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Fixed point theorems and their applications to theory of Nash equilibria

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Abstract

In this paper we prove fixed point theorems for set-valued mappings in products of posets. Applications to the theory of Nash equilibria are presented.

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1. Introduction

In this paper we apply fixed point results derived in [7] by a generalized iteration method to prove fixed point theorems for set-valued mappings in partially ordered sets (posets) and in their products. The main purpose is to develop tools for the study of Nash equilibria of the following multi-player game (cf. [15, Chapter 10]).

Let I be a set of players and T be a set of exogenous parameters reflecting the environment in which the players operate. Denote by X_i the *strategy set* of player i. Given a strategy $x = \{x_i\}_{i\in I} \in X = \prod_{i\in I} X_i$ of the players and $i \in I$, denote $x = (x_{-i}, x_i)$, where $x_{-i} = \{x_j\}_{j\in I\setminus\{i\}}$ is the strategy of other players. For all fixed $x \in X$ and $t \in T$, let a subset $X_i^{x,t}$

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of X_i denote the set of player i's *feasible replies* to (x, t), and let an element $u_i^t(x)$ of a poset $W_i = (W_i, \preccurlyeq_i)$ denote a *utility* to player $i \in I$. We say that a feasible reply y_i of player i to $(x, t) \in X \times T$ is *optimal* if y_i maximizes the utility $u_i^t(y)$ over $y_i \in X_i^{x,t}$ when $y_{-i} = x_{-i}$. A strategy $x = \{x_i\}_{i \in I}$ is called a *Nash equilibrium* for $t \in T$ if for each $i \in I$ the ith component x_i of x is an optimal reply of player i to (x, t).

Denoting by $F_i^t(x)$ the set of all optimal replies of player i to $(x, t) \in X \times T$, the above definitions imply that each Nash equilibrium for $t \in T$ is a fixed point of the set-valued mapping $F^t = \{F_i^t\}_{i \in I}$ whose components have values $F_i^t(x)$. Each fixed point of F^t is in turn a fixed point of a single-valued selection mapping $S^t = \{S_i^t\}_{i \in I} : X \to X$ with $S_i^t(x) \in F_i^t(x)$, $i \in I$, $x \in X$.

The above consideration yields the following plan of the paper. In Section 2 we provide basic abstract fixed point results for single-valued self-mappings of posets, proved in [7] by a generalized iteration method, and apply them to derive fixed point results for set-valued mappings F^t on posets, by assuming the existence of different kinds of selections for F^t. The results of Section 2 are then applied in Sections 3 and 4 to the case when X is a product of nonempty posets. Examples are given to show that the obtained results are not consequences of fixed point theorems in chain complete posets or in complete lattices (cf., e.g., [1–3,9,11–13,15,17]. In Section 5 we consider cases where fixed points are computable.

The fixed point results of Sections 3 and 4 will be so formulated that to every one of them there corresponds an existence result for Nash equilibria of the above defined multi-player game when the values $F_i^t(x)$ of each component of F^t are sets of all optimal replies of player i to $(x, t) \in X \times T$, provided that they are nonempty. In Section 6 we present conditions for the sets $X_i^{x,t}$ of feasible replies and for the utility mappings u_i^t which are sufficient for the existence of Nash equilibria. For instance, we prove that the greatest Nash equilibrium $\overline{x}^t$ exists and maximizes each utility $u_i^t(x)$ over all Nash equilibria for every $t \in T$ if each X_i is an order-bounded and order-closed subset of an ordered normed space E_i with regular order cone, if each $X_i^{x,t}$ is closed and directed upwards, if each mapping $x \mapsto X_i^{t,x}$ is increasing upwards, and if each utility mapping u_i^t is increasing.

Compared, e.g., to [10,14–16], the framework is more general in the sense that the strategy posets X_i's need not be complete lattices, not even lattices, and that the utility mappings u_i^t are poset-valued. These extensions provide new possibilities for the study of optimal equilibrium policies of various enterprises, for instance in economics. In our last example we apply an obtained theoretical result to a price game model presented in [15]. This example contains a numerical part, where an iteration method presented in Section 5 is applied.

2. Preliminaries

In this section we present basic fixed point results for single-valued mappings in posets, and apply them to derive fixed point results for set-valued mappings by reducing them to single-valued cases with the help of selection mappings.

A subset W of a poset Y is called *well-ordered* if each nonempty subset of W has the least element. W is said to be *inversely well-ordered* if each nonempty subset of W has the greatest element. In particular, each well-ordered or inversely well-ordered subset of Y is a chain. Given posets X and Y, we say that a mapping $G : Y \to X$ is *increasing* in a subset W of Y if $G(x) \leqslant G(y)$ in X whenever $x, y \in W$ and $x \leqslant y$ in Y.

In what follows we assume that $X = (X, \leqslant)$ is a poset. If $a, b \in X$ and $a \leqslant b$, denote

$$[a) := \{x \in X \mid a \leqslant x\}, \quad (b] := \{x \in X \mid x \leqslant b\} \quad \text{and} \quad [a, b] := [a) \cap (b].$$

The following lemma is a consequence of [7, Theorems 1.1.1 and 1.2.1].

Lemma 2.1. *Given $G : X \to X$ and $a \in X$, there is a unique well-ordered chain C in X, called a well-ordered (w.o.) chain of G-iterations of a, satisfying*

$$a = \min C \text{ and if } a < x \in X \text{ then } x \in C \text{ iff } x = \sup G[\{y \in C \mid y < x\}]. \tag{2.1}$$

If $a \leqslant G(a)$, if G is increasing in $[a)$, and if $x_* = \sup G[C]$ exists, then x_* is the least fixed point of G in $[a)$. Moreover,

$$x_* = \max C = \min\{y \in [a) \mid G(y) \leqslant y\}. \tag{2.2}$$

For the sake of completeness we state the dual result to Lemma 2.1.

Lemma 2.2. *If $G : X \to X$ and $b \in X$, there exists exactly one inversely well-ordered chain D, called an i.w.o. chain of G-iterations of b, such that*

$$b = \max D \text{ and if } X \ni x < b \text{ then } x \in D \text{ iff } x = \inf G[\{y \in D \mid x < y\}]. \tag{2.3}$$

If $G(b) \leqslant b$, if G is increasing in $(b]$, and if $x^* = \inf G[D]$ exists, then x^* is the greatest fixed point of G in $(b]$. Moreover,

$$x^* = \min D = \max\{y \in (b] \mid y \leqslant G(y)\}. \tag{2.4}$$

Remark 2.1. It follows from [7, Lemma 1.1.3], that the least elements of the w.o. chain C of G-iterations of a are the elements of the iteration sequences $(G^n(a_i))_{n=0}^{\infty}$ with $a_0 = a$ and $a_{i+1} = \sup_n G^n(a_i)$, $i = 0, 1, \ldots,$ as long as these sequences are strictly increasing. In particular, if in Lemma 2.1 $G^n(a_i) = G^{n+1}(a_i)$ for some $n, i \in \mathbb{N}$, then $x_* = G^n(a_i)$ is the least fixed point of G in $[a)$. Similarly, the greatest elements of the i.w.o. chain D of G-iterations of b are the elements of the iteration sequences $(G^n(b_i))_{n=0}^{\infty}$ with $b_0 = b$ and $b_{i+1} = \inf_n G^n(b_i)$, $i = 0, 1, \ldots,$ as long as these sequences are strictly decreasing. In particular, if in Lemma 2.2 $G^n(b_i) = G^{n+1}(b_i)$ for some $n, i \in \mathbb{N}$, then $x^* = G^n(b_i)$ is the greatest fixed point of G in $(b]$.

Example 2.1. Let $\mathscr{L}^\infty[0, 1]$ denote the space of all bounded and Lebesgue-measurable functions $x : [0, 1] \to \mathbb{R}$. Choose

$$X = \{x \in \mathscr{L}^\infty[0, 1] \mid 0 \leqslant x(t) \leqslant 1 \quad \text{for all } t \in [0, 1]\}, \tag{2.5}$$

and assume that X is ordered pointwise. Let $G : X \to X$ be an increasing mapping which satisfies the following hypothesis.

(G) The image $G[C]$ of the well-ordered chain C of G-iterations of the zero-function 0 contains a (finite or infinite) cofinal sequence.

Because G is increasing, then $G[C]$ is well-ordered, and $G[C]$ has by (G) a strictly increasing cofinal sequence $(x_n)_{n=0}^m$. If $m < \infty$, then $x_m = \max G[C] = \sup\, G[C]$. Assume next that $m = \infty$. Since a pointwise limit of each increasing sequence of X exists and is its supremum in X, then $y = \sup_n x_n$ exists in X. Because $(x_n)_{n=0}^m$ is a cofinal sequence of $G[C]$, then $y = \sup\, G[C]$.

The above proof shows that G satisfies the hypotheses of Lemma 2.1. Since the zero-function is the least element of X, it then follows from Lemma 2.1 that $x_* = \max C$ is the least fixed point of G.

Remark 2.2. The pointwise ordering of X, given by (2.5), is a lattice-ordering. But X is not a complete lattice. For instance, if A is a nonmeasurable subset of [0, 1], and ξ_t denotes the characteristic function of $\{t\}$, then $B = \{\xi_t \mid t \in A\}$ is a subset of X which does not have a supremum in X. Thus there exists by [3] an increasing mapping $G : X \to X$ which does not have any fixed point. This implies by Example 2.1 that the image $G[C]$ of a w.o. chain of G-iterations C of 0 cannot have a supremum. Because $G[C]$ is well-ordered and nonempty, the result of Example 2.1 does not follow from those fixed point theorems where all nonempty well-ordered chains of X are assumed to posses supremums. (cf., e.g., [1–3,9,11–13,15,17].

Next we apply the results of Lemmas 2.1 and 2.2 to set-valued cases. Denote by 2^X the set of all subsets of X. If Y is a nonempty set, we say that a mapping $S : Y \to X$ is a *selection* of a set-valued mapping $F : Y \to 2^X \setminus \emptyset$ if $S(x) \in F(x)$ for each $x \in Y$. If X is a poset, and if $S(x)$ is the least (respectively greatest) element of $F(x)$ for each $x \in Y$, we say that S is a *least* (respectively *greatest*) selection of F. If $Z \subseteq Y$, denote $F[Z] = \bigcup\{F(x) \mid x \in Z\}$. An element x of X is called a *fixed point* of a mapping $F : X \to 2^X \setminus \emptyset$ if $x \in F(x)$. If the set of fixed points of F in a subset Z of X has a minimum (respectively a maximum) in Z, we call it the *least fixed point* (respectively the *greatest fixed point*) of F in Z.

The first fixed point result of this section reads as follows.

Theorem 2.1. *Given a nonempty set T, assume that mappings $F^t : X \to 2^X \setminus \emptyset$, $t \in T$, satisfy the following hypotheses.*

(H0) *For each $t \in T$ there is an $a^t \in X$ such that $F^t(a^t) \subseteq [a^t)$.*

(H1) *Nonempty chains of $F^t[[a^t)]$ have supremums in X for each $t \in T$.*

(a) *If each F^t has an increasing selection $S^t : X \to X$, then a mapping S^t has for each $t \in T$ the least fixed point x_t in $[a^t)$, which is also a fixed point of F^t in $[a^t)$.*

(b) *If each F^t has the least selection, which is also increasing, then F^t has for each $t \in T$ the least fixed point x_t in $[a^t)$, and*

$$x_t = \min\{y \in [a^t) \mid \min F^t(y) \leqslant y\} \quad \textit{for each } t \in T. \tag{2.6}$$

(c) *If T is a poset, and if mappings $t \mapsto S^t(x)$, $x \in X$, and $t \mapsto a^t$ are increasing, then the mapping $t \mapsto x_t$ is increasing.*

Proof. (a) Let $t \in T$ be fixed, and let $S^t : X \to X$ be an increasing selection of F^t. Hypothesis (H0) implies that $S^t(a^t) \in F^t(a^t) \subseteq [a^t)$, whence $a^t \leqslant S^t(a^t)$. Let C be the w.o. chain of S^t-iterations of a^t. In view of [6, Corollary 12] we have $S^t[C] \subseteq C$, so that $S^t[C]$ is well-ordered and $S^t[C] \subseteq F^t[C] \subseteq F^t[[a^t)]$. Thus sup $S^t[C]$ exists by hypothesis (H1). Consequently, $G = S^t$ satisfies the hypotheses of Lemma 2.1, whence $x_t = \max C$ is the least fixed point of S^t in $[a^t)$. Since S^t is a selection of F^t, then $x_t = S^t(x_t) \in F^t(x_t)$, so that x_t is a fixed point of F^t. It follows from (2.2) when $G = S^t$ that

$$x_t = \min\{y \in [a^t) \mid S^t(y) \leqslant y\} \quad \text{for each } t \in T. \tag{2.7}$$

(b) Assume that each F^t has the least selection S^t, and that each S^t is also increasing. The above proof implies that for each $t \in T$ the least fixed point x_t of S^t in $[a^t)$ exists and is a fixed point of F^t, and that (2.7) holds. Because $S^t(y) = \min F^t(y)$ for each $y \in X$ and $t \in T$, then (2.6) follows from (2.7). If y is a fixed point of F^t in $[a^t)$, then $y \in F^t(y)$, whence $\min F^t(y) \leqslant y$. Thus (2.6) implies that $x_t \leqslant y$. This proves that x_t is the least fixed point of F^t in $[a^t)$.

(c) Assume now that mappings $t \mapsto S^t(x)$, $x \in X$, and $t \mapsto a^t$ are increasing. If $t, \bar{t} \in T$ and $t \leqslant \bar{t}$, then $S^t(x_{\bar{t}}) \leqslant S^{\bar{t}}(x_{\bar{t}}) = x_{\bar{t}} \in [a_{\bar{t}}) \subseteq [a^t)$, which means that $x_{\bar{t}}$ belongs to the set $\{y \in [a^t) \mid S^t(y) \leqslant y\}$. Thus (2.7) implies that $x_t \leqslant x_{\bar{t}}$. This result shows that the mapping $t \mapsto x_t$ is increasing. □

Applying Lemma 2.2 we obtain the following dual result to Theorem 2.1.

Proposition 2.1. *Let T be a nonempty set, and assume that mappings $F^t : X \to 2^X \setminus \emptyset$, $t \in T$, satisfy the following hypotheses.*

(H2) *For each $t \in T$ there is a $b^t \in X$ such that $F^t(b^t) \subseteq (b^t]$.*

(H3) *Nonempty chains of $F^t[(b^t]]$ have infimums in X for each $t \in T$.*

(a) *If each F^t has an increasing selection $S^t : X \to X$, then a mapping S^t has for each $t \in T$ the greatest fixed point x^t in $(b^t]$, which is also a fixed point of F^t.*

(b) *If each F^t has the greatest selection S^t, which is also increasing, then F^t has for each $t \in T$ the greatest fixed point x^t in $(b^t]$, and*

$$x^t = \max\{y \in (b^t] \mid y \leqslant \max F^t(y)\} \quad \textit{for each } t \in T. \tag{2.8}$$

(c) *If T is a poset, and if mappings $t \mapsto S^t(x)$, $x \in X$, and $t \mapsto b^t$ are increasing, then the mapping $t \mapsto x^t$ is increasing.*

3. Fixed point results in product spaces

Throughout this section we assume that T is a nonempty poset, and that X is a product space: $X = \prod_{i \in I} X_i$, where I is an index set and each $X_i = (X_i, \leqslant_i)$ is a poset. Define a partial ordering $\leqslant$ on X by

$$\{x_i\}_{i \in I} \leqslant \{y_i\}_{i \in I} \quad \text{iff } x_i \leqslant_i y_i \text{ for each } i \in I. \tag{3.1}$$

A nonempty subset W of X_i is called *chain subcomplete downwards* if each nonempty chain of W has an infimum in X_i, and it belongs to W. If W contains supremums all its nonempty chains, it is called *chain subcomplete upwards*. If W is chain subcomplete downwards and upwards, we say that W is *chain subcomplete*. W is called a *meet sublattice* if $\inf\{x, y\}$ exists in X_i and belongs to W for all $x, y \in W$. If $\sup\{x, y\}$ exists in X_i and belongs to W for all $x, y \in W$, we say that W is a *join sublattice*. A *sublattice* is both meet and join sublattice. We say that W is *upwards directed* if for all $x, y \in W$ there exists a $z \in W$ such that $x \leqslant_i z$ and $y \leqslant_i z$. If the reversed inequalities hold, we call W *downwards directed*. If W is both upwards and downwards directed, we say that W is *directed*.

A mapping $F_i : X \to 2^{X_i} \backslash \emptyset$ is called *increasing upwards* if $x \leqslant \overline{x}$ in X and $z \in F_i(x)$ imply that $z \leqslant_i \overline{z}$ for some $\overline{z} \in F_i(\overline{x})$. F_i is said to be *increasing downwards* if $x \leqslant \overline{x}$ in X and $\overline{z} \in F_i(\overline{x})$ imply the existence of $z \in F_i(x)$ such that $z \leqslant_i \overline{z}$. If F_i is increasing upwards and downwards, we say that F_i in *increasing*. F_i is called *weakly ascending* if (cf. [15]) $\sup\{z, \overline{z}\} \in F_i(\overline{x})$ or $\inf\{z, \overline{z}\} \in F_i(x)$whenever $x \leqslant \overline{x}$ in X, $z \in F_i(x)$ and $\overline{z} \in F_i(\overline{x})$.

As an application to Theorem 2.1 and [15, Chapter 4, Theorem 5] we shall prove the following fixed point result.

Theorem 3.1. *Assume that mappings $F_i^t : X \to 2^{X_i} \backslash \emptyset$, $i \in I$, $t \in T$, satisfy the following hypotheses.*

(h0) $(x, t) \mapsto F_i^t(x)$ *is weakly ascending, and chain subcomplete-valued for each $i \in I$.*

(h1) *For each $t \in T$ there is such an $a^t = \{a_i^t\}_{i \in I} \in X$ that $F_i^t(a^t) \subseteq [a_i^t)$, and that $t \mapsto a_i^t$ is increasing for each $i \in I$.*

(h2) *Nonempty chains of $F_i^t[[a^t)]$ have supremums in X_i for all $i \in I$ and $t \in T$. Then $F^t = \{F_i^t\}_{i \in I}$ has for each $t \in T$ such a fixed point x_t that $t \mapsto x_t$ is increasing.*

Proof. We shall show that hypotheses (H0) and (H1) of Theorem 2.1 are valid. Hypothesis (h1) and (3.1) imply that (H0) holds. To show that (H1) holds, let $t \in T$ be fixed, and let $W = \{W_i\}_{i \in I}$ be a nonempty well-ordered chain in $F^t[[a^t)]$. Given $i \in I$, then W_i is a nonempty subset of $F_i^t[[a^t)]$. To show that W_i is well-ordered, let A be a nonempty subset of W_i. Then the set $B = \{x = \{x_i\}_{i \in I} \in W \mid x_i \in A\}$ is a nonempty subset of W, whence $y = \{y_i\}_{i \in I} = \min B$ exists. This result and (3.1) imply that $y_i = \min A$. Thus $x_i = \sup W_i$ exists in X_i by hypothesis (h2). This result holds for each $i \in I$, whence $x = \{x_i\}_{i \in I}$ is the supremum of W in X. This shows that hypothesis (H1) of Theorem 2.1 is satisfied.

Hypothesis (h0) implies by [15, Chapter 4, Theorem 5] that each F_i^t has such a selection $S_i^t : X \to X_i$ that $S_i^t(x) \leqslant_i S_i^{\overline{t}}(\overline{x})$ whenever $i \in I$, $x \leqslant \overline{x}$ in X and $t \leqslant \overline{t}$ in T. These results and (3.1) imply that relation

$$S^t(x) = \{S_i^t(x)\}_{i \in I}, \quad x \in X \tag{3.2}$$

defines for each $t \in T$ such an increasing selection $S^t : X \to X$ of F^t that $t \mapsto S^t(x)$ is increasing for each $x \in X$. Thus the mapping S^t has by Theorem 2.1(a) for each $t \in T$ the least fixed point x_t in $[a^t)$, which is also a fixed point of F^t. Moreover, since $t \mapsto a^t$ is increasing by (h1) and (3.1), the mapping $t \mapsto x_t$ is increasing by Theorem 2.1(c). □

As a consequence of Proposition 2.1(a) and (c) we obtain the following result which is dual to the result of Theorem 3.1.

Theorem 3.2. *Assume that mappings* $F_i^t : X \to 2^{X_i} \setminus \emptyset$, $i \in I$, $t \in T$, *satisfy hypothesis* (h0) *of Theorem* 3.1 *and the following hypotheses.*

(h3) *For each* $t \in T$ *there is a* $b^t = \{b_i^t\}_{i \in I} \in X$ *such that* $F_i^t(b^t) \subseteq (b_i^t]$, *and* $t \mapsto b_i^t$ *is increasing for each* $i \in I$.

(h4) *Nonempty chains of* $F_i^t[(b^t]]$ *have infimums in* X_i *for all* $i \in I$ *and* $t \in T$.

Then $F^t = \{F_i^t\}_{i \in I}$ *has for each* $t \in T$ *such a fixed point* x^t *that* $t \mapsto x^t$ *is increasing.*

Proof. Hypotheses (h3) and (h4) imply that hypotheses (H2) and (H3) of Proposition 2.1 are valid, and $(x, t) \mapsto F^t(x)$ has an increasing selection $(x, t) \mapsto S^t(x)$ by (h0). It then follows from Proposition 2.1(a) that S^t has for each $t \in T$ the greatest fixed point x^t in $(b^t]$, which is also a fixed point of F^t. Moreover, the mapping $t \mapsto x^t$ is increasing by Proposition 2.1(c). □

The next result is a consequence of Theorem 2.1(b) and (c).

Theorem 3.3. *Assume that mappings* $F_i^t : X \to 2^{X_i} \setminus \emptyset$, $i \in I$, $t \in T$, *satisfy hypotheses* (h1) *and* (h2) *of Theorem* 3.1 *and the following hypothesis.*

(h5) *Each* F_i^t *is increasing downwards, and each* $F_i^t(x)$ *is downwards directed and contains a lower bound to each of its nonempty chains.*

Then $F^t = \{F_i^t\}_{i \in I}$ *has for each* $t \in T$ *the least fixed point* x_t *in* $[a^t)$, *and* (2.6) *holds. If* $t \mapsto F_i^t(x)$ *is increasing downwards for all* $i \in I$ *and* $x \in X$, *then the mapping* $t \mapsto x_t$ *is increasing.*

The above results hold also when (h5) *is replaced by the following hypothesis.*

(h6) *Each* F_i^t *is weakly ascending, and its values are chain subcomplete meet sublattices for each* $i \in I$.

Proof. Let $x \in X$, $t \in T$ and $i \in I$ be fixed. Hypothesis (h5) implies that nonempty inversely well-ordered chains of $F_i^t(x)$ have lower bounds in X_i, and that they belong to $F_i^t(x)$. It then follows from Zorn's Lemma that $F_i^t(x)$ has a minimal element z_i. Because $F_i^t(x)$ is downwards directed, then $z_i = \min F_i^t(x)$. This holds for all $x \in X$, whence the equation

$$S_i^t(x) = \min\, F_i^t(x), \quad i \in I, \ \ x \in X$$

defines the least selection $S^t = \{S_i^t\}_{i \in I}$ of $F_i^t = \{F_i^t\}_{i \in I}$. Moreover, each F_i^t is increasing downwards by (h5), whence each S_i^t is increasing. Thus, by (3.1), S^t is increasing, and is the least selection of F^t.

Hypotheses (h1) and (h2) imply that hypotheses (H0) and (H1) of Theorem 2.1 hold. Thus the assertions follow from Theorem 2.1(b) and (c).

If hypothesis (h6) holds, it follows from [15, Chapter 4, Theorem 5] that F_i^t has the least increasing selection S_i^t for each $i \in I$. Consequently, $S^t = \{S_i^t\}_{i \in I}$ is the least increasing selection of F^t. □

The next result is dual to Theorem 3.3, and it follows from Proposition 2.1(b) and (c) and [15, Chapter 4, Theorem 5].

Theorem 3.4. *Assume that mappings* $F_i^t : X \to 2^{X_i}\setminus\emptyset$, $t \in T$, $i \in I$, *satisfy hypotheses* (h3) *and* (h4) *of Theorem* 3.2 *and the following hypothesis*.

(h7) *Each* F_i^t *is increasing upwards, and each* $F_i^t(x)$ *is upwards directed and contains an upper bound to each of its nonempty chains*.

Then $F^t = \{F_i^t\}_{i\in I}$ *has for each* $t \in T$ *the greatest fixed point* x^t *in* $(b^t]$, *and* (2.8) *holds*. *If* $t \mapsto F_i^t(x)$, *is increasing upwards for all* $i \in I$ *and* $x \in X$, *then the mapping* $t \mapsto x^t$ *is increasing*.

The above results hold also when (h7) is replaced by the following hypothesis.

(h8) F_i^t *is weakly ascending*, *and its values are chain subcomplete join sublattices for each* $i \in I$ *and* $t \in T$.

4. Special cases

Hypotheses (h1) of Theorem 3.1 and (h3) of Theorem 3.2 can sometimes be replaced by properties of posets defined as follows.

Definition 4.1. Given a poset Y, we say that $c \in Y$ is a *sup-center* of Y if $\sup\{c, x\}$ exists in Y for all $x \in Y$, and an *inf-center* of Y if $\inf\{c, x\}$ exists in Y for all $x \in Y$. If c is both sup- and inf-center of Y, we say that c is an *order center* of Y. A subset Z of Y is called *closed upwards* if it contains supremums of all its supremum possessing well-ordered chains and *closed downwards* if it contains infimums of all its infimum possessing inversely well-ordered chains. Z is called *order-closed* if it is closed upwards and downwards.

In the following list we present posets which have inf-, sup- or order centers.

If Y is a meet lattice, then each point of Y is its inf-center.

If Y is a join lattice, then each point of Y is its sup-center.

If Y is a lattice, then each point of Y is its order center.

If $\min Y$ (respectively $\max Y$) exists, it is a sup-center (respectively an inf-center) of Y.

If the Euclidean m-space $\mathbb{R}^m$ is ordered coordinatewise, then the center of any ball Y of $\mathbb{R}^m$ is an order center of Y.

If E is a lattice-ordered normed space, and if $\|\sup\{0, x\}\| \leqslant \|x\|$ for each $x \in E$ (this holds, for instance, if E is a Riesz space or a Banach lattice), then the center of any ball Y of E is an order center of Y.

In what follows we assume that $T = (T, \leqslant)$ is a poset, that $X = \prod_{i\in I} X_i$ is a product of posets $X_i = (X_i, \leqslant_i)$, and that the order relation $\leqslant$ of X is defined by (3.1).

Theorem 4.1. *Assume that mappings* $F_i^t : X \to 2^{X_i}\setminus\emptyset$, $i \in I$, $t \in T$, *satisfy the following hypotheses*.

(ha) *Nonempty chains of each* $F_i^t[X]$ *have supremums and infimums in* X_i.

(hb) *Each mapping* $(x, t) \mapsto F_i^t(x)$ *is weakly ascending and has order-closed values*.

(a) *If each* X_i *has an inf-center, then* $F^t = \{F_i^t\}_{i \in I}$ *has for each* $t \in T$ *such a fixed point* x_t *that the mapping* $t \mapsto x_t$ *is increasing.*
(b) *If each* X_i *has a sup-center, then* $F^t = \{F_i^t\}_{i \in I}$ *has for each* $t \in T$ *such a fixed point* x^t *that the mapping* $t \mapsto x^t$ *is increasing.*

Proof. (a) Hypotheses (ha) and (hb) imply that hypothesis (h0) of Theorem 3.1 is valid. Thus each F^t has an increasing selection $S^t : X \to X$. Assume that each X_i has an inf-center c_i, and let $t \in T$ be fixed. It follows from (3.1) that $c = \{c_i\}_{i \in I}$ is an inf-center of the product space X. Then the relation

$$G(x) = \inf\{c, S^t(x)\}, \quad x \in X \tag{4.1}$$

defines an increasing mapping $G : X \to X$, and $G(c) \leqslant c$. Let D be the i.w.o. chain of G-iterations of c. Because S^t is increasing, then $S^t[D]$ is inversely well-ordered. Moreover, it is a subset of $F^t[X]$, whence $y = \inf S^t[D]$ exists in X by the hypothesis (ha) (cf. the proof of Theorem 3.1). Applying this result and the definition (4.1) of G it is easy to see that $\inf\{c, y\} = \inf G[D]$.

The above proof shows that G, defined by (4.1), satisfies the hypotheses of Lemma 2.2, whence $a^t := \min D$ is the greatest fixed point of G in $(c]$. Moreover, it follows from (2.4) and (4.1) that

$$a^t = \max\{y \in (c] \mid y \leqslant \inf\{c, S^t(y)\}\}. \tag{4.2}$$

In particular, $a^t \leqslant S^t(a^t)$. Let C be the w.o. chain of S^t-iterations of a^t. $S^t[C]$ is well-ordered and $S^t[C] \subseteq F^t[X]$. Thus $\sup S^t[C]$ exists by the hypothesis (ha). Consequently, $G = S^t$ satisfies the hypotheses of Lemma 2.1, whence $x_t = \max C$ is the least fixed point of S^t in $[a^t)$. Since S^t is a selection of F^t, then $x_t = S^t(x_t) \in F^t(x_t)$, so that x_t is a fixed point of F^t. It follows from (2.2) when $G = S^t$ that

$$x_t = \min\{y \in [a^t) \mid S^t(y) \leqslant y\} \quad \text{for each } t \in T. \tag{4.3}$$

Since $t \mapsto S^t(x)$ is by the proof of Theorem 3.1 increasing for each $x \in X$, it follows from (4.2) and (4.3) (cf. the proof of Theorem 2.1) that mappings $t \mapsto a^t$ and $t \mapsto x_t$ are increasing.

(b) Assume next that c is a sup-center of X. Given $t \in T$, define

$$G(x) = \sup\{c, S^t(x)\}, \quad x \in X \tag{4.4}$$

and let C be a w.o. chain of G-iterations of c. In this case one can show that $b^t = \sup G[C]$ exists. It then follows from Lemma 2.1 that $b^t = \max C$, that b^t is the least fixed point of G in $[c)$, and that

$$b^t = \min\{y \in [c) \mid \sup\{c, S^t(y)\} \leqslant y\}. \tag{4.5}$$

In particular, $S^t b^t \leqslant b^t$. If D denotes the i.w.o. chain of S^t-iterations of b^t, then $x^t = \inf S^t[D]$ exists in X. Thus, by Lemma 2.2, $x^t = \min D$, x^t is the greatest fixed point of S^t in $(b^t]$, and

$$x^t = \max\{y \in (b^t] \mid y \leqslant S^t(y)\} \quad \text{for each } t \in T. \tag{4.6}$$

Since $x^t = S^t(x^t) \in F^t(x^t)$, then x^t is a fixed point of F^t. The above relations obtained for b^t and x^t and the monotonicity of mappings $t \mapsto S^t(x)$, $x \in X$, imply that the mappings $t \mapsto b^t$ and $t \mapsto x^t$ are increasing. □

The following result is a modification of Theorem 3.3 to the case when hypothesis (h1) is replaced by the existence of inf-centers of X_i.

Theorem 4.2. *Assume that mappings $F_i^t : X \to 2^{X_i} \backslash \emptyset$, $i \in I$, $t \in T$, satisfy hypothesis* (ha) *of Theorem* 4.1 *and the following hypothesis.*

(hc) *Each F_i^t is increasing downwards, and each $F_i^t(x)$ is downwards directed and contains a lower bound to each of its nonempty chains.*

If each X_i has an inf-center c_i, then $F^t = \{F_i^t\}_{i \in I}$ has for each $t \in T$ the least fixed point x_t in $[a^t)$, where

$$a^t = \max\{y \in (c] \mid y \leqslant \inf\{c, \min F^t(y)\}\}, \quad c = \{c_i\}_{i \in I}. \tag{4.7}$$

If the mapping $t \mapsto F_i^t(x)$ is increasing downwards for all $i \in I$ and $x \in X$, then the mappings $t \mapsto a^t$ and $t \mapsto x_t$ are increasing.

Proof. Hypothesis (hc) implies by the proof of Theorem 3.3 that $\min F_i^t(x)$ exists for all $i \in I$, $t \in T$ and $x \in X$, and that the mapping $x \mapsto \min F_i^t(x)$ is increasing. In view of this result and (3.1) $\min F^t(x) = \{\min F_i^t(x)\}_{i \in I}$ exists for all $t \in T$ and $x \in X$, and the mapping $x \mapsto \min F^t(x)$ is increasing for each $t \in T$. Choosing $S^t := \min F^t$ in the proof of Theorem 4.1 a), we see that if each X_i has an inf-center c_i, then $\min F^t$ has the least fixed point x_t in $[a^t)$, where a^t is given by (4.7). As in the proof of Theorem 3.3 one can show that x_t is the least fixed point of F^t in $[a^t)$.

Assume next that the mapping $t \mapsto F_i^t(x)$ is increasing downwards for all $i \in I$ and $x \in X$. This hypothesis and (3.1) imply that mappings $t \mapsto \min F^t(x)$, $x \in X$, are increasing. Moreover, it follows from (4.3) that

$$x_t = \min\{y \in [a^t) \mid \min F^t(y) \leqslant y\} \quad \text{for each } t \in T.$$

Applying this result, (4.7) and the monotonicity of mappings $t \mapsto \min F^t(x)$, $x \in X$, it is easy to show that the mappings $t \mapsto a^t$ and $t \mapsto x_t$ are increasing. □

The next result is dual to that of Theorem 4.2, and its proof is similar.

Theorem 4.3. *Assume that mappings $F_i^t : X \to 2^{X_i} \backslash \emptyset$ satisfy the hypothesis* (ha) *of Theorem* 4.1 *and the following hypothesis.*

(hd) *Each F_i^t is increasing upwards, and each $F_i^t(x)$ is upwards directed and contains an upper bound to each of its nonempty chains.*

If each X_i has a sup-center c_i, then $F^t = \{F_i^t\}_{i \in I}$ has for each $t \in T$ the greatest fixed point x_t in $(b^t]$, where

$$b^t = \min\{y \in [c) \mid \sup\{c, \max F^t(y)\} \leqslant y\}, \quad c = \{c_i\}_{i \in I}. \tag{4.8}$$

If the mapping $t \mapsto F^t(x)$ is increasing upwards for each $x \in X$, then the mappings $t \mapsto b^t$ and $t \mapsto x^t$ are increasing.

If in Theorem 4.2 each X_i is a meet lattice, then $c = \{c_i\}_{i\in I}$ in (4.7) can be any element of X. This holds also in (4.8) if each X_i in Theorem 4.3 is a join lattice. In particular, we get as a consequence of Theorems 4.2 and 4.3 the following fixed point theorem.

Theorem 4.4. *Given a poset T and a product $X = \prod_{i\in I} X_i$ of lattices X_i, assume that mappings $F_i^t : X \to 2^{X_i}\setminus\emptyset$, $i \in I$, $t \in T$, satisfy hypotheses* (ha), (hc) *and* (hd). *Then for each choice of $c_i \in X_i$ and for each $t \in T$ the mapping $F^t = \{F_i^t\}_{i\in I}$ has least and greatest fixed points x_t and x^t in $[a^t, b^t]$, where a^t and b^t are given by* (4.7) *and* (4.8). *If the mapping $t \mapsto F^t(x)$ is increasing for each $x \in X$, the mappings $t \mapsto a^t$, $t \mapsto b^t$, $t \mapsto x_t$ and $t \mapsto x^t$ are increasing.*

Proof. Since each element of a lattice is its order center, the results are direct consequences of the results of Theorems 4.2 and 4.3. □

Example 4.1. Let $\mathscr{L}^\infty[0, 1]$ denote the space of all bounded and Lebesgue-measurable functions $x : [0, 1] \to \mathbb{R}$, equipped with a pointwise ordering. Given functions $\underline{x}_i, \overline{x}_i \in \mathscr{L}^\infty[0, 1]$, $\underline{x}_i \leqslant \overline{x}_i$, $i \in I$, choose $X_i := [\underline{x}_i, \overline{x}_i]$. Assume that the ordering $\leqslant_i$ of each X_i is a pointwise ordering, and that the ordering of the product space $X = \prod_{i\in I} X_i$ is defined by (3.1). Let T be a poset, and assume that mappings $F_i^t : X \to 2^{X_i}\setminus\emptyset$, $i \in I$, $t \in T$, satisfy hypotheses (hc) and (hd) of Theorems 4.2 and 4.3 and the following hypothesis.

(a) Nonempty well-ordered and inversely well-ordered chains of $F_i^t[X]$ posses cofinal (finite or infinite) sequences for all $i \in I$ and $t \in T$.

Hypothesis (a) implies by a reasoning used in Example 2.1 that hypothesis (ha) of Theorem 4.2 holds. Thus the hypotheses of Theorem 4.4 are valid. Since each X_i is lattice-ordered, it follows from Theorem 4.4 that for each choice of $c_i \in X_i$ and for each $t \in T$ the mapping $F^t = \{F_i^t\}_{i\in I}$ has least and greatest fixed points x_t and x^t in $[a^t, b^t]$, where a^t and b^t are given by (4.7) and (4.8). Moreover, if the mapping $t \mapsto F^t(x)$ is increasing for each $x \in X$, the mappings $t \mapsto x_t$ and $t \mapsto x^t$ are increasing.

Remark 4.1. In [15] fixed point results are derived for multifunctions in product spaces $X = \prod_{i\in I} X_i$ where each X_i is a complete lattice. As for other results dealing with the existence of increasing selections, see, e.g., [12]. In [5] necessary and sufficient conditions are introduced for the existence of least, greatest, minimal or maximal elements of posets.

Since the spaces X_i in Example 4.1 are not chain complete if $\underline{x}_i \neq \overline{x}_i$, its result does not follow from such fixed point theorems where X is assumed to be a chain complete poset or a complete lattice.

Next we shall study the validity of hypotheses (h2) of Theorem 3.1, (h4) of Theorem 3.2 and (ha) of Theorem 4.1. Let X_i be a nonempty closed subset of an ordered normed space E_i which has a regular order cone, or equivalently, all order-bounded and monotone sequences of E_i converge. It follows from [7, Proposition 1.3.2] that:

(a) Each order-bounded subset of X_i is chain complete in X_i.

This result holds also when the order cone of each E_i is fully regular, or equivalently, all norm-bounded and monotone sequences of E_i converge. In this case [7, Proposition 1.3.3] implies also the following result.

(b) Each norm-bounded subset of X_i is chain complete in X_i.

This result holds by [7, Proposition 1.3.6] also when E_i is a reflexive ordered Banach space. If E_i is a weakly sequentially complete Banach space whose order cone is normal, then [4, Theorem 2.4.5] implies that the order cone of E_i is fully regular, and hence also regular. Thus both (a) and (b) hold.

The above reasoning implies the following result.

Lemma 4.1. *Hypotheses* (h2) *of Theorem* 3.1, (h4) *of Theorem* 3.2 *and* (ha) *of Theorem* 4.1 *hold if each X_i is a nonempty order-closed subset of an ordered Banach space E_i, and if one the following hypotheses is valid.*

(hi) *The order cone of E_i is regular or fully regular, and $F_i^t[X]$ is order-bounded for all $i \in I$ and $t \in T$.*

(hii) *The order cone of E_i is fully regular, and $F_i^t[X]$ is order- or norm-bounded for all $i \in I$ and $t \in T$.*

(hiii) *E_i is reflexive, and $F_i^t[X]$ is norm-bounded for all $i \in I$ and $t \in T$.*

(hiv) *E_i is weakly sequentially complete, the order cone of E_i is normal, and $F_i^t[X]$ is order- or norm-bounded for all $i \in I$ and $t \in T$.*

Remark 4.2. Each of the following spaces equipped with a p-norm are ordered Banach spaces with fully regular order cone when $1 \leqslant p < \infty$.

(a) $\mathbb{R}^m$, ordered coordinatewise.
(b) l^p, ordered componentwise.
(c) $L^p(\Omega, E)$, ordered a.e. pointwise, where $\Omega = (\Omega, \mathscr{A}, \mu)$ is a measure space and E a Banach space with fully regular order cone (cf. [7, Proposition 5.8.7]).

The spaces R^n and l^p are also Banach lattices. If E in (c) is a Banach lattice, so is also $L^p(\Omega, E)$. Its order cone is regular whenever the order cone of E is regular (cf. [7, Proposition 5.8.7]).

As for further examples of ordered Banach spaces E whose order cones are regular (respectively fully regular), see, e.g., [4,7,8].

The spaces given in (a)–(c) above are also reflexive if $1 < p < \infty$, Ω is a domain in $\mathbb{R}^m$ and $E = \mathbb{R}$. Moreover, a.e. pointwise ordered Sobolev spaces $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$, $k \in \mathbb{N}$, are reflexive partially ordered Banach spaces. When $k = 1$, the center of each ball of these spaces is its order center.

5. On the computation of fixed points

Consider first the case when for some $t \in T$ both the chains D and C in the proof of Theorem 4.1 have a finite number of elements. In view of Remark 2.1 and the definition

(4.1) of G we can compute a^t and x_t in the case when X has an inf-center c by following successive approximations:

(1) a^t is the last element of the finite sequence of iterations
$a_0 = c$, and $a_{n+1} = \inf\{c, S^t(a_n)\}$, as long as $a_{n+1} < a_n$.
(2) x_t is the last element of the finite sequence of iterations
$x_0 = a^t$, and $x_{n+1} = S^t(x_n)$, as long as $x_n < x_{n+1}$.
Dually, if c is a sup-center of X, then
(3) b^t is the last element of the finite sequence of iterations
$b_0 = c$, and $b_{n+1} = \sup\{c, S^t(b_n)\}$, as long as $b_n < b_{n+1}$.
(4) x^t is the last element of the finite sequence of iterations
$y_0 = b^t$, and $y_{n+1} = S^t(y_n)$, as long as $y_{n+1} < y_n$.

Next we shall consider a special case when $\max F^t$ and/or $\min F^t$ can be determined without the use of the Axiom of Choice.

Given a nonempty subset A of a poset we say that a subset B of A is *dense upwards* in A if, whenever $B \ni x < y \in A$, then either y is a maximal element of A, or there exists a $z \in B$ such that $x < z$. B is called *dense downwards* in A if, whenever $A \ni x < y \in B$, then either x is a minimal element of A, or there exists a $z \in B$ such that $z < y$. If B is dense upwards and downwards in A we say that B is *order dense* in A.

Proposition 5.1. (a) *The results of Theorem* 4.2 *hold if hypothesis* (hc) *is replaced by the following hypothesis.*

(hj) *Each $F_i^t(x)$ is directed downwards and has a countable subset whose strictly decreasing sequences are finite, and which is dense downwards in $F_i^t(x)$.*

(b) *The results of Theorem* 4.3 *hold if hypothesis* (hc) *is replaced by*:

(hk) *Each $F_i^t(x)$ is directed upwards and has a countable subset whose strictly increasing sequences are finite, and which is dense upwards in $F_i^t(x)$.*

Proof. We shall prove only (b), because the proof of (a) is similar. It suffices to show that each $F_i^t(x)$ has a maximum. Let B be a countable subset of $F_i^t(x)$ whose strictly increasing sequences are finite, and which is dense upwards in $F_i^t(x)$. Because $F_i^t(x)$ is nonempty, we may assume that also B is nonempty. Then B can be represented as $B = \{z_n \mid 0 \leqslant n \leqslant m\}$, where $m \in \mathbb{N} \cup \{\infty\}$. Denote $x_0 := z_0$, and when x_k is chosen, and if $B_k = \{z_n \in B \mid x_k <_i z_n\}$ is nonempty, let x_{k+1} be that element z_n of B_k whose index n is least. The so obtained sequence (x_k) is strictly increasing. Thus (x_k) is finite by (hk). The above construction implies that $y = \max_k x_k$ exists and is a maximal element of B. If $y = \max F_i^t(x)$, the proof is complete. Otherwise, $x \not\leqslant_i y$ for some $x \in F_i^t(x)$. Since $F_i^t(x)$ is directed upwards, there exists a $z \in F_i^t(x)$ such that $x \leqslant_i z$ and $y \leqslant_i z$. Because $x \not\leqslant_i y$, then $y <_i z$. But then $z = \max F_i^t(x)$, for otherwise, B being dense upwards in $F_i^t(x)$, there exists a $w \in B$ such that $y <_i w$, which is impossible, since y is a maximal element of B. This concludes the proof. □

The above proof implies that if hypothesis (hj) (respectively (hk)) holds, then $\min F_i^t(x)$ (respectively $\max F_i^t(x)$) is computable for all $i \in I$, $t \in T$ and $x \in X$. Combining this result with the corollaries given below we have cases where some fixed points of F^t are computable.

Corollary 5.1. *Assume that hypothesis* (hj) *of Proposition* 5.1 *holds, that each X_i has an inf-center c_i, and that for some $t \in T$ strictly monotone sequences of* min $F^t[X]$ *are finite. Then a^t, given by* (4.7) *and the least fixed point x_t of F^t in $[a^t)$, can be computed in the following manner.*

a^t is the last element of the finite sequence of iterations

$a_0 = c$, *and* $a_{n+1} = \inf\{c, \min F^t(a_n)\}$, $c = \{c_i\}_{i \in I}$, *as long as* $a_{n+1} < a_n$.

x_t is the last element of the finite sequence of iterations

$x_0 = a^t$, *and* $x_{n+1} = \min\ F^t(x_n)$, *as long as* $x_n < x_{n+1}$.

Corollary 5.2. *Assume that hypothesis* (hk) *of Proposition* 5.1 *holds. If each X_i has a sup-center c_i, and if for some $t \in T$ strictly monotone sequences of* max $F^t[X]$ *are finite, then b^t, given by* (4.8) *and the greatest fixed point x_t of F^t in $(b^t]$ can be computed as follows.*

b^t is the last element of the finite sequence of iterations

$b_0 = c$, *and* $b_{n+1} = \sup\{c, \max F^t(b_n)\}$, $c = \{c_i\}_{i \in I}$ *as long as* $b_n < b_{n+1}$.

x^t is the last element of the finite sequence of iterations

$y_0 = b^t$, *and* $y_{n+1} = \max\ F^t(y_n)$, *as long as* $y_{n+1} < y_n$.

6. Applications to theory of Nash equilibria

To every fixed point result of Sections 3 and 4 there corresponds an existence result for Nash equilibria of the multi-player game introduced in the Introduction when the following conditions hold. A poset $X_i = (X_i, \leqslant_i)$ is the strategy set player i for each $i \in I$, T is a poset of exogenous parameters, and the values $F_i^t(x)$ of components of $F^t = \prod_{i \in I} F_i^t$ are sets of all optimal replies of player i to $(x, t) \in X \times T$, where $X = \prod_{i \in I} X_i$. By definition, $F_i^t(x)$ is the set of maximum points of the mapping $u_i^{x,t}$ from the set $X_i^{x,t}$ of feasible replies to a poset $W_i = (W_i, \preccurlyeq_i)$, defined by

$$u_i^{x,t}(y_i) := u_i^t(x_{-i}, y_i), \quad y_i \in X_i^{x,t}, \tag{6.1}$$

where $u_i^t : X \to W_i$ is player i's utility mapping. Recall that a strategy $x \in X$ is a Nash equilibrium for $t \in T$ if and only if x is a fixed point of F^t, i.e., $x \in F^t(x)$.

Because the hypotheses are so far expressed in terms of mappings $F_i^t : X \to 2^{X_i} \setminus \emptyset$, $i \in I$, $t \in T$, we are going to study in this section what conditions assumed for sets $X_i^{x,t}$ and for mappings $u_i^{x,t}$ are sufficient to ensure the validity of such hypotheses. The first requirement is that the sets $F_i^t(x)$ of optimal replies are nonempty. A necessary and sufficient condition for this is hypothesis (Ha) of Theorem 6.1. Necessity is obvious, and sufficiency is shown in the proof of Theorem 6.1.

A subset A of $X_i^{x,t}$ is called a *level set* of $u_i^{x,t}$ if $A = (u_i^{x,t})^{-1}(w_i)$, and *up-level set* of $u_i^{x,t}$ if $A = (u_i^{x,t})^{-1}[[w_i)]$ for some $w_i \in u_i^{x,t}[X_i^{x,t}]$. These sets contain $F_i^t(x)$, and are equal to it if $w_i = \max\ u_i^{x,t}[X_i^{x,t}]$.

Our first existence result is an application of Theorem 4.3.

Theorem 6.1. *Assume that the following hypotheses are valid*:

(Ha) *The range of each* $u_i^{x,t}$ *contains an upper bound to each of its nonempty chains and is directed upwards*.

(Hb) *Nonempty chains of* $\bigcup\{X_i^{x,t} \mid x \in X\}$ *have supremums and infimums in* X_i *for all* $i \in I$ *and* $t \in T$.

(Hc) *Every level set of each* $u_i^{x,t}$ *is closed upwards and directed upwards*.

(Hd) *If* $x < \overline{x}$ *in* X, $i \in I$ *and* $t \in T$, *there exist up-level sets* A *of* $u_i^{x,t}$ *and* B *of* $u_i^{\overline{x},t}$ *such that whenever* $y_i \in A$ *and* $z_i \in B$, *there exists a* $\overline{z}_i \in X_i^{\overline{x},t}$ *such that* $y_i \leqslant_i \overline{z}_i$, *and* $u_i^{\overline{x},t}(\overline{z}_i) \not<_i u_i^{\overline{x},t}(z_i)$.

If each X_i *has a sup-center* c_i, *then for each* $t \in T$ *there exists the greatest Nash equilibrium* x^t *in the order interval* $(b^t]$ *of* X, *where* b^t *is given by* (4.8). *Moreover*, *the mappings* $t \mapsto b^t$ *and* $t \mapsto x^t$ *are increasing if the following hypothesis holds*:

(He) *If* $t < \overline{t}$ *in* T, $i \in I$ *and* $x \in X$, *there exist up-level sets* A *of* $u_i^{x,t}$ *and* B *of* $u_i^{x,\overline{t}}$ *such that whenever* $y_i \in A$ *and* $z_i \in B$, *there exists a* $\overline{z}_i \in X_i^{x,\overline{t}}$ *such that* $y_i \leqslant_i \overline{z}_i$, *and* $u_i^{x,\overline{t}}(\overline{z}_i) \not<_i u_i^{x,\overline{t}}(z_i)$.

Proof. Let $i \in I$, $t \in T$ and $x \in X$ be fixed. The first condition of (Ha) implies by Zorn's Lemma that the set $u_i^{x,t}[X_i^{x,t}]$ has a maximal element, which by the second condition of (Ha) is the maximum of $u_i^{x,t}[X_i^{x,t}]$. Thus the set $F_i^t(x)$ is nonempty, and it contains the supremum of each of its nonempty chains by hypotheses (Hb) and (Hc). Hypothesis (Hc) implies also that $F_i^t(x)$ is directed upwards. Thus $\max F_i^t(x)$ exists.

To show that a mapping $x \mapsto F_i^t(x)$, $i \in I$, $t \in T$, is increasing upwards, assume that $x < \overline{x}$, and let $y_i \in F_i^t(x)$ be given. Choosing $z_i \in F_i^t(\overline{x})$, there exists by (Hd) a $\overline{z}_i \in X_i^{\overline{x},t}$ such that $y_i \leqslant_i \overline{z}_i$ and $u_i^{\overline{x},t}(\overline{z}_i) \not<_i u_i^{\overline{x},t}(z_i)$. This result and the choice of z_i imply that $\overline{z}_i \in F_i^t(\overline{x})$. Since $y_i \leqslant_i \overline{z}_i$, this proves that F_i^t is increasing upwards.

The above proof shows that the hypothesis (hd) of Theorem 4.3 holds. The hypothesis (Hb) implies that the hypothesis (ha) of Theorem 4.3 holds. Consequently, if each X_i has a sup-center c_i, then $F^t = \{F_i^t\}_{i \in I}$ has for each $t \in T$ the greatest fixed point x^t in the order interval $(b^t]$ of X, where b^t is given by (4.8). By definition, x^t is also the greatest Nash equilibrium for t in $(b^t]$.

The proof that $t \mapsto F_i^t(x)$ is increasing upwards for all $i \in I$ and $x \in X$ if (He) holds is similar to the above proof that F_i^t is increasing upwards. Thus the last conclusion of Theorem 4.3 implies that the mappings $t \mapsto b^t$ and $t \mapsto x^t$ are increasing. □

Next we present alternative hypotheses to Theorem 6.1.

Theorem 6.2. *Let hypothesis* (Hb) *of Theorem* 6.1 *and the following hypotheses hold*.

(Hf) *Each* $X_i^{x,t}$ *is closed upwards and directed upwards*.

(Hg) *Each mapping* $u_i^{x,t}$ *is increasing*.

(Hh) *Each mapping* $x \mapsto X_i^{x,t}$ *is increasing upwards*.

If each X_i *has a sup-center* c_i, *then for each* $t \in T$ *there exists the greatest Nash equilibrium* x^t *in the order interval* $(b^t]$ *of* X, *where* b^t *is given by* (5.8). *Moreover*, *the mappings* $t \mapsto b^t$ *and* $t \mapsto x^t$ *are increasing if the following hypothesis holds*.

(Hi) *Each mapping* $t \mapsto X_i^{x,t}$ *is increasing upwards*.

Proof. Let $i \in I$, $t \in T$ and $x \in X$ be fixed. Hypotheses (Hb) and (Hf) imply by Zorn's Lemma that $z_i = \max X_i^{x,t}$ exists. If $y_i \in X_i^{x,t}$, then $y_i \leqslant_i z_i$, whence $u_i^{x,t}(y_i) \preccurlyeq_i u_i^{x,t}(z_i)$ by (Hg). This holds for each $y_i \in X_i^{x,t}$, so that $z_i \in F_i^t(x)$. Moreover, $z_i = \max F_i^t(x)$ because $F_i^t(x) \subseteq X_i^{x,t}$ and $z_i = \max X_i^{x,t}$.

To show that a mapping F_i^t, $t \in T$, $i \in I$, is increasing upwards, assume that $x \leqslant \overline{x}$, and let $y_i \in F_i^t(x)$ be given. By (Hh) there exists a $z_i \in X_i^{\overline{x},t}$ such that $y_i \leqslant z_i$. Thus $y_i \leqslant \max X_i^{\overline{x},t} \in F_i^t(\overline{x})$. This proves that F_i^t is increasing upwards.

Applying hypothesis (Hi) and noticing that $\max F_i^t(x) = \max X_i^{x,t}$ for all $i \in I$, $t \in T$ and $x \in X$, we see, as in the above proof, that a mapping $t \mapsto F_i^t(x)$ is increasing upwards for all $i \in I$ and $x \in X$.

The above results are the only ones for whose derivation hypotheses (Ha), (Hc), (Hd) and (He) were used in the proof of Theorem 6.1, which implies the assertions. □

The following result is dual to Theorem 6.2.

Theorem 6.3. *Assume that hypothesis* (Hb) *of Theorem* 6.1 *and the following hypotheses are satisfied*:

(Hf′) *Each* $X_i^{x,t}$ *is closed downwards and directed downwards.*

(Hg′) *Each mapping* $u_i^{x,t}$ *is decreasing.*

(Hh′) *Each mapping* $x \mapsto X_i^{x,t}$ *is increasing downwards.*

If each X_i *has an inf-center* c_i, *then for each* $t \in T$ *there exists the least Nash equilibrium* x_t *in the order interval* $[a^t)$ *of* X, *where* a^t *is given by* (4.7). *Moreover*, *the mappings* $t \mapsto a^t$ *and* $t \mapsto x_t$ *are increasing if the following hypothesis holds.*

(Hi′) *Each mapping* $t \mapsto X_i^{x,t}$ *is increasing downwards.*

In the following propositions we give sufficient conditions for the existence of such Nash equilibria which maximize utilities of each player over all Nash equilibria.

Proposition 6.1. *Assume that the hypotheses of Theorem* 6.1 *or Theorem* 6.2 *and the following hypotheses are valid.*

(Hj) $\bigcup\{X_i^{x,t} \mid x \in X\}$ *is bounded from above for all* $i \in I$ *and* $t \in T$.

Then the greatest Nash equilibrium $\overline{x}^t$ *exists for every* $t \in T$. *Moreover*, *each* $\overline{x}^t$ *maximizes each utility* $u_i^t(x)$ *over all Nash equilibria for* t *if*

(Hk) $x_{-i} \mapsto u_i^t(x_{-i}, x_i)$ *is increasing in* $X_{-i} = \prod_{j \in I \setminus \{i\}} X_j$, *for all* $i \in I$, $x_i \in X_i$, $t \in T$.

Proof. Hypothesis (Hj) implies the existence of $b^t = \{b_i^t\}_{i \in I} \in X$ such that $F^t(x) \leqslant b^t$ for each $x \in X$ and $t \in T$. It is easy to show that the hypotheses of Theorem 3.4 hold. Thus the mapping F^t has for each $t \in T$ the greatest fixed point $\overline{x}^t$ in $(b^t]$. Since $F^t[X] \subseteq (b^t]$ for each $t \in T$, then each $\overline{x}^t$ is the greatest of all fixed points of F^t, and hence the greatest of all Nash equilibria for t. This proves the first assertion. This result and hypothesis (Hk) imply by [15, Chapter 10, Theorem 3] the second assertion. □

The next result, which is dual to Proposition 6.1, is a consequence of Theorem 3.3.

Proposition 6.2. *Assume that the hypotheses of Theorem* 6.3 *and the following hypothesis are valid.*

(Hj′) $\bigcup\{X_i^{x,t} \mid x \in X\}$ *is bounded from below for all* $i \in I$ *and* $t \in T$.

Then the least Nash equilibrium $\underline{x}^t$ *exists for every* $t \in T$. *Moreover, each* $\underline{x}^t$ *maximizes each utility* $u_i^t(x)$ *over all Nash equilibria for* t *if*

(Hk′) $x_{-i} \mapsto u_i^t(x_{-i}, x_i)$ *is decreasing in* $X_{-i} = \prod_{j\in I\setminus\{i\}} X_j$ *for all* $i \in I$, $x_i \in X_i$, $t \in T$.

Now we are able to prove a result which implies the one given in the Introduction.

Corollary 6.1. *The greatest Nash equilibrium* $\overline{x}^t$ *exists and maximizes each utility* $u_i^t(x)$ *over all Nash equilibria for every* $t \in T$, *if hypotheses* (Hf), (Hg) (Hh) *and* (Hk) *and the following hypothesis hold.*

(Hl) *All strategy sets* X_i *are order-bounded and order-closed subsets of ordered normed spaces with regular order cones.*

Proof. Hypothesis (Hl) and Remarks 4.1(a) imply that hypotheses (Hb) and (Hj) hold. Thus the assertions follow from Proposition 6.1. □

When X_i's are as in Corollary 6.1, hypotheses (Hd) and (He) of Theorem 6.1 can be replaced by more concrete hypotheses, as done in the following corollary.

Corollary 6.2. *Assume that the following hypotheses hold*:

(A) *All posets* X_i *and* W_i *are order-bounded and order-closed subsets of ordered normed spaces with regular order cones.*

(B) *Each* $X_i^{x,t}$ *is a chain, each of its level sets is closed upwards, and the range of each* $u_i^{x,t}$ *is closed upwards and directed upwards.*

Then each $F_i^t(x)$ *is nonempty and has the maximum. Assume moreover that*:

(C) $X_i^{\overline{x},t} \cap (z_i] = \emptyset$ or $z_i \in X_i^{\overline{x},t}$ *whenever* $i \in I$, $t \in T$, $z_i = \max F_i^t(x)$ *and* $x < \overline{x}$ *in* X.

(D) $u_i^t(\overline{x}_{-i}, y_i) \preccurlyeq_i u_i^t(\overline{x}_{-i}, z_i)$ *whenever* $z_i = \max F_i^t(x)$, $y_i \in X_i^{\overline{x},t} \cap (z_i]$, $i \in I$, $x \leqslant \overline{x}$ *in* X *and* $t \in T$.

Then the greatest Nash equilibrium $\overline{x}^t$ *exists for every* $t \in T$. *Moreover, each* $\overline{x}^t$ *maximizes each utility* $u_i^t(x)$ *over all Nash equilibria for* t *if* (Hk) *holds. Each* $t \mapsto \overline{x}^t$ *is increasing if the following hypotheses are also valid.*

(E) $X_i^{x,\overline{t}} \cap (z_i] = \emptyset$ or $z_i \in X_i^{x,\overline{t}}$ *whenever* $i \in I$, $x \in X$, $z_i = \max F_i^t(x)$ *and* $t < \overline{t}$ *in* T.

(F) $u_i^{\overline{t}}(x_{-i}, y_i) \preccurlyeq_i u_i^{\overline{t}}(x_{-i}, z_i)$ *whenever* $z_i = \max F_i^t(x)$, $y_i \in X_i^{x,\overline{t}} \cap (z_i]$, $i \in I$, $x \in X$ *and* $t \leqslant \overline{t}$ *in* T.

Proof. Hypotheses (A) and (B) imply that hypotheses (Ha), (Hb) and (Hc) of Theorem 6.1 hold, whence each $F_i^t(x)$ is nonempty and has the maximum. To show that hypothesis (Hd) of Theorem 6.1 is valid, let $i \in I$ and $t \in T$ be fixed, assume that $x < \overline{x}$ in X, and let $z_i = \max F_i^t(x)$. If $z_i \notin X_i^{\overline{x},t}$, it follows from (C) that $z_i <_i \overline{z}_i$ for all $\overline{z}_i \in X_i^{\overline{x},t}$. In particular, $z_i <_i \overline{z}_i$ for all $\overline{z}_i \in F_i^t(\overline{x})$. Assume next that $z_i \in X_i^{\overline{x},t}$. It follows from (C), (D) and (6.1) that $u_i^{\overline{x},t}(y_i) \preccurlyeq_i u_i^{\overline{x},t}(z_i)$for each $y_i \in X_i^{\overline{x},t} \cap (z_i]$. This result implies that $z_i \leqslant_i \overline{z}_i = \max F_i^t(\overline{x})$.

Consequently, hypothesis (Hd) of Theorem 6.1 holds when $A = F_i^t(x)$ and $B = F_i^t(\overline{x})$. Moreover, hypothesis (A) implies that hypothesis (Hk) of Proposition 6.1 holds. Thus the first two assertions follow from Proposition 6.1.

Hypothesis (A) implies that we can choose each b_i^t to be an upper bound of X_i, whence $t \mapsto b_i^t$ is increasing as a constant mapping. The proof that $t \mapsto F_i^t(x)$ is increasing upwards for all $i \in I$ and $x \in X$ if (E) and (F) hold is similar to the above proof. Thus the last conclusion of Theorem 4.4 implies that the mapping $t \mapsto \overline{x}^t$ is increasing. □

Remark 6.1. The proof of Proposition 5.1 implies that the sets $X_i^{x,t}$ and $u_i^{x,t}[X_i^{x,t}]$ have computable maximums if they are directed upwards and have countable subsets whose strictly increasing sequences are finite, and which are dense upwards. These results combined with those of Section 5 provide tools to compute those Nash equilibria whose existence follows from the above results.

The hypotheses of Theorems 6.2 and 6.3 impose only monotonicity properties for utility functions, whereas in Theorem 6.1 hypotheses given for level sets and up-level sets of the utility functions replace monotonicity. The following simple examples describe the differences of these cases.

Example 6.1. Assume that each strategy set X_i and each set $X_i^{x,t}$ of feasible strategies is a bounded and closed subset of $\mathbb{R}$, and that each $u_i^{x,t}$ is real-valued and upper semicontinuous, i.e.,

$$\limsup_{z_i \to y_i} u_i^{x,t}(z_i) \leqslant u_i^{x,t}(y_i) \quad \text{for each } y_i \in X_i^{x,t}. \tag{6.2}$$

As shown in [5, Example 1], hypothesis (Ha) of Theorem 6.1 holds. Hypothesis (Hb) is also satisfied because each X_i is bounded and closed. Assume moreover that each $u_i^{x,t}$ is left-continuous. If A is a level set of $u_i^{x,t}$, then the limit of each increasing and convergent sequence of A belongs to A. Thus A is closed upwards. As a set of real numbers A is also directed upwards. Consequently, hypothesis (Hc) of Theorem 6.1 is valid. It then follows from the proof of Theorem 6.1 that each set $F_i^t(x)$ is nonempty and has a maximum. Hypothesis (Hj) is also valid. If each function $x \mapsto \max F_i^t(x)$ is increasing, then also hypothesis (Hd) of Theorem 6.1 holds. Thus Proposition 6.1 implies that the greatest Nash equilibrium exists for each t.

If $X_i^{x,t}$ is finite, the above continuity hypotheses hold automatically for any function $u_i^{x,t}$. If $X_i^{x,t}$ is a compact interval, then the possible discontinuities of an upper semicontinuous function $u_i^{x,t}$ are jumps downward.

Example 6.2. Assume, as in Example 6.1, that each strategy set X_i and each set $X_i^{x,t}$ of feasible strategies is a bounded and closed subset of $\mathbb{R}$, and that each $u_i^{x,t}$ is real-valued and increasing. Then hypotheses (Hb), (Hf), (Hg) and (Hj) hold. If the functions $x \mapsto \max X_i^{t,x}$ are increasing, then also hypothesis (Hh) of Theorem 6.2 holds, and Proposition 6.1 implies that the greatest Nash equilibrium exists for each t.

Monotonicity hypotheses will remain the same also when $X_i^{x,t}$ is finite. On the other hand, no continuity is required from $u_i^{x,t}$ even in the case when $X_i^{x,t}$ is a compact interval.

Compared with Example 6.1 an interesting difference is that jumps upward are only possible discontinuities in the case when $u_i^{x,t}$ is increasing. Thus the directions of possible jumps are opposite to each other in Examples 6.1 and 6.2.

Our last example deals with a price game of firms (cf. [15, Chapter 10, Section 3]).

Example 6.3. Suppose a set I of firms i sell versions of a product that compete in a market, and that the unit cost that firm i incurs to manufacture the product is $c_i^{t_i^1} > 0$, where the parameter t_i^1 reflects changes of raw material prices, salaries, etc. The demand $d_i^{t_i^2}(p)$ that firm i experiences for the product is positive and depends on the prices $p = \{p_i\}_{i\in I}$ charged by the firms, and on the parameter t_i^2 which reflects the factors which change in the average purchasing capacity of all consumers (so t_i^2's can be considered as equal). Denoting $t = \{(t_i^1, t_i^2)\}_{i\in I}$, the profit $u_i^t(p)$ earned by each firm i is

$$u_i^t(p) = d_i^{t_i^2}(p)(p_i - c_i^{t_i^1}). \tag{6.3}$$

The possible prices form a finite set $X_i \subset [0, b_i]$ of nonnegative real numbers, the number of possible prices depending on the least unit of the currency and on the largest possible price b_i. Let a set of feasible prices of each firm i be $X_i^{p,t} = [c_i^{t_i^1}, b_i] \cap X_i$. In view of (6.1) and (6.3) we have

$$u_i^{p,t}(y_i) = u_i^t(p_{-i}, y_i) = d_i^{t_i^2}(p_{-i}, y_i)(y_i - c_i^{t_i^1}), \quad y_i \in X_i^{p,t}. \tag{6.4}$$

Since each set of feasible prices is finite, it follows from the considerations of Example 6.1 that hypotheses (A), (B) and (C) of Corollary 6.2 hold.

It is reasonable to assume that when the other firms rise their prices $p_{-i} = \{p_j\}_{j\neq i}$, the demand of the product of firm i increases. This means that the following hypothesis is valid:

(I) Each mapping $p_{-i} \mapsto d_i^{t_i^2}(p_{-i}, y_i)$ is increasing.

In particular, hypothesis (Hk) holds. Moreover, we make the following hypothesis.

(II) Each mapping $p_{-i} \mapsto d_i^{t_i^2}(p_{-i}, y_i)/d_i^{t_i^2}(p_{-i}, z_i)$ is decreasing when $y_i \leqslant z_i = \max F_i^t(p)$.

To show that hypothesis (D) of Corollary 6.2 holds, let $i \in I$ and t_i^j, $j = 1, 2$, be fixed, and assume that $p \leqslant \overline{p}$ in X and $y_i \leqslant z_i = \max F_i^t(p)$ in X_i. Since $p_i = z_i$ maximizes $u_i^t(p_{-i}, p_i)$, we have

$$u_i^t(p_{-i}, y_i) \leqslant u_i^t(p_{-i}, z_i),$$

which is by (6.4) equivalent to

$$d_i^{t_i^2}(p_{-i}, y_i)(y_i - c_i^{t_i^1}) \leqslant d_i^{t_i^2}(p_{-i}, z_i)(z_i - c_i^{t_i^1}),$$

or

$$(d_i^{t_i^2}(p_{-i}, y_i) - d_i^{t_i^2}(p_{-i}, z_i))(y_i - c_i^{t_i^1}) \leqslant d_i^{t_i^2}(p_{-i}, z_i)(z_i - y_i),$$

which can be rewritten as

$$(d_i^{t_i^2}(p_{-i}, y_i)/d_i^{t_i^2}(p_{-i}, z_i) - 1)(y_i - c_i^{t_i^1}) \leqslant z_i - y_i.$$

Applying hypothesis (II) to the above inequality we get

$$(d_i^{t_i^2}(\overline{p}_{-i}, y_i)/d_i^{t_i^2}(\overline{p}_{-i}, z_i) - 1)(y_i - c_i^{t_i^1}) \leqslant z_i - y_i,$$

or

$$(d_i^{t_i^2}(\overline{p}_{-i}, y_i)^{-1} - d_i^{t_i^2}(\overline{p}_{-i}, z_i))(y_i - c_i^{t_i^1}) \leqslant d_i^{t_i^2}(\overline{p}_{-i}, z_i)(z_i - y_i),$$

or

$$d_i^{t_i^2}(\overline{p}_{-i}, y_i)(y_i - c_i^{t_i^1}) \leqslant d_i^{t_i^2}(\overline{p}_{-i}, z_i)(z_i - c_i^{t_i^1}),$$

which is by (6.4) equivalent to

$$u_i^t(\overline{p}_{-i}, y_i) \leqslant u_i^t(\overline{p}_{-i}, z_i).$$

This proves the validity of hypothesis (D) of Corollary 6.2, whence the first part of Corollary 6.2 implies the following result.

(i) The greatest Nash equilibrium $\overline{p}^t$ of prices exists, and it maximizes each profit $u_i^t(p)$ over all Nash equilibria for each t.

The approximations for max $F_i^t(p)$, i.e., the maxima of the prices y_i which maximize the functions $u_i^{p,t}$ on $X_i^{p,t}$, can be computed when the demand functions $d_i^{t_i^2}$ have been estimated by a market research, and when all the manufacturing costs $c_i^{t_i^1}$ are known. If they satisfy hypotheses (I) and (II), an approximation to the greatest Nash equilibrium $\overline{p}^t = \{\overline{p}_i^t\}_{i\in I}$ for prices can be obtained as a last element of the finite sequence of the following iterations:

(IT) $y_0 = b := \{b_i\}_{i\in I}$ and $y_{n+1} = \max F^t(y_n) = \{\max F_i^t(y_n)\}_{i\in I}$ as long as $y_{n+1} < y_n$ in $X = \{X_i\}_{i\in I}$.

It is reasonable to assume that the demands of the products of firms decrease when their prices increase:

(III) Each mapping $y_i \mapsto d_i^{t_i^2}(p_{-i}, y_i)$ is decreasing.

Assume also that the parameters t_i^1 are so ordered that the unit costs of products of firms increase when the parameters t_i^1 increase, i.e., the following hypothesis is valid.

(IV) Each mapping $t_i^1 \mapsto c_i^{t_i^1}$ is increasing.

This implies that hypothesis (E) of Corollary 6.2 holds. Let the sets of parameters t_i^2 be partially ordered, and assume that the following hypothesis holds:

(V) Each mapping $t_i^2 \mapsto d_i^{t_i^2}(p_{-i}, y_i)/d_i^{t_i^2}(p_{-i}, z_i)$ is decreasing when $y_i \leqslant z_i = \max F_i^t(p)$.

To show that hypothesis (F) of Corollary 6.2 holds, let $i \in I$ and $p_{-i} \in X_{-i}$ be fixed, and assume that $t_i^j \leqslant \bar{t}_i^j$, $j = 1, 2$, and $y_i \leqslant z_i = \max F_i^t(p)$ in X_i. Since $p_i = z_i$ maximizes $u_i^t(p_{-i}, p_i)$, we have

$$u_i^t(p_{-i}, y_i) - u_i^t(p_{-i}, z_i),$$

which is by (6.4) equivalent to

$$\begin{aligned} d_i^{t_i^2}(p_{-i}, y_i)(y_i - c_i^{\bar{t}_i^1}) \leqslant\; & d_i^{t_i^2}(p_{-i}, z_i)(z_i - c_i^{\bar{t}_i^1}) \\ & + (d_i^{t_i^2}(p_{-i}, z_i) - d_i^{t_i^2}(p_{-i}, y_i))(c_i^{\bar{t}_i^1} - c_i^{t_i^i}). \end{aligned}$$

The last term in the right-hand side of the above inequality is nonpositive by hypotheses (III) and (IV), whence we obtain

$$d_i^{t_i^2}(p_{-i}, y_i)(y_i - c_i^{\bar{t}_i^1}) \leqslant d_i^{t_i^2}(p_{-i}, z_i)(z_i - c_i^{\bar{t}_i^1}),$$

or

$$(d_i^{t_i^2}(p_{-i}, y_i)/d_i^{t_i^2}(p_{-i}, z_i) - 1)(y_i - c_i^{\bar{t}_i^1}) \leqslant z_i - y_i.$$

Applying hypothesis (V) to the above inequality we get

$$(d_i^{\bar{t}_i^2}(p_{-i}, y_i)/d_i^{\bar{t}_i^2}(p_{-i}, z_i) - 1)(y_i - c_i^{\bar{t}_i^1}) \leqslant z_i - y_i,$$

or

$$d_i^{\bar{t}_i^2}(p_{-i}, y_i)(y_i - c_i^{\bar{t}_i^1}) \leqslant d_i^{\bar{t}_i^2}(p_{-i}, z_i)(z_i - c_i^{\bar{t}_i^1}),$$

which is by (6.4) equivalent to

$$u_i^{\bar{t}}(p_{-i}, y_i) \leqslant u_i^{\bar{t}}(p_{-i}, z_i).$$

Thus also hypothesis (F) of Corollary 6.2 is valid. As a consequence of the last conclusion of Corollary 6.2 we obtain the following result:

(ii) The mapping $t \mapsto \overline{p}^t$ is increasing.

To present a numerical example, assume that $|I| = 3$, and that the demands d_i, $i = 1, 2, 3$, are of the form

$$\begin{cases} d_1((p_2, p_3), p_1) := [400(p_2 + p_3) + (40(p_2 + p_3) - 400)(p_1)^2], \\ d_2((p_1, p_3), p_2) := [410(p_1 + p_3) + (40(p_1 + p_3) - 400)(p_2)^2], \\ d_3((p_1, p_2), p_3) := [420(p_1 + p_2) + (40(p_1 + p_2) - 400)(p_3)^2], \end{cases}$$

where $[z]$ denotes the greatest integer $\leqslant z$. It is easy to see that hypotheses (I) and (II) are valid.

(a) Choosing $c_1 := 1$, $c_2 := 1.1$, $c_2 := 1.2$ and $b_i := 2 * c_i$, $i = 1, 2, 3$, we obtain when the smallest price shift is .05 the following optimal equilibrium for the prices:

$$p_1^* = 1.80, \quad p_2^* = 1.85 \quad \text{and} \quad p_3^* = 1.90.$$

The corresponding profits are: $u_1^* = 552.00$, $u_2^* = 490.55$ and $u_3^* = 431.20$.

(b) Choosing $\overline{c}_1 := 1.2$, $\overline{c}_2 := 1.3$, $\overline{c}_2 := 1.4$ and $\overline{b}_i := 2 * \overline{c}_i$, $i = 1, 2, 3$, the optimal price equilibrium is

$$\overline{p}_1^* = 2.00, \quad \overline{p}_2^* = 2.05 \quad \text{and} \quad \overline{p}_3^* = 2.10.$$

The corresponding profits are: $\overline{u}_1^* = 579.20$, $\overline{u}_2^* = 516.80$ and $\overline{u}_3^* = 455.70$.

The above numerical results are obtained by calculating iterations (IT) with the help of a simple Maple-program.

Remark 6.2. The price model of Example 6.3 has been studied also in [15]. The 'supermultiplicativity' of demand functions assumed in [15] is replaced in Example 6.3 by hypotheses (I) and (II).

The theoretical results of this section differ from those derived, e.g., in [10,14–16] in the sense that the strategy posets need not be complete lattices, not even lattices, and the utility mappings u_i^t need not be chain-valued. These generalizations open new possibilities to apply the theory of Nash equilibria.

References

[1] S. Abian, A.B. Brown, A theorem on partially ordered sets, with applications to fixed point theorems, Can. J. Math. 13 (1961) 78–82.

[2] H. Amann, Order structures and fixed points, Atti 2° Semin. Anal. Funz. Appl. (1977).

[3] A.C. Davis, A characterization of complete lattices, J. Pacific J. Math. 5 (1955) 311–319.

[4] D. Guo, Y.J. Cho, J. Zhu, Partial Ordering Methods in Nonlinear Problems, Nova Science Publishers, New York, 2004.

[5] S. Heikkilä, Applications of a recursion method to maximization problems in partially ordered sets, Nonlinear Stud. 3 (2) (1996) 249–260.

[6] S. Heikkilä, On chain methods used in fixed point theory, Nonlinear Stud. 6 (2) (1999) 171–180.

[7] S. Heikkilä, V. Lakshmikantham, Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations, Marcel Dekker, New York, 1994.

[8] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces II, Function Spaces, Springer, Berlin, 1979.

[9] G. Markowsky, Chain-complete posets and directed sets with applications, Algebra Univ. 6 (1976) 53–68.

[10] P. Milgrom, C. Shannon, Monotone comparative statics, Econometrica 62 (1994) 157–180.

[11] F. Shuhong, Semilattice structure of fixed point set for increasing operators, Nonlinear Anal. 27 (7) (1996) 793–796.

[12] R.E. Smithson, Fixed points of order preserving multifunctions, Proc. Am. Math. Soc. 28 (1) (1971) 304–310.

[13] A. Tarski, A lattice theoretical fixpoint theorem and its applications, J. Pacific J. Math. 5 (1955) 285–309.

[14] D. Topkis, Equilibrium points in nonzero-sum n-person submodular game, SIAM J. Optim. Control 17 (1989) 773–787.

[15] A.F. Veinott Jr., Lattice Programming, Draft for OR 375 Seminar, Johns Hopkins University Press, Baltimore, London, 1992.

[16] X. Vives, Nash equilibrium with strategic complementaries, J. Math. Econ. (1990) 305–321.

[17] E. Zeidler, Nonlinear Functional Analysis and Its Applications, vol. I, Springer, Berlin, 1985.