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Existence and Uniqueness of Equilibrium in Distorted Dynamic Economies with Capital and Labor[1]

Manjira Datta

Arizona State University, Tempe, Arizona

Leonard J. Mirman

University of Virginia

and

Kevin L. Reffett

Arizona State University, Tempe, Arizona

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In this paper, we provide a set of sufficient conditions under which recursive competitive equilibrium exists and is unique for a large class of distorted dynamic equilibrium models with capital and elastic labor supply. We develop a lattice based approach to the problem. The class of economies for which we are able to obtain our existence and uniqueness result is considerably larger than those considered in previous work. We conclude by applying the new results to some important examples of monetary economies often used in applied work. *Journal of Economic Literature* Classification Numbers: C62, D51, D90, E10.

Key Words: recursive competitive equilibrium; distortion; monotone operators; Tarski's fixed point theorem.

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1. INTRODUCTION

This paper studied the existence of competitive equilibrium in a large class of dynamic recursive economies with capital accumulation and elastic labor supply. The type of equilibrium distortions considered encompasses a broad class of non-optimal environments including distortionary taxation policies, situations where firms face equilibrium production externalities, and various monetary economies where there exists a functional equivalence between monetary and real economies. This class of infinite horizon dynamic equilibrium models continues to be the standard setting used to model many issues in applied macroeconomics.[2] Relative to the existing literature concerning the existence of equilibrium for these models, we are able to make progress in a number of important directions. First, we are able to provide conditions under which there exists a competitive equilibrium for situations where both the equilibrium distortions and the period preferences are allowed to be quite general. To accomplish this, we develop a class of concave, monotone operator methods for these economies with elastic labor supply. In particular in our setting, period preferences and equilibrium distortions are allowed to take a more general form than in previous work.[3] In particular, the period preferences must

[2] Examples of environments in applied work with capital and elastic labor abound. See for example the monetary models built around Greenwood and Huffman [22] and Cooley and Hansen [13, 14] and Ohanian and Cole [8]; the multiple means of payment models such as Ireland [27] and Lacker and Schreft [32]; non-convex endogenous growth models such as Romer [37]; equilibrium models with market frictions such as the monopolistic competition models such as Hornstein [25] and Devereux, Head, and Lapham [15]; and the optimal taxation models Judd [28], Chari and Kehoe [7], and Judd [29].

[3] Greenwood and Huffman [23] study an environment with elastic labor supply similar to the one we study, but require $u(c, l) = h(c - g(l))$, where h is monotone increasing and $g(l)$ is concave. In this case, the marginal rate of substitution between leisure and consumption to be independent of consumption, which means the results for the case without elastic labor supply can be applied to this economy (after constructing an appropriate technology for the problem).

Coleman [11] also studies an environment related to ours, and he requires the preferences to have the restricted homothetic form $u(c, l) = h(g(c, l))$, where h is monotonically increasing, g homogeneous of degree one, and h strictly concave and satisfied Inada conditions (where obviously the latter restriction on h rules out many homothetic forms such as Cobb-Douglas preferences). Also, Coleman considers the distorted equilibrium prices to have the following homogeneous form: $r(K, \theta) = r(\frac{K}{N}, \theta)$ and $w(K, \theta) = w(\frac{K}{N}, \theta)$. See assumption 4 and its use in proving Lemma 2. For applied work, homogeneous distortion can be very problematic because of its implied restrictions on fiscal and monetary policy (e.g., the when studying issues associated with the welfare of optimal dynamic taxation like in Coleman [12]).

only exhibit a weak form of complementarity in consumption and leisure,[4] and the equilibrium distortions must only be weakly decreasing in the capital stock. Second, we are able to give the first global uniqueness results for these economies.

The issue of existence of a recursive competitive equilibrium in economies with capital and elastic labor supply has been studied in previous work by Greenwood and Huffman [23] and Coleman [11]. In these papers, the authors have successfully extended the monotone map methods developed in Lucas and Stokey [33] and Coleman [9] for distorted recursive economies in various settings without elastic labor supply to the situation where agents face a nontrivial labor–leisure choice. Although these results are quite interesting, the joint restrictions required on the economic primitives of taste and technology as well as the class of equilibrium distortions appear to rule out many interesting applications of the methods in applied general equilibrium problems. In particular for many interesting applications, these assumptions are often not met even for many examples of the models studied (e.g. for various versions of the monetary economies described in papers such as Cooley and Hansen [13] and Cole and Ohanian [8], for multiple means of payment models of Lacker and Schreft [32], or Ireland [27], or the optimal taxation problems studied in Chari and Kehoe [7] and Judd [29]).

In the next section of the paper, we describe the environment. Section 3 develops a sequence of decision problems needed to characterize the two stage budgeting approach we take to the equilibrium problem. Section 4 proves the existence of equilibrium, while Section 5 characterizes some properties of the equilibrium. Section 6 provides some examples of monetary frameworks which fit in our setting, and others which require some extensions. The final section concludes.

2. A MODEL ECONOMY WITH EQUILIBRIUM DISTORTIONS

The model is formulated as an infinite horizon stochastic growth model as described in Brock and Mirman [6] to allow for equilibrium distortions and elastic labor supply. Time is discrete. There is a continuum of infinitely-lived and identical household–firm agents. Uncertainty comes in the form of exogenous shocks $\theta \in \Theta$ which are assumed to evolve according

[4] For the situation where preferences are such that consumption and leisure are allowed to be substitutes, tight restrictions must be placed on the magnitude of the cross derivatives of both period preferences and production technologies as in Coleman [11]. Of course, all of these cases covered in his paper work on our setting also. We simply do not have any additional results along these lines.

to a first-order Markov process with stationary transition matrix $\chi(\theta, d\theta')$. For convenience, we assume that Θ has a finite number of elements.

Households each own an identical technology for producing output goods. Production takes place in the context of perfectly competitive markets for both output goods and the factors of production. Household preferences are constructed as follows. For each period and state, preferences are represented by a period utility index $u(z_t)$, where $z_t = (c_t, l_t) \in R_+ \times [0, 1]$. Letting $\theta^t = (\theta_1, ..., \theta_t)$ denote the history of the shocks until period t, the household's lifetime preference are additively separable and defined over infinite sequences indexed by dates and histories $z = (z_{\theta^t})$ and are given as follows

$$U(\mathbf{z}) = E_o \left\{ \sum_{t=0}^{\infty} \beta^t u(z_t) \right\}. \tag{1}$$

Here the integral in (1) is with respect to the probability structure of future histories of the shocks θ^t given the transition matrix χ. The assumptions on the period utility function u: $R \times [0, 1] \mapsto R$ are as follows,

Assumption 1. (i) The period utility function is bounded above, continuously differentiable, strictly increasing, and strictly concave in (c, l).

(ii) The partial derivatives $u_c(c, l)$ and $u_l(c, l)$ satisfy the Inada conditions:

$$\lim_{c \to 0} u_c(c, l) = \infty, \qquad \lim_{c \to \infty} u_c(c, l) = 0, \qquad \lim_{l \to 0} u_l(c, l) = \infty.$$

(iii) The second partials satisfy the conditions:

$$\frac{u_{cc}}{u_c} \leqslant \frac{u_{lc}}{u_l}, \qquad \frac{u_{ll}}{u_l} \leqslant \frac{u_{cl}}{u_c}.$$

The assumptions on period utility are standard. Note that Assumption 1(iii) is normality. It also means that the marginal rate of substitution u^l/u_c is non-decreasing in c and u_l/u_c is non-increasing in l. And this is slightly stronger than quasi-concavity of the period utility function (we assumed it to be strictly concave) because it implies

$$u_c^2 u_{ll} + u_l^2 u_{cc} \leqslant 2 u_c u_l u_{cl},$$

which is a necessary condition for quasi-concavity. This condition is automatically satisfied if $u_{cc}(c, l) < 0$, $u_{ll}(c, l) < 0$ and $u_{cl}(c, l) \geqslant 0$. if the cross-partial is negative, this condition restricts its magnitude. In most of the situations studied previously in the literature it is assumed that the cross-partial of period utility is either positive or not "too negative". These assumptions can be shown to hold for both the case of the quasi-linear preferences discussed in Greenwood and Huffman [23] and the restricted homothetic class (strictly concave transformation of homogeneous of degree one functions) studied in Coleman [11].

Each household is endowed with a unit of time and enters into a period with an individual stock of capital k. We assume a decentralization where firms do not face dynamic decisions problems. Households own firms as well as both the factors of production and rent these factors of productions in competitive markets. In addition, to allow for externalities in the production process, we allow the production technologies of the firms to depend on per capita aggregates. Each period, firms rent capital k and labor n from households, sell output goods in competitive markets, and then return all profits to households at the end of the period. Let f: $\mathbb{K} \times [0, 1] \times \mathbb{K} \times [0, 1] \times \Theta$ summarize the production possibilities for the firm in any given period; $\mathbb{K}$ is a compact set to be described in more detail later. Assume also that technology satisfies the following assumption:

Assumption 2. The production function satisfies,

(i) $f(0, 0, K, N, \theta) = 0$ for all $(K, N, \theta) \in \mathbb{K} \times [0, 1] \times \Theta$,

(ii) $f(k, n, K, N, \theta)$ is continuous, increasing, differentiable; in addition, it is concave and homogeneous of degree one in (k, n).[5]

(iii) $f(k, n, K, N, \theta)$ also satisfies the standard Inada conditions in (k, n) for all $(K, N, \theta) \in \mathbb{K} \times [0, 1] \times \Theta$; i.e.:

$$\lim_{k \to 0} f_k(k, n, K, N, \theta) = \infty,$$

$$\lim_{n \to 0} f_n(k, n, K, N, \theta) = \infty,$$

$$\lim_{k \to \infty} f_k(k, n, K, N, \theta) = 0.$$

(iv) There exists a $\hat{k}(\theta) > 0$, such that $f(\hat{k}(\theta), 1, \hat{k}(\theta), 1, \theta) + (1 - \delta)\, \hat{k}(\theta) = \hat{k}(\theta)$ and $f(k, 1, k, 1, \theta) < k$ for all $k > \hat{k}(\theta)$, for all $\theta \in \Theta$.[6]

[5] We could dispense with the constant returns to scale assumption in (k, n) as in Greenwood and Huffman [23] and allow for decreasing returns to scale without changing the results. Unfortunately, such models allow firms to make non-zero profits in equilibrium. Since in such an equilibrium, organizational issues become important, we only consider the case of constant returns to scale in (k, n).

[6] The restrictions on technology in (iii) are made to conform with the standard assumptions made in the stochastic growth literature (e.g., Brock and Mirman [6]). Weaker Inada conditions are possible on f. In our problem, an alternative version of (iii) has $f(\hat{k}, 1, \hat{k}, 1, \theta) + (1 - \delta)\, \hat{k} \leqslant \hat{k}$ for all θ, with equality for some θ. Then without the Inada condition, a sufficient condition for $\hat{k} > 0$ has $f_k(0, 1, K, 1, \theta) + (1 - \delta)\, \hat{k} > 1$, and for some small $\hat{k} < \infty$.

Assumption 2 is standard in the stochastic growth literature (cf., Brock and Mirman [6]). With this assumption, we can define $\bar{k} = \sup_\theta \hat{k}(\theta)$. Then the state space for the capital stock and output can be defined on the compact set $\mathbb{K} \subseteq [0, \bar{k}]$. Let $\mathbb{K}_+$ denote the set of strictly positive values for k.

3. DECISION PROBLEMS AND EQUILIBRIUM

3.1. *Overview of the Approach*

We being by briefly describing our approach to the existence problem. We develop an application of a two-stage budgeting approach where in the first stage, households make all of their intratemporal choices (in this case, consumption and leisure) while facing an appropriately parameterized aggregate economy. Then, in the second stage, they choose the optimal dynamics. The approach is closely related to the approach taken in Greenwood and Huffman [23] and generates an equilibrium problem of the structure of that studied in Coleman [11]. Multistage budgeting techniques are not new. (See for example, Arrow [2], but more formally the work of Gorman [20, 21]). Our two-stage approach basically follows the approach described in Gorman [20]. As for the decentralization, we follow what is standard in the literature (e.g., Coleman [11]) and assume that the distorted prices are imposed on households and in general the competitive factor prices for the economy, evaluated in equilibrium, are allowed to be different between firms and households. The firms hire inputs and pay according to their marginal social products while the households face distortion; thus, the wage and rental rates they receive are allowed to differ from the marginal social product.

3.1.1. *First-stage decision problems.* Imagine a consumer faced with a choice problem in a single good, leisure. The objective is to maximize the difference between the level of utility and the expenditure to obtain that level of utility (See Topkis [41, Chap. 2] for a similar consumer choice problem). Normalizing on the price of consumption goods, consumers take the price of leisure $w(K, \theta)$, the level of per capita consumption C, and the per capita leisure level $L(C, K, \theta)$, as given. Here $C \in \mathbb{K}$, w: $\mathbb{S} \to R_{++}$, L: $\mathbb{K} \times \mathbb{S} \to [0, 1]$, and L is a continuously once-differentiable function where $\mathbb{S} := \mathbb{K} \times \Theta$. Households then solve, given w,

$$v(C, K, L, \theta) = \sup_{l \in [0, A]} \frac{u(C, l)}{u_c(C, L)} - wl,$$

for each $(C, K, L, \theta) \in \mathbb{K}^2 \times [0, 1] \times \Theta$. Given the assumptions on u, standard arguments, using the theorem of the maximum, establish that the value function v is well defined and continuous (e.g., see Berge [3, p. 115]). Furthermore, by the strict concavity of period utility in Assumption 1 the optimal policy correspondence associated with v is a singleton. The necessary condition for this first-stage maximization problem is

$$\frac{u_l(C, l^*(C, K, \theta))}{u_c(C, L)} = w(K, \theta). \tag{2}$$

To finish our description of the first stage, we need to determine equilibrium factor prices as functions of the aggregate state variable. We do this from the representative firm's static production problem. Assume that firms maximize profits under perfect competition; i.e., the firms maximize profits subject to given factor prices, say $\bar{r}(K, \theta)$ and $\bar{w}(K, \theta)$, the rental rate for capital, and the wage rate, respectively. The factor prices are continuous functions of the aggregate state variable. The representative firm's maximum profit is

$$\Pi(\bar{r}, \bar{w}, K, N, \theta) = \sup_{k, n} f(k, n, K, N, \theta) = \bar{r}k - \bar{w}n. \tag{3}$$

Anticipating the standard definition of competitive equilibrium with $k = K$ and $n = N(S)$, for $S \in \mathbb{S}$, prices in the factor markets are

$$\bar{r} = f_k(K, N(S), \theta), \tag{4}$$

$$\bar{w} = f_n(K, N(S), \theta). \tag{5}$$

Notice that, given the assumed structure on the firm's decision problem, the theorem of the maximum implies that Π is a continuous function and that solutions to the firm's problem exist.

3.1.2. *The second-stage decision problem.* In the second stage, the household solves a dynamic capital accumulation problem. To describe this problem, we parametrize the aggregate economy facing a typical decision maker. Define $\mathbb{G}$ to be the space of bounded, continuous functions with domain $\mathbb{S}$ and range $\mathbb{R}_+$. To parametrize the household's decision problem, we first describe the aggregate economy.

If the aggregate per capita capital stock is K, then households assume that per capita consumption decisions C, and per capita labor supply N, and the recursion of the capital stock K' is given by

$$K' = \kappa(S); \qquad C = C(S); \qquad N = N(S); \qquad C, \kappa, N \in \mathbb{G}.$$

Further using the solution to the household's first-stage decision problem (and imposing equilibrium on the labor market), define the per capita aggregate labor supply $N(S) = 1 - l^*(C, K, \theta)$. Then the aggregate economy consists of functions $\Omega = (w, r, \kappa, C, N)$ from a space of functions with suitable restrictions needed to parametrize the household's decision problem in the second stage. Assume that the policy-induced equilibrium distortions have the following standard form,

$$r = [1 - \pi_k(S)]\, \bar{r}, \qquad w = [1 - \pi_n(S)]\, \bar{w},$$

where $\pi = [\pi_k, \pi_n]$ is a continuous mapping $\mathbb{S} \to [0, 1) \times [0, 1)$. We assume regularity conditions on the distorted prices w and r.

Assumption 3. The vector of distortions $\pi = [\pi_k, \pi_n]$ is such that the distorted wage $w = (1 - \pi_n(K, \theta))\, \bar{w}(K, N(K, \theta), \theta)$ and the distorted rental rate $r = (1 - \pi_n(K, \theta))\, \bar{r}(K, N(K, \theta), \theta)$ satisfy the following:

(i) $w: \mathbb{K} \times \Theta \to \mathbb{R}_+$ is continuous, at least once-differentiable, and (weakly) increasing in K,

(ii) $r: \mathbb{K}_+ \times \Theta \to \mathbb{R}_+$ is continuous and decreasing in K such that

$$\lim_{K \to 0} r(K, \theta) \to \infty.$$

In other words, we assume that the distorted wage and rental rates behave as the non-distorted rates $\bar{w}$, $\bar{r}$ or the marginal products of labor and capital, respectively.[7] Assumptions 3(i) and 1(iii) imply that leisure increases with higher consumption and decreases with larger capital accumulation.

Next define the lump-sum transfers to each agent: $d(S) = \pi_k K + \pi_n N(K, \theta)$. The household's total income is $y(s) = rk + wN + (1 - \delta)\, k + \Pi$, the sum of distorted rental and wage incomes, undepreciated capital, and profits where s is the individual household's state, $s = (k, S) = (k, K, \theta)$.[8]

[7] Note that the distorted prices restrict the types of production externalities that are consistent with the equilibrium we construct. This restriction is require to eliminate problems with the local determinacy of equilibrium. See Boldrin and Rustichini ([5], Theorem 3.1) for discussion.

[8] It is important to note that the methods discussed here and in Coleman [11] can be restated following the approach in Greenwood and Huffman [23]. Instead of endowing the household with an income constraint and distorted prices, as Coleman [11] does, we could endow the agent with the technological constraint, $\Lambda(k, K, \theta) = f(k, K, N(K, \theta), \theta)(1 - \pi_k(S)) + (1 - \delta)\, k + \pi_k(S)\, f(K, N(K, \theta), \theta)$.

Solving the modified planning problem they describe using this technology delivers an integral equation, which will need to be restricted so that it satisfies the assumptions of the technology in Greenwood and Huffman [23] concerning derivatives of Λ. These restrictions take into account the presence of N and how it changes in equilibrium (and thereby effecting the shape of Λ) similar to the externality that is allowed in their paper. Also for reference below in our construction of the fixed point problem, notice that in equilibrium, $\Lambda(K, \theta) = f(K, N, K, N, \theta) + (1 - \delta)\, K$.

Note that $y(s)$ is a continuous function. We now define the household's feasible correspondence $\Psi(s)$ for the distorted economy which consists of the set $(c, k') \in \mathbb{R}^2_+$ that satisfies the following constraint,

$$c + wl^*(C, K, \theta) + k' = y, \tag{6}$$

given $(k, K, \theta) \gg 0$. Notice that $\Psi(s)$ is well behaved. In particular since Π is continuous, Ψ is a non-empty, compact and convex-valued, continuous correspondence.

Next, we state the second-stage decision problem for the household. At the beginning of any period the aggregate state for the economy is given by $S \in \mathbb{S}$. Each household enters the period with its individual capital stock $k \in \mathbb{K}$, so its individual state is $s \in \mathbb{K} \times \mathbb{S}$. Then the household's dynamic decision problem is summarized by the Bellman equation

$$J(s) = \sup_{(c, k') \in \Psi(s)} u(c, l^*(C, K, \theta)) + \beta \int_{\Theta} J(z')\, \chi(\theta, d\theta'). \tag{7}$$

Standard arguments show the existence of $J \in \mathbb{J}$ that satisfies the functional equation, where $\mathbb{J}$ is the space of bounded, continuous functions with the uniform norm. In addition, since u is strictly concave in c, standard arguments also establish that J is strictly concave in its first argument, k. Following arguments in Mirman and Zilcha [36], the concavity of J also implies that the envelope theorem applies and J is once differentiable in k.

We are now prepared to define equilibrium

DEFINITION 1. A (recursive) competitive equilibrium for this economy consists of sequence functions r, w, d, and κ; a value function for the household $J(s)$ and the associated individual decisions $c^*(s)$ and $n^*(s)$ such that (i) given r, w, d, and κ, $J(s)$ satisfies the household's Bellman equation; (ii) $c^*(s)$ solves the right-hand side optimization in Bellman's equation, $l^*(s) = 1 - n^*(s)$ solves the first-stage utility maximization; (iii) all markets clear: i.e., $k' = \kappa(S) = K'$, $n^*(s) = N(S)$, $c^*(s) = C(S)$ and the government budget constraint holds, i.e., $d = \pi_k k + \pi_n n^*$.

4. EXISTENCE OF EQUILIBRIUM

Before we state the existence problem, we define a number of functions. In equilibrium, $c(s) = C(S)$, $k = K$, $n = N(S)$, then $y(s) = F(K, \theta) = f(K, 1 - l^*(C(S), K, \theta), \theta) + (1 - \delta) K$. The next period capital stock, in equilibrium, is given as $K' = y - C$. Also, for later reference, define $\hat{l}(S)$ as the solution to

$$\frac{u_l(f(K, 1-\hat{l}(S), \theta), \hat{l}(S))}{u_c(f(K, 1-\hat{l}(S), \theta), \hat{l}(S))} = (1-\pi_n(S))\, f_n(K, 1-\hat{l}(S), \theta). \tag{8}$$

Notice that $\hat{l}$ is the amount of leisure that is compatible with no household investment in the first-stage utility maximization. At any (aggregate) state S, the maximum possible amount of consumption occurs if $c = f$ and, i.e., if there is no investment. In general, the amount of consumption is less than f and leisure, which is positively related to consumption, is therefore less than $\hat{l}(S)$. That is, for a given state S, $1-\hat{l}(S)$ is the lower bound for the amount of labor supplied. In addition, $\hat{l}(S)$ is differentiable with respect to K, by the implicit function theorem, since the marginal utilities, technology, and distorted wage is differentiable in K.[9] Moreover, for the special case, $u_{cl} \geqslant 0$, $\hat{l}(S)$ is increasing in K. $\hat{l}(S)$ is also increasing in K, for the case $u_{cl} < 0$, if

$$u_{ll} - f_n u_{cl} < 0, \qquad u_{cl} - f_n u_{cc} > 0.$$

Using standard arguments, the Euler equation, associated with the right side of the Bellman Eq. (7) above, can be rewritten as

$$u_c(c, l^*(c, K, \theta)) = \beta \int_{\Theta} u_c(c(K', \theta'), l^*(c', K', \theta'))\, r(K', \theta')\, \chi(\theta, d\theta'). \tag{9}$$

Here the $'$ notation refers to next period value of the particular variable. Given a candidate function $c(S)$, we rewrite the Euler Eq. (9) in equilibrium as

$$\begin{aligned} u_c(c, l^*(c, K, \theta)) = \beta \int_{\Theta} & u_c(c(F_c - c, \theta'), l^*(c(F_c - c, \theta'), K', \theta')) \\ & \times r(F_c - c, \theta')\, \chi(\theta, d\theta'), \end{aligned} \tag{10}$$

where $F_c = f(K, 1 - l^*(c(K, \theta), K, \theta), \theta) + (1-\delta) K$. We can use Eq. (10) to define a nonlinear operator that yields a strictly positive fixed point in the space of consumption functions. This fixed point is an equilibrium for the economy.

[9] To globalize the implicit function argument, zero must be a regular value for every point on the interior of $\mathbf{K} \times \Theta$ for the mapping $\Delta(\hat{l}(S), K, \theta) = \frac{u_l(f(K, 1-\hat{l}(S), \theta), \hat{l}(S))}{u_c(f(K, 1-\hat{l}(S), \theta), \hat{l}(S))} - (1-\pi_n(S))$ $f_n(K, 1-\hat{l}(S), \theta)$, which is then used to construct a differentiable structure for the manifold $M = int[0, 1] \times \mathbf{K} \times \Theta$. Then noting that Δ is proper, a global version of the implicit function theorem is available.

Define $F^u(S) = F^u(K, \theta) = f(K, 1 - \hat{l}(K, \theta), \theta) + (1 - \delta) K$ and the following space of functions,

$$\mathbf{H} = \{h: \mathbb{S} \to \mathbb{K};\ h \text{ is continuous, } h(S) \in [0, F^u(S)] \text{ and}$$
$$u_c(h(S), l^*(h(S), S)) \text{ is decreasing in } h,$$
$$u_c(h(S), l^*(h(S), S)) \text{ is decreasing in } K.\}$$

Equip $\mathbf{H}$ with the sup norm. Note that the assumption the marginal utility of consumption is decreasing in h means that the space $\mathbf{H}$ differs from the space of consumption functions studied in Coleman [11]. It is easily verified that for the preferences considered in that paper, the restriction u_c decreasing in h is implied. However, since the class of preferences studied in this paper is larger than that studied in Coleman, additional restriction is necessary on the space of consumption functions.

Define the mapping Z: $\mathbf{H} \times \mathbf{Y} \times \mathbf{K} \times \mathbf{Z} \to \mathbb{R}$, where $\mathbf{Y} \subset \mathbf{R}_+$, as

$$Z(h, \zeta, K, \theta) = \Psi_1(\zeta, K, \theta) - \Psi_2(h, \zeta, K, \theta), \tag{11}$$

$$\Psi_1 = u_c(\zeta, l^*(\zeta, K, \theta)), \tag{12}$$

$$\Psi_2 = \beta \int_\Theta u_c(h(F_\zeta - \zeta, \theta'), l^*(h(F_\zeta - \zeta, \theta'), F_\zeta - \zeta, \theta')) \times r(F_\zeta - \zeta, \theta')\, \chi(\theta, d\theta'). \tag{13}$$

Here $F_\zeta = f(K, 1 - l^*(\zeta, K, \theta) + (1 - \delta) K)$. Then define the nonlinear operator A: $\mathbf{H} \to \mathbf{H}'$ as

$$Ah(K, \theta) = \{\zeta \text{ such that } Z(h, \zeta, K, \theta) = 0,\ h > 0;\quad Ah(K, \theta) = 0 \text{ elsewhere}\}, \tag{14}$$

where $\mathbf{H}'$ at this point is an appropriate Banach space.

We now discuss some properties of the operator A, as defined by equations (11)–(14).

PROPOSITION 1. *Under Assumptions* 1–3 *for any* $h \in \mathbf{H}$, *there exists a unique* $Ah = h$ *such that* $Z(h, \tilde{h}, K, \theta) = 0$, *for any* (K, θ).

Proof. Take $h \in \mathbf{H}$ and fix $(K, \theta) \in S$. Notice first that $F_{\tilde{h}} - \tilde{h}$ is decreasing in $\tilde{h}$ from the fact that $N = 1 - l^*(\tilde{h}, K, \theta)$ is decreasing in $\tilde{h}$. Given Assumptions 1 and 3(ii), the second term in Z [see Eq. (13)] is strictly increasing in $\tilde{h}$ since h is such that $u_c(h(K, \theta), l^*(h(K, \theta), K, \theta))$ is decreasing in K. Furthermore, under Assumption 1(iii), the first term in Z [see Eq. (12)] is also strictly decreasing in $\tilde{h}$. So Z is strictly decreasing in $\tilde{h}$.

Assumption 1(ii) implies that as $\tilde{h} \mapsto f$, the second term in Z approaches infinity, while the first term remains finite. Therefore Z tends towards $-\infty$. Likewise, under Assumption 1(ii), as $\tilde{h} \to 0$, the first term of Z approaches ∞ while the second term remains finite; therefore $Z \to -\infty$. Therefore given continuity assumptions on preferences and distorted prices, $\tilde{h}$ exists and is unique. Continuity implies that $Ah_n \to Ah$ if $h_n \to h$. ▮

Proposition 1 implies that for all states, the operator A is well defined and continuous. To study the fixed points of A, we first establish that A is a self map: i.e., A: $\mathbf{H} \to \mathbf{H}$. In order to prove this it will be convenient to assume.

Assumption 4. $u_{cl} \geqslant 0$.

Greenwood and Huffman [23] only consider the case where $u_{cl} = 0$. Coleman [11] allows for $u_{cl} \geqslant 0$ and also some cases where $u_{cl} < 0$. However, he considers a restricted homothetic class of preferences (see the discussion in footnote 2) and, in addition, imposes more restrictions (jointly on utility, production functions, and distortions) to study the case of negative cross partials of u. The same case of negative cross-partials of u can be handled in our setting also. At this stage, we are unable to capture more general cases of negative cross partials of u than Coleman [11]; therefore, we focus only on the $u_{cl} \geqslant 0$ case. And, we have the following:

THEOREM 1. *Under Assumptions* 1–4 $Ah \subset \mathbf{H}$.

Proof. Under the continuity assumptions on preferences, technologies, and distorted prices, continuity of Ah is obvious. Also it is straightforward that Ah belongs to the interval $[0, F^u(K, \theta)]$, otherwise the equality, $Z(h, Ah, K, \theta) = 0$, cannot be met since the second term in Z is not defined.

To prove $Ah \subset \mathbf{H}$, we need to check two more properties of Ah:

(I) Take $h_2 \geqslant h_1$, h_1 and h_2 in $\mathbf{H}$; we first show $u_c(Ah, l^*(Ah, K, \theta))$ is decreasing in Ah, or, $u_c(Ah_2, l^*(Ah_2, K, \theta)) \leqslant u_c(Ah_1, l^*(Ah_1, K, \theta))$.

Define $\tilde{h}_1 = Ah_1$ such that $Z(h_1, Ah_1, K, \theta) = 0$ and $\tilde{h}_2 = Ah_2$ such that $Z(h_2, Ah_2, K, \theta) = 0$. Recall that

$$\Psi_2(h, \tilde{h}, K, \theta) = \beta \int_{\Theta} u_c(h(F_{\tilde{h}} - \tilde{h}, \theta'), l^*(h(F_{\tilde{h}} - \tilde{h}, \theta), F_{\tilde{h}} - \tilde{h}, \theta')) \times r(F_{\tilde{h}} - \tilde{h}, \theta')\, \chi(\theta, d\theta'), \tag{15}$$

where $F_{\tilde{h}} = f(K, 1 - l^*(\tilde{h}, K, \theta), \theta) + (1 - \delta) K$. Since h_2 is in $\mathbf{H}$, $\Psi_2(h_2, \tilde{h}_2, K, \theta) \leqslant \Psi_2(h_1, \tilde{h}_1, K, \theta)$. That is, Ψ_2 is decreasing h. Since Ψ_1 is independent of h, the solution $\tilde{h}_2$ must be such that $\Psi_1(\tilde{h}_2, K, \theta) \leqslant \Psi_1(\tilde{h}_1, K, \theta)$.

Or, $u_c(Ah_2, l^*(Ah_2, K, \theta)) \leqslant u_c(Ah_1, l^*(Ah_1, K, \theta))$, for all (K, θ). This verifies that $u_c(Ah, l^*(Ah, K, \theta))$ is decreasing in Ah.

Finally, to complete the proof we need to show that Ah is such that $u_c(Ah(K, \theta), l^*(Ah(K, \theta), K, \theta))$ is decreasing in K.

(II) Here we show that for any $K_2 \geqslant K_1$, $u_c(\tilde{h}_2, l^*(\tilde{h}_2, K_2, \theta)) \leqslant u_c(\tilde{h}_1, l^*(\tilde{h}_1, K_1, \theta))$ where $\tilde{h}_1 = Ah_1$ for (K_1, θ) and $\tilde{h}_2 = Ah_2$ for (K_2, θ), that is, $Z(h_1, \tilde{h}_1, K_1, \theta) = 0$ and $Z(h_1, \tilde{h}_2, K_2, \theta) = 0$. Note that $F_{\tilde{h}} - \tilde{h}$ is increasing in K, since $l^*(\tilde{h}, K, \theta)$ is decreasing in K, the marginal products of capital and labor are positive and also by Assumption 3(ii) r is decreasing in K. From Eq. (15), therefore, $\Psi_2(h_1, \tilde{h}_1, K_2, \theta) \leqslant \Psi_2(h_1, \tilde{h}_1, K_1, \theta)$ and from the definition of Z, $\Psi_2(h_1, \tilde{h}_1, K_1, \theta) = \Psi_1(\tilde{h}_1, K_1, \theta)$. Also, $l^*(\tilde{h}, K, \theta)$ decreasing in K and $u_{cl} \geqslant 0$ imply that $\Psi_1(\tilde{h}_1, K_2, \theta) \leqslant \Psi_1(\tilde{h}_1, K_1, \theta)$. Therefore, as K increases both Ψ_1 and Ψ_2 decrease. Using the fact that $\Psi_1(\tilde{h}_2, K, \theta) \leqslant \Psi_1(\tilde{h}_1, K, \theta)$, for all (K, θ), from part I above we get

$$\Psi_1(\tilde{h}_2, K_2, \theta) \leqslant \max\{\Psi_1(\tilde{h}_1, K_2, \theta), \Psi_2(h_1, \tilde{h}_1, K_2, \theta)\} \leqslant \Psi_1(\tilde{h}_1, K_1, \theta),$$

which verifies $u_c(\tilde{h}_2, l(\tilde{h}_2, K_2, \theta)) \leqslant u_c(\tilde{h}_1, l(\tilde{h}_1, K_1, \theta))$. ∎

Notice that $\mathbf{H}$ is a non-empty, convex subset of a space of continues, bounded real-valued functions but it not equicontinuous and is therefore not compact. Since it is well known that the space of all continuous functions on a compactum, denoted by $\mathfrak{I}(X)$, with the sup-norm metric is a Banach lattice, $\mathbf{H}$ is a sublattice in $\mathfrak{I}(X)$. Now, a closed subset of continuous, bounded real-valued functions (on a compact domain) equipped with sup-norm metric is compact if and only if is equicontinuous. The theorem of Arzela and Ascoli (see Dieudonne [16, pp. 136–137]) says that a set of equicontinuous, pointwise compact subsets of the continuous functions is relatively compact. A set is relatively compact if its closure is compact.

Define the following subset of $\mathbf{H}$:

$$\begin{aligned}\bar{\mathbf{H}} = \{h \in \mathbf{H} \text{ such that } 0 \leqslant |h(K_2, \theta) - h(K_1, \theta)| \\ \leqslant |F(K_2, l^*(h(K_2, \theta), K_2, \theta)) - F(K_1, l^*(h(K_1, \theta), K_1, \theta))|, \\ \text{for all } K_2 \geqslant K_1\}\end{aligned}$$

A standard argument shows that the space of consumption functions $\bar{\mathbf{H}} \subset \mathbf{H}$ is a closed, pointwise compact, and equicontinuous set of functions. Then by a standard application of Arzela–Ascoli, $\bar{\mathbf{H}}$ is a compact order convex, order interval in **H**.[10] Notice that the restriction on consumption in the space $\bar{\mathbf{H}}$ that distinguishes it from **H** implies that the investment function $K' = F_h - h$ is an increasing functions of the current capital stock K which follows because F_h is increasing in K (since l^* is decreasing in K, the marginal products of capital and labor are positive).

We now define the standard pointwise partial order on $\mathbf{H}$: $h' \geqslant h$ if $h'(S) \geqslant h(S)$ for all $S \in \mathbb{K} \times \Theta$, and we adopt the same order on the subspace $\bar{\mathbf{H}}$. We now show that the operator A defined on complete lattice $\bar{\mathbf{H}}$ in monotone increasing. We now have the following lemma

LEMMA 1. *$\bar{\mathbf{H}}$ is a complete lattice.*

Proof. The interval topology on a partially ordered set X is that topology that has the property that each closed set is either X, the empty set, or can be represented as the intersection of sets that are finite unions of closed intervals in X. A lattice X is complete if every subset set of X has a supremum. Frink [17] shows that a lattice is compact in its interval topology if and only if it is a complete lattice. The converse is shown in Birkhoff [4]. Notice that $\bar{\mathbf{H}}$ is compact in its interval topology; therefore, it is a complete lattice. ∎

We now show that the operator A maps **H** into the complete lattice of functions, $\bar{\mathbf{H}}$.

THEOREM 2. $Ah \subset \bar{\mathbf{H}}$.

Proof. By Theorem 1, $Ah \subset \mathbf{H}$. Then we only need to show that for all $K_2 \geqslant K_1$,

$$\begin{aligned} 0 &\leqslant |Ah(K_2, \theta) - Ah(K_1, \theta)| \\ &\leqslant |F(K_2, l^*(Ah(K_2, \theta), K_2, \theta)) - F(K_1, l^*(Ah(K_1, \theta), K_1, \theta))|. \end{aligned}$$

Take $K_2 \geqslant K_1$. Since $Ah \in \mathbf{H}$, $\Psi_1(Ah(K_2, \theta), K_2, \theta) \leqslant \Psi_1(Ah(K_1, \theta), K_2, \theta)$. But then $\Psi_2(h, Ah(K_2, \theta), K_2, \theta) \leqslant \Psi_2(h, Ah(K_1, \theta), K_2, \theta) \leqslant \Psi_2(h, Ah(K_1, \theta), K_1, \theta)$ for a solution at $Z = 0$. This requires $0 \leqslant |Ah(K_2, \theta) - Ah(K_1, \theta)| \leqslant |F(K_2, l^*(Ah(K_2, \theta), K_2, \theta)) - F(K_1, l^*(Ah(K_1, \theta), K_1, \theta))|$. ∎

Since $\bar{\mathbf{H}}$ is a compact subset of **H**, and A is a continuous mapping of **H** into $\bar{\mathbf{H}}$, we can restrict A to the domain $\bar{\mathbf{H}}$. We could then establish the

[10] Note that it is here that the assumption of a countable number of states in Θ is meaningful. To generalize to a continuum of states, one can add assumptions concerning the existence and direction of the derivative $f_\theta(k, n, k, n\theta)$ (namely that f_θ exists, is nonnegative, and has a uniform bound). Then the equicontinuity argument can be adapted using the derivative f_θ or the uniform continuity of f also.

existence of a fixed point for A by appearing to Schauder's theorem (see, for example, Hutson and Pym [26, p. 208, Theorem 8.2.3.]). Unfortunately, as noted in Coleman [9], such an existence result is not useful since the zero consumption plan is a (trivial) fixed point, which is not an equilibrium: there does not exist a strictly concave value function that satisfied the household's Bellman equation that is associated with the zero consumption plan. Therefore, we do not follow this line of argument for existence of an equilibrium. Instead, we use Tarski's theorem that proves that an increasing self-map on a complete lattice has a fixed-point.

In fact,

THEOREM 3. *Under Assumptions* 1–4, *A is monotone on* $\bar{\mathbf{H}}$.

Proof. For monotonicity, take $h' \geqslant h$, $h, h' \in \bar{\mathbf{H}}$. Consider Eqs. (11)–(14). Note that u_c is strictly decreasing in h, the second term in Z, Ψ_2 is decreasing in h, while the first term Ψ_1 is independent of h. Also, $Z(h, Ah, K, \theta) = 0$ by definition of the operator A; therefore $Z(h', Ah, K, \theta) \geqslant 0$. Z is decreasing in its second argument by Assumptions 1(i), 2(ii) and 3(ii); hence the solution Ah' must be such that $Ah' \geqslant Ah$. ▮

Tarski's theorem states that the set of fixed points of the monotone operator, mapping on a complete lattice $\bar{\mathbf{H}}$ into itself is non-empty. Further, if there exists an order interval in $\bar{\mathbf{H}}$ say $[a, b]$, such that $a \leqslant Aa$, and $Ab \leqslant b$, then one can compute the minimal and maximal fixed points of Ah on $\bar{\mathbf{H}}$. As in Coleman [9] and Greenwood and Huffman [23], an element in $\bar{\mathbf{H}}$ that the operator A maps down in F. Taking successive iterations from that A^nF we show that the limit, $\lim_{n \to \infty} A^nF$, is the maximal fixed point of A. And zero consumption at every state is the minimal fixed point in $\bar{\mathbf{H}}$.

PROPOSITION 2. *Under Assumptions* 1–4, *among the set of fixed points of* A: $\bar{\mathbf{H}} \to \bar{\mathbf{H}}$ *exists a maximal fixed point* $Ah^* \in \bar{\mathbf{H}}$ *such that* $\lim_{n \to \infty} A^nF \to Ah^* = h^*$, *uniformly. Further, the maximal fixed point has* $h^* > 0$.

Proof. The first result (existence and convergence) follows from the fact that $\bar{\mathbf{H}}$ is complete and A is a monotone self-map of $\bar{\mathbf{H}}$. Note $AF \leqslant F$; therefore, by application of Tarski's theorem, the operator A has a fixed point. And, in addition, A^nF converges to a maximal fixed point Ah^* in the set $\mathbf{H}_I = \{h \mid h \in \bar{\mathbf{H}}, h \leqslant F\}$. Since $\bar{\mathbf{H}}$ and S are compact, the convergence is uniform. The second property (positivity) follows from an obvious modification of the main theorem in Greenwood and Huffman [23]. ▮

In a paper on distorted infinite horizon models without elastic labor supply, Coleman [9] proves that for the case of CES preferences, there exists at most one strictly positive fixed point of A. In Coleman [10], an alternative method of proof is provided which allows consideration of smooth concave utility for the inelastic labor supply case. We now provide the proof of uniqueness of equilibrium for the elastic labor supply case.[11] The method of proof consists of several steps and, in particular, considers a different space of functions in which to look for the unique equilibrium and this space is related to **H**.

Before we prove uniqueness of equilibrium, we establish some useful results. First, we define a function $f^u(K, \theta) = f(K, 1 - \hat{l}(K, \theta), \theta)$. We then can define a set of functions **M** as follows:

$$\mathbf{M} = \Big\{ m(K, \theta) \mid m\colon \mathbf{K} \times \Theta \to \mathbf{K} \text{ is continuous;}$$

$$0 \leqslant m(K, \theta) \leqslant \frac{1}{u_c(f^u(K, \theta), \hat{l}(K, \theta))} \text{ for } K > 0;$$

$$m(K, \theta) = 0 \text{ for } K = 0; \text{ and } \frac{r(K', \theta)}{m(K', \theta)} < \frac{r(K, \theta)}{m(K, \theta)} \text{ for } K' > K \Big\}.$$

The space **M** can be interpreted as the space of the reciprocal of the marginal utility functions. Since $r(K, \theta)$ in our problem is continuous on a compact set, it is uniformly continuous. Therefore it is straightforward to verify that **M** is an equicontinuous, pointwise compact subset of the space of continuous functions on a compact topological space, namely $\Im(X)$. Therefore h and m are related. To see this, we define a suitable operator on the space **M** and find a unique strictly positive fixed point of this operator. First, define the function $H(m, K, \theta)$ for each m**M** implicitly as follows (the following lemma makes sure that this definition is meaningful):

$$u_c(H(m(K, \theta), K, \theta), l(H(m(K, \theta), K, \theta), K, \theta)) = \frac{1}{m(K, \theta)}, \quad m > 0;$$

$$\text{and} \qquad H(m, K, \theta) = 0, \quad m = 0.$$

Note that then $H(m(K, \theta), K, \theta) = h(K, \theta)$ pointwise.

LEMMA 2. *The mapping $H(m, K, \theta)$ is well defined for each m, K, θ.*

[11] It is useful to note that given the construction of a strictly positive fixed point of A in the previous proposition, the compactness of **H** is not essential to the existence of C^0 (continuous, but not differentiable) equilibrium. In future work, we shall study the smoothness of the equilibrium manifold for this environment. In such a setting, for reasons relating to continuity of A and to handle issues associated with uniform convergence in the standard uniform C^1 topology, compactness plays a more fundamental role.

Proof. As $m \to 0$, $H(m, K, \theta) \to 0$, for all (K, θ) and as $m \to 1/(u_c(f^u(K, \theta), \hat{l}(K, \theta)))$, $H(m, K, \theta) \to f(K, 1-\hat{l}, \theta)$. Also, note that $H(1/(u_c(c, l(c, K, \theta))), K, \theta) = c$, for all (K, θ), and H is continuous. Therefore, H is well defined. ▮

To characterize $H(m, K, \theta)$, take $m' \geqslant m$ in the pointwise partial order on **M**. Define $h_2 = H(m', K, \theta)$ and $h_1 = H(m, K, \theta)$. Notice when $m' \geqslant m$, we have $h_2 \geqslant h_1$. We can now show that $f(k, 1 - l(H(m, K, \theta), K, \theta)) - H(m, K, \theta)$ is decreasing in m by the definition of $H(m, K, \theta)$. Define

$$\Delta(h, f_h - h, \theta) = \beta \int u_c(h(f_h - h, \theta'), l(h(f_h - h, \theta'), f_h - h, \theta')) \times r(f_h - h, \theta')\, \chi(\theta, d\theta').$$

Then for $m' \geqslant m$, we have the following inequality

$$u_c(Ah_1, l(Ah_1, K, \theta)) = \Delta(h_1, f_{Ah_1} - Ah_1, \theta) \geqslant \Delta(h_2, f_{Ah_1} - Ah_1, \theta).$$

Therefore, for such a perturbation of h, the mapping Z used in the definition of Ah is now nonnegative. Therefore, the first term in the definition of Z must decrease and the second term must increase in a solution Ah_2. The latter implies $f_{Ah_2} - Ah_2 \leqslant f_{Ah_1} - Ah_1$. But then by the definition of $H(m, K, \theta)$, $f(K, 1 - l(H(m, K, \theta), K, \theta)) - H(m, K, \theta) = f_{H(m)} - H(m)$ must be decreasing in m. Since m' and m were arbitrary, that completes the proof of the claim.

Now, define the mapping

$$\hat{Z}(m, \tilde{m}, K, \theta) = \frac{1}{\tilde{m}} - \beta \int_{\Theta} \frac{r(f_{\tilde{m}} - H(\tilde{m}, K, \theta), \theta')}{m(f_{\tilde{m}} - H(\tilde{m}, K, \theta), \theta')} \chi(\theta, d\theta'), \tag{16}$$

where $f_m - H(m, K, \theta) = f(K, 1 - l(H(m, K, \theta), K, \theta), \theta) - H(m, K, \theta)$ and we are ready to define the operator:

$\hat{A}(m) = \{\tilde{m} \in \mathbf{M} \mid \hat{Z}(m, \tilde{m}, K, \theta) = 0$, for $m > 0$; 0 elsewhere$\}$.

Defining the standard partial order on **M**, that is, $m' \geqslant m$, $m', m \in \mathbf{M}$ if and only if $m'(K, \theta) \geqslant m(K, \theta)$ for all (K, θ). Finally, if $m'(K, \theta) > m(K, \theta)$, $m, m' \in \mathbf{M}$, H must be such that $u_c(H(m, K, \theta), l(H(m, K, \theta), K, \theta))$ is decreasing in m for each (K, θ). Since $h \in \mathbf{H}$, $u_c(c, l(c, K, \theta)$ is decreasing in c, and there exists $h, h' \in \mathbf{H}$ such that $h' = H(1/(u_c(h', l(h', K, \theta))), K, \theta) = H(m', K, \theta)$ and $h = H(1/(u_c(h, l(h, K, \theta))), K, \theta) = H(m, K, \theta)$.

If the operator $\hat{A}m$ is well defined, we will be able to relate orbits of the operator $\hat{A}^n m_0 \in \mathbf{M}$ to the operator $A^n h_0 \in \bar{\mathbf{H}}$ by the following construction.

Consider some $h_0 \in \bar{\mathbf{H}}$. For such an h_0, there exists an $m_0 = 1/(u_c(h_0, l(h_0, K, \theta))) \in \mathbf{M}$ such that $H(1/(u_c(h_0, l(h_0, K, \theta))))= h_0$. By definition, $\hat{Z}(m_0, \hat{A}m_0, K, \theta) = \hat{Z}(H(1/(u_c(h_0, l(h_0, K, \theta))), K, \theta), \hat{A}H(1/(u_c(Ah_0, l(Ah_0, K, \theta)))), K, \theta) = Z(h_0, Ah_0, K, \theta)$. Therefore, $h_1 = Ah_0 = H(1/(u_c(Ah_0, l(Ah_0, K, \theta)))) = H(\hat{A}m_0)$. A similar argument establishes $A^n h_0 = H(\hat{A}^n m_0)$, $n = 1, 2, \ldots$. We now show the operator $\hat{A}m$ is well defined.

LEMMA 3. *The operator $\hat{A}$ is a well-defined self-map.*

Proof. Recall $f_m - H(m, K, \theta)$ decreasing in m. Also that $\hat{Z}$ is strictly increasing in m, and strictly decreasing in $\tilde{m}$. Also for fixed $m > 0$, $K > 0$, $\tilde{m} \to 0$ implies that $\hat{Z} \to -\infty$. Similarly, as $\tilde{m} \to 1/(u_c(f, \hat{l}(K, \theta)))$, $\hat{Z} \to \infty$. Therefore there is a unique $\hat{A}m$ for each $m > 0$, $K > 0$, all θ. Note $\hat{A}m = 0$ elsewhere.

Now, we show $\hat{A}m \subset \mathbf{M}$. Note also that if $K' > K$, $\hat{Z}(m, \tilde{m}, K', \theta) > \hat{Z}(m, \tilde{m}, K, \theta)$. Therefore, $\hat{A}m(K', \theta) > \hat{A}m(K, \theta)$, when $K' > K$. Since r decreasing in K, $\frac{r}{m}$ is decreasing in K. Therefore $\hat{A}m \subset \mathbf{M}$. ▮

We now can prove the last step of our argument.

LEMMA 4. *$\hat{A}$ has a strictly positive fixed point.*

Proof. We know that $m = 0$ is a fixed point of $\hat{A}$. To verify that $\hat{A}$ has strictly positive fixed points in $\mathbf{M}$, consider the trajectory of $\hat{A}$ from $m_0 = 1/(u_c(f^u, \hat{l}(K, \theta)))$. It is easily verified that $0 < \hat{A}m_0 \leqslant m_0$. Compute $h_0(K, \theta) = H(m_0(K, \theta)) = H(1/(u_c(h_0, l(h_0, K, \theta)))) \leqslant H(1/(u_c(f^u, \hat{l}(K, \theta))))$. Notice that $h_0(K, \theta)$ is equal to an orbit of the operator $Ah \in \bar{\mathbf{H}}$, namely, it is the same as $Af^u(K, \theta)$. Therefore, h_0 is the optimal plan associated with a one-period distorted dynamic economy. similar calculations show $h_n(K, \theta) = H(A^{n-1} m_0(K, \theta))$ is the optimal plan associated with an n-period economy. Since A and $\hat{A}$ are continuous, and they both map compact sets to compact sets, $h^* = \lim_{n \to \infty} h_n(K, \theta) = \lim_{n \to \infty} H(A^n m_0(K, \theta)) \in \bar{\mathbf{H}}$. Therefore following an argument in Coleman [11], we can associate a value function with each orbit $A^n f^u$ which is strictly concave. Therefore $h^* > 0$, and $m^* = \lim_{n \to \infty} H(A^n m_0) > 0$. So $\hat{A}$ has a strictly positive fixed point. ▮

For uniqueness of this strictly positive fixed point m^*, we will show that $\hat{A}m$ is a K_0-monotone and a pseudo-concave operator, and therefore it has at most one strictly positive fixed point. First the definitions:

DEFINITION 2. A monotone operator $\hat{A}$ is pseudo-concave if for any strictly positive $m \in \mathbf{M}$, and $t \in (0, 1)$, $\hat{A}tm > t\hat{A}m$ for all (K, θ).

DEFINITION 3. An operator $\hat{A}$: $\mathbf{M} \to \mathbf{M}$ is said to be K_0-monotone if it is monotone and if there exists, for any strictly positive fixed point m_1 of $\hat{A}$, a strictly positive K_0 such that for any $K_1 \in [0, K_0]$ and any $m_2 \in \mathbf{M}$, $m_2 \leqslant m_1$, $\hat{A}m_2(K, \theta) \leqslant \hat{A}m_1(K, \theta)$, for all $K \geqslant K_1$.

THEOREM 4. *Under Assumptions* 1–4, $h^* > 0$ *is the unique equilibrium.*

Proof. Since $\hat{Z}$ is increasing in m, and decreasing in $\tilde{m} = \hat{A}m$, $\hat{A}m_1 \geqslant \hat{A}m_2$ for $m_1 \geqslant m_2$. Also the Inada condition in Assumption 2 is sufficient for $\hat{A}$ to be K_0-monotone (see Coleman [9, Lemmas 9, 10]). Finally, a sufficient condition for pseudo-concavity is

$$\hat{Z}(tm, t\hat{A}m, K, \theta) > \hat{Z}(tm, \hat{A}tm, K, \theta). \tag{17}$$

This inequality follows since $m \in \mathbf{M}$, and r decreasing in K. Hence,

$$\hat{Z}(tm, t\hat{A}m, K, \theta) = \frac{1}{\tilde{m}} - \beta \int_{\Theta} \frac{r(f_{\tilde{m}} - H(t\tilde{m}), \theta')}{m(f_{\tilde{m}} - H(t\tilde{m}), \theta')} \chi(\theta, d\theta') > 0, \tag{18}$$

and $\hat{Z}(tm, \hat{A}tm, K, \theta) = 0$. Therefore by Theorem 8 in Coleman [9]), $\hat{A}$ has at most one strictly positive fixed point. Also note that this implies that Ah has at most one strictly positive fixed point. If not, there exists another strictly positive fixed point, say h^{**} such that $\lim_{n \to \infty} A^n h'_0 = h^{**}$ and $h^{**} = \lim_{n \to \infty} H(\hat{A}^n m'_0)$ where $\lim_{n \to \infty} \hat{A}^n m'_0 \to m^{**}$. But that contradicts the fact that $\hat{A}m$ has at most one strictly positive fixed point in $\mathbf{M}$. Therefore $Ah^*(K, \theta) = H(m^*(K, \theta)) > 0$, and there is only one equilibrium in $\mathring{\mathrm{H}}$. ∎

5. DISCUSSION

5.1. *Relating the Results to Those in the Existing Literature*

To relate our results to those existing in the literature, we consider the dynamic capital accumulation model with elastic labor supply as considered in Coleman [11]. For this economy, we show that existence and uniqueness of equilibrium can be established applying the theorems in Coleman [9, 10], which are models with inelastic labor supply. This section, therefore, highlights the limited nature of the environments admissible in Coleman [11]. Our construction exploits the homogeneous structure of the environment assumed in that paper. The homogeneity allows us (after a simple transformation of the commodity space and the economic primitives) to state the existence problem for an artificial class of Brock–Mirman models without elastic labor supply, indexed by a "scaling" parameter. For each scale parameter, existence and uniqueness of equilibrium can be studied using the methods of Coleman [9]. The equilibrium scale parameter is then determined by solving a simple side condition that governs the optimal consumption–leisure mix.

We first describe the restrictions on the economic primitives of u, f, r, w that are maintained in this section of the paper:

Assumption 5. (i) The period preference u is homothetic, i.e., $u(c, l) = v(g(c, l))$ where g is homogeneous of degree one and v is a monotonic increasing transformation; and $u_{cl} \geqslant 0$ and that implies $\Psi(\frac{c}{l}) = (g_l(c, l))/(g_c(c, l))$ has $\Psi' > 0$. In addition, g satisfies the Inada conditions.

(ii) The equilibrium distortions are homogeneous, i.e., $w(K, \theta) = w(\frac{K}{N}, \theta)$ and $r(K, \theta) = r(\frac{K}{N}, \theta)$.

Note that these assumptions are more general than those studied in Coleman [11]. We allow period utility to be homothetic, as opposed to the restricted homothetic form studied in that paper.

To formalize the side conditions used in the model, imagine households facing a similar environment to that in Section 2. The household's first stage problem under Assumptions 1–5 generates the following first order condition,

$$\Psi\left(\frac{c}{l}\right) = w(K_n, \theta),$$

where $K_n = \frac{K}{N}$. Also, $w(K_n, \theta) = (1 - \pi_n(K_n, \theta))\, \bar{w}$ where $\bar{w} = f_n(K_n, \theta) = f_2(K_n, 1, K_n, 1, \theta)$. From Assumptions 2 and 3 w is increasing in K_n and is C^1.[12] Define the space $\mathbf{H}^c$ to be the closure of the space of continuous functions $h\colon S \to R_+$ such that $h \in [0, f(K_n, \theta)]$ and $h(K'_n, \theta) - h(K_n, \theta) \leqslant f(K'_n, \theta) - f(K_n, \theta)$.

Standard arguments show that $\mathbf{H}^c$ is a convex and an equicontinuous set of functions. Therefore $\mathbf{H}^c$ is compact. Define the parameter $z(K, \theta; h)$ implicitly for each, $h \in \mathbf{H}^c$, from the equation

$$\Psi\left(\frac{h}{1 - z(K, \theta; h)}\right) = w\left(\frac{K}{z(K, \theta; h)}, \theta\right). \tag{19}$$

Now using the homogeneity assumptions we can rewrite the decision problem for the firm in terms of $k_n = \frac{k}{n}$ for a given z. Here z is interpreted as the level of labor supply, i.e., $f(k, n, K, N, \theta) = f(k_n, K_n, \theta)\, z$. Then, for given continuous functions describing the competitive factor prices, say

[12] Since in this paper and the work of Coleman [11], $w(K_n, \theta)$ is a continuous and monotone in K_n, it is well known that w is therefore almost everywhere differentiable in K_n. So the assumption that w is C^1 differs from the assumptions used in Coleman [11] on a set of Lebesgue measure zero.

$\bar{r}(K_n, \theta)$ and $\bar{w}(K_n, \theta)$ for the rental price of capital and the wage rate, each firm solves

$$\Pi_n(\bar{r}, \bar{w}, K_n, \theta) = \sup_{k_n \in \mathbf{K}} f(k_n, K_n, \theta) - \bar{r}k_n - \bar{w}. \tag{20}$$

Anticipating the standard definition of competitive equilibrium with $k = K$, $n = N$ prices in the factor markets are simply

$$\bar{r} = f_k(K_n, \theta)$$

and

$$r = (1 - \pi_k(K_n, \theta))\, f_k(K_n, \theta). \tag{21}$$

Notice that the firms' problem is well defined and the theorem of the maximum implies the value function Π_n is a continuous function.

We now study existence and uniqueness using, basically, the techniques of Coleman [9]. To understand how the economy studied in Coleman [11] can be viewed as a model without a labor–leisure choice, but involving a scaling parameter z, first consider the case that households in the actual economy have period preferences $v(g(c, l)) = g(c, l)$, i.e., the case that period preferences are homogeneous of degree one. Notice that this case is not covered by the assumptions of Coleman [11], since here v is not strictly concave and does not satisfy the Inada condition.

Let the preferences for the class of Brock–Mirman models without a labor–leisure choice be $g(c_l)\,\frac{1-z}{z}$ where $g(c_l) = g(\frac{c}{l}, 1)\, l$. Here, we eliminate the labor–leisure choice (with $l = 1$), and introduce the scale parameter z. Define $\mathbf{L}$ as the space of functions $z(K, \theta)$ such that z: $\mathbb{S} \to (0, 1)$ is continuous and $\frac{1-z}{z}$ is non-decreasing in K. Imagine households as solving the following dynamic program. Given $z \in (0, 1)$, and κ in $\mathbb{G}$, $\bar{r}$, r, π_n and d_n, where $\mathbb{G}$ is the space of bounded, continuous, non-negative, real-valued functions. The household solves the Bellman equation

$$J(s_n) = \sup_{c_l,\, k'_n \in \Gamma(s_n)} \left\{ g(c_l)\,\frac{1-z}{z} + \beta \int_{\Theta} J(s'_n)\, \chi(\theta, d\theta') \right\}, \tag{22}$$

where $\Gamma(s_n) = \{c_l, k'_n \,|\, \frac{1-z}{z} c_l + k'_n \leqslant y_n;\quad y_n = rk_n + d_n + \Pi_n,\quad (c_l, k_n) \geqslant 0,$ $d_n(K_n, \theta) = \pi_k K_n + \pi_n;\ \Pi_n(\bar{r}, \bar{w}, K_n, \theta) = \sup_{k_n \in \mathbf{K}} f(k_n, K_n, \theta) - \bar{r}k_n - \bar{w},$ $s_n = (k_n, S_n)\}$.

Notice a couple of things. First d_n is exactly the same restriction as in Coleman [11]. Also, if z equals the equilibrium labor supply in sate (K, θ), then y_n is exactly the same y in Coleman [11]. Further in such an equilibrium, the transformation surface used to describe social feasibility for the actual economy $f(K, N, \theta) = C + K'$ is exactly equal to the transformation surface for the artificial economy $f(K_n, \theta) = \frac{1-z}{z} C_l + K'_n$.

Defining a recursive equilibrium for this scaled artificial economy, following Coleman [9], using standard arguments to describe the equilibrium conditions, and explicitly noting the influence of the labor supply decision on the factor prices (and recognizing that in equilibrium these decisions are functions of the aggregate state variable) one can show that the set of equilibrium restrictions for this economy with homogenous preferences is summarized by the following two equations. I.e., in equilibrium we have, for each $z \in \mathbf{L}$,

$$g_c(C_l) = \beta \int g_c(C'_l)\, r(K'_n, \theta')\, \chi(\theta, d\theta'), \tag{23}$$

$$\frac{1 - z(K, \theta)}{z(K, \theta)} C_l + K'_n = F(K_n, \theta). \tag{24}$$

We solve the functional Eq. (23), using the methods in Coleman [9], for a unique $C_l^*(K_n, \theta; z)$.

COROLLARY 1 (Coleman [9]). *Under Assumptions* 1–5, *there exists a unique strictly positive equilibrium in the model with a labor-leisure choice.*

Proof. Consider the following operator $Z(h, \tilde{h}, K, \theta; z)$: $\mathbf{H}^c \times \mathbf{H}^c \times \mathbf{S} \times \mathbf{L} \to \mathbf{R}$ as

$$Z(h, \tilde{h}, K_n, \theta; z) = \Psi_1(\tilde{h}, K_n, \theta) - \Psi_2(h, \tilde{h}, K_n, \theta; z), \tag{25}$$

where in this case,

$$\Psi_1(\tilde{h}, K_n, \theta) = g_c(\tilde{h}),$$

$$\Psi_2(h, \tilde{h}, K_n, \theta; z) = \beta \int_{\Theta} g_c\left(h\left(f - \frac{1-z}{z}\tilde{h}, \theta'\right)\right) \times r\left(f - \frac{1-z}{z}\tilde{h}, \theta'\right) \chi(\theta, d\theta').$$

Here $f = f(K_n, \theta)$. Use Z to define a nonlinear operator on $\mathbf{H}^c$:

$$A_1 h = \{\tilde{h} \text{ such that } Z(h, \tilde{h}, K_n, \theta; z) = 0,\ h > 0;\ 0 \text{ elsewhere};\ z \in \mathbf{L}\}.$$

Then for any such $z \in \mathbf{L}$, the fact that A_1: $\mathbf{H}^c \to \mathbf{H}^c$, A_1 is monotone, A_1 is k_o-monotone, and A_1 is pseudo-concave follows from the main theorems in Coleman [9]. Hence, there exists a unique strictly positive $C_l^*(K_n, \theta; z)$

which is a fixed point of A_1, and therefore the equilibrium for this fictitious economy is (C_l^*, κ_n) where $K_n' = \kappa_n(K_n, \theta; z)$ is unique. ∎

Remark 1. Notice that in Corollary 1, we could study the equilibrium problem on the exact same set of closure of the relatively compact functions used in Coleman [9]. Therefore, the consumption–leisure ratio is monotonically increasing in K_n.

Now, to construct the equilibrium in the actual economy, from the equilibrium of the artificial economy we rescale. Multiply C_l^* by $1-z$ to get $C^*(K_n, \theta; z)$. The side condition, given the function $C^*(K_n, \theta; z)$, is

$$\Psi\left(\frac{C^*\left(\frac{K}{z(K,\theta)}, \theta; z(K,\theta)\right)}{1-z(K,\theta)}\right) = \left(1-\pi_n\left(\frac{K}{z(K,\theta)}, \theta\right)\right) f_n\left(\frac{K}{z(K,\theta)}, \theta\right), \tag{26}$$

which can be used to compute a unique root $z^*(K, \theta) \in (0, 1)$. From this $z^*(K, \theta)$, we can compute the scales of the $\frac{C}{l}$ and $\frac{K}{n}$ as needed for the original economy. To do this, use $z^*(K, \theta)$ to compute $C_l^*(K_n, \theta; z^*) = C_l^*(K/z^*, \theta; z^*) = C_l^*(K, \theta)$. Then again multiplying by $(1-z^*)$, the actual consumption level in this solution is $C^*(K/z^*, \theta; z^*) = C^*(K, \theta)$. Note that the social feasibility condition for the artificial economy $\frac{1-z}{z} C_l + K_n' = f(K_n, \theta)$, for any z, and the feasibility for the actual economy, $f(K, z, \theta) = C + K'$, are equivalent. Upon substitution into the operator Z,

$$Z(C_l^*(K,\theta), C_l^*(K,\theta), K, \theta; z^*(K,\theta)) = \Psi_1(C_l^*, K, \theta) - \Psi_2(C_l^*, C_l^*, K, \theta; z^*(K,\theta)), \tag{27}$$

where

$$\Psi_1(C_l^*, K, \theta) = u_c(C_l^*(K, \theta)),$$

$$\Psi_2(C_l^*, C_l^*, K, \theta; z^*) = \beta \int_\Theta u_c(C_l^*(f(K, z^*, \theta) - C^*(K,\theta), \theta')) \times r(f(K, z^*, \theta) - C, \theta')\, \chi(\theta, d\theta').$$

Therefore, the operator Z is zero for such a C^* and C_l^*. Thus, the triplet (C^*, z^*, κ) with $K' = \kappa(S)$, $S = (K, \theta)$ satisfies all the restrictions of the original economy, and is therefore equilibrium for the actual economy and is also unique.

Now, to allow for the entire class of homothetic preferences, we only modify the above argument slightly. For the actual economy, we have the following set of restrictions in equilibrium (see, e.g., Coleman [11]),

$$v'(g(C, 1-N))\, g_c\left(\frac{C}{1-N}\right) = \beta \int_{\Theta} v'(g(C', 1-N'))\, g_c\left(\frac{C'}{1-N'}\right) \times r(K', \theta')\, \chi(\theta, d\theta'), \tag{28}$$

$$C + K' = f(K, \theta), \tag{29}$$

$$\Psi\left(\frac{C}{1-N}\right) = w(K, \theta), \tag{30}$$

with $\Psi' > 0$.

Following the same procedure as for the model with $v = 1$, for a "scaled" Brock–Mirman model without a labor–leisure choice, we can generate an equivalent set of equilibrium restrictions on the three functions (C, κ, L). Define a subset of $\mathbf{L}$, say $\mathbf{L}'$, as the space of functions $z \in \mathbf{L}$ such that for any $C_l \in \mathbf{H}^c$, $v'(g(C_l, 1-z))\, g_c(C_l)$ is decreasing in K_n. Then given $z \in \mathbf{L}'$, κ in $\mathbb{G}$ and $\bar{r}$, r, π_n, d_n, the household solves the Bellman equation

$$J(s) = \sup_{c_l,\, k'_n \in \Gamma(s_n)} v'(g(C_l)\{1-z\})\, g(c_l)\, \frac{1-z}{z} + \beta \int_{\Theta} J(s'_n)\, \chi(\theta, d\theta'), \tag{31}$$

where again $\Gamma(s_n) = \{c_l, k'_n \mid \frac{1-z}{z} c_l + k'_n \leqslant y_n;\ y_n = rk_z + d_n + \Pi_n,\ (c_l, k_n) \geqslant 0,$ $d_n(K_n, \theta) = \pi_k K_n + \pi_n;\ \Pi_n(\bar{r}, \bar{w}, K_n, \theta) = \sup_{k_n \in \mathbf{K}} f(k_n, K_n, \theta) - \bar{r}\, k_n - \bar{w},\ s_n = (k_n, S_n)\}$. Standard arguments prove that the value function has all the desirable properties. Therefore, after defining a recursive competitive equilibrium in the standard manner noting that $C_l = c_l$, for each $z \in \mathbf{L}'$,

$$v'(g(C_l)[1-z])\, g_c(C_l) = \beta \int v'(g(C'_l)\{1-z'\})\, g_c(C'_l) \times r(K'_n, \theta')\, \chi(\theta, d\theta'), \tag{32}$$

$$\frac{1-z(K, \theta)}{z(K, \theta)}\, C_l + K'_n = F(K_n, \theta). \tag{33}$$

From Corollary 1, for any $z \in \mathbf{L}'$, there exists a unique strictly positive equilibrium in a model without a labor-leisure choice. Thus to complete the argument, we must show that there exists some $z^* \in \mathbf{L}'$ that satisfies the side condition

$$\Psi\left(\frac{C^*\left(\frac{K}{z^*(K, \theta)}, \theta; z^*(K, \theta)\right)}{1 - z^*(K, \theta)}\right) = w\left(\frac{K}{z^*(K, \theta)}, \theta\right). \tag{34}$$

That is, there must be a $z^* \in \mathbf{L}'$ such that it satisfies

$$\frac{g_l\left(C^*\left(\frac{K}{z(K,\theta)},\theta;z^*(K,\theta)\right)\right)}{g_c\left(C^*\left(\frac{K}{z(K,\theta)},\theta;z^*(K,\theta)\right)\right)} = \left(1-\pi_n\left(\frac{K}{z^*(K,\theta)},\theta\right)\right)f_n\left(\frac{K}{z^*(K,\theta)},\theta\right). \tag{35}$$

From Remark 1, C_l^* is monotone increasing in K_n. Therefore, we must only show that there exists a continuous functions z^* such that $u_c(C^*(\frac{K}{z^*},\theta), 1-z^*(K,\theta))$ is decreasing in C_l^*. One function that satisfies this condition is the constant function (i.e., independent of the capital stock) $z^*(K,\theta)=z^*(\theta)$, that belongs to $\mathbf{L}'$.

5.2. *Applications to Monetary Economies*

In this section, we construct some examples in monetary theory where the methods in the previous sections can be applied. The examples are interesting for a couple of reasons. The first example is a case where even though the structure of the economy is very simple, the methods in the existing literature fail. Indeed, the first example makes the point that functional equivalence between monetary and real economies is not necessarily sufficient to use the methods in the existing literature, even if one is willing to impose the restrictions on taste and technology apparently required to use these methods. The problem with directly applying the existing methods for studying real equilibrium with distortionary taxation to the monetary economy is that in general the structure of the equilibrium distortion in the monetary economy falls within the class of the types of equilibrium distortion in the tax economy for which the nonlinear operator used to construct equilibrium has the desired properties. Using our methods, we show that no addition assumptions are needed on the equilibrium distortion.[13] The second example is interesting because it shows that even after constructing a equivalent tax economy to interpret the equilibrium distortions in the monetary economy, additional restrictions on the monetary are often required to deliver distortionary inflation taxes which have the monotonicity structure required for the applying our methods.

[13] This is because of the way money demand is modeled in an exogenous cash-credit model. The structural relationship between income and interest rates in the money demand in such a model depends completely on preferences. In an endogenous cash-credit model, this is not the case. Instead, the money demand function depends critically on how one characterizes transactions technologies.

5.2.1. *Exogenous cash–credit models.* Consider an exogenous cash–credit model of the type describe in Cooley and Hansen [13]. In their model, there is a cash good (consumption) and a credit good (leisure).[14] The aggregate money price of consumption goods take the following homogeneous form,

$$p(S) = \frac{P(S)}{M},$$

where p is assumed to be a bounded, continuous function, $P(S)$ is continuous, and M is the per capita money stock at the beginning of the period.[15] The prices in the factor markets associated with the firms are given using the standard argument,

$$\bar{w}(S) = f_n(K, N, \theta); \qquad \bar{r}(S) = f_k(K, N, \theta),$$

where $S = (K, \theta)$. Finally assume that the per-capita capital stock and the level of per capita labor supply as

$$K' = \kappa(S); \qquad N = N(S).$$

Since this is a cash-in-advance economy, purchases of consumption goods (and possibly new investment goods) are subjected to cash-in-advance constraints of the form,

$$p(c + \xi k') \leqslant m^d + j, \tag{36}$$

where $j = \frac{J(S)}{M}$ is the per-capita, lump-sum money transfer at the beginning of the period, ξ is an indicator variable indicating whether new investment goods are cash goods or credit goods (see Abel [1]), and $m^d = M^d/M$. Households choose next periods money position $M^{d'}$ to satisfy the recursion

$$p(c + x) + \frac{M^{d'}}{M} \leqslant p\{y + (1 - \delta k)\} + m^d + j, \tag{37}$$

where $y = \bar{r}k + \bar{w}N + d + \Pi$. For the monetary economy, we use the equilibrium factor prices from the production sector. We assume that the per-capita money rule takes the following form

$$M' = h_m(S', S)\, M, \tag{38}$$

[14] Because of its tractability, the literature is replete with examples of cash-credit type environments used to study monetary policy. Recent examples include Col and Ohanian [8] and Chari and Kehoe [7].

It should also be mentioned that for the example economy actually studied in Cooley and Hansen [13] using numerical methods (where preferences are logarithmic, technologies are Cobb-Douglas, and the monetary rule does not depend on the endogenous state variable), the methods in Coleman [11] apply.

[15] See Cooley and Hansen [13] for a more complete description of the environment.

where h_m is assumed to be continuous, bounded, and $\beta \int \frac{1}{h_m} \chi(\theta, d\theta') \leqslant 1$. The latter part of this assumption is standard for cash-in-advance models, and forces the nominal interest rate to be nonnegative.

Then the decision problem for a representative agent in this economy is, given p, $\bar{W}$, $\bar{R}$, h, N, ξ, and G, find a value function v with associated policies $q=(c, l, k', m^{d\prime}) \in \Psi(s)$. Here the set $\Psi(s)$ consists of the q that satisfy (36) and (37) with $(c, l, k') \geqslant 0$ and $l+n=1$ and satisfies

$$J(s)= \sup_{q \in \Psi(s)} u(c, l)+\beta \int_{\Theta} J(s')\, \chi(\theta, d\theta'). \tag{39}$$

We can define equilibrium in the standard manner. It is well known that the equilibrium of this economy can be rewritten as a tax problem with the following equilibrium restrictions.

Define a state dependent tax $\tau(S)=\frac{\phi(S)}{\lambda(S)+\phi(S)}$. Then when $\xi=0$, a standard argument shows that an inflation tax in this model is a tax on capital, subsidy to leisure, and the equilibrium conditions correspond with the restrictions in (36) and (37). When $\xi=1$, then the versions of (36) and (37) correspond to a subsidy on leisure and a tax on capital income. Then to make the existence and characterization arguments apply to this monetary economy, we need to solve for a money rule $h_m(S', S)$ that implements inflation taxes (or subsidies) τ that generate prices r and w, where τ generates h given solutions C and N in

$$(1-\tau(S))\, u_c(C(S), 1-N(S))=\beta \int_{\Theta} \frac{u_c(C(S'), 1-N(S'))}{h_m(S', S)}\, \chi(\theta, d\theta'), \tag{40}$$

where all of the quantities are evaluated at their equilibrium values for the hypothetical tax economy.

5.2.2. *Multiple means of payment models.* Many multiple means of payments models are functionally equivalent to the costly credit model described in Lacker and Schreft [32], so we will study a version of their model with capital accumulation.[16] To review briefly the details of the trading environment in this model, note that there are continuum of locations indexed on a circle of unit circumference with a continuum of agents, a continuum at each location. Households reside at one of these locations, say $\eta \in [0, 1]$, and consist of a shopper, a worker, and a producer. Letting $i \in [0, 1]$ index locations, we can study the behavior of a representative household at a particular location, say for the sake of convenience, location

[16] Multiple means of payment models which can be shown to be n the same class as Lacker and Schreft [32] (assuming monetary policy is distortionary and technologies are convex) include Gillman [18] and Ireland [27], although none of these models include capital accumulation.

$\eta = 0$, and impose a symmetric equilibrium on the economy. Suppose that this agent likes goods from every location, but is endowed with technology for producing only "home" location goods, i.e., location 0 goods. Let period preferences be represented by $u(g(\mathbf{c}), l)$, where $\mathbf{c}$ is a vector of consumption goods c_i, $i \in [0, 1]$. Take the aggregator $g(\mathbf{c}) = \inf_{z \in [0,1]}\{c_z\}$, and assume that u: $\mathbf{R} \times [\mathbf{0}, \mathbf{1} \to \mathbf{R}]$ satisfies Assumption 1.[17] The household lifetime preferences are given by

$$E_0\left[\sum \beta^t \int_{z \in [0,1]} u(g(c_z), l)\, dz\right], \tag{41}$$

where $\beta \in (0, 1)$, the z here refers to an arbitrary location, and the expectation is taken at the initial period with respect to states of the household. For production technologies, assume that new vintages of capital are a composite good that are built by transforming goods from all locations in this period into local capital goods next period. For simplicity, we study the aggregator $x = \inf_{z \in [0,1]}\{x_z\}$. Let (local) capital evolve according to the recursion $k' = (1-\delta)\, k + x$. Finished goods output in any location then depends on both capital and local labor supply. Assume that each technology hires home inputs, and receives a common technology shock, so output $y_i = f(K_i, N_i, \theta)$ where f satisfies our Assumption 2.

Anticipating the symmetric equilibrium, note that from the assumptions on preferences and technologies, we can study equilibrium independent of i. We can now discuss the nature of the costly claims in the model following Lacker and Schreft [32]. Let $\zeta_c(z)$ be the cost of issuing a private claim to finance consumption purchases in a location z units away from the household's home location on the unit circle, and similarly define $\zeta_x(z)$ as the cost of financing investment goods. Assume for simplicity that ζ_c, ζ_x are monotone increasing and convex in z and $\zeta_c(0) = 0 = \zeta_x(0)$. Moreover, as $z \to \frac{1}{2}$ both $\zeta_c(z) \to \infty$ and $\zeta_x(z) \to \infty$.[18] Then given a nominal interest rate $i(S)$, there exists a location $z_\iota^*(i(S))$ such that

$$1 + \zeta_\iota(z_\iota^*(S)) = 1 + i(S), \tag{42}$$

where $\iota = c$, x, and z^* is the location at which households are indifferent between costly claims and fiat money to finance purchases. Assume that, as

[17] Note that aggregator $g(\mathbf{c})$ is neither strictly monotone or differentiable, yet this does not complicate the application of the main theorems in Section 4 at all. All we require is that u satisfy Assumption 1.

[18] Weaker assumptions could be made on this transactions cost function. We could, for example, place restriction on monetary policy h such that monetary equilibrium exist. We use this assumption only because we seek to recover a tax economy from the underlying monetary economy, and our argument exploits the fact that monetary equilibrium exist. Similar arguments could be made if we allow for pure credit equilibrium.

in the standard cash-in-advance model, j is a continuous function for the money transfer, the factor prices w and r are continuous functions for the distorted wage rate and rental rate of capital, respectively, and p is the scaled money price of a consumption good. Also assume that functions $h_m(S', S')$, $\kappa(S)$ and $N(S)$ describe the monetary growth rate, the recursion on the per-capita capital stock, and the per capita labor supply function.

Then, to develop the household's decision problem, let $m^{d\prime}$ be the desired level of money holdings next period prior to the transfer. Given that within z_c^* and z_x^* units away from the household's home location, the household is using intermediated claims, and noting that the investment and consumption aggregators have the convenient property that x_z and c_z are independent of z, the endogenous cash-in-advance constraint on consumption and investment good purchases are

$$p(1-2z_c^*(S))\, c \leqslant n_c, \tag{43}$$

$$p(1-2z_x^*(S))\, x \leqslant n_x, \tag{44}$$

$$\sum n_\iota = \hat{m}. \tag{45}$$

Here $\hat{m}$ is the normalized money position of the household at the beginning of the period after all outstanding intermediated claims are settled, and the money transfer j has been received. Therefore, if we let γ_ι be the accounts receivable being paid to the household for goods sold on credit last period, then $\hat{m}$ is simply

$$\hat{m} = \frac{m^d + j}{M} + \frac{\int_0^{z_c^*} (1+\zeta_c(i))\, di(\gamma_c - \bar{c}) + \int_0^{z_x^*} (1+\zeta_x(i))\, di(\gamma_x - \bar{x})}{h_m(S', S)}, \tag{46}$$

where $\bar{c}$ and $\bar{x}$ are last period's value of consumption and new investment. Then, given current period purchases, next periods desired cash holdings $m^{d\prime}$ are determined by

$$\frac{m^{d\prime}}{M} = p(y + (1-\delta)\, k) + \hat{m}, \tag{47}$$

where $y = \bar{r}k + \bar{w}N + d + \Pi - \int_0^{z_c^*} (1+\zeta_c(i))\, di - \int_0^{z_x^*} (1+\zeta_x(i))\, di$, and again the factor prices in the monetary economy are given from the firm's problem.

The decision problem for a representative agent in this economy can be developed as follows. Given p, $\bar{W}$, $\bar{R}$, h, N, ξ, ζ_c, ζ_x, i, and G, find a value function v with associated polices $q = (c, l, k', m^{d\prime}, n_c, n_x) \in \Psi(s)$ where the set $\Psi(s)$ consists of the q that satisfy (43)–(47) with z^* given by (42), and with both $(c, l, k, n_c, n_x) \geqslant 0$, with $l+n=1$, which satisfies

$$J(s) = \sup_{q \in \Psi(s)} \int_{z \in [0,1]} \{u(g(c_z), l)\}\, ds + \beta \int_{\Theta} J(s')\, \chi(\theta, d\theta'). \tag{48}$$

Let $\lambda_1 - \lambda_5$ be the multipliers on (46), (47), and (43)–(45), respectively. Then the first-order conditions along with market clearing, evaluated in equilibrium deliver

$$\begin{aligned} u_c(c, 1-n) = {} & p[(1-2z_c^*)(\lambda_2+\lambda_3)] \\ & + p\beta \int_0^{z_c^*} (1+\zeta_c(i))\, di \int_{\Theta} \frac{\lambda_1'}{h_m(S', S)} \chi(\theta, d\theta') \end{aligned} \tag{49}$$

$$u_l(c, l) = p\bar{w}\lambda_2 \tag{50}$$

$$\begin{aligned} p(\lambda_2+\lambda_4)(1-2z_x^*) = {} & \beta \int_{\Theta} \left\{ \lambda_2' p' \bar{r}' - p \int_0^{z_x^*} (1+\zeta_x(i))\, di \right. \\ & \left. \times \int_{\Theta} \frac{\lambda_1'}{h_m(S', S)} \chi(\theta, d\theta') \right\} \end{aligned} \tag{51}$$

$$\lambda_2 + \lambda_5 - \lambda_1 \leqslant 0 \qquad (= \text{if } \hat{m} > 0) \tag{52}$$

$$-\lambda_2 + \beta \int_{\Theta} \frac{\lambda_1'}{h_m(S', S)} \chi(\theta, d\theta') \leqslant 0 \qquad (= \text{if } m^{e\prime} > 0) \tag{53}$$

$$\begin{aligned} -\lambda_5 + \lambda_3 \leqslant 0 & \qquad (= \text{if } n_c > 0) \\ -\lambda_5 + \lambda_4 \leqslant 0 & \qquad (= \text{if } n_x > 0). \end{aligned} \tag{54}$$

To construct this equilibrium as the equilibrium of a tax problem, define the functions

$$\phi_c(S) = \left\{ -2z_c^* + \int_0^{z_c^*} (1+\zeta_c(i))\, di \right\} \lambda_2 + (1-2z_c^*)\, \lambda_5, \tag{55}$$

$$\phi_x(S) = \left\{ -2z_x^* + \int_0^{z_x^*} (1+\zeta_x(i))\, di \right\} \lambda_2 + (1-2z_x^*)\, \lambda_5. \tag{56}$$

Then the resulting economy is similar in structure to the exogenous cash–credit economy, except now the money demand function depends on nominal interest rates and the primitives of the transactions costs technologies via the specification of the transactions technologies (as opposed to preference primitives in the exogenous cash–credit model). To map this problem into our framework, simply define $\tau_c = \phi_c/(\lambda_2 + \phi_c)$, $\phi_n = (\phi_x - \phi_c)$, and $\tau_x = \phi_n/(\lambda_2 + \phi_n)$. Then the equilibrium conditions for the monetary economy in (49)–(54) can be written as

$$\frac{u_l(c, l)}{u_c(c, l)} = (1 - \tau_c(K, \theta))\, \bar{w}(K, \theta), \tag{57}$$

$$[1 + \tau_n(S)]\, u_c(c, l) = \beta \int_{\Theta} u_c(c', l')(1 - \tau_c(K', \theta'))\, \bar{r}(K', \theta')\, \chi(\theta, d\theta'), \tag{58}$$

where if $\zeta_c = \zeta_x$, $\tau_n = 0$ for all $S \in \mathbb{S}$. We can use an analogous procedure as in the above example to solve the first equation (again using the budget constraint for the first period problem) to obtain c^* and l^*. In solving the equilibrium problem by an operator defined from the second equation, a sufficient condition for our methods to work has τ_n being restricted in a manner that has the right side of second equation decreasing in K. In the case that the transactions costs technologies are equal, this condition is trivially met.

In the case that the transactions technologies are not equal, a couple of things are important to note. First, in general this operator does not map into a space of functions where the u_c is decreasing in K. For this to occur, one has to place additional restrictions on the primitives, transactions technologies, and monetary policies so τ_n behaves as required. This point is an important one. It is clear from these examples that often in applied work, one must be careful that the assumptions on the structure of the equilibrium distortion in Assumption 3 are met. Even though there is often an observationally equivalent way of representing the equilibrium of a monetary economy as a model with a state contingent income tax, this does not mean that the proof of existence and uniqueness follows from the equivalent tax economy. Additional structure is required. This often places restrictions on the types of monetary policies that can be considered.

6. CONCLUSION

In this paper we have presented a set of sufficient conditions for the existence of equilibrium in representative agent distorted dynamic economies with capital and elastic labor supply. What is important here is that we require none of the homogeneity assumptions for equilibrium distortions and preferences that have been required in the previous literature. In many applications, these assumptions prove problematic. In addition, we are able to characterize sets of sufficient conditions under which the equilibrium policies vary in a monotone manner with respect to the parameters of policy and technology. We feel that given the importance of applied work in general equilibrium of distorted stochastic growth models with both capital and elastic labor supply (many with production externalities), the added level of generality is important.

There are many additional interesting questions to ask in future research. One issue concerns the existence and characterization of equilibrium in representative agent models with multisector production and meaningful fluctuations in relative prices within each period. Such economies (with many goods and many sectors) offer a wide range of questions concerning the robustness of our results. In addition, similar questions can be asked in multiagent settings. Work by Kehoe, Romer, and Woodford [30] provides some initial results on this problem, but their methods do not exploit either the recursive nature of these economies or the monotonicity of operators which can be used to pose the existence of equilibrium problem. In this work (as in the work of Santos and Vigo-Aguiar [38]), the differentiability of optimal plans are both potentially very important for the existence and characterization of equilibrium. Applications of (and generalizations of) the implicit function theorem to non-smooth environments appear to be critical if progress is going to be made along these lines (see Shannon [39] for a discussion of regular, nonsmooth methods for solving systems of equations). More important, since order is what is critical when conducting comparative analytic statements, a purely latticed based approach based upon the concerns of supermodularity and lattice programming such as in the work of Topkis [40, 41] along with the use of a more general ordered based fixed point theory adapted for correspondences found in Zhou [42] might provide a very powerful alternative to the topological approaches often take to such problems. Such an approach would allow for concerns of monotonicity to stand paramount, as opposed to assumptions concerning the strong concavity of the primitive economic data such as found in Coleman [9]. The results in Mirman, Morand, and Reffett [35] show that such an order-based alternative is available for the class of models considered in Coleman [9]. In future work, we plan to pursue these issues in multiagent settings and dynamic games more generally.

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