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Monotone Methods for Markovian Equilibrium in Dynamic Economies

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Abstract. In this paper, we provide an overview of an emerging class of "monotone map methods" in analyzing distorted equilibrium in dynamic economies. In particular, we focus on proving the existence and characterization of competitive equilibrium in non-optimal versions of the optimal growth models. We suggest two alternative methods: an Euler equation method for a smooth, strongly concave environment, and a value function method for a non-smooth supermodular environment. We are able to extend this analysis to study models that allow for unbounded growth or a labor–leisure choice.

1. Introduction

The basic tool for studying macroeconomic dynamics is the stochastic optimal growth model, a model that has been used in almost all macroeconomic modeling in the last two decades. Initiated with the work of Brock and Mirman [11], the study of this model and its many variants has mostly relied on the assumptions of continuity, concavity and regularity of the representation of the underlying primitive data of the model to jointly imply differential characterization of equilibrium. Hence, it has been natural to apply topological methods to derive all of the fundamental properties of these models. In particular, one can appeal to topological methods to show the existence of a unique Markovian equilibrium and to develop a class of Euler equation methods to characterize this equilibrium.

However, the use of these models to analyze even deeper questions of macroeconomics have resulted in classes of growth models that do not possess the properties required for applying these topological tools. This is the case, for instance, when the convexity assumptions typically assumed for production technologies are relaxed. For example, in the endogenous growth literature, the role of human capital, central to explaining the differences in long-term growth rates among developed countries, constitute a form of market failure since private decisions of agents fail to incorporate the external effects of human capital. This, in turn, implies that the second welfare theorem is not applicable. There are many other examples worth noting. In public finance and monetary models, when fiscal and monetary agents play a role in determining the equilibrium dynamics governing capital accumulation, investment, consumption and output, the second welfare theorem is no longer applicable. As a consequence, existing topological methods are of limited applicability; questions concerning the existence and characterization of equilibrium become complicated to study and may require a non-topological method of analysis.

In this survey, we discuss an emerging class of methods, which we identify as the "monotone map methods" pioneered by Coleman [12–14] and extended by Greenwood and Huffman [20] and Datta, Mirman and Reffett [16]. In a related but parallel field of macroeconomics, lattice theory techniques were introduced by Amir, Mirman and Perkins [6] and extended by Hopenhayn and Prescott [21]. These two strands of the literature has recently been amalgamated by Datta et al. [15], Mirman, Morand and Reffett [33], and Morand and Reffett [35]. These new methods are powerful, and are based on results developed in the mathematics literature studying fixed-point operators on partially ordered spaces. We are particularly interested in studying the optimal policies of the agents in these models and showing the existence of equilibrium using these properties, as well as characterizing the comparative statics of equilibrium solutions.[1] Such methods are powerful, and are based upon some important results developed in the literature in mathematics studying fixed points of operators on partially ordered spaces. Results in this literature vary from primarily topological, as in the works of Krasnoselskii [24,25] and Amann [5], to lattice based, as in the works of Tarski [47], Topkis [51,52], Vives [55] and Zhou [56]. Some of the methods are topological and often require the underlying domain of the continuous compact operators to be subsets of partially ordered Banach spaces with particularly desirable properties; other methods are primarily lattice based and require very little, if any, concerning the continuity of operators. However, in that case, very strong completeness properties of the domains of the operators are required. These monotone methods (whether topological or lattice based), unlike methods based upon fixed-point theorems by Brouwer, Schauder, or Fan-Glicksberg, are constructive and can therefore be used as the basis for a systematic study of the theoretical properties of numerical methods that are generally used in the applied literature to compute numerical solutions to such models. As the quality of

[1] In a recent paper by Santos [41], an example of non-existence of continuous Markov equilibrium is presented for an economy with a decreasing tax rate. What happens in that example is that the tax is sufficiently decreasing to destroy the complementarity between the households asset choice decision and the aggregate capital stock. This means that, in equilibrium, generally the Markov equilibrium (if it existed) would not be monotone in the endogenous state variables. This is the basis of the counterexample. It should be noted that using the methods of section 3 in this paper, some economies with decreasing taxes can have monotone Markovian equilibrium. The key is that the tax cannot decrease "too much" where "too much" refers to the tax being such that $u(f(k, K, z; t) - y)$ is no longer has a single crossing property in $(y; k, K)$ for each z for the t under consideration.

numerical approximations of Markov equilibrium is becoming an emerging concern in much of the applied macroeconomics and public finance literature, monotone methods are particularly appealling as they provide a globally stable class of iterative methods that practioners can implement in their numerical work.[2]

We begin by considering a class of environments that are basically non-optimal versions of the model studied in Brock and Mirman [11] (e.g., single sector production, identical agents, uncertainty, rational expectations, inelastic labor supply). That is, we deal with growth models in this survey that are amended to allow for possible violations of the second welfare theorem (e.g., situations where there are taxes, distortionary monetary policy or production externalities associated with capital accumulation). These are the prototype of models considered in Coleman [12] and Greenwood and Huffman [20]. We first present the monotone map methods that are typically used in the existing literature and that are built around equilibrium versions of the household's Euler equations (as in Coleman [12]). We then discuss a new monotone map approach, which is not based on equilibrium Euler equations, but is based on the operator defined from Bellman's equation. Thus, the value function approach generates a sequence that has fixed points on a complete lattice of functions. This operator also uses information about the value function to obtain additional characterizations of the equilibrium that are not available using a pure Euler equation approach. We then discuss how to develop an alternative version of the monotone map procedure that does not involve any smoothness considerations (e.g., that does not use Euler equations). This new method is developed in a recent paper of Mirman, Morand and Reffett [33]. We conclude by showing how the methods discussed in the paper can be extended to more general specifications of the primitive economic data. In particular, we consider models with endogenous labor supply, models that allow for unbounded equilibrium growth, and models with some limited form of altruism.[3]

[2] It is true that for Brouwer's fixed point theorem, one can use Scarf's algorithm to approximate the fixed point, and therefore in some sense this fixed point theorem is constructive. But a couple of remarks are in order. First, for the class of problems we study, in general Markov equilibria are elements of function lattices that are no finite dimensional (i.e., not compact), and therefore Scarf's procedure will not apply. As there are not versions of Scarf's procedure for infinite dimensional operators, the theorems in the Brouwer–Schauder–Fan-Glicksberg class are not constructive. Second, Scarf's algorithm is silent on which fixed point is actually computed. Therefore stability is a question, as is also sensitivity of any numerical procedure based upon it. For example, changes in initial conditions for any computational procedure is a concern, as would be studying the comparative statics of any such fixed point algorithm in a parameter. Finally, even in the rare case that the equilibrium in a finite-dimensional space (e.g., an equicontinuous closed subset of continuous function), it is not clear how one can enumerate the dimension of the space, and therefore its basis. Therefore its not clear how one would even implement Scarf's method even for cases where Markov equilibrium set in compact subsets of bounded continuous function on a compactum.

[3] One should note that another approach to Markov equilibrium is a topological approach based upon the Negishi problem proposed in the work of Kehoe, Levine and Romer [22,23]. This method is a very interesting alternative to monotone methods. Unfortunately, to date, it has not been applied to a very broad class of non-optimal economies. One problem with this approach is that it would appear the tools of differential topology and degree theory are critical for its application. In much of this work, the equilibrium manifold must be the graph of a appropriately smooth function, and this requirement seems problematic

The paper is organized as follows. In section 2 we discuss the methods of Coleman [12], Greenwood and Huffman [20] and Datta, Mirman and Reffett [16]. We call this a monotone map method that utilizes the Euler equation, and, therefore, applies to smooth and strongly concave environments. In section 3 we present an alternative monotone map method based upon the properties of the best response map of each agent, and construct an operator directly from Bellman's equation involving a value function iteration that exploits the supermodularity of the environment. We then compare the two methods. Specifically, we show that as opposed to the topological constructions that underlie the approaches in Coleman [12,14], the analysis of operators constructed from the best response mapping relies only on the order structure and properties of a set of functions, and that we can directly apply Tarski's fixed-point theorem to a complete lattice of functions. As a result, our work generalizes some of the results in Hopenhayn and Prescott [21]. Section 4 shows how the methods of Coleman [12] can be extended to models with unbounded growth, endogenous labor supply, and nonpaternalistic altruism. Section 5 concludes.

2. The Euler equation method for strongly concave environments

2.1. The primitive data

Time is discrete and indexed by $t \in T = \{0, 1, 2, \ldots\}$, and there is a continuum of infinitely-lived and identical household/firm agents. In each period, households are endowed with a unit of time, which they supply inelastically to competitive firms. Uncertainty comes in the form of a finite state, first-order Markov process denoted by $z_t \in \mathbb{Z}$, with stationary transition probabilities $\chi(z, z')$.[4] Let the set $\mathbb{K} \subset \mathbb{R}_+$ contain all the feasible values for the aggregate endogenous state variable K, i.e., the per capita capital to labor ratio, and define the product space $\mathbb{S}$: $\mathbb{K} \times \mathbb{Z}$. Since the household also enters each period with an individual level of the endogenous state variable k, the individual capital to labor ratio, we denote the state of a household by the vector $s = (k, S) \in \mathbb{K} \times \mathbb{S}$.

For each period and state, the preferences are represented by a period utility index $u(c_i)$, where $c_i \in \mathbb{K} \subset \mathbb{R}_+$ is period i consumption. Letting $z^i = (z_1, \ldots, z_i)$ denote the history of the shocks until period i, a household's lifetime preferences are defined over infinite sequences indexed by date and history $\mathbf{c} = (c_{z^i})$ and are given by:

$$U(\mathbf{c}) = E_0\left\{\sum_{i=0}^{\infty} \beta^i u(c_i)\right\},$$

where the summation is with respect to the probability structure of the shocks. We make the following assumption:

(as many economies do not possess smooth Markov equilibrium). It is important to remember that these tools are complementary to monotone methods, especially when studying strongly concave models.

4 To simplify the exposition, we use a finite state-space for the exogenous shocks. See Hopenhayn and Prescott [21] for a discussion on handling of shock processes with more general state-spaces, and the additional restrictions that are placed on transition processes for the shocks.

Assumption 1. The utility function $u : \mathbb{K} \mapsto \mathbb{R}$ is bounded, twice continuously differentiable, strictly increasing, strictly concave. In addition, $u'(c)$ satisfies the standard Inada conditions:

$$\lim_{c \to 0} u'(c) = \infty \quad \text{and} \quad \lim_{c \to \infty} u'(c) = 0.$$

We assume that output available to the household in the current period is represented by the function $f(k, K, z; t)$, where t is a continuous mapping that represents distortions and thus influences technology. The underlying technology exhibits constant returns to scale in private inputs k and elastic or inelastic labor supply, and satisfies the following assumption.

Assumption 2.

(i) $f(0, K, z; t) = 0$ for all $K \in \mathbb{K}$, $z \in \mathbb{Z}$ and $t \in \mathbb{T}$.

(ii) f is uniformly continuous, twice continuously differentiable, strictly increasing in (k, K) and strictly concave in its first argument.

(iii) $\lim_{k \to 0} f_1(k, K, z) = \infty$ for all $k > 0$.

Assumption 3. There exist $\hat{k}(z) > 0$ such that $f(\hat{k}(z), K, z; t) + (1 - \delta)\hat{k}(z) = \hat{k}(z)$ and $f(k, K, z; t) < k$ for all $k > \hat{k}(z)$ and for all $z \in \mathbb{Z}$.

The restrictions on the primitives in assumptions 1 and 2 are made to apply the Euler equation approach developed in the next section. They are also consistent with the standard assumptions made in the stochastic growth literature (e.g., Brock and Mirman [11]). Assumptions 1 and 2 jointly imply that the utility function $\Upsilon(k, K, k', z; t) = u[f(k, K, z; t) - k']$ is strongly concave in (k, k') for each (K, z, t). Assumption 3 is a standard feature in stochastic growth literature (see Brock and Mirman [11]), and implies that $\sup_z \hat{k}(z, t)$ exists. As a consequence, the state-space for the endogenous variable k (and for output) can be taken to be the compact interval $\mathbb{K} = [0, k]$, where k is the larger of the two quantities, the initial stock k_0 and $\sup_z \hat{k}(z, t)$. This implies that the boundedness assumption on utility can be relaxed, since a continuous utility defined on a compact space is necessarily bounded.

The parameter vector t may represent the actions of a government and can be interpreted in many ways. For instance, in an economy with a state contingent capital income tax (as in Coleman [12]) the modified technology can be written as follows:

$$f(k, K, z; T) = \big(1 - t_1(K, z)\big)g(k, K, z) + t_2(K, z),$$

where g is the undistorted production function, $t_1(K, z) : \mathbb{S} \to [0, 1]$, and $t_2(K, z)$ is interpreted as a lump sum transfer. If we define the standard lexicographic partial order on the set of parameter vectors $t \in \mathbb{T}$ as $t'(K, z) \succeq t(K, z)$ if either $t_1'(K, z) < t_1(K, z)$ for all $S \in (K, z) \in \mathbb{K} \times \mathbb{Z}$, or $t_1'(K, z) = t_1(K, z)$ and $t_2'(K, z) \geqslant t_2(K, z)$, then $f(k, K, z; t)$ is increasing in t. Many other cases of distorted economies can also be handled in this framework. In each period, we impose the condition that the government budget is in equilibrium and the revenues exactly match the expenditures, i.e., $t_2(K, z) = g_1(K, K, z)t_2(K, z)$.

The dynamic decision problem for the household is simple. For a given $t \in \mathbb{T}$, we define the household's feasible correspondence $\Gamma(k, K, z; t)$ for the distorted economy as the set of actions (c, k') satisfying the following constraints:

$$c + k' = f(k, K, z; t) \quad \text{and} \quad c, k' \geqslant 0.$$

Under assumption 1, $\Gamma(k, K, z; t)$ is a well-behaved correspondence for each $s = (k, K, z) \in \mathbb{K} \times \mathbb{S}$ for fixed $t \in \mathbb{T}$. In particular, since f and t are assumed to be continuous, Γ is a non-empty, compact and convex-valued, continuous correspondence for each state s. Also, since t is increasing in K, for each z, the correspondence Γ is expanding in (k, K). Also note that the household's feasible correspondence is expanding in $t \in \mathbb{T}$; that is, if $t' \geqslant t$ in the partial order structure on $\mathbb{T}$, then $\Gamma(k, K, z; t) \subseteq \Gamma(k, K, z; t')$ for all (k, K, z).

To construct the household's decision problem, consider that households assume that the per capita capital stock evolves according to:

$$K' = \kappa(K, z; t),$$

where for any given $t, \kappa(\cdot, \cdot; t) : \mathbb{S} \to \mathbb{K}$ is continuous in both its arguments and is increasing in K for each (z, t). Then the household solves the dynamic decision problem summarized in the following Bellman equation:

$$J(s) = \sup_{(c,k') \in \Gamma(s;t)} \left\{ u(c) + \beta \int_{\mathbb{Z}} J(s') \chi(z, \mathrm{d}z') \right\}. \tag{1}$$

Standard arguments show the existence of a $J \in \mathbb{V}$ that satisfies this functional equation, where $\mathbb{V}$ is the Banach space of bounded, continuous, real valued functions with the sup norm (see, for instance, Stokey, Lucas and Prescott [45]). In addition, standard arguments also establish that J is strictly concave in k, and following Mirman and Zilcha [34], the concavity of J also implies that J is once differentiable in k.

We define an equilibrium for the state contingent income tax economy as follows:

Definition. A (recursive) competitive equilibrium for this economy consists of a parameter vector (t_1, t_2), a value function for the household $J(s)$, and the associated individual decisions c and k' such that: (i) $J(s)$ satisfies the household's Bellman equation (1), and c, k' solve the optimization problem in the Bellman's equation given t; (ii) all markets clear: i.e., $k' = \kappa(S) = K'$ and (iii) the government budget balances.

2.2. Existence of equilibrium

Since the parameter vector t generally induces distortions, we cannot construct an equilibrium solution based upon the second welfare theorem. We adopt an alternative strategy, the "Euler equation approach"[5] and construct an equilibrium by iterating an operator constructed from the household's decision problem. Specifically, we look for a fixed point of a monotone operator defined implicitly by the Euler equation on the space of policy functions. An operator A on a partially ordered set $(\mathbb{X}, \geqslant)$ is monotone (or order-preserving) if $h' \geqslant h$ implies $Ah' \geqslant Ah$, for all $(h, h') \in \mathbb{X}$. With this in mind, we can now state and prove a version of a topological fixed point theorem, due to Amann [5], for increasing maps that directly applies to our problem.

Theorem 1. Let E be an equicontinuous set of functions defined on a compact set X and equipped with the sup norm and the pointwise partial order, and $[\bar{y}, \hat{y}]$ be a closed order interval in E. Suppose that $A : [\bar{y}, \hat{y}] \to [\bar{y}, \hat{y}]$, is an increasing, continuous map. Then A has a maximal fixed point $\hat{x}$ and $\hat{x} = \lim_{n\to\infty} A^n \hat{y}$, and the sequence $\{A^n \hat{y}\}_{n=0}^{\infty}$ is decreasing.

Proof. Necessarily $A\hat{y} \leqslant \hat{y}$ and, recursively, the sequence $\{A^n \hat{y}\}_{n=0}^{\infty}$ is a decreasing sequence of equicontinuous functions. By the Arzela–Ascoli theorem, there exists a convergent subsequence in E. First, since the sequence is decreasing, the converging subsequence is the sequence itself.

Second, for any $k \in X$, the sequence $\{A^n \hat{y}(k)\}_{n=0}^{\infty}$ is a monotone sequence, bounded below by $\bar{y}(k)$, and therefore converges to its inf. Thus, the limit of the sequence $\{A^n \hat{y}\}_{n=0}^{\infty}$ is $\inf_{n\in N}\{A^n \hat{y}\}$ which we denote $\hat{x}$, and $\hat{x} \in E$. Further, since $\bar{y} \leqslant A^n \hat{y} \leqslant \hat{y}$, then $\bar{y} \leqslant \hat{x} = \inf_{n\in N}\{A^n \hat{y}\} \leqslant \hat{y}$, and $\hat{x} \in [\bar{y}, \hat{y}]$.

Third, by continuity of A, $A\hat{x} = \hat{x}$ which implies that $\hat{x}$ is a fixed point. If x is an arbitrary fixed point in $[\bar{y}, \hat{y}]$ such that $\bar{y} \geqslant x \geqslant \hat{x}$, then for all n, $A^n \bar{y} \geqslant x \geqslant \hat{x}$, and therefore $\hat{x} = \inf\{A^n \bar{y}\} \geqslant x \geqslant \bar{x}$ so that $x = \hat{x}$, which implies that $\hat{x}$ is the maximal fixed point.

Finally, since the sequence $\{A^n \hat{y}\}_{n=0}^{\infty}$ converges pointwise to a continuous function, the convergence is uniform by Dini's theorem. (Note that if we relax the assumption of compactness of X, then the convergence is uniform on any compact subset of X.) □

We apply this theorem to an operator defined in Coleman [12] to prove existence of a Markovian equilibrium.[6] As for establishing uniqueness of Markovian equilibrium, unfortunately as shown in Coleman [12], our operator has desirable concave properties only for specific environments (i.e., CES utility) and, thus, cannot be used to establish uniqueness in the most general setup. However, in section 2.3, we discuss a second operator, similar to the operator studied in Coleman [14], whose fixed points coincide with the fixed points of the first operator. We then show that this second operator has

[5] This is in contrast to the "value function" or the "Bellman equation" approach, in which one looks for a fixed point of the Bellman's operator in the space of value functions. In a non-smooth environment, the Bellman equation approach is useful while the Euler equation approach need not be.

[6] Coleman considers an application of a theorem in Dugundji and Granas [19] for order continuous operators. When considering Coleman's application of this theorem, it is important to recall that a topologically continuous operator is not necessarily order continuous.

at most one strictly positive fixed point. This same line of argument is developed for a model with labor–leisure choice in Datta, Mirman and Reffett [16].

To apply the above theorem, we need to define a candidate operator A whose fixed points coincide with the Markovian equilibrium. The Euler equation associated with the optimal policy function from the right-hand side of the Bellman equation in (1) (after appealing to the envelope condition) is:

$$u'\big(c(K,z)\big) = \beta \int_{\mathbb{Z}} u'\big[c\big(F(K,z)-c(K,z),z'\big)\big] r\big(F(K,z)-c(K,z),z'\big)\chi(z,\mathrm{d}z'). \quad (2)$$

Here, $F(K,z) = f(K,K,z;t)$ and $r(K,z) = f_1(K,K,z;t)$ for notational simplicity, that is, the equilibrium condition $K = k$ has been imposed.

Definition. Consider the space, denoted $\mathbb{H}^0$, of consumption functions h such that:

(i) $h: \mathbb{S} \to \mathbb{K}$;

(ii) $0 \leqslant h(K,z) \leqslant F(K,z)$ for all $(K,z) \in \mathbb{S}$;

(iii) $0 \leqslant h(K',z) - h(K,z) \leqslant F(K',z) - F(K,z)$ for all $K' \geqslant K$ and all z.

We equip $\mathbb{H}^0$ with the standard sup norm and the partial order $\geqslant$ defined as $h' \geqslant h$ if and only if $h'(K,z) \geqslant h(K,z)$ for all $(K,z) \in \mathbb{S}$. Notice that condition (iii) imposes that both h and $f-h$ are increasing. Also, since f is assumed to be continuous, implies that the functions in $\mathbb{H}^0$ are continuous. The following lemma summarizes some important properties of $\mathbb{H}^0$.

Lemma 2. Under assumptions 1 and 2, the set $\mathbb{H}^0$ is a closed order interval of an equicontinuous set of functions.

Proof. Condition (ii) implies that h is chosen in the closed-order interval $[0, F]$. Equicontinuity is induced by the required double monotonicity in condition (iii) of the elements of $\mathbb{H}^0$, in conjunction with the uniform continuity of $F(K,z)$ (recall from assumption 2 that $F(K,z) = f(K,K,z;t)$ is continuous in k and therefore uniformly continuous since $\mathbb{K}$ is compact). The assumption of uniform continuity of F on its domain implies that:

$$\forall \varepsilon > 0, \exists \delta > 0 \quad \left|K' - K\right| < \delta \quad \Longrightarrow \quad \left|F\big(K',z\big) - F(K,z)\right| < \varepsilon.$$

For all h in $\mathbb{H}^0$, property (iii) implies: for all $K' \geqslant K$,

$$0 \leqslant h\big(K',z\big) - h(K,z) \leqslant F\big(K',z\big) - F(K,z).$$

Combining this last inequality with the uniform continuity of F leads to:

$$\forall \varepsilon > 0, \exists \delta > 0 \quad \left|K' - K\right| < \delta \quad \Longrightarrow \quad \left|h\big(K',z\big) - h(K,z)\right| < \varepsilon, \quad \forall c \text{ in } \mathbb{H}^0_+,$$

which demonstrates equicontinuity of the set of functions. □

Notice that this result does not require compactness of the state-space $\mathbb{S}$. This will be useful when studying the case of unbounded growth in the last section of the paper that discusses extensions. To construct Markovian equilibrium, we simply define a nonlinear operator based upon an equilibrium version of the household Euler equation. Therefore, for any $h \in \mathbb{H}^0$, $h > 0$, and any (K, z), define $Ah(K, z)$ as the solution, for y of the following equation:

$$u'(y) = \beta \int_Z u'\big(h(F - y, z'), z'\big) r(F - y, z') \chi(z, \mathrm{d}z').$$

For $h = 0$, we set $Ah(K, z) = 0$. The following lemma lists key properties of the operator A:

Lemma 3. Under assumptions 1 and 2:

1. For any $h \in \mathbb{H}^0$, and for any (k, z), there exists a unique $Ah(k, z)$.
2. A maps $\mathbb{H}^0$ into itself.
3. A is a monotone operator on $\mathbb{H}^0$.

In addition, if assumption 3 holds:

4. A is a continuous operator on $\mathbb{H}^0$.
5. There exists a maximal fixed point $h^* \in \mathbb{H}^0$ and the sequence $\{A^n F\}$ converges uniformly to h^*.
6. The maximal fixed point is strictly positive.

Proof. The proofs of (1)–(4) are in Coleman [12]. It is important to note that neither (1), (2), nor (3), rely on compactness of the state-space, and are therefore valid under assumptions 1 and 2 only. Part (5) follows directly from the topological fixed point theorem stated above, and provides an important algorithm for computational procedures. Notice that this theorem does not rule out the possibility for the zero consumption to be the only fixed point of A. However, it is easy to show that the zero consumption plan is not optimal, a feature of the model that crucially relies on the assumption of unbounded marginal utility at zero. □

We can now state our existence result.

Proposition 4. Under assumptions 1–3, there exists an equilibrium.

Proof. That the set of fixed points of the operator A for any t is non-empty follows from lemma 3. Further, any strictly positive fixed point is decentralizable as competitive equilibrium, and lemma 3(6) guarantees that the maximal fixed point is strictly positive. □

2.3. *Uniqueness of equilibrium*

Coleman [12] establishes the uniqueness of the fixed point of the mapping A by restricting the utility function (see assumption 7 in Coleman [12]). We build on the work of Coleman [13] to demonstrate uniqueness for the general class of utility functions satisfying assumption 1 by introducing another operator, denoted $\widehat{A}$, which we show is pseudo-concave and K_0-monotone. Note that, as stated in the next theorem, pseudo-concave and K_0-monotone operators have at most one strictly positive fixed point. Recalling that $(X, \geqslant)$ is a arbitrary partially ordered set, we define the following:

Definition. An operator $\widehat{A}$ on X is pseudo-concave if for any strictly positive function c in X any $0 < \lambda < 1$, and for all $(K, z) \in S$, $(\widehat{A}\lambda c)(K, z) > \lambda(\widehat{A}c)(K, z)$.

Definition. An operator $\widehat{A}$ on X is K_0-monotone if it is monotone and if for any strictly positive fixed point c_1 of $\widehat{A}$ and any $c_2 \in X$, there exists some $K_0 > 0$ such that $c_1(K, z) \geqslant c_2(K, z)$ imply $c_1(K, z) \geqslant (\widehat{A}c_2)(K, z)$ for all $K \geqslant K_1$, $0 \leqslant K_1 \leqslant K_0$ and all z.

Theorem 5. An operator $\widehat{A} : X \to X$ that is pseudo-concave and K_0-monotone has at most one strictly positive fixed point.

Proof. See Coleman [12]. □

We construct the operator $\widehat{A}$ as follows. First define the set of functions, denoted by M, endowed with the standard partial pointwise order.

M is the collection of $m : \mathbb{R}_+ \times \mathbb{Z} \to \mathbb{R}$ such that:

(i) m is continuous;

(ii) for all $(K, z) \in \mathbb{R}_+ \times \mathbb{Z}$, $0 \leqslant m(K, z) \leqslant F(K, z)$;

(iii) for any $K = 0$, $m(K, z) = 0$.

Notice that the set of consumptions functions $\mathbb{H}^0$ consistent with some $m \in M$ forms a strict subset of M. Indeed the set of consumption functions consistent with M is much larger than $\mathbb{H}^0$. For any $m \in M$, consider the function $\Psi(m(K, z))$ implicitly defined by:

$$u'\big[\Psi\big(m(K, z)\big)\big] = \frac{1}{m(K, z)}, \quad \text{for } m > 0, \qquad 0 \text{ elsewhere.}$$

Clearly, Ψ is continuous, increasing, $\lim_{m\to 0} \Psi(m) = 0$, and $\lim_{m\to F(K,z)} \Psi(m) = F(K, z)$. Using the function Ψ, for any $m > 0$ we denote by $\widehat{A}m(K, z)$ the solution for y to the equation:

$$\widehat{Z}(m, y, K, z) = \frac{1}{y} - \beta E_z\left[\frac{r(F(K, z) - \Psi(y), z')}{m(F(K, z) - \Psi(y), z')}\right] = 0,$$

and set $\widehat{A}m = 0$ when $m = 0$.

Since $\widehat{Z}(m, y, K, z)$ is strictly decreasing and continuous in y, and $\lim_{y\to 0} \widehat{Z}(m, y, K, z) = \infty$ and $\lim_{y\to F(K,z)} \widehat{Z}(m, y, K, z) = -\infty$ for each $m(K, z) > 0$, with $K > 0$, and $z \in \mathbb{Z}$, there exists a unique $\widehat{A}m(K, z)$.

It is easy to show that to each fixed point of the operator A corresponds a fixed point of the operator $\widehat{A}$. Indeed, consider x such that $Ax = x$ and define $y = 1/u'(x)$ (or, equivalently $\Psi(y) = x$). It is also easy to verify that $Am \subset M$ and is monotone on M. By definition, x satisfies:

$$u'\big(x(K,z)\big) = \beta E_z\big\{r\big(F(K,z) - x(K,z), z'\big) \times u'\big(x\big(F(K,z) - x(K,z), z'\big)\big)\big\}$$

for all (K, z). Substituting the definition of y into this expression yields:

$$\frac{1}{y} = \beta E_z\left\{\frac{r(F(K,z) - \Psi(y(K,z)), z')}{y(F(K,z) - \Psi(y(K,z), z'))}\right\},$$

which shows that y is a fixed point of $\widehat{A}$.

Lemma 6. The operator $\widehat{A}$ is pseudo-concave and K_0-monotone, and therefore has at most one strictly positive fixed point.

Proof. Recall that $\widehat{A}$ is pseudo-concave if, for any strictly positive m and any $0 < t < 1$, $\widehat{A}tm(K, z) > t\widehat{A}m(K, z)$ for all $K > 0$ and for all $z \in Z$. Since $\widehat{Z}$ is strictly decreasing in its second argument, a sufficient condition for pseudo-concavity is that:

$$\widehat{Z}\big(tm, t\widehat{A}m, K, z\big) > \widehat{Z}\big(tm, \widehat{A}tm, K, z\big) = 0. \tag{3}$$

By definition:

$$\widehat{Z}\big(tm, t\widehat{A}m, K, z\big) = \frac{1}{t\widehat{A}m} - \beta E_z\left\{\frac{r(F(K,z) - \Psi(t\widehat{A}m(K,z)), z')}{tm(F(K,z) - \Psi(t\widehat{A}m(K,z)), z')}\right\},$$

so that:

$$t\widehat{Z}\big(tm, t\widehat{A}m, K, z\big) = \frac{1}{\widehat{A}m} - \beta E_z\left\{\frac{r(F(K,z) - \Psi(t\widehat{A}m(K,z)), z')}{m(F(K,z) - \Psi(t\widehat{A}m(K,z)), z')}\right\}.$$

Since Ψ is increasing and $r(K', z')/m(K', z')$ is decreasing in K':

$$\begin{aligned}&\frac{1}{\widehat{A}m} - \beta E_z\left\{\frac{r(F(K,z) - \Psi(t\widehat{A}m(K,z)), z')}{m(F(K,z) - \Psi(t\widehat{A}m(K,z)), z')}\right\}\\ &\quad > \frac{1}{\widehat{A}m} - \beta E_z\left\{\frac{r(F(K,z) - \Psi(\widehat{A}m(K,z)), z')}{m(F(K,z) - \Psi(\widehat{A}m(K,z)), z')}\right\} = 0\end{aligned}$$

and $\widehat{Z}(tm, t\widehat{A}m, K, z) > 0$ so that condition (3) is obtained.

The condition $\lim_{k\to 0} f_1(k, K, z) = \infty$ for all $K > 0$, all z in assumption 2(iii) implies that $r(0, z') = \infty$ for all z'. Given that $\widehat{A}$ is monotone, this latter condition is sufficient for the operator $\widehat{A}$ to be K_0-monotone (lemmas 9 and 10 in [12]). □

All fixed points of A, and at least one of them is strictly positive, are also fixed points of $\widehat{A}$, which has at most one fixed point. Thus, necessarily, the strictly positive fixed point of A is unique, and we can state our existence result.

Proposition 7. Under assumptions 1–3, there exists a unique strictly positive equilibrium $\mathbb{H}^0$.

Proof. The existence of a unique strictly positive fixed point for $\widehat{A}$ follows from lemma 6, along with the theorem of Coleman [12] for pseudo-concave, K_0-monotone operators. As any trajectory of the operator A is equivalent to a trajectory of the operator $\widehat{A}$ by a standard argument in Coleman [14], the strictly positive fixed point of $\widehat{A}$ coincides with the unique strictly positive Markovian equilibrium in $\mathbb{H}^0$. □

3. A value function iteration method for a class of superextremal economies

In this section, we relax the smoothness restrictions on the primitives assumed in the previous section, and use lattice theoretic methods along with supermodularity to study existence and characterization of equilibrium. The new set of assumptions encompasses a different approach than the one considered before, although our method of analysis also applies to the setup of the previous section. Abandoning the assumption of differentiability and smoothness implies that we cannot work with the Euler equation, and therefore, we build a method based on the Bellman's equation. The approach in this section of the paper can therefore be viewed broadly as an extension of the work of Hopenhayn and Prescott [21] for distorted economies. It builds upon the pioneering work of Veinott [54], Topkis [50,52], LiCalzi and Veinott [26], and Milgrom and Shannon [31], and develops a superextremal approach to characterizing the fixed point operator. Then using order theoretic fixed point arguments due to Tarski [47], we provide an alternative approach to the existence and characterization of Markovian equilibrium. It should be noted that for the case of optimal growth with general Markov shocks, the more general results have been obtained in Mirman, Morand and Reffett [32]. In this paper, the authors show how to use a theorem in Topkis (along with a new collection of commodity lattices) to generalize the previous results for Leontief production obtained in the Markov shock case in Hopenhayn and Prescott [21].

Mirman, Morand and Reffett [33] extend the analysis further and show that these value function methods can be generalized to distorted economies in which the strict concavity assumptions on the primitives are relaxed. However, without strict concavity the analysis is complicated by the fact that the optimal policy need no longer be a single-valued mapping, but may be a correspondence. Mirman, Morand and Reffett [33] generate existence and characterization results by constructing an argument based on a generalization of Tarski's fixed-point theorem to correspondences.

In this section, we make the following assumptions on the primitives of taste and technology (contrast these assumptions with assumptions 1–3 made in the previous section):

Assumption 4. The primitive economic data satisfy the following:

(i) period utility function $u : \mathbb{K} \mapsto \mathbb{R}$ is bounded, continuous, strictly concave, and strictly increasing in c;

(ii) for any $t \in \mathbb{T}$, the production function $f(\cdot; t)$: $\mathbb{K} \times \mathbb{K} \times \mathbb{Z}$ is continuous and non-decreasing in all of its arguments, and satisfies $f(0, K, z; t) = 0$ for all $(K, z, t) \in \mathbb{K} \times \mathbb{Z} \times \mathbb{T}$;

(iii) there exist $\hat{k}(z, t) > 0$ such that $f(\hat{k}(z, t), k(z, t), z, t) + (1 - \delta)\hat{k}(z, t) = \hat{k}(z, t)$ and $f(k, k, z, t) < k$, for all $k > \hat{k}(z, t)$ and for all $z \in \mathbb{Z}$ and $t \in \mathbb{T}$;

(iv) the period utility function u, the production function f, and the parameter $t \in \mathbb{T}$ are such that $\Upsilon(y, k, K, z, t(K, z)) = u(f(k, K, z; t) - y)$ is supermodular in (x, y) where $x = (k, K)$ for each (z, t);

(v) $f(k, K, z; t)$ is increasing in t in the sense that if $t' \geqslant_{\mathbb{T}} t$, then for all $(k, K, z) \in K \times Z \times T$, $f(k, K, z; t') \geqslant f(k, K, z, t)$.

Finally, we restrict the shock process as in Hopenhayn and Prescott [21] (so that the supermodularity is preserved under the operation of integration):

Assumption 5. The transition function $\chi(\cdot, \cdot)$ is increasing in its first argument in the stochastic order sense.

Assumption 5 assumes that the transition function is totally positive of order 2 (or equivalently log supermodular), and guarantees that the operation of integration preserves supermodularity on a Euclidean lattice for the parameter set.

3.1. The household decision problem

For a given $t \in \mathbb{T}$ the household's feasible correspondence $\Gamma(k, K, z, t)$ is the set of actions (c, k') satisfying:

$$c + k' = f(k, K, z; t) \quad \text{and} \quad c, k' \geqslant 0,$$

and is a well-behaved correspondence for each (k, K, z), $t \in \mathbb{T}$ under assumption 4. In particular, Γ is again under assumption 4, a non-empty, compact-valued, continuous correspondence for each state $s = (k, S)$. Since t is increasing in K, the correspondence is expanding in (k, K), while also expanding in $t \in \mathbb{T}$.

To complete the description of the aggregate economy, we assume that households take as given the following recursion on the per capita aggregate capital stock K:

$$K' = h(K, z)$$

where $h \in \mathbb{H} \subset \mathbb{C}$ and $\mathbb{C}$ is the set of bounded functions defined on a compact $\mathbb{S}$, and $\mathbb{H}$ is a subset of $\mathbb{C}$ for which we also require h to be socially feasible, i.e., for each $t \in \mathbb{T}$,

$0 \leqslant h(K, z) \leqslant f(K, K, z; t)$ for all (K, z), upper semicontinuous and non-decreasing in its first argument for each z, i.e., for the latter condition, $h(K', z) - h(K, z) \geqslant 0$ when $K' \geqslant K$. Equipped with the sup norm and the standard pointwise order structure, $\mathbb{H}$ is a subset of the Banach lattice $\mathbb{C}$ of bounded functions defined on $\mathbb{S}$. Notice that we do not require h to be continuous. Indeed, we place no restrictions on f, other than continuity and monotonicity. Therefore, we cannot expect the policy function to be continuous. The following proposition establishes that sufficient structure for applying Tarski-related fixed-point theorems exists.

Proposition 8. $\mathbb{H}$ is a complete sublattice of the Banach lattice of bounded functions $\mathbb{C}$.

Proof. Consider the standard pointwise order on $\mathbb{H}$, and any family of functions of $\mathbb{H}$. First, the sup and inf of such a family are both non-decreasing functions that belong to $[0, f]$. Second, the lower pointwise envelope of this family (i.e., the inf) is usc. Since $\mathbb{H}$ has a top element (f), by Davey and Priestley [17, theorem 2.16, p. 34], $\mathbb{H}$ is a complete lattice, and therefore a sublattice of the Banach lattice of bounded functions $\mathbb{C}$. □

We are now ready to represent the typical household's decision problem for a decentralized competitive equilibrium. Consider a household entering the period in state (k, K, z) for a given h and t. For any given $h \in \mathbb{H}$ and $t \in \mathbb{T}$, the value function $v(k, K, z; t, h)$ necessarily satisfies:

$$v(k, K, z; t, h) = \sup_{c, k' \in \Gamma(k, K, z; t)} \left\{ u(c) + \beta \int_{\mathbb{Z}} v\big(k', h(K, z), z'; t, h\big) \chi(z, \mathrm{d}z') \right\}. \quad (4)$$

Defining the operator T^C as:

$$T^C v(k, K, z; t, h) = \sup_{0 \leqslant y \leqslant f(k, K, z, t)} \left\{ u\big(f(k, K, z, t) - y\big) + \beta \int_{\mathbb{Z}} v\big(y, h(K, z), z'; t, h\big) \chi(z, \mathrm{d}z') \right\},$$

and applying the standard version of the theorem of the maximum (see Berge [10]) and the contraction mapping theorem (e.g., Stokey, Lucas and Prescott [45]) applied to the space $\mathbb{V}$ of functions $v(k, K, z; t, h)$ that are bounded, continuous in k, increasing in all arguments equipped with the pointwise partial order and the uniform metric, it is easy to show that T^C delivers a unique value function $v^*(k, K, z; t, h)$ for each pair (t, h), as stated in the following lemma:

Lemma 9. For $h \in \mathbb{H}$ and $t \in \mathbb{T}$, under assumptions 4 and 5, there exists a unique function v^* bounded, weakly increasing, concave and continuous in its first argument satisfying Bellman's functional equation (4).

To further characterize $v^* \in V$, let

$$\gamma(k, K, z; t, h) = \left\{ y \;\middle|\; y = \operatorname*{arg\,max}_{0 \leqslant y \leqslant f(k,K,z,t)} \left\{ u\big(f(k, K, z, t) - y\big) + \beta \int_Z v^*\big(y, h(K, z), z'; t, h\big) \chi(z, \mathrm{d}z') \right\} \right\},$$

the optimal policy associated with the value function $v^*(k, K, z; t, h)$. Since the right-hand side in the definition of γ is strictly concave in y, the optimal policy exists for each (k, K, h, t). We next show that the Bellman's operator T^C maps a closed subspace $\mathbb{V}'$ of $\mathbb{V}$ into itself.

Definition. Denote $\mathbb{V}'$ as the set of functions $v \in \mathbb{V}$ such that $v(k, K, z; t, h)$ is:

(i) supermodular in (k, K) for each (z, t);

(ii) supermodular in (k, h) for each (K, z, t);

(iii) supermodular in (k, t) for each (K, z, h).

By a supermodular function, we mean the following. Say X is a lattice, and define the function $f : X \to R$. Let $x, y \in X$. Then we say that f is a super* function from a lattice to a chain where "$*$" is a binary operation on the chain R if f satisfies the following inequality: $f(x) * f(y) \leqslant f(x \vee y) * f(x \wedge y)$. If $*$ is "$+$", we say the function is supermodular (or equivalently superadditive). If $*$ is either "$\wedge$" or "$\vee$" we say f is superextremal. If $*$ is "$\cdot$", we say f is log-supermodular (or equivalently log-superadditive). LiCalzi and Veinott [26] discuss the class of super* functions, and also use this notion to construct the class of superextremal functions. Additionally, they provide a complete characterization of the superextremal class. The monotonicity results within the superextremal class can be shown to be closed under increasing transformation, and in this sense are ordinal. Of course, supermodularity is a cardinal property of a function. A supermodular function is superextremal. In this section, we will deal with supermodular functions, although Mirman, Morand and Reffett [33] discuss possible extensions of these results to the ordinal case of superextremal functions.

Notice that the space of supermodular functions (and actually superextremal functions) is closed under pointwise limits. Therefore $\mathbb{V}'$ is closed in $\mathbb{V}$. It can then be shown that the Bellman operator is actually a self map and a contraction on $\mathbb{V}'$, and that then provides the needed monotonicity results to generalize the Euler methods of Coleman [12]. By assumption 4, and given that h is increasing in K for each z, then the right-hand side of the Bellman's equation is (i) supermodular in (y, k, K) for each (z, t), (ii) supermodular in (y, h) for each (k, K, z, t), and (iii) supermodular in (y, t) for each (k, K, z, h). As a consequence, the set of optimal solutions associated with some $v \in \mathbb{V}'$ has γ increasing in the strong set order in (i) (k, K) for each (z, t), (ii) increasing in h for each (k, K, z, t), and (iii) increasing in t for each (k, K, z, h) by a result in Topkis [52, theorem 2.8.1]. Therefore by the monotonicity theorem in Milgrom and Shannon [31],

Tv is quasisupermodular in its appropriate arguments. To show that $T^C v \in \mathbb{V}'$, simply use the generalized envelope result for non-concave value functions in k (when the policies are monotone) due to Amir [7] to obtain $\partial T^C v = u' \circ f_1$. Then following an argument in Mirman, Morand and Reffett [33], it can be shown that this envelope is (i) increasing in K each (k, z, t, h), (ii) increasing in h for each (k, K, z, t), and (iii) increasing in t for each (k, K, z, h) for each $v \in \mathbb{V}'$. Therefore $T^C v \in \mathbb{V}'$. By a standard reapplication of the contraction mapping theorem, $T^C v^* \in \mathbb{V}'$.

We therefore have the following result:

Theorem 10. Given any $t \in \mathbb{T}$ and $h \in \mathbb{H}$, under assumptions 4 and 5, the value function $v(k, K, z)$ has an optimal investment policy $\gamma(k, K, z)$ that is increasing in (k, K) for each $z \in \mathbb{Z}$.

3.2. Existence of equilibrium

Some of the assumptions required in the proof of existence in section 2 have been relaxed in this section, so existence has to be established through a different path. Our strategy is to construct a nonlinear correspondence A which maps a complete lattice of functions $\mathbb{H}$ into itself, and to show that this correspondence is monotone increasing on $\mathbb{H}$ in the pointwise partial order. Since $\mathbb{H}$ is a complete lattice, it follows from Tarski's theorem that the set of fixed points of A is not empty.[7] We rely on the following version of Tarski's theorem[8] [47]:

Theorem 11. Let $(X, \geqslant)$ be a complete lattice $A : X \to X$ an increasing mapping. The set of fixed points of A is a non-empty complete lattice.[9] Further, the sets of excessive and deficient points (respectively $s \geqslant A(s)$ and $s \leqslant A(s)$) are non-empty complete lattices, and the greatest (respectively least) fixed point is the greatest deficient point (respectively least excessive).

We construct the operator A as follows. For a given h, and a given t, we define Ah as the optimal policy along a candidate equilibrium trajectory, that is, we impose the equilibrium condition $k = K$ in the optimal policy so that:

$$Ah(K, z) = \gamma(K, K, z; t, h).$$

This construction defines an operator A with the following properties:

[7] Note that under a more general setup relaxing the assumption of strict concavity, Zhou [56] develops a version of Tarski's theorem for correspondence that can be applied to demonstrate existence of equilibrium (see Mirman, Morand and Reffett [33]).

[8] For a proof, see Veinott [54, theorem 11].

[9] For the set of fixed points to be non-empty, monotonicity of the operator together with chain completeness of the domain are sufficient conditions. If, in addition, there exists a deficient point, then there exists a maximal fixed point (Knaster–Tarski theorem; see Aliprantis and Border [3, p. 14]).

Lemma 12. Given any $t \in \mathbb{T}$, under assumptions 4 and 5, for any $h \in \mathbb{H}$, $Ah \in \mathbb{H}$ and A is increasing in h in the partial pointwise order.

The following proposition is then a direct consequence of Tarski's theorem.

Theorem 13. Under assumptions 4 and 5, the set of equilibrium is a non-empty complete lattice.

Because a complete lattice is, a fortiori, chain complete, and the operator A has an excessive point (the zero consumption, which satisfies $A0 \geqslant 0$) and a deficient point (the production function, which satisfies $Af \leqslant f$), from Tarski's theorem the mapping A has a minimal and a maximal fixed point. These order-based methods also suggest algorithms to compute the minimal and maximal fixed points, as demonstrated in Mirman, Morand and Reffett [33].

Theorem 14. For each $t \in \mathbb{T}$, the maximal fixed point of the operator A can be computed as $h_u(t) = \lim_{n\to\infty} A^n f(K, K, z; t)$.

3.3. Monotone comparative analysis

Comparative analysis for this economy describes how the set of fixed points $E(t) \in P(\mathbb{H})$ of the operator A changes with respect to changes in the parameter $t \in \mathbb{T}$. Recall that $\mathbb{T}$ is endowed by the pointwise partial order. When comparing the elements of $P(\mathbb{H})$, we will use the two order relationships that are defined below (see Veinott [54] for a discussion of the ordinal structure of the various partial orders on $P(\mathbb{H})$).

Let X be a partially ordered set and $P(X)$ the power set of X:

(i) Weak induced set order (see Shannon [44] or Topkis [52]): The weak induced set order on $P(X) \setminus \emptyset$, denoted by $\geqslant_w$, is such that $B \geqslant_w B'$ if for each $x \in B$, there exists $x' \in B'$ such that $x \succeq x'$, and for each $x' \in B'$ there exists $x \in B$ such that $x \succeq x'$.

(ii) Induced set ordering (discussed in Topkis [52] in section 2.4): The induced set ordering on $P(X) \setminus \emptyset$, denoted by $\geqslant_s$, is such that $B \geqslant_s B'$ if for each $x \in B$ and each $x' \in B'$, $x' \wedge x \in B'$ and $x' \vee x \in B$.

Clearly, when X is a lattice, the set order $\geqslant_s$ is a stronger ordering than the weak order $\geqslant_w$ in the sense that $B \geqslant_s B'$ implies that $B \geqslant_w B'$. While it is clear that $\geqslant_s$ implies $\geqslant_w$, we were unable to obtain sufficient conditions to generate comparative analysis results in $\geqslant_s$.

Our strategy for comparative analysis is simple: We show that A is increasing in the strong set order in the parameter t on $\mathbb{T}$, where the order we consider on $\mathbb{T}$ is simply the pointwise partial order. This, in effect, implies that the return on capital for each state (k, K, z, h) is increasing in t in the pointwise order. That is, we obtain a single crossing property (actually increasing differences) in (y, t) for each (k, K, z, h). Therefore, we have the following:

Theorem 15. Under assumptions 4 and 5, for all $t' \succeq t$, the set of fixed points $E(t)$ of the non-linear operator A satisfies: (i) $E(t') \geqslant_w E(t)$, and (ii) $E(t') \geqslant_s E(t)$ on $\mathbb{H}$.

One important implication of the above theorem is that for strongly concave economies, the comparative analysis available for Markovian equilibrium in a parameter are very strong. In particular, as equilibrium within a class of equicontinuous functions are unique, the theorem implies that the equilibrium correspondence is a monotone function in $t \in T$.

4. Some extensions of the Euler equation method

The Euler equation method described in section 2 of this paper can be tailored to various environments. First, we show how to establish existence and uniqueness of equilibrium in models where the state-space is not compact (i.e., assumption 3 in section 2 does not hold). Without the compactness of the space of candidate equilibrium functions, which crucially depends on the compactness of the state-space, we must abandon the topological fixed-point theorem of section 2 in favor of an order based fixed-point theorem (see Morand and Reffett [35] for complete details). Second, we apply the Euler equation method to models in which labor is an input to the production of the consumption good, as well as capital. There is labor–leisure choice and distortions that take the form of taxes on labor and capital income. This modification of the primitives increases the dimension of the choice set of households and requires defining a more complicated set of functions as the domain of our monotone increasing operator constructed from the Euler equations (see Datta, Mirman and Reffett [16]). Finally, we note that the Euler equation method provides a quick and elegant proof of existence and uniqueness of equilibrium for a simple version of the altruistic model studied in Ray [36].

4.1. Unbounded growth

When the state-space is unbounded, and therefore not compact, the fixed-point theorem used in section 2 cannot be directly applied. Thus, the Euler equation method developed in section 2 must be modified for models in which assumption 3 does not hold, which is the case, for instance, in endogenous growth models with externalities for which there are no stationary representation, as in Greenwood and Huffman [20]. The difficulty arises from the fact that the set of functions on which A operates is much larger than in the compact state-space case, since it includes some unbounded elements. Fortunately, the space of candidate equilibrium functions $\mathbb{H}^0$ is a complete lattice whether or not the state space is compact. We exploit this feature of $\mathbb{H}^0$, together with the monotonicity of A, and follow the order based argument developed in Morand and Reffett [35] to prove existence of equilibrium. Uniqueness rests on the same argument as in section 2.

Recall that the space $\mathbb{H}^0$ defined in section 2 is endowed with the point-wise partial order. Consider the lattice operations $\vee$ and $\wedge$

$$(h \vee g)(K, z) = \max\{h(K, z), g(K, z)\}$$

and

$$(h \wedge g)(K, z) = \min\{h(K, z), g(K, z)\}$$

for each $(K, z) \in \mathbb{S}$. Since the inf and sup of any family of increasing functions are obviously increasing functions, we have the following result:

Lemma 16. $\mathbb{H}^0$ is a complete lattice.

Proof. Recall that a lattice $\mathbb{H}^0$ is complete if any subset G of $\mathbb{H}^0$ is such that G has a sup and an inf. Consider any family G of elements of $\mathbb{H}^0$. Clearly (i) $0 \leqslant \sup G \leqslant F$, (ii) $\sup G$ is weakly increasing in K, and (iii) $F - \sup G = \inf\{F - g\}_{g \in G}$ is also weakly increasing in K. A similar argument applies for $\inf G$. Thus $\mathbb{H}^0$ is a complete lattice. □

Recalling from section 2 that the operator A is monotone (whether or not the space is compact), Tarski's theorem cited above applies.

Proposition 17. Under assumptions 1 and 2, the set of fixed points is non-empty, and there exist greatest and least fixed points.

Proof. Standard application of Tarski's fixed-point theorem (stated in the previous section). □

Additional characterization of the maximal fixed point rests on the property that A is order continuous at the pointwise limit of the sequence $\{A^n F\}_{n=0}^{\infty}$. We refer the reader to Morand and Reffett [35] for the complete analysis, and simply state the following result.

Proposition 18. The sequence $\{A^n F\}_{n=0}^{\infty}$ converges to the maximal fixed point, and the convergence is uniform on any compact subset of the state-space.

Finally, it is important to note that relaxing the boundedness assumption on utility is not a trivial matter, since there is no general theory that guarantees existence of a value function, let alone existence of an Euler equation, when an unbounded return function is combined with a unbounded state space.[10]

4.2. Models with capital and elastic labor supply

We alter the model presented in section 2 to incorporate elastic labor supply, allowing for state contingent wage and capital taxes and production externalities, as in Datta, Mirman and Reffett [16]. Production is characterized by perfectly competitive markets for both output and the factors of production, as in the previous sections, but households

[10] There are some results on existence of a value function for some primitives generating unbounded return functions (see Alvarez and Stokey [4], Miao [29], and Morand and Reffett [35]).

have preferences defined over both consumption and leisure, so their unit of time will no longer be supplied inelastically. Specifically, households maximize:

$$U(\mathbf{x}) = E_0\left\{\sum_{t=0}^{\infty} \beta^t u(x_t)\right\},$$

where $x_t = (c_t, l_t) \in \mathbb{R}_+ \times [0, 1]$. We now consider changing assumptions 1–3 in section 2 of the paper. For elastic labor supply, consider the following assumption on the period utility $u : \mathbb{R}_+ \times [0, 1] \mapsto \mathbb{R}$:

Assumption 6.

(i) The period utility function u is bounded, twice continuously differentiable, strictly increasing, and strictly concave in (c, l).

(ii) The partial derivatives $u_c(c, l)$ and $u_l(c, l)$ satisfy the Inada conditions:

$$\lim_{c\to 0} u_c(c, l) = \infty, \quad \lim_{c\to\infty} u_c(c, l) = 0, \quad \lim_{l\to 0} u_l(c, l) = \infty.$$

(iii) $u_{cl}(c, l) \geqslant 0$.

Assumption 6(iii) is particularly important, as we require the marginal rate of substitution u_l/u_c to be non-decreasing in c and non-increasing in l to construct a monotone operator. One should note, however, that assumption 6(iii) is not sufficient enough to have the period utility function supermodular in (c, l, k, K).[11]

Let f: $\mathbb{K} \times [0, 1] \times \mathbb{K} \times [0, 1] \times \mathbb{Z}$ summarize the production possibilities for the firm in any given period, where $\mathbb{K}$ is a compact set defined similarly as in section 2. The technology allows for externalities in the production process (that is, f also depends on per capita aggregates).

Assumption 7. The production function satisfies:

(i) $f(0, 0, K, N, z) = 0$ for all $(K, N, z) \in \mathbb{K} \times [0, 1] \times \mathbb{Z}$.

(ii) $f(k, n, K, N, z)$ is continuous, increasing, differentiable; in addition, it is concave and homogeneous of degree one in (k, n).

(iii) $f(k, n, K, N, z)$ also satisfies the standard Inada conditions in (k, n) for all $(K, N, z) \in \mathbb{K} \times [0, 1] \times \mathbb{Z}$; i.e., $\lim_{k\to 0} f_k(k, n, K, N, z) = \infty$; $\lim_{n\to 0} f_n(k, n, K, N, z) = \infty$; $\lim_{k\to\infty} f_k(k, n, K, N, z) = 0$.

[11] The setup is more general than Greenwood and Huffman [20], which only consider the case where $u_{cl} = 0$. Coleman [13] allows for $u_{cl} \geqslant 0$ and also some cases where $u_{cl} \leqslant 0$ but considers a restricted homothetic class of preferences and imposes more restrictions (jointly on utility, production functions and distortions) to study the case of negative cross partials of u. The same case of negative cross-partials of u can also be handled in our setting.

Assumption 8. There exists a $\hat{k}(z) > 0$, such that $f(\hat{k}(z), 1, \hat{k}(z), 1, z) + (1-\delta)\hat{k}(z) = \hat{k}(z)$ and $f(k, 1, k, 1, z) < k$ for all $k > \hat{k}(z)$, for all $z \in \mathbb{Z}$.

Assuming that firms maximize profits under perfect competition, and denoting $\bar{r}(K, z)$ as the rental rate on capital and $\bar{w}(K, z)$ as the wage rate, these factor prices are continuous functions of the aggregate state variable. The representative firm's maximum profit is:

$$\Pi(\bar{r}, w, K, N, z) = \sup_{k,n}\big\{f(k, n, K, N, z) - \bar{r}k - \bar{w}n\big\}.$$

Anticipating the standard definition of competitive equilibrium with $k = K$ and $n = N(S)$, for $S \in \mathbb{K} \times \mathbb{Z} = \mathbb{S}$, prices in the factor markets are $\bar{r} = f_k$ and $\bar{w} = f_n$. Also, given the assumed structure on the firm's decision problem, the Theorem of the Maximum [10] implies that Π is a continuous function, and that solutions to the firm's problem exist.

The household solves a standard dynamic capital accumulation problem, which we describe by parametrizing the aggregate economy faced by a typical decision maker. If the aggregate per capita capital stock is K, then households assume that per capita consumption decisions C, and per capita labor supply N, and the recursion of the capital stock K' is given by,

$$K' = \kappa(S); \qquad C = C(S); \qquad N = N(S).$$

The aggregate economy consists of functions $\Omega = (w, r, \kappa, C, N)$ from a space of functions with suitable restrictions needed to parameterize the household's decision problem in the second stage. Assume that the policy-induced equilibrium distortions have the following standard form,

$$r = \big[1 - \pi_k(S)\big]\bar{r} \quad \text{and} \quad w = \big[1 - \pi_n(S)\big]\bar{w},$$

where $\pi = [\pi_k, \pi_n]$ is a continuous mapping $\mathbb{S} \to [0, 1) \times [0, 1)$. We now place one additional regularity condition on the distorted prices w and r.

Assumption 9. The vector of distortions $\pi = [\pi_k, \pi_n]$ is such that the distorted wage $w = (1 - \pi_n(K, z))\bar{w}(K, N(K, z), z)$ and the distorted rental rate $r = (1 - \pi_n(K, z))\bar{r}(K, N(K, z), z)$ satisfy:

(i) $w : \mathbb{K} \times \mathbb{Z} \to \mathbb{R}_+$ is continuous, at least once-differentiate and (weakly) increasing in K,

(ii) $r : \mathbb{K}_+ \times \mathbb{Z} \to \mathbb{R}_+$ is continuous and decreasing in K such that,

$$\lim_{K\to 0} r(K, z) \to \infty.$$

In other words, we assume that the distorted wage and rental rates behave geometrically as the non-distorted rates $\bar{w}, \bar{r}$ (which are the marginal products of labor and capital, respectively).

Let the lump-sum transfers to each agent be given by a function $d(S) = \pi_k K + \pi_n N(K, z)$, so that the household's total income (taking into account the elastic nature of labor supply) is $y(s) = rk + wN + (1 - \delta)k + \Pi$, the sum of distorted rental and wage incomes, undepreciated capital and profits, where $s = (k, S) = (k, K, z)$ is the individual household state variable. Note that $y(s)$ is a continuous function. The household feasible correspondence $\Psi(s)$ for the distorted economy then consists of the set of $(c, k') \in \mathbb{R}^2_+$ and $l \in [0, 1]$ satisfying the following constraint:

$$c + w(1 - l) + k' = y,$$

given $(k, K, z) \gg 0$. Notice that $\Psi(s)$ is well behaved: in particular, since Π is continuous, Ψ is a non-empty, compact and convex-valued, continuous correspondence.

The household's dynamic decision problem is summarized by the Bellman equation:

$$J(s) = \sup_{(c,l,k') \in \Psi(s)} u(c, l) + \beta \int_{\mathbb{Z}} J(s') \chi(z, \mathrm{d}z').$$

The first-order condition for the household's problem can be written as:

$$\frac{u_l(c, l)}{u_c(c, l)} = \big(1 - \pi_n(S)\big) f_n(K, N, z), \tag{5}$$

which determines the equilibrium (intra-period) relationship between consumption and leisure, and:

$$u_c(c, l) = \beta \int_{\mathbb{Z}} u_c\big(c\big(K', z'\big), l\big(K', z'\big)\big) r\big(K', z'\big) \chi(z, \mathrm{d}z'), \tag{6}$$

which governs the relationship between consumption in two consecutive time periods.

Under assumptions 6 and 7, one can easily establish that the solution $l^*(c, K, z)$ to the (intra-period) first-order condition (5):

$$\frac{u_l(c, l^*(c, K, z))}{u_c(c, l^*(c, K, z))} = \big(1 - \pi_n(S)\big) f_n\big(K, 1 - l^*(c, K, z), K, z\big)$$

is decreasing in c and increasing in K. Given a candidate equilibrium function $c(S)$, we rewrite the Euler equation (6) in equilibrium using l^* as,

$$u_c\big(c, l^*(c, K, z)\big) = \beta \int_{\mathbb{Z}} u_c\big(c(F_c - c, z'), l^*\big(c(F_c - c, z'), K', z'\big)\big) r(F_c - c, z') \chi(z, \mathrm{d}z'),$$

where $F_c = f(K, 1 - l^*(c, K, z), z) + (1 - \delta)K$, and we use this last equation to define a nonlinear operator that yields a strictly positive fixed point in the space $\overline{\mathbf{H}}$ of candidate consumption functions.

Since the class of preferences considered in this model is larger than in section 2, the space $\overline{\mathbf{H}}$ differs from $\mathbb{H}^0$. It is defined as follows. First, define $\hat{l}(S)$ as the solution to,

$$\frac{u_l(f(K, 1 - \hat{l}(S), z), \hat{l}(S))}{u_c(f(K, 1 - \hat{l}(S), z), \hat{l}(S))} = \big(1 - \pi_n(S)\big) f_n\big(K, 1 - \hat{l}(S), z\big),$$

which implies that $\hat{l}(S)$ is the amount of leisure if everything is currently consumed. Since, in general, the amount of consumption is less than f, leisure, which is positively related to consumption, is therefore less than $\hat{l}(S)$: $\hat{l}$ thus defines an upper bound on output. It is used to define a candidate set of consumption functions. Second, define $F^u(S) = F^u(K, z) = f(K, 1 - \hat{l}(K, z), z) + (1 - \delta)K$.

Definition. Consider the space $\overline{\mathbf{H}}$ of consumption functions $h : \mathbb{S} \to \mathbb{R}$ such that

(i) h is continuous;

(ii) $h(S) \in [0, F^u(S)]$;

(iii) $u_c(h(S), l^*(h(S), S))$ is decreasing in h;

(iv) $u_c(h(S), l^*(h(S), S))$ is decreasing in K;

(v) for any $K_2 \geqslant K_1$, h satisfies:

$$\begin{aligned}0 &\leqslant \big|h(K_2, z) - h(K_1, z)\big| \\ &\leqslant \big|F\big(K_2, l^*\big(h(K_2, z), K_2, z\big)\big) - F\big(K_1, l^*\big(h(K_1, z), K_1, z\big)\big)\big|.\end{aligned}$$

Equip $\overline{\mathbf{H}}$ with the sup norm and the standard pointwise partial order. A standard argument shows that the space of consumption functions $\overline{\mathbf{H}}$ is a closed, pointwise compact, and equicontinuous set of functions. By the Arzela–Ascoli theorem (see Royden [38]) $\overline{\mathbf{H}}$ is a compact.

As in section 2, given any $h \in \overline{\mathbf{H}}$, the operator A is defined implicitly from the Euler equation: $Ah(K, z)$ is the solution for y of:

$$\begin{aligned}u_c\big(y, l^*(y, K, z)\big) = \beta \int_{\mathbb{Z}} & u_c\big(h(F_y - y, z'), l^*\big(h(F_y - y, z'), F_y - y, z'\big)\big) \\ & \times r(F_y - y, z')\chi(z, \mathrm{d}z').\end{aligned}$$

We refer to Datta, Mirman and Reffett [16] for the proof of the following lemma.

Lemma 19. Under assumptions 6–9, A maps $\overline{\mathbf{H}}$ into itself and is monotone and continuous.

Proposition 20. Under assumptions 6–9, the set of fixed points of $A : \overline{\mathbf{H}} \to \overline{\mathbf{H}}$ is nonempty and there exists a maximal fixed point $h^* \in \overline{\mathbf{H}}$. The sequence $\{A^n F\}_{n=0}^{\infty}$ converges uniformly to h^*, uniformly. Further, the maximal fixed point is strictly positive ($h^* > 0$).

Proof. The first result follows from the fact that A is monotone and $\overline{\mathbf{H}}$ is compact and therefore chain complete. Note $AF \leqslant F$ and $A0 \geqslant 0$, therefore by application of Tarski's theorem the operator A has a fixed point. By continuity of A, the sequence $\{A^n F\}_{n=0}^{\infty}$ converges to the maximal fixed point h^* and since S is compact, the convergence in uniform. The second property (positivity) follows from an obvious modification of the main theorem in Greenwood and Huffman [20].

Uniqueness of equilibrium can be established through an argument related to that in section 2 of the paper. Consequently $h^* > 0$ is the unique equilibrium. □

4.3. Non-paternalistic altruism

The Euler equation method can be applied to a simple version of the non-paternalistic altruism models studied in Ray [36], where altruism is confined to one's immediate successor and takes a separable form. Distortions can be introduced in the setup in the same manner as in section 2. For simplicity, we discuss the application of the Euler equation method in the absence of distortions. There is a continuum of agents each living for one period. An agent in period t divides his output between consumption and investment, and, at the end of the period, gives birth to a new agent and dies. Output (before taxes and subsidies) of his descendant in period $t + 1$ is obtained through a production technology represented by the function f, which exhibits constant returns to scale in private inputs. Production takes place in the context of perfectly competitive markets for both the output good and the factors of production.

Let V_s be the utility of generation s. The preferences of each generation are represented by a simple non-paternalistic utility function v which maps, for each t, (c_t, V_{t+1}) into utilities of generation t. We assume that v is time separable, and therefore takes the form:

$$v(c_t, V_{t+1}, z_t) = u(c_t) + \beta E_t V_{t+1},$$

where $0 < \beta < 1$. Also, assume that period utility u satisfies assumption 1 of section 2.

Agents are endowed with a unit of time which they supply inelastically to competitive firms. In the absence of distortions, let $f(k, z)$ be the output available for an agent to share between his consumption and the bequest to his descendant. We make the following assumptions:

Assumption 10. $f(k, K, z)$ is continuous and strictly increasing in its argument, continuously differentiable and strictly concave in its first arguments. In addition, $f(0, z) = 0$ and $\lim_{k\to 0} f_1(0, z) = \infty$ for all $z \in \mathbb{Z}$.

Assumption 11. There exists a $\hat{k}(z) > 0$, such that $f(\hat{k}(z), z) = \hat{k}(z)$ and $f(k, z) < k$ for all $k > \hat{k}(z)$, for all $z \in \mathbb{Z}$.

The maximization problem associated with an agent in period t is:

$$\max_c \{u(c) + \beta E_z[V_t(f(K, z) - c, z')]\},$$

in which c is chosen in the compact interval $\Gamma(K, z) = [0, f(K, z)]$. Our concept of equilibrium follows the one in Ray [36].

Definition. A stationary equilibrium is a vector (V^*, c^*) where V^* is an indirect utility function, or value function, and c^* a consumption policy, such that:

(i) The value function V^* satisfies:

$$V^*(K, z) = \sup_{c \in \Gamma(K,z)} \big\{u(c) + \beta E_z\big[V^*\big(f(K, z) - c, z'\big)\big]\big\}.$$

(ii) The optimal consumption policy satisfies:

$$c^*(K, z) = \arg\max_{c \in \Gamma(K,z)} \big\{u(c) + \beta E_z\big[V^*\big(f(K, z) - c, z'\big)\big]\big\}.$$

To study Markovian equilibrium for this economy, note that existence of a unique bounded, continuous, strictly increasing, strictly concave in its first argument value function V^* with an envelope condition for its first argument is a standard result. Consequently, the optimal policy function necessarily satisfies the Euler equation:

$$u'\big(c^*(K, z)\big) = \beta E_z\big\{u'\big[c^*\big(f(K, z) - c^*(K, z), z'\big)\big] f_1\big(f(K, z) - c^*(K, z), z'\big)\big\}.$$

An equilibrium consumption is then a strictly positive solution $c^*(K, z) > 0$ to this Euler equation. Therefore for economies operating under assumptions 10 and 11, a similar analysis to that of section 2 applies to this problem, and the sequence $\{A^n f\}_{n=0}^{\infty}$ converges uniformly to the unique Markovian equilibrium.

5. Suggestions for future research

In this paper we have discussed methods of constructing competitive equilibrium for a broad class of strongly concave infinite horizon economies. The methods restrict attention to continuous Markovian equilibrium. In the smooth case of section 2, we construct a closed and equicontinuous subset of bounded continuous functions in the C^0 topology. This is a compact set (in the C^0 topology). One open question is whether the equilibrium function is actually once differentiable? For Pareto optimal environments, this question is answered in the work of Araujo and Scheinkman [8] and Santos [39]. It remains an open question whether the equilibrium in Coleman [12] is once differentiable. The answer to this question is potentially of great interest.

One reason for an interest in the existence of smooth Markov equilibrium is that when equilibrium are smooth, the error bounds of Santos and Vigo [43] and Santos [42] can be shown to apply. Further, one logical method to obtain a bound in the variation of the derivative of functions governing the Markov equilibrium is to use the variation in the derivative of the production function. In such a case, one might be able to produce a natural estimate of the error bounds for numerical solutions to the problem. Second, when estimating non-linear rational expectations models with simulated methods of moments procedures, conducting asymptotic analysis of the estimators is facilitated if the equilibrium is at least once differentiable in the parameters of the model. Topological methods will most likely need to be employed to address issues concerning the smoothness of Markov equilibrium, as spaces of once differentiable functions do not have a lattice structure.

In the Euler equation methods, the role of strict concavity is crucial. When the value function is appropriately concave, the (optimal) best responses of each agent are single valued continuous mappings, and this greatly simplifies the analysis. If we relax the strict concavity property in competitive environments the methods have to be changed. Indeed, the Euler equation method in section 2 is likely to fail, for once we lose strict concavity of the value function, continuous selections in general no longer exist in the best response mappings. In this case, therefore, the Euler equation method has little applicability. Mirman, Morand and Reffett [33] relax strict concavity and show existence by extending the value function iteration method of section 3 by using Zhou's generalization of Tarski's theorem (see Zhou [56]). Further, when using the so-called induced set order of Topkis [52], the monotone comparative analysis like that obtained in section 3 is once again available. However, having lost uniqueness, the strong comparative analysis results, like those discussed in Milgrom and Shannon [31] are not available. One interesting way of extending the results in section 3 is to find new orders under which the single crossing property between the controls and the parameters can be investigated. In the present work, only Euclidean orders are studied. It is possible though that more general results might be forthcoming by employing a richer class of orders for the parameters (which are functions in our case), thereby yielding additional monotone iterative procedures useful for computing and constructing Markovian equilibrium.

Finally, the monotone methods discussed in this paper could prove very useful in studying dynamic games (for instance policy games between two countries). It is well known that in these environments, even if the return functions are strictly concave, the value functions for such dynamic games are generally not concave. While topological methods based upon single valued operators seem difficult to apply, the value function iteration method of section 3, however, appears very promising because it is based on supermodularity and does not need concavity. For, at least the symmetric equilibrium case discussed in Sundaram [46], it appears that the methods in Mirman, Morand and Reffett [33] might work. This is part of our agenda for future research.

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