ON EXTREMAL NASH EQUILIBRIA FOR MIXED EXTENSIONS OF FINITE NORMAL-FORM GAMES

S. Heikkilä[a,*] and K. Reffett[b]

[a]Department of Mathematical Sciences, University of Oulu, Box 3000, FIN-90014 University of Oulu, Finland
[b]Department of Economics, WP Carey School of Business, Arizona State University, Tempe AZ 85287, U.S.A.

Abstract. In this paper derive conditions under which a mixed extension of a normal-form game has extremal Nash equilibria in pure strategies, and that either of them gives best utilities among all mixed Nash equilibria when strategy spaces are finite posets and the values of utility functions are in ordered vector spaces.

Keywords: normal-form game, pure strategy, mixed strategy, Nash equilibrium, least, greatest

AMS Subject Classification: 91A06, 91A10, 91A44

1. Introduction

Recently it is shown (cf. [1]) that when the strategies of a supermodular normal-form game are real numbers, the mixed extension of that game is supermodular, its equilibria form a non-empty complete lattice, and its extremal equilibria are in pure strategies when mixed strategies are ordered by first order stochastic dominance. The problems which arise when the strategies are not in $\mathbb{R}$ are described in Chapter 3 of [1] as follows: "When the strategy spaces are multidimensional, the set of mixed strategies is not a lattice. This implies that we lack the mathematical structure needed for the theory of complementarities."

In this paper we derive existence and comparison results for Nash equilibria of normal-form games $G = \{S_1, \dots, S_n, u_1, \dots, u_n\}$ whose strategy spaces S_i are finite posets and utility functions u_i are vector-valued. In particular, a mixed extension of any supermodular game G whose strategy spaces are finite lattices and utility functions are real-valued is shown to posses least and greatest Nash equilibria in pure strategies. If the utilities $u_i(s_1, \dots, s_n)$ are also increasing (respectively decreasing) in s_j, $j \neq i$, the utilities of the greatest (respectively the least) Nash equilibrium majorize the utilities of all pure and mixed Nash equilibria. Consequently, the lack of the lattice structure of the spaces of mixed strategies of such games don't prevent from proving that their extremal and also most profitable Nash equilibria exists and are in pure strategies.

* Corresponding author. E-mail: sheikki@cc.oulu.fi, Phone: +35885531736, Fax: +35885531730

2. Preliminaries

By a *partially ordered set* (*poset*) we mean a nonempty set X equipped with a reflexive, antisymmetric and transitive order relation '$\le$'. If X is a real vector space, and if $x \le y$ implies that $x + z \le y + z$ for all $z \in X$ and $\alpha x \le \alpha y$ for all $\alpha \ge 0$, we say that X is an *ordered vector space*. If $x \le y$ and $x \ne y$, denote $x < y$. A function f from a poset $X = (X, \le)$ to another poset $Y = (Y \preceq)$ is called *increasing* if $f(x) \preceq f(y)$ whenever $x \le y$, and *decreasing* if $f(y) \preceq f(x)$ whenever $x \le y$.

Let S be a nonempty subset of a poset X. If $z \in X$ and $x \le z$ for all $x \in S$, then z is called an *upper bound* of S. If $z \le y$ for all other upper bounds y of S, we say that z is the *least upper bound* of S, and denote $z = \sup S$. If $z \in S$ and $x \le z$ for all $x \in S$ we say that z is the *greatest element* of S, and denote $z = \max S$. A lower bound, the greatest lower bound $\inf S$ and the least element $\min S$ of S are defined similarly.

We say that a nonempty subset S of a poset X is *directed upward* if to each pair x, y of elements of S there corresponds a $z \in S$ such that $x \le z$ and $y \le z$. If the reversed inequalities hold, we say that S is *directed downward*. If S is both upward and downward directed, we call S *directed*.

S is called a *sublattice* if $x \vee y := \sup\{x, y\}$ and $x \wedge y := \inf\{x, y\}$ exist in X and belong to S all x, $y \in S$. S is a *chain* if $x \le y$ or $y \le x$ for all x, $y \in S$. Every chain is a sublattice and every sublattice is directed.

Definition 1. We say that $G = \{S_1, \dots, S_n, u_1, \dots, u_n\}$ is a *finite normal-form game* of players i, $i = 1, \dots, n$, if each *strategy set* S_i for player i is a finite nonempty subset of a poset $X_i = (X_i, \le_i)$ and u_i is a *utility function* of player i, defined on $S = S_1 \times \cdots \times S_n$, and having values in an ordered vector space $E_i = (E_i, \le_i)$.

The *mixed extension* Γ of a finite normal-form game G is obtained when players i are allowed to choose independently randomizations $\sigma_i \in \Sigma_i$ of strategies of S_i, that is, each *mixed strategy* σ_i is a probability measure over S_i. The utilities $\mathcal{U}_i$ are the expected values:

$$\mathcal{U}_i(\sigma_1, \dots, \sigma_n) = \sum_{(s_1,\dots,s_n)\in S} \sigma_1(\{s_1\}) \cdots \sigma_m(\{s_n\}) u_i(s_1, \dots, s_n),\ i = 1, \dots, n. \tag{1}$$

Strategies of Σ_i of the form $\delta_{s_i}(x_i) = \begin{cases} 1,\ x_i = s_i \\ 0,\ x_i \ne s_i \end{cases}$, $s_i \in S_i$, are called *pure strategies*. The set of pure strategies of Σ_i is denoted by Δ_i. In the following we use notations: $s_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$, $u_i(s_1, \dots, s_n) = u_i(s_i, s_{-i})$, $\sigma_{-i} = (\sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_n)$, $\mathcal{U}_i(\sigma_1, \dots, \sigma_n) = \mathcal{U}_i(\sigma_i, \sigma_{-i})$, $S_{-i} = (S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_n)$,
$\Sigma_{-i} = (\Sigma_1, \dots, \Sigma_{i-1}, \Sigma_{i+1}, \dots, \Sigma_n)$ and $\Delta_{-i} = \Delta_1 \times \cdots \times \Delta_{i-1} \times\ \Delta_{i+1} \times \cdots \times \Delta_n$.

Definition 2. Let $G = \{S_1, \dots, S_n, u_1, \dots, u_n\}$ be a finite normal-form game. We say that mixed strategies $\sigma_1^*, \dots, \sigma_n^*$ form a *Nash equilibrium* for the mixed extension Γ of G if

$$\mathcal{U}_i(\sigma_i^*, \sigma_{-i}^*) = \max_{\sigma_i \in \Sigma_i} \mathcal{U}_i(\sigma_i, \sigma_{-i}^*) \ \text{ for all } \ i = 1, \dots, n. \tag{N}$$

3. An existence result

In what follows we assume that $G = \{S_1, \dots, S_n, u_1, \dots, u_n\}$ is a finite normal-form game. All products of posets are assumed to be ordered by product ordering. Assume also that for each fixed $i = 1, \dots, n$ the set Σ_i of probability measures on S_i is ordered by first order stochastic dominance $\preceq_i$, defined as follows:

(SD) $\sigma_i \preceq_i \tau_i$ iff $\sigma_i(A) \le \tau_i(A)$ for such $A \subseteq S_i$ that $x_i \in A$ and $x_i \le_i y_i$ imply $y_i \in A$.

It follows from (1) that

$$\begin{aligned} \mathcal{U}_i(\sigma_i, \sigma_{-i}) &= \sum_{s_i \in S_i} \sigma_i(\{s_i\}) U_i(s_i, \sigma_{-i}), \quad i = 1, \dots, n, \text{ where} \\ U_i(s_i, \sigma_{-i}) &= \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(\{s_{-i}\}) u_i(s_i, s_{-i}),\ \sigma_{-i}(\{s_{-i}\}) = \prod_{j \ne i} \sigma_j(\{s_j\}). \end{aligned} \tag{2}$$

Since each S_i is finite, and hence each function $s_i \mapsto U_i(s_i, \sigma_{-i})$ has only a finite number of values, then the hypothesis

(H0) *For every fixed $i = 1, \dots, n$ and $\sigma_{-i} \in \Sigma_{-i}$ the set $\{U_i(s_i, \sigma_{-i}) \mid s_i \in S_i\}$ is directed upward.*

ensures that the sets

$$F_i(\sigma_{-i}) = \{x_i \in S_i \mid U_i(x_i, \sigma_{-i}) = \max_{s_i \in S_i} U_i(s_i, \sigma_{-i})\}, \quad i = 1, \dots, n, \tag{3}$$

are nonempty. The following hypothesis

(H1) *For every fixed $i = 1, \dots, n$ and $\sigma_{-i} \in \Sigma_{-i}$ the set $F_i(\sigma_{-i})$ is directed upward.*

implies that the maximums of $F_i(\sigma_{-i})$ exist. Assume that they have the following properties.

(H2) *Each set $\{\max F_i(\sigma_{-i}) \mid \sigma_{-i} \in \Sigma_{-i}\}$ has an upper bound in S_i.*

(H3) *If $\sigma_{-i} \le \tau_{-i}$ in Σ_{-i} and $\tau_{-i} \in \Delta_{-i}$, then $\max F_i(\sigma_{-i}) \le_i \max F_i(\tau_{-i})$, $i = 1, \dots, n$.*

Lemma 1. *Let the hypotheses (H0)–(H3) hold. Then for all fixed $i \in \{1, \dots, n\}$ and $\sigma_{-i} \in \Sigma_{-i}$ the set*

$$\mathcal{F}_i(\sigma_{-i}) = \{\tau_i \in \Sigma_i \mid \mathcal{U}_i(\tau_i, \sigma_{-i}) = \max_{\sigma_i \in \Sigma_i} \mathcal{U}_i(\sigma_i, \sigma_{-i})\} \tag{4}$$

is nonempty, and

$$\max \mathcal{F}_i(\sigma_{-i}) = \delta_{\overline{s}_i(\sigma_{-i})}, \text{ where } \overline{s}_i(\sigma_{-i}) = \max F_i(\sigma_{-i}),\ i = 1, \dots, n,\ \sigma_{-i} \in \Sigma_{-i}. \tag{5}$$

Proof. Let $i \in \{1, \dots, n\}$ and $\sigma_{-i} \in \Sigma_{-i}$ be fixed. Denoting $c_i = \max_{s_i \in S_i} U_i(s_i, \sigma_{-i})$, it follows from (2) and (3) that for each $\tau_i \in \Sigma_i$,

$$\mathcal{U}_i(\tau_i, \sigma_{-i}) = c_i \cdot \tau_i(F_i(\sigma_{-i})) + \sum_{s_i \in S_i \setminus F_i(\sigma_{-i})} \tau_i(\{s_i\}) U_i(s_i, \sigma_{-i}).$$

This result implies that $\mathcal{U}_i(\tau_i, \sigma_{-i}) \leq_i c_i$, and that equality holds if $\tau_i(F_i(\sigma_{-i})) = 1$. Thus $\tau_i \in \mathcal{F}_i(\sigma_{-i})$ iff $\mathcal{U}_i(\tau_i, \sigma_{-i}) = c_i$. Consequently, if $\tau_i \in \mathcal{F}_i(\sigma_{-i})$, then

$$0 = c_i - \mathcal{U}_i(\tau_i, \sigma_{-i}) = \sum_{s_i \in S_i \setminus F_i(\sigma_{-i})} \tau_i(\{s_i\})(c_i - U_i(s_i, \sigma_{-i})).$$

Since $U_i(s_i, \sigma_{-i}) <_i c_i$ for each $s_i \in S_i \setminus F_i(\sigma_{-i})$, the above result implies that $\tau_i(\{s_i\}) = 0$ for each $s_i \in S_i \setminus F_i(\sigma_{-i})$, whence $\tau_i(F_i(\sigma_{-i})) = 1$.

The above proof shows that

$$\mathcal{F}_i(\sigma_{-i}) = \{\tau_i \in \Sigma_i \mid \tau_i(F_i(\sigma_{-i})) = 1\}, \quad i = 1, \dots, n, \ \sigma_{-i} \in \Sigma_{-i}. \tag{6}$$

It follows from (6) and from the definition (SD) of $\preceq_i$ that (5) holds. □

Now we are ready to prove our first existence result.

Theorem 1. *Under the hypotheses (H0)–(H3) there exist such pure strategies $\overline{\sigma}_1^*, \dots, \overline{\sigma}_n^*$ that for every $i = 1, \dots, n$ the strategy $\overline{\sigma}_i^*$ is the greatest of those mixed strategies σ_i of player i which maximize the utility $\mathcal{U}_i(\sigma_i, \sigma_{-i})$ when the strategy vector of other players is $\overline{\sigma}_{-i}^* = (\overline{\sigma}_1^*, \dots, \overline{\sigma}_{i-1}^*, \overline{\sigma}_{i+1}^*, \dots \overline{\sigma}_n^*)$. In particular, the strategies $\overline{\sigma}_1^*, \dots, \overline{\sigma}_n^*$ form a Nash equilibrium for Γ. It can be obtained by a finite number of successive approximations.*

Proof. The hypothesis (H3), the definition (SD) of $\preceq_i$ and (5) imply that

$$\delta_{\overline{s}_i(\sigma_{-i})} \preceq_i \delta_{\overline{s}_i(\hat{\sigma}_{-i})} \text{ whenever } \sigma_{-i} \leq \hat{\sigma}_{-i} \text{ in } \Delta_{-i} \text{ for all } \ i = 1, \dots, n. \tag{7}$$

By the hypothesis (H2) we can choose upper bounds $b_i \in S_i$ for the sets $\{\max F_i(\sigma_{-i}) \mid \sigma_{-i} \in \Sigma_{-i}\}$, $i = 1, \dots, n$. Denote

$$\sigma_i^0 := \delta_{b_i} \ \text{ and } \sigma_{-i}^0 := (\delta_{b_1}, \dots, \delta_{b_{i-1}}, \delta_{b_{i+1}}, \dots, \delta_{b_n}), \ i = 1, \dots, n.$$

The above notations, (5) and the definition (ST) of $\preceq_i$ imply that

$$\sigma_i^1 := \delta_{\overline{s}_i(\sigma_{-i}^0)} \preceq_i \delta_{b_i} = \sigma_i^0 \ \text{ for all } i = 1, \dots, n.$$

By this result we also have $\sigma_{-i}^1 := (\sigma_1^1, \dots, \sigma_{i-1}^1, \sigma_{i+1}^1, \dots, \sigma_n^1) \leq \sigma_{-i}^0$ in Δ_{-i} for all $i = 1, \dots n$, which by (7) implies that

$$\sigma_i^2 := \delta_{\overline{s}_i(\sigma_{-i}^1)} \preceq_i \delta_{\overline{s}_i(\sigma_{-i}^0)} = \sigma_i^1 \ \text{ for all } i = 1, \dots, n.$$

According to this result we then have $\sigma_{-i}^2 := (\sigma_1^2, \dots, \sigma_{i-1}^2, \sigma_{i+1}^2, \dots, \sigma_n^2) \leq \sigma_{-i}^1$ in Δ_{-i} for all $i = 1, \dots n$. It then follows from (7) that

$$\sigma_i^3 := \delta_{\overline{s}_i(\sigma_{-i}^2)} \preceq_i \delta_{\overline{s}_i(\sigma_{-i}^1)} = \sigma_i^2 \ \text{ for all } i = 1, \dots, n, \text{ e.t.c.}$$

Because the set $\{\max F_i(\sigma_{-i}) \mid \sigma_{-i} \in \Delta_{-i}\}$ is a subset of a finite set S_i for each $i = 1, \ldots, n$, then, by (5), the set $\{\delta_{\overline{s}_i(\sigma_{-i})} \mid \sigma_{-i} \in \Delta_{-i}\}$ is finite. Thus, continuing the above reasoning a finite number of times, say k times, we get the following result:

$$\sigma_i^{k+1} := \delta_{\overline{s}_i(\sigma_{-i}^k)} = \delta_{\overline{s}_i(\sigma_{-i}^{k-1})} = \sigma_i^k \quad \text{for all } i = 1, \ldots, n.$$

Denoting $\overline{\sigma}_i^* := \sigma_i^k$ and $\overline{\sigma}_{-i}^* := (\sigma_1^k, \ldots, \sigma_{i-1}^k, \sigma_{i+1}^k, \ldots, \sigma_n^k)$, $i = 1, \ldots, n$, we then have

$$\overline{\sigma}_i^* = \delta_{\overline{s}_i(\overline{\sigma}_{-i}^*)} = \max \mathcal{F}_i(\overline{\sigma}_{-i}^*) \quad \text{for all } i = 1, \ldots, n. \tag{8}$$

According to this result the strategies $\overline{\sigma}_i^*$ are the greatest strategies which maximize the utility $\mathcal{U}_i(\sigma_i, \sigma_{-i})$ when $\sigma_{-i} = \overline{\sigma}_{-i}^*$. In particular,

$$\mathcal{U}_i(\overline{\sigma}_i^*, \overline{\sigma}_{-i}^*) = \max_{\sigma_i \in \Sigma_i} \mathcal{U}_i(\sigma_i, \overline{\sigma}_{-i}^*), \quad i = 1, \ldots, n,$$

so that the strategies $\overline{\sigma}_1^*, \ldots, \overline{\sigma}_n^*$ satisfy the Nash equilibrium condition (N). Moreover, they are pure strategies by (8). □

4. Comparison results

In the proof of our main comparison results we shall apply the following Lemma.

Lemma 2. *Let the hypotheses (H0)–(H3) hold and let $\mathcal{F}_i(\sigma_{-i})$ be defined by (4). Then the following result is valid.*

$$\text{If } \sigma_i \preceq_i \max \mathcal{F}_i(\sigma_{-i}) \text{ for each } i = 1, \ldots, n, \text{ then } \sigma_i \preceq_i \overline{\sigma}_i^* \text{ for each } i = 1, \ldots, n. \tag{9}$$

Assume moreover, that

(H4) $u_i(s_i, s_{-i}) \leq_i u_i(s_i, \hat{s}_{-i})$ *whenever* $s_{-i} \leq \hat{s}_{-i}$ *in* S_{-i}, $s_i \in S_i$ *and* $i = 1, \ldots, n$.

Then for all $i = 1, \ldots, n$,

$$U_i(s_i, \sigma_{-i}) \leq_i U_i(s_i, \tau_{-i}) \quad \text{whenever } \sigma_{-i} \leq \tau_{-i} \text{ in } \Sigma_{-i},\ \tau_{-i} \in \Delta_{-i} \text{ and } s_i \in S_i. \tag{10}$$

Proof. The hypothesis (H3), the definition (SD) of $\preceq_i$ and (5) imply that

$$\max \mathcal{F}_i(\sigma_{-i}) \preceq_i \max \mathcal{F}_i(\tau_{-i}) \text{ whenever } \sigma_{-i} \leq \tau_{-i} \text{ in } \Sigma_{-i} \text{ and } \tau_{-i} \in \Delta_{-i},\ i = 1, \ldots, n. \tag{11}$$

Assume that $\sigma_i \preceq_i \max \mathcal{F}_i(\sigma_{-i})$ for each $i = 1, \ldots, n$. According to the notations used in the proof of Theorem 1 and the hypothesis (H2) we have

$$\sigma_i \preceq_i \max \mathcal{F}_i(\sigma_{-i}) \preceq_i \delta_{b_i} = \sigma_i^0 \quad \text{for all } i = 1, \ldots, n.$$

By this result we have also $\sigma_{-i} \leq \sigma_{-i}^0$ for all $i = 1, \ldots n$, which implies by the above result and (11) that

$$\sigma_i \preceq_i \max \mathcal{F}_i(\sigma_{-i}) \preceq_i \max \mathcal{F}_i(\sigma_{-i}^0) = \sigma_i^1 \quad \text{for all } i = 1, \ldots, n.$$

This result implies that $\sigma_{-i} \le \sigma^1_{-i}$ for all $i = 1, \dots n$. By the above result and (11) we then have

$$\sigma_i \preceq_i \max \mathcal{F}_i(\sigma_{-i}) \preceq_i \max \mathcal{F}_i(\sigma^1_{-i}) = \sigma^2_i \text{ for all } = 1, \dots, n, \text{ e.t.c.}$$

Repeating the above process k times, where k is as in the proof of Theorem 1, we get

$$\sigma_i \preceq_i \max \mathcal{F}_i(\sigma_{-i}) \preceq_i \max \mathcal{F}_i(\sigma^k_{-i}) = \sigma^k_i = \overline{\sigma}^*_i \quad \text{for all } i = 1, \dots, n.$$

This proves the first assertion.

To prove that (10) holds, assume that the hypothesis (H4) is valid. Let $i \in \{1, \dots, n\}$ and $s_i \in S_i$ be given, and assume that $\sigma_{-i} \le \tau_{-i}$ in Σ_{-i}, and that
$\tau_{-i} = (\delta_{\hat{s}_1}, \dots, \delta_{\hat{s}_{i-1}}, \delta_{\hat{s}_{i+1}}, \dots, \delta_{\hat{s}_n}) \in \Delta_{-i}$. Since $\sigma_{-i} \le \tau_{-i}$ in Σ_{-i}, then $\sigma_j \preceq_j \delta_{\hat{s}_j}$ for $j \ne i$. If $s_j \in S_j$ and $s_j \not\le_j \hat{s}_j$, then $\hat{s}_j \notin A_j = \{x_j \in S_j \mid s_j \le_j x_j\}$. Since A_j is increasing, it follows from the definition (SD) of $\preceq_j$ that $\sigma_j(\{s_j\}) \le \sigma_j(A_j) \le \delta_{\hat{s}_j}(A_j) = 0$. Thus $s_j \le_j \hat{s}_j$ for all $s_j \in S_j$ for which $\sigma_j(\{s_j\}) > 0$. This result and the hypothesis (H4) imply that

$$U_i(s_i, \sigma_{-i}) = \sum_{s_{-i} \in S_{-i}} \prod_{j \ne i} \sigma_j(\{s_j\}) u_i(s_i, s_{-i}) \le_i \sum_{s_{-i} \in S_{-i}} \prod_{j \ne i} \sigma_j(\{s_j\}) u_i(s_i, \hat{s}_{-i})$$
$$\le_i u_i(s_i, \hat{s}_{-i})) = U_i(s_i, \tau_{-i}).$$

This proves second assertion. □

As a consequence of Lemma 2 we get our main comparison result.

Theorem 2. *Let the hypotheses (H0)–(H3) hold, and let $\overline{\sigma}^*_1, \dots, \overline{\sigma}^*_n$ be the Nash equilibrium constructed in Theorem 1. If $\sigma^*_1, \dots, \sigma^*_N$ is a Nash equilibrium for Γ, then $\sigma^*_i \preceq_i \overline{\sigma}^*_i$ for each $i = 1, \dots, N$. Assume moreover that (H4) holds. Then $\mathcal{U}_i(\sigma^*_i, \sigma^*_{-i}) \le_i \mathcal{U}_i(\overline{\sigma}^*_i, \overline{\sigma}^*_{-i})$ for all $i = 1, \dots, N$.*

Proof. Let $\sigma^*_1, \dots, \sigma^*_n$ be a Nash equilibrium for Γ. Then $\sigma^*_i \in \mathcal{F}_i(\sigma^*_{-i})$, whence $\sigma^*_i \preceq_i \max \mathcal{F}_i(\sigma^*_{-i})$ for each $i = 1, \dots n$. This result implies by (9) that $\sigma^*_i \preceq_i \overline{\sigma}^*_i$ for each $i = 1, \dots, n$, which proves the first assertion.

Assume next that also the hypothesis (H4) holds. Since $\sigma^*_i \preceq_i \overline{\sigma}^*_i$ for all $i = 1, \dots, n$, then also $\sigma^*_{-i} \le \overline{\sigma}^*_{-i}$ in Σ_{-i} for all $i = 1, \dots, n$. It then follows from (10) and from the equilibrium condition (N) that

$$\mathcal{U}_i(\sigma^*_i, \sigma^*_{-i}) = \sum_{s_i \in S_i} \sigma^*_i(\{s_i\}) U_i(s_i, \sigma^*_{-i}) \le_i \sum_{s_i \in S_i} \sigma^*_i(\{s_i\}) U_i(s_i, \overline{\sigma}^*_{-i}) =$$
$$\mathcal{U}_i(\sigma^*_i, \overline{\sigma}^*_{-i}) \le_i \mathcal{U}_i(\overline{\sigma}^*_i, \overline{\sigma}^*_{-i}) \quad \text{for all} \quad i = 1, \dots, n.$$

This proves the second assertion. □

5. Dual results

In this section we derive existence and comparison results for the least Nash equilibrium of a finite normal-form game $G = \{S_1, \dots, S_n, u_1, \dots, u_n\}$. The hypotheses (H1)–(H3) will be replaced by the following hypotheses.

(Ha) *For every fixed $i = 1, \dots, n$ the sets $F_i(\sigma_{-i})$, $\sigma_{-i} \in \Sigma_{-i}$, are directed downward.*

(Hb) *Each set $\{\min F_i(\sigma_{-i}) \mid \sigma_{-i} \in \Sigma_{-i}\}$ has a lower bound in S_i.*

(Hc) *If $\sigma_{-i} \le \tau_{-i}$ in Σ_i and $\tau_i \in \Delta_{-i}$, then $\min F_i(\sigma_{-i}) \le_i \min F_i(\tau_{-i})$.*

The following existence result is dual to that of Theorem 1.

Theorem 3. *Under the hypotheses (H0), (Ha), (Hb) and (Hc) there exist such pure strategies $\underline{\sigma}_1^*, \dots, \underline{\sigma}_n^*$ that for every $i = 1, \dots, n$ the strategy $\underline{\sigma}_i^*$ is the least of those mixed strategies σ_i of player i which maximize the utility $\mathcal{U}_i(\sigma_i, \sigma_{-i})$ when the strategy vector of other players is $\underline{\sigma}_{-i}^* = (\underline{\sigma}_1^*, \dots, \underline{\sigma}_{i-1}^*, \underline{\sigma}_{i+1}^*, \dots \underline{\sigma}_n^*)$. In particular, the strategies $\underline{\sigma}_1^*, \dots, \underline{\sigma}_n^*$ form a Nash equilibrium for Γ. It can be obtained by a finite number of successive approximations.*

Proof. Since each S_i is finite, it follows from the hypothesis (Ha) that for each $i = 1, \dots, n$ the set $F_i(\sigma_{-i})$ of maximizers of the function $s_i \mapsto U_i(s_i, \sigma_{-i})$ has the least element. Applying the result (6) and the definition (SD) of $\preceq_i$ we see that

$$\delta_{\underline{s}_i(\sigma_{-i})} = \min \mathcal{F}_i(\sigma_{-i}), \text{ where } \underline{s}_i(\sigma_{-i}) = \min F_i(\sigma_{-i}), \ i = 1, \dots, n. \tag{12}$$

The hypothesis (Hc), the definition (SD) of $\preceq_i$ and (12) imply that

$$\delta_{\underline{s}_i(\sigma_{-i})} \preceq_i \delta_{\underline{s}_i(\hat{\sigma}_{-i})} \text{ whenever } \sigma_{-i} \le \hat{\sigma}_{-i} \text{ in } \Delta_{-i} \text{ for all } \ i = 1, \dots, n. \tag{13}$$

By the hypothesis (Hb) we can choose lower bounds $a_i \in S_i$ for the sets $\{\min F_i(\sigma_{-i}) \mid \sigma_{-i} \in \Sigma_{-i}\}$, $i = 1, \dots, n$. Denote

$$\sigma_i^0 := \delta_{a_i} \text{ and } \sigma_{-i}^0 := (\delta_{a_1}, \dots, \delta_{a_{i-1}}, \delta_{a_{i+1}}, \dots, \delta_{a_n}), \ i = 1, \dots, n.$$

The above notations, (12) and (13) imply that

$$\delta_{a_i} = \sigma_i^0 \preceq_i \sigma_i^1 := \delta_{\underline{s}_i(\sigma_{-i}^0)} \text{ for all } i = 1, \dots, n.$$

By this result we also have $\sigma_{-i}^0 \le \sigma_{-i}^1 := (\sigma_1^1, \dots, \sigma_{i-1}^1, \sigma_{i+1}^1, \dots, \sigma_n^1)$ in Δ_{-i} for all $i = 1, \dots n$, which by (12) and (13) implies that

$$\sigma_i^1 = \delta_{\underline{s}_i(\sigma_{-i}^0)} \preceq_i \sigma_i^2 := \delta_{\underline{s}_i(\sigma_{-i}^1)} \text{ for all } i = 1, \dots, n, \text{ e.t.c.}$$

Because the set $\{\min F_i(\sigma_{-i}) \mid \sigma_{-i} \in \Delta_{-i}\}$ is a subset of a finite set S_i for each $i = 1, \dots, n$, then, by (12), the set $\{\delta_{\underline{s}_i(\sigma_{-i})} \mid \sigma_{-i} \in \Delta_{-i}\}$ is finite. Thus, continuing the above reasoning a finite number of times, say j times, we get the following result:

$$\sigma_i^j = \delta_{\underline{s}_i(\sigma_{-i}^{j-1})}) = \sigma_i^{j+1} := \delta_{\underline{s}_i(\sigma_{-i}^j)} \text{ for all } i = 1, \dots, n.$$

Denoting $\underline{\sigma}_i^* := \sigma_i^j$ and $\underline{\sigma}_{-i}^* := (\sigma_1^j, \dots, \sigma_{i-1}^j, \sigma_{i+1}^j, \dots, \sigma_n^j)$, $i = 1, \dots, n$, we then have

$$\underline{\sigma}_i^* = \delta_{\underline{s}_i(\underline{\sigma}_{-i}^*)} = \min \mathcal{F}_i(\underline{\sigma}_{-i}^*) \quad \text{for all } i = 1, \dots, n. \tag{14}$$

According to this result the strategies $\underline{\sigma}_i^*$ are the least strategies which maximize the utility $\mathcal{U}_i(\sigma_i, \sigma_{-i})$ when $\sigma_{-i} = \underline{\sigma}_{-i}^*$. In particular,

$$\mathcal{U}_i(\underline{\sigma}_i^*, \underline{\sigma}_{-i}^*) = \max_{\sigma_i \in \Sigma_i} \mathcal{U}_i(\sigma_i, \underline{\sigma}_{-i}^*), \quad i = 1, \dots, n,$$

so that the strategies $\underline{\sigma}_1^*, \dots, \underline{\sigma}_n^*$ satisfy the Nash equilibrium condition (N). Moreover, they are pure strategies by (14). □

The following Lemma is dual to Lemma 2, and its proof is similar.

Lemma 3. *Under hypotheses (H0), (Ha), (Hb) and (Hc) the following result is valid.*

$$\text{If } \min \mathcal{F}_i(\sigma_{-i}) \preceq_i \sigma_i \text{ for each } i = 1, \dots, n, \text{ then } \underline{\sigma}_i^* \preceq_i \sigma_i \text{ for each } i = 1, \dots, n. \tag{15}$$

Assume moreover that

(Hd) $u_i(s_i, \hat{s}_{-i}) \le_i u_i(s_i, s_{-i})$ *whenever* $s_{-i} \le \hat{s}_{-i}$ *in* S_{-i}, $s_i \in S_i$ *and* $i = 1, \dots, N$.

Then

$$U_i(s_i, \tau_{-i}) \le_i U_i(s_i, \sigma_{-i}) \quad \text{whenever } \sigma_{-i} \le \tau_{-i} \text{ in } \Sigma_{-i},\ \sigma_{-i} \in \Delta_{-i} \text{ and } s_i \in S_i. \tag{16}$$

As a consequence we get the following comparison results which are dual to the results of Theorem 2.

Theorem 4. *Let G be a finite normal-form game. Assume that the hypotheses (H0), (Ha), (Hb) and (Hc) hold, and let $\underline{\sigma}_1^*, \dots, \underline{\sigma}_N^*$ be the Nash equilibrium for Γ constructed in Theorem 3. If $\sigma_1^*, \dots, \sigma_N^*$ is a Nash equilibrium for Γ, then $\underline{\sigma}_i^* \preceq_i \sigma_i^*$ for each $i = 1, \dots, N$. Assume moreover that (Hd) holds. Then $\mathcal{U}_i(\sigma_i^*, \sigma_{-i}^*) \le_i \mathcal{U}_i(\underline{\sigma}_i^*, \underline{\sigma}_{-i}^*)$ for all $i = 1, \dots, N$.*

Proof. Let $\sigma_1^*, \dots, \sigma_n^*$ be a Nash equilibrium for Γ. Then $\sigma_i^* \in \mathcal{F}_i(\sigma_{-i}^*)$, whence $\min \mathcal{F}_i(\sigma_{-i}^*) \preceq_i \sigma_i^*$ for each $i = 1, \dots n$. This result implies by (15) that $\underline{\sigma}_i^* \preceq_i \sigma_i^*$ for each $i = 1, \dots, n$, which proves the first assertion.

Assume next that also the hypothesis (Hd) holds. Since $\underline{\sigma}_i^* \preceq_i \sigma_i^*$ for each $i = 1, \dots, N$, then also $\underline{\sigma}_{-i}^* \preceq_i \sigma_{-i}^*$ for each $i = 1, \dots, N$. It then follows from (16) and (N) that

$$\begin{aligned} \mathcal{U}_i(\sigma_i^*, \sigma_{-i}^*) &= \sum_{s_i \in S_i} \sigma_i^*(\{s_i\}) U_i(s_i, \sigma_{-i}^*) \le_i \sum_{s_i \in S_i} \sigma_i^*(\{s_i\}) U_i(s_i, \underline{\sigma}_{-i}^*) \\ &= \mathcal{U}_i(\sigma_i^*, \underline{\sigma}_{-i}^*) \le_i \mathcal{U}_i(\underline{\sigma}_i^*, \underline{\sigma}_{-i}^*) \quad \text{for all } \ i = 1, \dots, N. \end{aligned}$$

This proves the second assertion. □

6. Consequences

In this section we apply the results of Section 3, 4 and 5 to finite supermodular normal-form games whose *utility functions are vector-valued.*

Definition 3. A normal-form game $G = \{S_1, \dots, S_n, u_1, \dots, u_n\}$ is a *finite supermodular game* if for all $i = 1, \dots, n$,

(h1) S_i is a finite sublattice of a poset $X_i = (X_i, \le_i)$, and the values of u_i are in an ordered vector space $E_i = (E_i, \le_i)$;

(h2) $u_i(s_i, s_{-i})$ is *supermodular* in s_i, i.e., if $x_i, y_i \in S_i$ and $s_{-i} \in S_{-i}$, then $u_i(x_i, s_{-i}) + u_i(y_i, s_{-i}) \le_i u_i(x_i \wedge y_i, s_{-i}) + u_i(x_i \vee y_i, s_{-i})$;

(h3) $u_i(s_i, s_{-i})$ *has increasing differences* in (s_i, s_{-i}), i.e., if $x_i <_i y_i \in S_i$ and $s_{-i} < \hat{s}_{-i}$ in S_{-i}, then $u_i(y_i, s_{-i}) - u_i(x_i, s_{-i}) \le_i u_i(y_i, \hat{s}_{-i}) - u_i(x_i, \hat{s}_{-i})$.

Lemma 4. *The hypotheses (H1)–(H3), (Ha), (Hb) and (Hc) are valid for the mixed extension of any finite supermodular normal-form game.*

Proof. Let $G = \{S_1, \dots, S_n, u_1, \dots, u_n\}$ be a finite supermodular normal-form game. To prove the validity of (H1), let $i \in \{1, \dots, n\}$, $\sigma_{-i} \in \Sigma_{-i}$ and x_i, $y_i \in F_i(\sigma_{-i})$ be given. Applying (2), (3) and condition (h2) we obtain

$$0 \le U_i(x_i, \sigma_{-i}) - U_i(x_i \wedge y_i, \sigma_{-i}) = \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(\{s_{-i}\})(u_i(x_i, s_{-i}) - u_i(x_i \wedge y_i, s_{-i}))$$

$$\le_i \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(\{s_{-i}\})(u_i(x_i \vee y_i, s_{-i}) - u_i(y_i, s_{-i})) = U_i(x_i \vee y_i, \sigma_{-i}) - U_i(y_i, \sigma_{-i}) \le 0.$$

The above result and (3) imply that $x_i \wedge y_i$ and $x_i \vee y_i$ belong to $F_i(\sigma_{-i})$. Thus $F_i(\sigma_{-i})$ is a sublattice. In particular the hypotheses (H1) and (Ha) hold.

Since each S_i is a finite sublattice, then $a_i = \min S_i$ and $b_i = \max S_i$ exist; b_i is an upper bound of $\{\max F_i(\sigma_{-i}) \mid \sigma_{-i} \in \Sigma_{-i}\}$ in S_i, and a_i is a lower bound of $\{\min F_i(\sigma_{-i}) \mid \sigma_{-i} \in \Sigma_{-i}\}$ in S_i. This proves that the hypotheses (H2) and (Hb) are valid.

To prove that (H3) holds, let $i \in \{1, \dots, n\}$ be given, and assume that $\sigma_{-i} \le \tau_{-i}$ in Σ_{-i}, and that $\tau_{-i} = (\delta_{\hat{s}_1}, \dots, \delta_{\hat{s}_{i-1}}, \delta_{\hat{s}_{i+1}}, \dots, \delta_{\hat{s}_n}) \in \Delta_{-i}$. Since $\sigma_{-i} \le \tau_{-i}$ in Σ_{-i}, then $\sigma_j \preceq_j \delta_{\hat{s}_j}$ for $j \ne i$. As in the proof of Lemma 2 one can show that $s_j \le_j \hat{s}_j$ for all $s_j \in S_j$ for which $\sigma_j(\{s_j\}) > 0$. Denoting $x_i = \max F(\sigma_{-i})$ and $y_i = \max F(\tau_{-i})$ and applying conditions (h2) and (h3) of Definition 3 we get

$$0 \le_i U_i(x_i, \sigma_{-i}) - U_i(x_i \wedge y_i, \sigma_{-i}) = \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(\{s_{-i}\})(u_i(x_i, s_{-i}) - u_i(x_i \wedge y_i, s_{-i}))$$

$$\le_i \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(\{s_{-i}\})(u_i(x_i, \hat{s}_{-i}) - u_i(x_i \wedge y_i, \hat{s}_{-i}))$$

$$= u_i(x_i, \hat{s}_{-i}) - u_i(x_i \wedge y_i, \hat{s}_{-i}) \le_i u_i(x_i \vee y_i, \hat{s}_{-i}) - u_i(y_i, \hat{s}_{-i})$$

$$= \sum_{s_{-i} \in S_{-i}} \tau_{-i}(\{s_{-i}\})(u_i(x_i \vee y_i, s_{-i}) - u_i(y_i, s_{-i})) = U_i(x_i \vee y_i, \tau_{-i}) - U_i(y_i, \tau_{-i}) \le 0.$$

Thus all the inequalities above are equalities. In particular, $U_i(x_i \vee y_i, \tau_{-i}) = U_i(y_i, \tau_{-i})$, so that $x_i \vee y_i \in F(\tau_{-i})$. Since $y_i = \max F_i(\tau_{-i})$, then $x_i \vee y_i = y_i$, whence $x_i \leq y_i$. This proves that the hypothesis (H3) is valid. The proof that (Hc) holds is similar. □

The next result is a consequence of Lemma 4, Theorems 1–4.

Proposition 1. *Let $G = \{S_1, \ldots, S_n, u_1, \ldots, u_n\}$ be a finite supermodular normal-form game. If the hypothesis (H0) holds, then mixed extension Γ of G has least and greatest Nash equilibria in pure strategies. If also the hypothesis (H4) (respectively (Hd)) holds, then the utilities of the greatest (respectively the least) Nash equilibrium majorize the utilities of all other Nash equilibria for Γ.*

The hypothesis (H0) holds if u_i:s are real-valued, whence we obtain the results stated in the Introduction.

Corollary 1. *A mixed extension of any supermodular normal-form game whose strategy sets are finite lattices, and whose utilities $u_i(s_i, s_{-i})$ are real-valued and increasing (respectively decreasing) in s_{-i}, has least and greatest Nash equilibria in pure strategies, and the greatest (respectively the least) one gives the best utilities among all mixed Nash equilibria.*

Remark 1. The proof Lemma 4 shows that each $F(\sigma_i)$ is a sublattice. But *this does not imply that $\mathcal{F}(\sigma_i)$ is a sublattice.*

Quasisupermodularity or log-supermodularity (cf., e.g., [3]) are not available because the utility functions u_i are vector-valued, and not chain- or real-valued, in general.

Consider next the case when the utility functions and are of the form

$$\begin{cases} u_i(s_i, s_{-i}) = (u_{i1}(s_{i1}, s_{-i}), \ldots, u_{im_i}(s_{im_i}, s_{-i})), \\ s_i = (s_{i1}, \ldots, s_{im_i}) \in S_i = \times_{j=1}^{m_i} S_{ij},\ s_{-i} \in S_{-i},\ \text{where} \\ u_{ij}(s_{ij}, s_{-i}) = d_{ij}(s_{ij}, s_{-i}) q_{ij}(s_{ij}),\ i = 1, \ldots, n,\ j = 1, \ldots, m_i, \\ d_{ij} \colon S_{ij} \times S_{-i} \to \mathbb{R},\ \text{and}\ \ q_{ij} \colon S_{ij} \to \mathbb{R}_+. \end{cases} \tag{17}$$

The following hypotheses are imposed.

(ha) *Each S_{ij} is a finite subset of $\mathbb{R}$.*

(hb) *Each $d_{ij}(s_{ij}, s_{-i})$ has increasing differences in (s_{ij}, s_{-i}) .*

Proposition 2. *Let $G = (S_1, \ldots, S_N, u_1, \ldots, u_N)$ be a normal-form game whose utilities are given by (17). Assume that the hypotheses (ha) and (hb) hold, and that all the functions q_{ij} and $d_{ij}(s_{ij}, \cdot)$ are increasing (respectively decreasing). Then G admits a mixed extension Γ which has least and greatest Nash equilibria in pure strategies, and the greatest (respectively the least) Nash equilibrium gives the best utilities among all Nash equilibria for Γ.*

Proof. We shall show that G is a supermodular game. Let $i \in \{1, \ldots, N\}$ be fixed. The hypothesis (ha) and the definitions of u_i imply that condition (h1) of Definition 3 holds when

$E_i = \mathbb{R}^{m_i}$, ordered coordinatewise. Since each function is supermodular in its real variable, the component functions $u_{ij}(s_{ij}, s_{-i})$ of $u_i(s_i, s_{-i}) = (u_{i1}(s_{i1}, s_{-i}), \ldots, u_{im_i}(s_{im_i}, s_{-i}))$ are supermodular in s_{ij} Thus $u_i(s_i, s_{-i})$ is supermodular in $s_i = (s_{i1}, \ldots, s_{im_i})$.

Assume next that the functions q_{ij} and $d_{ij}(s_{ij}, \cdot)$ are all increasing or all decreasing, and that $s_{ij} < \hat{s}_{ij}$ and $s_{-i} < \hat{s}_{-i}$. Since each $d_{ij}(s_{ij}, s_{-i})$ has increasing differences in (s_{ij}, s_{-i}), we have

$$
\begin{aligned}
&u_{ij}(\hat{s}_{ij}, \hat{s}_{-i}) - u_{ij}(s_{ij}, \hat{s}_{-i}) - (u_{ij}(\hat{s}_{ij}, s_{-i}) - u_{ij}(s_{ij}, s_{-i}))\\
&= d_{ij}(\hat{s}_{ij}, \hat{s}_{-i})h_{ij}(\hat{s}_{ij}) - d_{ij}(s_{ij}, \hat{s}_{-i})h_{ij}(s_{ij}) - d_{ij}(\hat{s}_{ij}, s_{-i})h_{ij}(\hat{s}_{ij}) + d_{ij}(s_{ij}, s_{-i})h_{ij}(s_{ij})\\
&= (d_{ij}(\hat{s}_{ij}, \hat{s}_{-i}) - d_{ij}(\hat{s}_{ij}, s_{-i}))(h_{ij}(\hat{s}_{ij}) - h_{ij}(s_{ij}))\\
&+ (d_{ij}(\hat{s}_{ij}, \hat{s}_{-i}) - d_{ij}(\hat{s}_{ij}, s_{-i}) + d_{ij}(\hat{s}_{ij}, s_{-i}) - d_{ij}(s_{ij}, s_{-i}))h_{ij}(s_{ij}) \ge 0.
\end{aligned}
$$

This proves that each $u_{ij}(s_{ij}, s_{-i})$ has increasing differences in (s_{ij}, s_{-i}).

The above proof shows that conditions (h2) and (h3) of Definition 3 hold. The given hypotheses and the definition of u_i ensure that also the hypothesis (H0) holds. Thus the assertions follow from Proposition 1. □

7. An Application

Assume that $p_{ij} \ge 0$, $i = 1, \ldots, n$ and that

$$
\begin{cases}
u_i(s_i, s_{-i}) = (u_{i1}(s_{i1}, s_{-i}), \ldots, u_{im_i}(s_{im_i}, s_{-i})), \text{ where}\\
u_{ij}(s_{ij}, s_{-i}) = d_{ij}(s_{ij}, s_{-i})h_{ij}(s_{ij})(s_{ij} - p_{ij}),\\
d_{ij}\colon S_{ij} \times S_{-i} \to \mathbb{R}_+,\ h_{ij}\colon S_{ij} \to \mathbb{R}_+,\ i = 1, \ldots, n,\ j = 1, \ldots, m_i.
\end{cases}
\tag{18}
$$

Players i are assumed to stand for firms, each of which sell a number m_i of same or substitute products e_{ij}, i.e. $e_{ij} = e_{kj}$ or $e_{ij} \approx e_{kj}$ when $j = 1, \ldots, \min\{m_i, m_k\}$. The selling prices s_{ij} per unit are assumed to form a subset S_{ij} of $\mathbb{R}$ which is bounded from below by a unit purchase price p_{ij} of e_{ij}. The utilities u_i defined by (18), whose values belong to $\mathbb{R}^{m_i}$, are considered as net profit vectors of firms i. The functions

$$D_{ij}(s_{ij}, s_{-i}) = d_{ij}(s_{ij}, s_{-i})h_{ij}(s_{ij})$$

stand for demands of products e_{ij}.

The utilities u_i defined in (18) are of the form (17), where

$$q_{ij}(s_{ij}) = h_{ij}(s_{ij})(s_{ij} - p_{ij}),\ i = 1, \ldots, n,\ j = 1, \ldots, m_i. \tag{19}$$

As a consequence of Proposition 2 we obtain the following result.

Corollary 2. *Let G be a price game whose utilities are given by (18), and let the hypotheses (ha) and (hb) hold. Assume also that all the functions q_{ij} given by (19) and all the functions $d_{ij}(s_{ij}, \cdot)$ are increasing. Then the mixed extension Γ of G has a least and greatest Nash equilibria in pure strategies. Moreover, the utilities of the greatest Nash equilibrium majorize the utilities of all other Nash equilibria for Γ.*

Consider next the special case:

$$d_{ij}(s_{ij}, s_{-i}) = f_{ij}(s_{ij}) + g_{ij}(s_{-i}),\ s_{ij} \in S_{ij},\ s_{-i} \in S_{-i},\ i = 1, \ldots, n,\ j = 1, \ldots, m_i, \tag{20}$$

where $f_{ij}: S_{ij} \to \mathbb{R}$, and $g_{ij}: S_{-i} \to \mathbb{R}$. In this case

$$d_{ij}(\hat{s}_{ij}, \hat{s}_{-i}) - d_{ij}(s_{ij}, \hat{s}_{-i}) = f_{ij}(\hat{s}_{ij}) - f_{ij}(s_{ij}) = d_{ij}(\hat{s}_{ij}, s_{-i}) - d_{ij}(s_{ij}, s_{-i})$$

for all fixed $i = 1, \ldots, n$ and $j = 1, \ldots, m_i$. Thus the hypotheses given for d_{ij} in Corollary 2 hold if each g_{ij} is increasing.

Example 1. Two Supermarkets 1 and 2 sell both two different coffee brands. Purchasing prices per package in dollars are $(p_{11}, p_{12}) = (1, 1.10)$ for 1 and $(p_{21}, p_{22}) = (1.10, 1)$ for 2. Possible selling prices s_{ij} vary from $2p_{ij}$, with five cents differences, to $3p_{ij}$. Assume that demands per month are $d_{ij} = (d_{i1}, d_{i2})$ where the components d_{ij}, $i, j = 1, 2$, are defined by ($[\cdot]$ denotes the greatest integer function)

$$\begin{cases} d_{11} = 30(20s_{11} - 40s_{11}^2 + [200s_{21} + s_{22}]), \ d_{12} = 30(30s_{12} - 40s_{12}^2 + [s_{21} + 220s_{22}]), \\ d_{21} = 30(40s_{21} - 40s_{21}^2 + [210s_{11} + s_{12}]), \ d_{22} = 30(20s_{22} - 40s_{22}^2 + [s_{11} + 180s_{12}]). \end{cases} \tag{21}$$

Determine the least and greatest Nash equilibria for selling prices and calculate the corresponding utilities.

Solution. The given problem can be converted to a normal-form game $G = (S_1, S_2, u_1, u_2)$, where utilities are

$$u_i = (d_{i1}(s_{i1} - p_{i1}), d_{i2}(s_{i2} - p_{i2})), \ i = 1, 2,$$

with $p_{11} = p_{22} = 1$, $p_{12} = p_{21} = \frac{11}{10}$, and strategy sets are $S_i = S_{i1} \times S_{i2}$, where

$$S_{11} = S_{22} = \{\frac{j}{20}, 40 \le j \le 60\}, \ S_{12} = S_{21} = \{\frac{k}{20} \mid 44 \le k \le 66\}.$$

In this case $X_i = E_i = \mathbb{R}^2$, $i = 1, 2$. The demand functions d_{ij} are of the form (20). Thus the mixed extension Γ of G has by Corollary 2 least and greatest strategies in pure strategies. Applying the methods of successive approximation used in the proofs of Theorems 1 and 3 one can calculate, by using a simple Maple programming, the least and the greatest Nash equilibria for prices:

$$\overline{s}_1^* = (2.75, 2.85), \ \overline{s}_2^* = (2.95, 2.60), \ \underline{s}_1^* = (2.70, 2.80), \ \underline{s}_2^* = (2.90, 2.55).$$

Corresponding profits are

$$\begin{cases} \overline{u}_1^* = (18086.25, 17566.50), \ \overline{u}_2^* = (19419.45, 14236.80), \\ \underline{u}_1^* = (17564.40, 17003.40), \ \underline{u}_2^* = (18824.40, 13805.85). \end{cases}$$

Since the demand functions d_{ij} are increasing with respect to s_{-i}, then the profits $(\overline{u}_i^*, \overline{u}_2^*)$ majorize by Corollary 2 the profits obtained by all pure and mixed pricing strategies.

Remark 2. In the special case when $h_{ij}(s_{ij}) \equiv 1$ and $m_i = 1$ in (18) we obtain the model considered in [3, Subsection 4.4.1]).

In [2] existence and comparison results are derived for games G with arbitrary nonempty set I of players, and when the strategy set of player i depends on the strategy vector s_{-i} of other players.

References

[1] F. Echenique, *Mixed equilibria for games of strategic complementarities*, Economic Theory (2003), 33–44.

[2] S. Heikkilä and K. Reffett, *Fixed point theorems and their applications to theory of Nash equilibria*, Nonlinear Anal. (to appear).

[3] Donald Topkis, *Supermodularity and complementarity*, Princeton University Press, New Jersey, 1998.