)$ *is a countable chain complete poset with minimal and maximal elements. In addition, for any increasing* (*decreasing*) *sequence* $\{\mu_n\}_{n\in\mathbb{N}}$:

$$\mu_n \Longrightarrow \vee\{\mu_n\}_{n\in\mathbb{N}} \quad (\text{respectively, } \mu_n \Longrightarrow \wedge\{\mu_n\}_{n\in\mathbb{N}}).$$

Proof. Clearly $(\Lambda(S, \mathcal{B}(S)), \geq_s)$ is a poset with minimal and maximal elements (the singular measures $\delta_{(0,z_{\min})}$ and $\delta_{(x_{\max},z_{\max})}$, respectively). Since all monotone sequences of $([0, k_{\max}] \times [z_{\min}, z_{\max}], \geq)$ (and therefore order bounded) converge, by Lemma 1 in Heikkilä and Salonen (1996) all monotone sequences $\{\mu_n\}_{n\in\mathbb{N}}$ in $(\Lambda(S, \mathcal{B}(S)), \geq_s)$ (and thus order bounded) weakly converge.[18] For a monotone increasing sequence, it is very easy to verify that this weak limit is precisely the lowest upper bound (and the greatest lower bound for a monotone decreasing sequence). Therefore, $(\Lambda(S, \mathcal{B}(S)), \geq_s)$ is a countable chain complete poset with minimal and maximal elements.

4.2. Existence of SME

Given a $\mathcal{B}(S)$-measurable function h in $(W, \leq)$ and with no restrictions on Q, it is well known that the state vector $s = (x, z) \in S$ follows a first-order Markov process with transition function P_h defined by:

$$\forall A \times B \in \mathcal{B}(S),\ P_h(x, z; A, B) = \begin{cases} Q(z, B), & \text{if } h(x, z) \in A \\ 0, & \text{otherwise.} \end{cases}$$

[16] Following a different argument, Hopenhayn and Prescott (1992) prove that $(\Lambda(S, \mathcal{B}(S)), \geq_s)$ is a chain complete lattice with minimal and maximal elements.

[17] Erikson et al. (2005) consider the space of probability measures on a compact subset S of the real line, which is shown to be a complete latice when endowed with the partial order of first-order stochastic dominance. When $S \subset \mathbb{R}^n$, this is no longer the case (see Morand, 2006 for further discussion of these issues).

[18] This result is also a consequence of Helly's Theorem (see, for instance, Corollary 1 to Theorem 12.9 in Stokey et al. (1989)).

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(see, for instance, Stokey et al., 1989). Equivalently we have the following Markov operator[19]:

$$P_h(x, z; A, B) = \int_B I_A(h(x, z))Q(z, \mathrm{d}z'),$$

where $I_A(h(x, z))$ is the indicator function of A (i.e., $I_A(h(x, z)) = 1$ if and only if $h(x, z) \in A$, and 0 otherwise).

In the notations of Duffie et al. (1994) (S, P_h) is a time homogenous Markov equilibrium, more precisely a nonsunspot THME (see Wang, 1994, Definition 3.4). Associated with the transition function P_h are the operators $T_h : \mathbf{M}(S, \mathcal{B}(S)) \to \mathbf{M}(S, \mathcal{B}(S))$ defined by:

$$\forall f \in \mathbf{M}(S, \mathcal{B}(S)),\ Tf(s) = \int_S f(s')P_h(s, \mathrm{d}s'),$$

and $T_h^* : \Lambda(S, \mathcal{B}(S)) \to \Lambda(S, \mathcal{B}(S))$ defined by:

$$\forall D \in \mathcal{B}(S),\ \mu_{t+1}(D) = T_h^*\mu_t(D) = \int_S P_h(s; D)\mu_t(\mathrm{d}s).$$

The quantity $T_h^*\mu_t(D)$ is thus the probability that the next period value of the state vector lies in the set D if the current period state vector is drawn according to the probability measure μ_t, and if all agents follow the optimal decision rule h. This leads us to define a *pure stationary Markov equilibrium*[20] (in short, pure SME) associated with the MEDP h as follows:

Definition 13. A pure SME associated with the MEDP h is a probability measure $\mu \in \Lambda(S, \mathcal{B}(S))$ such that:

$$\text{For all } D \in \mathcal{B}(S),\ \mu(D) = T_h^*\mu(D) = \int_S P_h(s; D)\mu(\mathrm{d}s).$$

In light of this definition, the question of the existence, uniqueness, and computation of SME is identical to that of obtaining the fixed points of T_h^* in $\Lambda(S, \mathcal{B}(S))$.[21]

We first remind the reader of the two classes of existing results concerning the existence of SME obtained as fixed points of the operator T_h^* in the theorem immediately below. Recall that the equilibrium Markov operator P_h has the Feller property if, $T(\mathbf{C}(S, \mathcal{B}(S))) \subset \mathbf{C}(S, B(S))$. This is equivalent to T_h^* being weakly continuous, that is, for all sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ in $\Lambda(S, \mathcal{B}(S))$:

$$\lambda_n \Longrightarrow \lambda \text{ implies } T_h^*\lambda_n \Longrightarrow T_h^*\lambda.$$

where convergence here is weak convergence.

Theorem 14.

(i) *If* $h \in (G_{\mathrm{m}}, \leq)$ *and* Q *satisfies* Assumption 3, *then there exists a SME associated with* h.
(ii) *If* P_h *has the Feller property, then there exists a SME associated with* h.

Proof. (i) Both h and Q increasing imply that T_h^* is an increasing self map on $(\Lambda(S, \mathcal{B}(S)), \leq)$ (and S has the minimal element $(0, z_{\min})$), and the result follows from Corollary 2 in Hopenhayn and Prescott (1992). (ii) See Stokey et al. (1989), Theorem 12.10. □

[19] When h is a MEDP, we will refer to this operator as the equilibrium Markov operator.
[20] Since this SME is associated with a known law of motion (the MEDP h), we may call it a "pure" SME, in contrast to Duffie et al. (1994) and Wang (1994) where there is no such information.
[21] Note that in Wang (1994) terminology, (S, P_h, μ) is a SME.

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As shown in the theorem above, current arguments establishing the existence of a SME through a fixed-point approach require either (i) the isotonicity of both Q and h, or (ii) P_h satisfying the Feller property and the MEDP h being continuous. An important contribution of our work is to demonstrate the new result that the $\mathcal{B}(S)$-measurability of the MEDP h is sufficient for the existence of a SME without isotonicity or continuity properties of h, but as long as Q is quasi-compact.

Theorem 15. *If Q satisfies* Assumption 3′, *then for any $\mathcal{B}(S)$-measurable function h in W there exists a SME associated with h.*

Proof. Note that by Assumption 3′, P_h necessarily satisfies Doeblin's condition (D). That is, given any arbitrary $\rho \in \Lambda(X, \mathcal{B}(X))$, consider the probability measure $\gamma' = \rho \otimes \gamma \in \Lambda(S, \mathcal{B}(S))$, where γ is the probability measure in Assumption 3′. By definition $P_h(x, z; A \times B) \leq Q(z, B)$, and Assumption 3′, this then implies that there exists $\varepsilon > 0$ such that:

$$\forall A \times B \in \mathcal{B}(S),\ \gamma'(A \times B) \leq \gamma(B) \leq \varepsilon \text{ implies that}$$

$$\forall z \in Z,\ P_h(x, z; A \times B) \leq Q(z, B) \leq 1 - \varepsilon,$$

which is precisely condition (D) for P_h. By Theorem 11.9 Stokey et al. (1989), if P_h satisfies Doeblin's condition, then for any $\mu_0 \in \Lambda(S, \mathcal{B}(S))$, the sequence $\{\lambda_n = (1/n)\sum_{i=0}^{n-1} T_h^{*i}\mu_0\}_{n\in\mathbb{N}}$ converges in the total variation norm and its limit is an invariant of T_h^*. It is therefore a SME associated with h. □

4.3. Computation of extremal SME

We turn now to the problem of computing the extremal SME through successive approximations, for which we first state the existing results in a theorem below. This theorem shows that when the equilibrium Markov operator P_h cannot be shown to have the Feller property, not much can be said about the construction of extremal SME unless P_h satisfies a monotone mixing condition. This is a serious concern since the Feller property of P_h in practice generally relies on the continuity of the MEDP h (as well as the Feller property for Q), a condition on the MEDP generally difficult to obtain in many problems including the stochastic OLG economies of this paper.[22]

We address this problem by showing that under the assumption of isotonicity of h and assuming Q is stochastically increasing, the quasi-compactness of Q (a weaker requirement than that of the Feller property) is sufficient to permit the construction of extremal SME by successive approximations on the equilibrium Markov operator. Before establishing this result, we first present the existing computational results in the theorem below.

Theorem 16.

(i) *If $h \in (G_m, \leq)$ and Q satisfies* Assumption 3, *and P_h has the Feller property, then the sequences $\{T_h^{*n}\delta_{(k_{\max}, z_{\max})}\}_{n\in\mathbb{N}}$ converges weakly to the maximal SME associated with h.*
(ii) *If $h \in (G_m, \leq)$ and P_h satisfies the monotone mixing condition, then for all μ in $\Lambda(S, \mathcal{B}(S))$, the sequence $\{T_h^{*n}\mu\}_{n\in\mathbb{N}}$ converges weakly to the unique SME associated with h.*

[22] There are results giving sufficient conditions for the existence of a continuous selection out of a usc correspondence, but these are of very little use for computational purposes.

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Proof. (i) Since $\delta_{(x_{\max},z_{\max})}$ (the probability measure associating mass one to the maximal point of S) is the maximal element of $\Lambda(S,\mathcal{B}(S))$, then, necessarily:

$$T_h^*\delta_{(k_{\max},z_{\max})} \leq_s \delta_{(k_{\max},z_{\max})}.$$

If T_h^* is increasing, then the sequence $\{T_h^{*n}\delta_{(k_{\max},z_{\max})}\}_{n\in\mathbb{N}}$ is decreasing, and therefore weakly convergent (to its infimum, which we denote μ). The weak continuity (i.e., the Feller property) of T_h^* then implies that $\mu = T_h^*\mu$. Next, consider any fixed point λ of T_h^*. Since T_h^* is increasing:

$$\lambda \leq \delta_{(k_{\max},z_{\max})} \Rightarrow \lambda = T_h^{*n}\lambda \leq_s T_h^{*n}\delta_{b(k_{\max},z_{\max})}, \quad \text{for all } n\in\mathbb{N},$$

and λ is therefore a lower bound for $\{T_h^{*n}\delta_{(k_{\max},z_{\max})}\}_{n\in\mathbb{N}}$, which implies that $\lambda \leq_s \mu$. Symmetric arguments can be made for the increasing sequence $\{T_h^{*n}\delta_{(0,z_{\min})}\}_{n\in\mathbb{N}}$. (ii) See Hopenhayn and Prescott (1992), Theorem 2. □

The next result concerns the existence of SME without the topological assumption of the Feller property, and the construction of extremal SME under the additional isotonicity assumptions. This is an important result, and a significant addition to the constructive theorem above, since it implies that the continuity of h is irrelevant for the construction of extremal fixed points of T_h^*.

Proposition 17. (i) *For any h in $(H_{\mathrm{m}},\leq)$ and Q satisfying* Assumption 3′, T_h^* *has a fixed point*. (ii) *If $h\in(G_{\mathrm{m}},\leq)$ and Q satisfies* Assumptions 3 *and* 3′, *the sequences* $\{T_h^{*n}\delta_{(k_{\max},z_{\max})}\}_{n\in\mathbb{N}}$ *converges weakly to the maximal SME associated with the MEDP h.*

Proof. (i) Since P_h satisfies condition (D) by Theorem 11.9 in Stokey et al. (1989) the sequence $\{\lambda_n = (1/n)\sum_{i=0}^{n-1} T_h^{*i}\mu_0\}_{n\in\mathbb{N}}$ converges in the total variation norm (and therefore weakly converges), and its (weak) limit denoted μ is an invariant measure of T_h^*. Thus, $T_h^*\mu=\mu$. This implies that the weak limit of any increasing sequence $\{T_h^{*n}\mu_0\}_{n\in\mathbb{N}}$ in $\Lambda(S,\mathcal{B}(S))$ is a fixed point of T_h^*. (ii) Consider any SME μ associated with the MEDP h. Since $\mu \leq_s \delta_{(k_{\max},z_{\max})}$, by isotonicity of $T_h^*\mu \leq_s T_h^{*n}\delta_{(k_{\max},z_{\max})}$ for all n and therefore $\mu \leq_s \wedge_{\Lambda(S,\mathcal{B}(S))}\{T_h^{*n}\delta_{(k_{\max},z_{\max})}\}_{n\in\mathbb{N}}$. □

4.4. Existence of nontrivial SME

As noted before, one problem concerning the existence and computation of extremal SME is the possibility that the set of SME reduces to a single "trivial" probability measure, where by "trivial" we mean a probability measure for which all mass is concentrated at $k=0$. Indeed for a MEDP h satisfying $h(x,z)<x$ for all $(x,z)\in S^*$, any SME is necessarily trivial, and an obvious case when this happens is if when $w(x,z)<x$ for all (x,z) in S^*.

We therefore identify sufficient conditions to prevent obtaining a (unique) trivial SME, and state these conditions in terms of the primitives of our particular problem, unlike Galor and Ryder (1989) or Wang (1993, 1994). In particular, we show that under Assumption 6 below, there exists a minimal MEDP $h_{\min}$ and a nontrivial SME associated with this $h_{\min}$. Furthermore, under the assumption of isotonicity of Q, we show that there exists a nontrivial minimal SME associated with the minimal MEDP $h_{\min}$, and we give an algorithm converging monotonically to it.

We first make the following technical assumption:

Assumption 6.

I There exists a right neighborhood $\Delta\subset X$ of 0 such that for all $k\in\Delta w(k,z_{\min})\geq k$.

II The following inequality holds:

$$\lim_{k\to 0^+} \frac{U_2(w(k, z_{\min}) - k, r(k, z_{\max})k)r(k, z_{\min})}{U_1(w(k, z_{\min}) - k, r(k, z_{\max})k)} > 1$$

Note that for log time separable utility, Assumption 6(II) is equivalent to:

$$\lim_{k\to 0^+} \left(\frac{w(k, z_{\min})}{k}\right) > 2,$$

and is trivially satisfied for the standard Cobb–Douglass production function with multiplicative shocks. It is easy to verify that Assumption 6(I) is also satisfied for this case.

We now have the following important result, the first part of which does not rely on the stochastic isotonicity of Q. In the second part, it is the stochastic isotonicity of Q that leads to the possibility of constructing the minimal SME by successive approximations.

Proposition 18. *Under* Assumptions 1, 2, 2′, 3′ *and* 6, *there exist a nonzero minimal MEDP* $h_{\min}$ *in* $(G_{\mathrm{m}}, \leq)$ *and a nontrivial minimal SME associated with* $h_{\min}$. *Further, under* Assumptions 1, 2, 2′, 3′, 3 *and* 6, *there exists* $k_0 \in X^*$ *such that the minimal SME has support in the set* $E = [k_0, x_{\max}] \times Z$ *and can be obtained as* $\vee\{T^{*n}_{h_{\min}}\delta_{(k,z_{\min})}\}_{n\in\mathbb{N}}$ *for any* $k \in \Theta =]0, k_0[$.

Proof. The proof of the existence of $h_{\min}$ and of the associated nontrivial minimal SME is in Appendix B.

The state space $S = [0, k_{\max}] \times [z_{\min}, z_{\max}]$ can be partitioned in three disjoint sets with different ergodic properties. The three sets are $\{0\} \times Z$, $\Theta \times Z$ (the interval $\Theta = (0, k_0)$ is introduced in Appendix B), and $E = [k_0, k_{\max}] \times Z$.

(i) The set $\{0\} \times Z$ is obviously ergodic since $h(0, z) = 0$ for any MEDP h.

(ii) The set $\Theta \times Z$ is transient since given any MEDP h and associated P_h there is a positive probability of leaving that set and no probability of returning in it. Indeed, consider any $x_0 \in \Theta$. The sequence $\{x_n\}_{n\in\mathbb{N}}$ recursively defined by $x_{n+1} = h_{\min}(x_n, z_{\min})$ for $n \in \mathbb{N}$, is strictly increasing, bounded above by $x_{\max}$, and therefore convergent. If x is its limit, then the lower semicontinuity of $h_{\min}(., z_{\min})$ implies that $h_{\min}(x, z_{\min}) = x$, so that $x > k_0$ necessarily since $h_{\min}(k, z_{\min}) > g_0(k, z_{\min}) > k$ for $k \in \Theta$. This (together with the isotonicity of $h_{\min}$ in z) implies that for any $s \in \Theta \times Z$ there exists $n \in \mathbb{N}$ such that $P^n_{h_{\min}}(s; \Theta \times Z) = 0$, which implies that $P^n_h(s; \Theta \times \times Z) = 0$ for any MEDP h since $h \geq h_{\min}$. Also, for all $(x, z) \in E h_{\min}(x, z) > g_0(x, z) = k_0$, which implies that for all MEDP h:

$$P_h(s; \Theta \times Z) = 0, \quad \text{for all } s \in E,$$

(and $P_h(0, z; [k_0, k_{\max}] \times Z) = 0$ as well).

(iii) The set $E = [k_0, k_{\max}] \times Z$ is invariant since for all $s \in E$, $h(s) \in [k_0, k_{\max}]$ and therefore:

$$P_h(s; E) = 1.$$

Next, for any given $k \in \Theta$, $\vee\{T^{*n}_{h_{\min}}\delta_{(k,z_{\min})}\}_{n\in\mathbb{N}}$ is a fixed point of $T^*_{h_{\min}}$ whose support belongs to E since $\Theta \times Z$ is transient and $\{0\} \times Z$ is ergodic. Consider any $k' \in \Theta$. Clearly, $\delta_{(k',z_{\min})} \leq_s \vee\{T^{*n}_{h_{\min}}\delta_{(k,z_{\min})}\}_{n\in\mathbb{N}}$ and by isotonicity of $T^*_{h_{\min}}$:

$$\vee\{T^{*n}_{h_{\min}}\delta_{(k',z_{\min})}\}_{n\in\mathbb{N}} \leq_s \vee\{T^{*n}_{h_{\min}}\delta_{(k,z_{\min})}\}_{n\in\mathbb{N}}.$$

Also, $\delta_{(k,z_{\min})} \leq_s \vee\{T^{*n}_{h_{\min}}\delta_{(k',z_{\min})}\}_{n\in\mathbb{N}}$ and therefore:

$$\vee\{T^{*n}_{h_{\min}}\delta_{(k,z_{\min})}\}_{n\in\mathbb{N}} \leq_s \vee\{T^{*n}_{h_{\min}}\delta_{(k',z_{\min})}\}_{n\in\mathbb{N}},$$

which implies that $\vee\{T^{*n}_{h_{\min}}\delta_{(k,z_{\min})}\}_{n\in\mathbb{N}} = \vee\{T^{*n}_{h_{\min}}\delta_{(k',z_{\min})}\}_{n\in\mathbb{N}}$.

Finally, consider any other SME λ with support included in E. Since $\delta_{(k,z_{\min})} \leq_s \lambda$ for all $k \in \Theta$, by isotonicity of $T^*_{h_{\min}}$:

$$\vee\{T^{*n}_{h_{\min}}\delta_{(k,z_{\min})}\}_{n\in\mathbb{N}} \leq_s \lambda,$$

which prove that $\vee\{T^{*n}_{h_{\min}}\delta_{(k,z_{\min})}\}_{n\in\mathbb{N}}$ is the smallest SME with support included in E. □

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Appendix A

This appendix presents the proof of Lemma 8, that is, the proof of existence of a nonzero element in $(G_{\mathrm{m}}, \leq)$ mapped strictly up by the operator A.

First, fix $k \in X^*$. For any $z \in Z$ (recalling that $s = (k, z)$),

$$\lim_{x\to 0^+}\int_Z U_1(w(s)-x, r(x,z')x)Q(z,\mathrm{d}z') = \int U_1(w(s),0)Q(z,\mathrm{d}z') \leq U_1(w(k,z_{\min}),0).$$

Thus, there exists $\Psi = (0, \bar{x}]$, such that, for all $x \in \Psi$ and all $z \in Z$:

$$\int_Z U_1(w(s)-x, r(x,z')x)Q(z,\mathrm{d}z') < 2U_1(w(k,z_{\min}),0). \tag{E0}$$

Secondly, under Assumptions 6 and 1(III) the expression $U_2(w(k, z_{\min}) - x, r(x, z_{\max})x)$ can therefore be made arbitrarily large in a right neighborhood of 0. Thus, there exists $\Omega = (0, \bar{k}]$ with $0 < \bar{k} \leq w(k, z_{\min})$ and $M > 0$ such that, for all $x \in \Omega$,

$$U_2(w(k,z_{\min})-x, r(x,z_{\max})x) > M.$$

For any $x \in \Omega$:

$$\begin{aligned}
&\int_Z U_2(w(s)-x, r(x,z')x)r(x,z')Q(z,\mathrm{d}z')\\
&\quad\geq \int_Z U_2(w(k,z_{\min})-x, r(x,z_{\max})x)r(x,z')Q(z,\mathrm{d}z')\\
&\quad\geq \int_Z Mr(x,z')Q(z,\mathrm{d}z') \geq Mr(x,z_{\min})
\end{aligned} \tag{E1}$$

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where the first inequality stems from $U_{12} \geq 0$ and U_2 decreasing, and the second from above. This last expression can be made arbitrarily large, (independently of z) by choosing x in Ω sufficiently close to 0. That is, it is always possible to choose x (independently of z) sufficiently small in $\Omega \cap \Psi$ so that

$$Mr(x, z_{\min}) \geq 2U_1(w(k, z_{\min}), 0) \tag{E2}$$

Pick such an x and set $\delta_0(s) = x$ for all $z \in Z$. Combining (E0)–(E2), $x = \delta_0(s)$ necessarily satisfies, for all $z \in Z$, the following inequality:

$$\int_Z U_2(w(s) - x, r(x, z')x)r(x, z')Q(z, \mathrm{d}z') > \int_Z U_1(w(s) - x, r(x, z')x)Q(z, \mathrm{d}z') \tag{E3}$$

That is, by construction, we have:

$$\int_Z U_2(w(s) - \delta_0(s), r(\delta_0(s), z')\delta_0(k, z))r(\delta_0(s), z')Q(z, \mathrm{d}z')$$
$$> \int_Z U_1(w(s) - \delta_0(s), r(\delta_0(s), z')\delta_0(s))Q(z, \mathrm{d}z')$$

By repeating the same operation for each k in X^*, and setting $\delta_0(0, z) = 0$, we thus construct a function $\delta_0 : X \times Z \to X$ constant in z, and therefore increasing in z.

Note that if $k' \geq k$ then $\delta_0(s)$ necessarily satisfies:

$$\int_Z U_2(w(k', z) - \delta_0(s), r(\delta_0(s), z')\delta_0(s))r(\delta_0(s), z')Q(z, \mathrm{d}z')$$
$$> \int_Z U_1(w(k', z) - \delta_0(s), r(\delta_0(s), z')\delta_0(s))Q(z, \mathrm{d}z').$$

Consequently, the function $p_0 : X \times Z \to X$ defined as:

$$p_0(k, z) = \min_{k' \geq k}\{\delta_0(k', z)\}.$$

also satisfies (E3) for any $(k, z) \in X^* \times Z$, while being increasing in k for all z, and constant in z for all k (and thus continuous in z for all k). Finally, the function h_0 defined as follows:

$$h_0(s) = \begin{cases} \sup_{0<k'<k} p_0(k', z), & \text{for } k \in X^*, z \in Z \\ 0, & \text{for } k = 0, z \in Z \end{cases}$$

is smaller than p_0 (and therefore than δ_0, hence it satisfies (E3)), increasing in k for all z, constant in z for all k, and lower semicontinuous in k for any given $z \in Z$.

We now prove that $\forall s = (k, z) \in X^* \times Z$, $Ah_0(s) > h_0(s) > 0$. Since $h_0(s) > 0$ by construction, suppose then that there exists $k \in X^*$ and $z \in Z$ such that $Ah_0(s) \leq h_0(s)$. As a result:

$$\int_Z U_1(w(s) - h_0(s), r(h_0(s), z')h_0(s))Q(z, \mathrm{d}z')$$
$$< \int_Z U_2(w(s) - h_0(s), r(h_0(s), z')h_0(s))r(h_0(s), z')Q(z, \mathrm{d}z')$$
$$\leq \int_Z U_2(w(s) - Ah_0(s), r(h_0(s), z')Ah_0(s))r(Ah_0(s), z')Q(z, \mathrm{d}z')$$
$$= \int_Z U_1(w(s) - Ah_0(s), r(h_0(s), z')Ah_0(s))Q(z, \mathrm{d}z').$$

where the first inequality stems from (E3), the second from the assumptions on the primitives, and the equality follows from the definition of $Ah_0(s)$. Summarizing, we have:

$$\int_Z U_1(w(s) - h_0(s), r(h_0(s), z')h_0(s))Q(z, \mathrm{d}z') < \int_Z U_1(w(s) - Ah_0(s), r(h_0(s), z')Ah_0(s))Q(z, \mathrm{d}z').$$

which is contradicted by the hypothesis that $U_{11} \leq 0$ and $U_{12} \geq 0$. Thus, necessarily, $Ah_0(s) > h_0(s)$ and A maps h_0 strictly up.

Finally, from a remark above, recall that for a given $s = (k, z) \in X^* \times Z$, any x such that $0 < x < h_0(s) <$ necessarily satisfies:

$$\int_Z U_1(w(s) - x, r(x, z')x)Q(z, \mathrm{d}z') < \int_Z U_2(w(s) - x, r(x, z')x)r(x, z')Q(z, \mathrm{d}z'),$$

and it must therefore be the case that $Ax > x$. □

Appendix B

This appendix presents the proof of the first part of Proposition 18, thus demonstrating the existence of a nonzero minimal MEDP h_{m} and of an associated nontrival SME.

The proof is in two parts. Appendix B.1 establishes the existence of $g_0 : X \times Z \to X$, isotone and $\mathcal{B}(S)$-measurable that is mapped up by the operator A and Appendix B.2 shows the existence of a probability measure μ_0 that is mapped up T_h^*, where h is any fixed point of A in the interval $[g_0, w]$.

B.1

By continuity of all functions in k, the inequality in Assumption 6(2) implies that there exists of $\Theta = (0, k_0] \subset \Delta \subset X$ such that, $\forall k \in \Theta$:

$$U_1(w(k, z_{\min}) - k, r(k, z_{\max})k) < U_2(w(k, z_{\min}) - k, r(k, z_{\max})k)r(k, z_{\min}).$$

Consequently, $\forall k \in \Theta = [0, k_0]$:

$$\int_Z U_1(w(s) - k, r(k, z')k)Q(z, \mathrm{d}z') \leq U_1(w(k, z_{\min}) - k, r(k, z_{\max})k) < U_2(w(k, z_{\min}) - k, r(k, z_{\max})k)r(k, z_{\min}) \leq \int_Z U_2(w(s) - k, r(k, z')k)r(k, z')Q(z, \mathrm{d}z').$$

Next, consider $g_0 \in (G_{\mathrm{m}}, \leq)$ defined as:

$$g_0(k, z) = \begin{cases} 0, & \text{if } k = 0, z \in Z \\ k, & \text{if } 0 < k \leq k_0, z \in Z. \\ k_0, & \text{if } k \geq k_0, z \in Z \end{cases}$$

Clearly g_0 is isotone in (k, z), continuous in z for all k (since constant in z), and continuous in k for all z and therefore $\mathcal{B}(S)$-measurable (since it is a Caratheodory function). We show now that $Ag_0(s) > g_0(s)$ for all $s \in S^*$. Suppose first that there exists $0 < k \leq k_0$ and $z \in Z$ such that

$Ag_0(k,z) \le g_0(k,z) = k$. Then:

$$\int_Z U_1(w(s)-k, r(k,z')k)Q(z,\mathrm{d}z') < \int_Z U_2(w(s)-k, r(k,z')k)r(k,z')Q(z,\mathrm{d}z')$$
$$\le \int_Z U_2(w(s)-Ag_0(s), r(k,z')Ag_0(s))r(Ag_0(s),z')Q(z,\mathrm{d}z')$$
$$= \int_Z U_1(w(s)-Ag_0(s), r(k,z')Ag_0(s))Q(z,\mathrm{d}z').$$

where the first inequality stems from a result just above, the second inequality from $U_{22} \le 0$, $U_{12} \ge 0$ and r decreasing in its first argument, and the equality follows from the definition of Ag_0. Thus, we have:

$$\int_Z U_1(w(s)-k, r(k,z')k)Q(z,\mathrm{d}z') < \int_Z U_1(w(s)-Ag_0(s), r(k,z')Ag_0(s))Q(z,\mathrm{d}z').$$

which is contradicted by the hypothesis that $U_{11} \le 0$ and $U_{12} \ge 0$. It must therefore be the case that, for all $(k,z) \in (0,k_0] \times Z$, $Ag_0(k,z) > g_0(k,z) = k$. Also, for any $k > k_0$ and $z \in Z$:

$$Ag_0(k,z) \ge Ag_0(k_0,z) > g_0(k_0,z) = k_0 = g_0(k,z),$$

since Ag_0 is isotone. Thus, $Ag_0 > g_0$ on S^*, and since $[g_0, w]$ is an order interval of the countable chain complete lattice $(G_\mathrm{m}, \le)$, an argument similar to that of Proposition 6 in the paper establishes that there exists a nonempty set of fixed points of A in $([g_0,w],\le) \subset (G_\mathrm{m},\le)$, and that the minimal fixed point of A in $([g_0,w],\le) \subset (G_\mathrm{m},\le)$ is $\vee_{G_\mathrm{m}}\{A^n g_0\} = h_{\min}$. Note also that the increasing sequence $\{A^n g_0\}_{n\in\mathbb{N}}$ of functions in $(G_\mathrm{m},\le) \subset (H_\mathrm{m},\le)$ is such that:

$$\vee_{H_\mathrm{m},G_\mathrm{m}}\{A^n g_0\}_{n\in\mathbb{N}}(s) = \lim_{n\to\infty} A^n g_0(s).$$

Since g_0 is continuous in k for all z, $A^n g_0$ has the same property for all $n \in \mathbb{N}$, and $h_{\min}$ is therefore lsc in k for all z as the upper envelope of a family of continuous functions.

The proof that there cannot be a fixed point of A in $(G_\mathrm{m},\le)$ or in $(H_\mathrm{m},\le)$ that is smaller than $h_{\min}$ other than 0 follows essentially the same argument as the proof that $Ag_0(s) > g_0(s)$ for all $s \in S^*$. (Assuming that there exists $0 < y < g_0(k,z) = k$ with $Ay \le y$ leads to a contradiction). This implies that $h_{\min}$ is the minimum nonzero fixed point of A in $(G_\mathrm{m},\le)$ or in $(H_\mathrm{m},\le)$.

B.2

Consider $\mu_0 = \delta_{(k_0/2, z_{\min})}$ which concentrates all the mass at $(k_0/2, z_{\min})$. Since $h_{\min}(k_0/2, z_{\min}) > k_0/2$ the support of the measure $T^*_{h_{\min}}\mu_0$ (denoted by $\mathrm{SUP}T^*_{h_{\min}}\mu_0$) is included in $(k_0/2, x_{\max}] \times Z$, which implies that $T^*_{h_{\min}}\mu_0 \ge_s \mu_0$ (in fact $T^*_{h_{\min}}\mu_0 >_s \mu_0$). If T^*_h is isotone, then we know that $\vee\{T^{*n}_{h_{\min}}\mu_0\}_{n\in\mathbf{N}}$ is a SME. If T^*_h is not isotone, it is easy to prove recursively that:

$$\mathrm{SUP}T^{*n}_{h_{\min}}\mu_0 \subset \left(\frac{k_0}{2}, x_{\max}\right] \times Z,$$

using the property that $h(x,z) > k_0/2$ for all $(x,z) \in (k_0/2, x_{\max}] \times Z$. As a result, $T^{*n}_{h_{\min}}\mu_0 \ge_s \mu_0$ for all $n \in \mathbb{N}$, and $\lambda_n = (1/n)\sum_{i=0}^{n-1} T^{*i}_h \mu_0 \ge_s \mu_0$ Recall that the weak limit μ of the sequence $\{\lambda_n = (1/n)\sum_{i=0}^{n-1} T^{*i}_h \mu_0\}_{n\in\mathbb{N}}$ is a SME so that $\mu \ge \mu_0$ since $\lambda_n \ge_s \mu_0$ for all n.

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