1. Isotone Recursive Methods: The Case of Homogeneous Agents

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1.1 Introduction

A foundation of modern macroeconomics is the stochastic growth model originally introduced in the seminal work of Brock and Mirman [15]. Their original model is an infinite horizon economy with a continuum of identical households, each with access to a complete set of financial markets that insure them against all sources of idiosyncratic risk. There is single sector production that employs capital and labor whose returns are summarized by a stochastic neoclassical production function representing an aggregate convex production set with identical private and social returns to inputs. There is also aggregate risk taking the form of a collection of identically and independently distributed (i.i.d.) random variables, the agents in the economy face no frictions in information acquisition (i.e., there is no learning), labor supply is inelastic, and there are no equilibrium distortions. The authors characterize the unique *Markovian Equilibrium Decision Process* (MEDP) and its associated unique (non-trivial) long-run equilibrium dynamics, in particular, the *Stationary Markovian Equilibrium* (SME).

[1] We are deeply indebted to Len Mirman and Olivier Morand for numerous lengthy discussions concerning many issues discussed in this survey. Many of the results presented in this paper were developed originally in some form during our joint work with Len and Olivier over the last five years. We dedicate this paper to Len Mirman on the occasion of his sixty-fifth birthday. Indeed, this paper would not have been written without Len's ongoing pioneering work on equilibrium growth under uncertainty. We also thank Elena Antoniadou, Hector Chade, John Coleman, Jeremy Greenwood, Seppo Heikkila, Ken Judd, Tom Krebs, Cuong Le Van, Robert Lucas, Jr., Jianjun Miao, Chris Shannon, John Stachurski, Yiannis Vailakis, Charles Van Marrewijk, Jean-Marie Viaene, Itzhak Zilcha, and especially Robert Becker and Manuel Santos for many helpful conversations over the past years. All mistakes remain our own.

Their methodological approach was pioneering, and relied heavily on recursive methods. Implicitly, it exploits the validity of a second welfare theorem and one can interpret the economic outcomes of the fictional social planner's problem from the perspective of a decentralized economic system. A fully decentralized recursive formulation of the Brock-Mirman framework is put forward by Prescott and Mehra[66] (see also, Stokey, Lucas, with Prescott[78]).

Over the last three decades, extensions of this model have become the foundation for the systematic study of many diverse issues in quantitative dynamic macroeconomic theory. Applications include models of economic fluctuations and business cycles, production-based asset pricing, the positive and normative implications of incomplete financial markets and public goods, the wealth inequality, the dynamic structure of altruistic economies, stochastic life-cycle models, models with physical and human capital, and the role of activist fiscal and/or monetary policy etc. However, many recent applications emphasize economic environments where the second welfare theorem is *not* available. These modifications create serious complications for a systematic study of the underlying structure of the MEDPs and the SME. A prevalent approach is to develop extensive applications of numerical methods to characterize MEDPs and the SME. From a mathematical perspective, many of these approaches have been *ad hoc* as they cannot be developed rigorously without providing characterizations of qualitative structure of the MEDPs and/or the SME.

An important question naturally emerges from this apparent disconnect between mathematical principle and macroeconomic practice: can one provide sharp and *constructive* characterizations of the MEDPs or the SME for generalized Brock-Mirman environments where the second welfare theorem fails? The most significant advance in providing an affirmative answer to this question has been the recent literature on "monotone methods" (also known as "monotone map" methods or "isotone recursive methods"). The pioneering work of Coleman [18][19][20][21], Greenwood and Huffman[34], Datta, Mirman, and Reffett[22] and Morand and Reffett[62] provide the genesis of the study of isotone recursive methods over the last fifteen years (they refer to them, as the "monotone-map" method). These papers present the first set of conditions under which constructive methods can be applied for studying the structure of a decentralized Markovian equilibrium in economies with or without non-classical production technologies.[2] An important generalization of this monotone-map approach is found in Mirman, Morand, and Reffett[59]. Here, a new and more general isotone map approach is presented (with the Coleman-Greenwood-Huffman approach as a special case) and can be applied

[2] The literature on monotone map methods is vast, and also includes the papers of Lucas and Stokey [56], Bizer and Judd [14] etc. An interesting alternative monotone method is developed in Becker and Foias [9].

For non-existence of a continuous MEDP, see Santos [73] and Krebs [49]. Mirman, Morand, and Reffett [59] show that although the Santos [73] example is robust to a large class of economies, in many case MEDPs are semi-continuous and isotone.

to a larger collection of dynamic economies with production nonconvexities (in the reduced-form production function). In this setting, sets of sufficient conditions for the existence of semicontinuous, continuous, Lipschitz continuous, and once-differentiable MEDPs are given. Since sufficient conditions for MEDPs to be differentiable are presented, therefore the error bounds constructed in Santos and Vigo[75] and Santos[74] apply. Finally a theory of *ordered* MEDPs is developed applying the seminal work in operations research on lattice programming and the qualitative study of equilibrium introduced in Veinott[84][85] and Topkis[81][82][83].

The chapter is organized as follows: in the next section, we introduce some useful terminology. Section 1.3 provides a survey of the existing literature on fixed point theory in order spaces. This fixed point theory is critical in the development of isotone recursive methods. In Section 1.4, we consider homogeneous agent economies with classical production technology and infinite horizon. In this section, we develop an "Euler equation" approach to isotone recursive methods. We discuss the case studied in Coleman[19] for nonoptimal homogeneous agent economies. In Section 1.5, we discuss the generalizations found in Mirman, Morand and Reffett[59]. Section 1.6 considers the case of elastic labor supply as in Coleman[20] and Datta, Mirman and Reffett[22]. In section 1.7, we conclude with a brief discussion of new frontiers in monotone recursive methods, to models with heterogeneous agents including the overlapping generations models with stochastic production (e.g., Erikson, Morand and Reffett[31]), models with unbounded stochastic nonoptimal growth (e.g., Morand and Reffett[62]), Ramsey-type models with heterogeneous agents (e.g., Datta, Mirman, Morand and Reffett[23], and the mixed monotone recursive methods discussed in Reffett[68] and Mirman, Reffett and Stachurski[60].

1.2 Preliminaries

1.2.1 Ordered Spaces

We begin with some useful terminology. For a more complete accounting of the ideas in this section, see Birkhoff[13], Veinott[85], and Davey and Priestley[24].

Posets: In our subsequent discussion, we shall respect two notational conventions: (i) we write "$\geq$" in place of "$\geq_X$" when the order relation $\geq_X$: $X \times X \rightarrow X$ is clearly implied; and (ii) for two elements of X, say a and b, the order relation "$a \geq b$" can also be written as "$b \leq a$". Let X be a set. We say X is *partially ordered set (or Poset)* if X is equipped with an order relation $\geq_X$: $X \times X \rightarrow X$ that is reflexive, antisymmetric and transitive. If every element of a poset X is comparable, then we say X is a *totally ordered set* or *chain*. Every chain has an inherent lattice structure.

Lattices. Let X be a poset equipped with a partial order $\geq$. An *upper (*respectively, *lower) bound* for a set $B \subset X$ is an element x^u(respectively, x^l) $\in B$ such that for any other element $x \in B$, $x \leq x^u$ (respectively, $x^l \leq x$)

for all $x \in B$. If there is a point x^u (respectively, x^l) such that x^u is the least element in the subset of upper bounds of $B \subset X$ (respectively, the greatest element in the subset of lower bounds of $B \subset X$), we say x^u (respectively, x^i) is the *supremum* (respectively, *infimum)* of B. Clearly if they exist, both the supremum (or, sup) and infimum (or, inf) must be unique. We say X is a *lattice* if for any two elements x and x' in X, X is closed under the operation of infimum in X , denoted $x \wedge x'$, and supremum in X, denoted $x \vee x'$.The former is referred to as "the meet", while the latter is referred to as "the join" of the two points, $x, x' \in X$. A subset B of X is a *sublattice* of X if it contains the sup and the inf (with respect to X) of any pair of points in B. A lattice is *complete* if any subset B of X has a least upper bound $\vee B$ and a greatest lower bound $\wedge B$ in B. If this completeness property only holds for countable subsets X_c, the lattice is $\sigma-$*complete.* If every chain $C \subset X$ is complete, then X is referred to as a *chain complete poset (or equivalent, a complete partially ordered set* or *CPO).* A set C is *countable* if it is either finite or there is a bijection from the natural numbers onto C. If every chain $C \subset X$ is countable and complete, then X is referred to as a *countably chain complete poset.* Finally, a subset A of a set $C \subset P$ is *cofinal* if for each $x \in C$, there is a $y \in A$ such that $x \leq y$.

Ordered vector spaces and cones. A *partially ordered vector space* or *linear semi-ordered space* is a poset X that is real vector space equipped with a partial order $\geq$ that is compatible with the following algebraic structure: (i) if $x \geq x'$, then $x + z \geq x' + z$, for all $z \in X$; (ii) if $x \geq x'$, then $\alpha x \geq \alpha x'$ for all $\alpha \geq 0$. Any partially ordered vector space that is also a lattice is called a *vector lattice*. If the space has a norm $\| x \|_X$ which satisfies whenever $| x | \geq | x' |$ in X, $\| x \| \geq \| x' \|$, we say X has a *lattice norm*. A complete normed vector space is a *Banach space.* A *normed vector lattice* is a vector lattice equipped with a lattice norm. A normed vector lattice X that is complete in the Cauchy sense, and is endowed with a lattice norm is referred to as a *Banach lattice.*

Let X be a topological space. The set $X^+ = \{x \in X,\ x \geq 0\}$ is the *order cone* of X if X is nonempty convex closed set that has the following two properties: *(i)* $x \in X^+ \Longrightarrow \alpha x \in X^+$ for $\alpha \geq 0$; *(ii)* if x and -x in $X^+, x = 0$ where 0 denote the zero of the cone. The partial order induced by the cone structure of X^+ has $x_1 \geq x_2$ if $x_1 - x_2 \in X^+$. Now, assume X is a real Banach space. A cone X^+ of X is *normal* if there exists a constant m such that for any $x_1, x_2 \in X^+, \| x_1 + x_2 \| \geq m, \| x_i \| = 1$ for $i = 1, 2$. Intuitively, the restriction of normality of the cone geometrically bounds the angle between any two unit vectors away from π, so a normal cone cannot become "too large". An increasing sequence in the cone $\{x_t\}_{t=1}^{t=\infty}$, $x_t \in X^+$ is a sequence that satisfies $x_1 \leq x_2 \leq ... \leq x_n \leq ...$ We say a cone X^+ is *regular* if if every increasing and bounded order sequence in X^+ has a limit in X^+. We say X^+ is *fully regular* if every increasing and norm bounded sequence in X^+ has a limit in X^+. A fully regular cone is also regular. A regular cone is normal. (See Guo and Lakshmikantham[35], Theorem 1.2.1). A cone X^+ is *solid* if its interior $\mathring{X}^+$ is nonempty.

Let $[a) = \{x|x \in X, x \geq a\}$ be the *upperset* of a, $(b] = \{x|x \in X, x \leq b\}$ the *lowerset* of b. X is an *ordered topological space* if X is equipped with a partial order and topology that implies $[a)$ and $(b]$ are closed in the topology on X. An *order interval* is defined to be $[a,b] = [a) \cap (b]$, $a \leq b$. Therefore, in an ordered topological space, all $[a,b] \subset X$ (e.g., order intervals) are closed in the topology of X. In our work, we often study fixed point problems where the domain/range is a compact, order interval in a normal and solid cone of positive continuous functions $X^+ = C^+(S)$ endowed with the C^0 uniform norm topology (where each function itself is defined on compactum S). Such a space is not a regular cone. We will often work on a transformation space that is a compact suborder interval in $C^+(S)$ (where compactness will be used to compensate for the loss of regularity in the cone $C^+(S)$).

1.2.2 Mappings

We now define some important properties of mappings, especially those defined on lattices and posets:

Isotone (or Order Preserving) Mappings on a Poset: Let $(X, \geq_X)$ and $(Y, \geq_Y)$ be Posets. A *mapping* is a relational statement between two spaces, say X and Y. We consider both "point-to-point" and "point-to-set" mappings. In the case of a "point-to-point" mapping, we refer to the mapping as a *function (*or equivalently as an *operator*). A function $m : X \to Y$ is said to be *isotone* on X if it is "order-preserving", i.e., $m(x') \geq_Y m(x)$, when $x' \geq_X x$, for $x, x' \in X$. If $m(x') >_Y m(x)$ when $x' >_X x$ for x, $x' \in X$, we say the function m is *increasing*. If $m(x') >_Y m(x)$ when $x' \geq_X x$, $x' \neq x$, we say the function m is *strictly increasing*. We say $m(x)$ is *antitone* (or, *order-reversing*) if $m(x) \geq_Y m(x')$ if $x' \geq_x x$. A function that is either isotone or antitone is *monotone*. When the mapping $m(x)$ is a self-mapping on X, we also refer to $m(x)$ as a *transformation* of X, and the set X as a *transformation set.* If our concern is the fixed points of a transformation $m(x)$ on X, we refer to the transformation set X as the *fixed point space*.

Notions of monotonicity are also available for multifunctions or correspondences. By a *correspondence or multifunction,* we always refer to a nonempty-valued mapping $M : X \to 2^Y$, e.g., a nonempty-valued "point-to-set" mapping. We say a correspondence or multifunction is *ascending* in the set relation S (denoted by $\geq_S$) if $M(x') \geq_S M(x)$, when $x' \geq_X x$ where $(X, \geq_X)$ is a partially ordered space. If this set relation $\geq_S$ induces a partial order on the powerset 2^Y (or, perhaps, $2^Y \backslash \emptyset$), we refer the ascending correspondence also as an *isotone correspondence*.

To make concrete the notion of an isotone versus ascending correspondence, we discuss some particular set relations; some that induce partial orders on 2^Y (or, $2^Y \backslash \varnothing$), others that do not.[3] The set relations we consider are each

[3] For a more detailed discussion, we refer to the classic references of Smithson [77] and Veinott [85].

compatible with pointwise set comparisons, and, therefore, closely related to the sufficient conditions under which correspondences admit isotone selections. We focus primarily on four such set relations. Let Y be a set, and $A, B \in 2^Y$. We define : (i) the *Veinott-Weak Set relation* $\geq_w$on $2^Y \backslash \varnothing$: $A \geq_w B$, if for any $a \in A$, $b \in B$, either $a \wedge b \in B$, or, $a \vee b \in A$; (ii) the *Veinott-Strong Set Order* $\geq_s$ on $2^Y \backslash \varnothing$: $A \geq_a B$, if for any $a \in A$, $b \in B$, $a \wedge b \in B$ and $a \vee b \in A$; (iii) the *Smithson−Weak Set relation* $\geq_{as}$on 2^Y : $A \geq_{as} B$ if we have either (C1) for any $b \in B$, there exists an $a \in A$ such that $a \geq b$; or, (C2) for any $a \in A$, there exists an $b \in B$ such that $a \geq b$; (iv) the *Pointwise Strong Set Order* $\geq_{ss}$on $2^Y \backslash \varnothing$: $A \geq_{ss} B$ if and only if $a \in A$, $b \in B$, then $a \geq b$ in the partial order structure on A, for all a, b.A final classic partial order on the powerset 2^Y is commonly referred to as set inclusion. We say a subset $A \geq_{SI} B$ under *set inclusion* $\geq_{SI}$ if $B \subset A$.

Fixed points. Let $\mu : X \to 2^X$ be a non-empty valued correspondence for each $x \in X$. The correspondence μ is said to have a *fixed point* if there exists an x such that $x \in \mu(x)$. Therefore, if μ is a function, then a fixed point is an x^* such that $x^* = \mu(x^*)$. A fixed point x^* is *minimal (*respectively, *maximal)* if there does not exist another fixed point, say y^*, such that $y^* \leq x^*$ (respectively, $x^* \leq y^*$). If a fixed point is either minimal or maximal, we say it is *extremal.*

1.3 Fixed Point Theory In Ordered Spaces

In this section, we provide an account of fixed point theory in ordered spaces. For a more extensive discussion, see excellent surveys in Amann[4], Guo and Lakshmikantham[35], veinott [85], Heikkila and Lakshmikantham[39] and Jachymski[44].

1.3.1 Existence

First, we discuss the existence and characterization of solutions for two prototypical classes of parameterized fixed point (or, transformation) problems often encountered in economic applications. Consider X is a poset, T is an ordered topological space. The two problems are stated as Problem 1 and Problem 2.

Problem 1: To characterize the fixed points of the mapping,

$$f(x, t) : X \times T \to X \text{ and } f \text{ is isotone on } X \text{ for all } t \in T.$$

Problem 2: To characterize the fixed points of the mapping,

$$F(x, t) : X \times T \to 2^X, F \text{ is ascending } (\geq_{as}) \text{ in (C1) or (C2) on } X \text{ for all } t \in T,$$

Recall that $\geq_{as}$ denotes Smithson's weak set relation on the powerset 2^X. We denote the fixed point correspondence, in either Problem 1 or Problem 2, as $G(t)$.

Lattice Theoretic Fixed Point Theorems. A classical case of Problem 1 occurs when X is a nonempty, complete lattice. This is the case studied in the seminal work of Tarski[79][80] in the early 1940s, see also Kantorovich[47].[4] We say a space Y has a *fixed point property* for isotone functions (or, more compactly, *fpp)* if and only if each isotone transformation of Y, say $f : Y \to Y$, has a fixed point. We state Tarski's theorem adapted to Problem 1:

Proposition 1.3.1. *(Tarski[80], Theorem 1): Fix $t \in T$, and let the mapping $f(x,t) : X \times T \to X$ be isotone in x for each t, where X is a complete lattice. Then the fixed point correspondence $G(t)$ is a nonempty, complete lattice for each $t \in T$.*

We make a few remarks on this result. First, the theorem does not say $G(t)$ is subcomplete in X . In general, it is not. Second, the operator f is assumed to have no continuity properties on X (e.g., we assume no order or topological continuity properties for $f(x,t)$).

Often in economic applications, because of the absence of sufficient concavity in the agent's decision problem along equilibrium trajectories, equilibrium fixed point problems cannot be posed in terms of a single valued operator such as in Problem 1; rather, they must be posed in a more abstract setting of the fixed point of multifunctions, as in Problem 2. For the general case, a key generalization of Tarski was obtained by Veinott[84] in the 1970s, see also Veinott[85] (Chapter 4, Theorem 14).[5]

Proposition 1.3.2. *(Veinott[85]):Let $F(x,t) : X \times T \to 2^X \backslash \varnothing, X$ be a complete lattice, T be a set. For any fixed $t \in T$, assume that $F(x,t)$ is a nonempty, isotone in Veinott's strong set order, closed, and sublattice-valued correspondence on X. If $G(t)$ is the fixed point correspondence for $F(x,t)$ at $t \in T$, then $G(t)$ is a nonempty complete lattice for each $t \in T$.*

Propositions 1.3.1 and 1.3.2 provide sufficient conditions for the existence of a complete lattice of fixed points for an isotone and/or ascending transformations of a complete lattice X. An interesting question is necessity: i.e., can one obtain a complete characterization of a complete lattice using the *fixed point property?* Davis[25] (Theorem 1) provides the converse to Tarski's theorem: a lattice X is complete if and only if every isotone transformation $f : X \to X$ has a fixed point. In the context of Problem 2, the Davis characterization of a complete lattice X is also provided. Smithson[77] (corollary 1.8) proves the following: if X is a lattice and $F(x)$ is a multifunction then X is complete if and only if the correspondence $F(x)$ is (a) ascending in the Smithson-weak set

[4] Tarski's original result dates from around 1942 and is available in Tarski [79]. It is a generalization of a result he developed with Knaster in 1921 (for isotone correspondences under set inclusion). A related result for semi-ordered linear spaces is in Kantorovich [47].

[5] Zhou [87] proves it independently in Theorem 1.

relation (C1) (respectively, ascending in the Smithson-weak set relation (C2)), and (b) the least upper bound $(F(x,t)) \in F(x,t)$ (the greatest lower bound $(F(x,t)) \in F(x,t)$) for all $x \in X, t \in T$, and $G(t)$ is nonempty for each $t \in T$.

Other useful characterizations of complete lattices are available, and we use them in the sequel, as needed. For example, one can characterize a complete lattice X in terms of its interval topology (Frink[32]). Recall, the *interval topology* for a set X takes all the closed intervals $[a,b]$ as a subbasis for the closed sets of X. Frink[32] provides the following characterization of a complete lattice X: X is a complete lattice if and only if X is compact in its interval topology (see also Birkhoff [13], Chapter 10, Theorem 20). Another very useful characterization of a complete lattice is in Davey and Priestly[24] (Theorem 2.31). Their result provides the following characterization of a complete lattice X : let X be a nonempty ordered set; then the following statements are equivalent (i) X is a complete lattice; (ii) for any subset $S \subset X$, $\inf(S) \in X$; and X has a top element and $\inf(S) \in X$ for every nonempty subset of X. These two characterizations of a complete lattice X are used repeatedly in this chapter.

Fixed Point Theory in Complete Partially Ordered Sets. Next, we now consider Problems 1 and 2 when the fixed point space X is *not* a complete lattice. A natural set of regularity conditions for an ordered set X to have the fixed point property turns out to be chain-completeness. Recall a set X is chain complete if for any chain C, $\inf(C)$ and $\sup(C)$ are in X. A set X has a *bottom element* a (respectively, *top element* b) if for every $x \in X$, $a \leq x$ (respectively, $x \leq b$). A set X is a *complete partially ordered set* (or, *CPO*) if and only if (i) X has a bottom element, and (ii) for each directed net $D \subset X$, we have a sup $D \in X$. A set X is a CPO if and only if every chain C in X has a least upper bound, $\sup(C) \in X$. (Davey and Priestley[24], Theorem 8.11). Therefore, the notion of a set X being "chain-complete" is equivalent to the space X being a CP0. We often use this terminology when discussing chain-completeness.

Chain completeness is a natural condition to check in applications. For example, every relatively compact chain C in an ordered topological space has an infimum and a supremum, $\inf(C)$ and $\sup(C)$. See Amann[4], Lemma 3.1. Therefore, every compact, ordered topological space is chain complete (Amann[4], Corollary 3.2). One of the earliest results on the existence of a fixed point for a self map on a poset is obtained in Bourbaki[16]. As a consequence of Zorn's lemma, it is shown that if X is an ordered set such that every chain has an upper bound (respectively, a lower bound), and $f(x)$ on X is increasing in the following sense: for all $x \in X$, $x \leq f(x)$ (respectively, $f(x) \leq x$), then f has at least one fixed point. An improvement on this result is given in Abian and Brown[1] (Theorems 2,3,4) and Pelczar[65].The version of the theorem that we state is due to Amann[4] (see also Zeidler [88], Section 11.9 for a proof):

Proposition 1.3.3. *(Amann[4], Theorem 1.4): Let* X *be a CPO,* $f(x,t) : X \times T \to X$ *be isotone in* X *for each* $t \in T$*. Suppose that for each* $t \in T$*, there exists* *a pair* $(x_L(t), x^U(t)) \in X \times X$, $x_L(t) \leq x^U(t)$ *such that* $x_L(t) \leq f(x_L(t), t)$ *and* $f(x^U(t), t) \leq x^U(t)$*. Then f has a minimal and a maximal fixed point in* $[x_L(t), x^U(t)]$.

We next consider a converse to this theorem.[6] That is, as in the case of a complete lattice, we ask if one can obtain a characterization of a CPO X using the fixed point property relative to isotone transformations. Clearly, an arbitrary ordered set X does not have a fixed point property; but it turns out that if X is an ordered set, and for each isotone operator $f(x)$ on X, $f(x)$ has a least fixed point, then X is a CPO. Alternatively, if G is the set of fixed points of an isotone self-map $f(x)$, and X is a CPO, then G is a CPO. A collection of remarkable necessity results are found in Markowsky ([**?**], Theorems 9-11); we present the following result of Markowsky's that is summarized in Davey and Priestley [24]:

Proposition 1.3.4. *(Davey and Priestley[24], Propositions 8.25, 8.26). Let* X *and fix* $t \in T$*. Then we have the following: (i) if every isotone map in* $X, f(x,t) : X \times T \to X$*, has a least fixed point* $x^*(t)$*, then X is a CPO; (ii) if* $f(x,t) : X \to X$ *is isotone on X for each* $t \in T$*, and* $G(t)$ *denotes the set of fixed points of* $f(x,t)$*at t; then if X is a CPO,* $G(t)$ *is a CPO.*

Next, we discuss generalizations of Proposition 1.3.3 to the case of multifunctions. The seminal references are Smithson[77] and Muenzenberger and Smithson[64]. Let $\mathbf{X}$ and $\mathbf{Y}$ be CPOs, $F(x) : \mathbf{X} \to \mathbf{2}^{\mathbf{Y}} \backslash \varnothing$ be a nonempty correspondence, and $X \subset \mathbf{X}$ a subchain. If for any isotone function $f(x) : X \to \mathbf{Y}$ such that $f(x) \in F(x)$, for $x_0 = \sup X$, we have $f(x_0) \leq y(x_0) \in F(x_0)$, we say the mapping $F(x)$ has the property of *Majorizing Chain Subcompleteness (MCSC)*. For correspondences that are ascending in Smithson's weak set relation (C1) or (C2), and that satisfy MCSC, we have the following generalization of Amann[4]:

Proposition 1.3.5. *(Smithson[77], Theorem 1.1): Let X be a CPO, and suppose* $F(x,t)$ *is isotone in the Smithson-weak set relation, (C1) and/or (C2), and satisfies Condition MCSC. If for each* $t \in T$ *there is a point* $x_L(t) \in X$ *and a point* $y \in F(x_L, t)$ *such that* $x_L(t) \leq y$*, then* $F(x,t)$ *has a fixed point for each t.*

Note that Smithson[77] (Proposition 1.6) obtains a generalization of Abian and Brown's[1] fixed point theorem for the case the X is a CPO. In recent work, Heikkilä and Hu [38] and Heikkila and Reffett[40] have generalized it further.

[6] An important converse to the Bourbaki fixed point principle (due to Zermelo) related to the fixed point result in the Abian-Brown-Pelczar theorem is in Jachymski[45].

1.3.2 Computational Fixed Point Theory

Recall that an operator $f(x) : X \to Y$ is *order-continuous* if for any countable chain $\{x_n\}$ having a supremum, we have $\sup f(x_n) = f(\sup x_n)$. If operators are order-continuous in Problem 1, we can weaken the conditions on the transformation space X, and also obtain stronger results on computing extremal fixed points by successive approximation on an operator from lower solutions x_L (e.g., a point x_L that has $x_L \leq f(x_L)$) and upper solutions x_U (e.g., a point x_U that has $f(x_U) \leq x_U$). The successive approximations indexed on the natural numbers can be shown to converge to extremal fixed points. If the underlying space is an ordered metric space, numerical implementations of our methods via Krasnosel'skii et al[48] (Chapter 4) can be shown to provide a *posteriori* error bounds in the underlying metric on X. This is particularly useful in our work, as many of the fixed point spaces we use (the economies studied in Sections 1.4-1.6) have uniform metric topologies.

We next discuss a result due to Kantorovich[47]. This result is available in a number of places in the literature (e.g., Dugundji and Granas[30] Theorem 4.2, Vulikh[86] Theorem XII.2.1, and Davey and Priestley[24] Theorem 8.15). We have the following result for a special case of Problem 1:

Proposition 1.3.6. *(Kantorovich[47]): Let X be a poset, $D = [a, b] \subset X$ countably chain complete. Assume for each $t \in T$, $f(x, t) : X \times T \to X$ is order continuous in x, such that $a(t) \leq f(a(t), t)$ and $f(b(t), t) \leq b(t)$. Let $G(t)$ be the fixed point correspondence of $f(x, t)$ for $t \in T$.Then (i) $G(t)$ is nonempty, and (ii) $\lim_n f^n(a(t); t) \to \inf G(t)$ (respectively, $\lim_n f^n(b(t); t) \to \sup G(t)$).*

An alternative setting that is common in economic applications of Problem 1 has the following structure: (i) the domain $D \subset X$ is a compact, order interval in a normal cone of positive continuous functions $C(X)$, where $X \subset R^n$ is also compact, and (ii) the operator $f(x, t)$ continuous and compact (e.g., completely continuous) in x for each $t \in T$. This is true in case of Coleman[19] and Datta et al[22] for the fixed point problem that constructs MEDPs. In this case, one can apply an important theorem due to Amann[3]:

Proposition 1.3.7. *(Amann[3], Theorem 6.1; corollary 6.2): Let X be an ordered Banach space, $[x_L(t), x^U(t)]$ an order interval with $x_L(t), x^U(t) \in X$, $x_L(t) \leq x^U(t)$, $f(x, t) : X \times T \to X$ is isotone on $[x_L(t), x^U(t)]$, compact and continuous in x, such that for each $t \in T$, $x_L(t) \leq f(x_L(t), t)$ and $f(x^U(t), t)) \leq x^U(t)$. Let $G(t)$ be the set of fixed points of $f(x, t)$ at $t \in T$.Then for each $t \in T$, (i) $G(t)$ is nonempty; (ii) $\lim_{n\to\infty} f_n(x_L(t); t) = \inf G(t)$ and $\lim_{n\to\infty} f_n(x^U(t), t) = \sup G(t)$ and the sequences $\{f_n(x_L(t), t)\}_{n=0}^{\infty}$ and $\{f_n(x^U(t), t)\}_{n=0}^{\infty}$ are increasing and decreasing, respectively.*

For Proposition1.3.6 and Proposition1.3.7, it is important that we obtain sufficient conditions that allow one to tie directly the computation of extremal fixed points to well-known numerical approximation algorithms in the existing literature (e.g., Krasnosel'skii et al[48] and Judd[46]). In some cases, such indexation on the natural number are not sufficient to show that successive approximation from lower or upper solutions for a particular set of fixed points actually computes an extremal fixed point. See the example in Davey and Priestley[24], section 8.16 or Heikkilä and Lakshmikantham[39], example 1.1.1. In these cases, one can define iterations on well-defined index sets that are subsets of chains. Heikkilä and Lakshmikantham [39] address this issue and deliver a generalized iterative method on a chain. A critical advantage of their approach is that it does not require either the axiom schema of replacement or the axiom of choice.

Proposition 1.3.8. *(Heikkila and Lakshmikantham[39], lemma 1.1.1): Let D be the set of subsets of P, P a poset with $\varnothing \in D$ and $f : D \to P$, there is a unique well-ordered chain C so that $x \in C$ if and only if $x = f\{y \in C | y < x\}$. If $f(C)$ exists, it is not a strict upper bound of C.*

We discuss the elements in the chain C.Standard transfinite iterations are contained: let $x_0 = f(\varnothing)$, $x_{n+1} = f(\{x_0, x_1, ..., x_n\})$ for $x_n < x_{n+1}$, $x_w = f(\{x_n\}_{n=0}^{n=\infty})$ with x_w a strict upper bound of $\{x_n\}_{n=0}^{n=\infty}$, then x_w is a next successor element of C, and so forth. When establishing conditions in applications under which the generalized iterations of the mapping f can be indexed on countable sets, it is useful to recall that by Zorn's lemma, if each well-ordered chain C in P has an upper bound in P, then P has a maximal element. From Heikkilä and Lakshmikantham[39], Lemma 1.1.2, we know that each chain of any poset contains a well-ordered cofinal chain. Further, by another lemma in Heikkilä and Lakshmikantham[39] Lemma 1.1.4, a well-ordered chain C in a poset P is countable if its subchains possess countable cofinal chains. Finally, a monotone sequence in an ordered topological space X converges if each of its subsequences has a cluster point. A natural question concerns sufficient conditions under which iterations on f from some lower solution x_L converge to fixed points on a countable indexation of iterations. One set of sufficient conditions are as follows:

Proposition 1.3.9. *Heikkila and Lakshmikantham[39], Lemma 1.1.7; Proposition 1.1.5; Proposition 1.1.6: (i) If a chain C in an ordered topological space X has a separable cofinal subset A, and if each nondecreasing sequence of A has a cluster point in X, then C contains a nondecreasing sequence that converges to $\sup C$; (ii) a well ordered chain of X is countable if the following occurs: (a) X is first countable, and each subchain of C is relatively compact; (b) each subset of C is separable and each nondecreasing sequence of C has a cluster point; (iii) If C is a chain in an ordered metric space X, and if each nondecreasing sequence of C has a cluster point, then C contains a nondecreasing sequence which converges to $\sup C$, and C is countable if each nondecreasing sequence of C has a cluster point.*

1.3.3 Monotone Selections and the Equilibrium Correspondence

In Problem 1 and Problem 2, a natural question to analyze is the existence of monotone comparison theorems on the space of parameters T.[7] Let $G(t) : T \rightarrow 2^X \backslash \varnothing$ denote the fixed point correspondence. We say the fixed point Problem 1 or 2 exhibits a *strong comparative structure (SCS)* if the fixed point correspondence $G(t)$ is an isotone correspondence from $T \rightarrow 2^X \backslash \varnothing$. We say Problem 1 or 2 exhibits a *weak comparative structure (WCS)* if its fixed point correspondence $G(t)$ admits an isotone selection. First, consider the SCS. Known sufficient conditions for $G(t)$ to be consistent with SCS involve the fixed point space X be a complete lattice, ordering the range of $G(t)$ using Veinott's strong set order on $2^X \backslash \varnothing$, and proving that $G(t)$ has a sublattice structure in $2^X \backslash \varnothing$. For example, if $G(t)$ is isotone from T to $2^X \backslash \varnothing$ in Veinott's strong set order, one immediately has the extremal selections $\sup G(t)$ and $\inf G(t)$ as isotone operators on T . The most general version of the result we discuss is due to Veinott[85] (Chapter 4, Theorem 14) and Topkis[83] (Theorem 2.5.2). The Veinott-Topkis Monotone Selection Theorem is stated as follows (see Topkis[83], Theorem 2.5.2 for a proof):

Proposition 1.3.10. *(Veinott[85]; Topkis[83]): Suppose X is a nonempty complete lattice, T a poset, $F(x,t) : X \times T \rightarrow 2^X \backslash \varnothing$ for each $(x,t) \in X \times T$, and assume that the correspondence $F(x,t)$ is isotone in Veinott's strong induced set order on $X \times T$. Let $G(t)$ be the fixed point correspondence of $F(x,t)$ at $t \in T$; then (a) for each $t \in T$, $\sup G(t)$ and $\inf G(t)$ exist; (b) $\sup G(t)$ and $\inf G(t)$ are isotone in $t \in T$; (c) If, in addition, $\sup G(t) < \inf G(t\prime)$ for $t < t\prime$, then $\sup G(t)$ and $\inf G(t)$ are strictly increasing in t on T.*

Second, consider the case of WCS. There are many alternative sufficient conditions under which fixed point problems exhibit WCS. Different forms of sufficient conditions are provided in Veinott[85] and Smithson[77]. We consider some additional isotone selection theorems that prove useful in the study of WCS in economic applications. These theorems apply in cases where the range of the fixed point correspondence does not necessarily possess the sublattice structure required to apply the Veinott-Topkis monotone selection theorem. For the first proposition, instead of assuming that the correspondence is isotone in Veinott's strong set order jointly in (x,t), we assume that $F(x,t)$ is ascending in Veinott's weak set relation in x for each $t \in T$. We also assume that the fixed point correspondence has the following structure: (i) $G(t)$:$T \rightarrow 2^Y \backslash \varnothing$ is a nonempty and chain subcomplete, and (ii) $G(t)$ is ascending in Veinott's weak set order. We now state Veinott's weak monotone selection theorem:

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[7] A well-known reference for monotone comparative statics in economics is Milgrom and Shannon [58]. However, their results built on prior results in operations research and reported in Veinott [84] and Topkis [81]. See Veinott's [85] lecture notes and Topkis [83].

Proposition 1.3.11. *(Veinott[85], Theorem 5) Let X be a lattice, T be a partially ordered set. Assume that $G(t) : T \to 2^X \backslash \varnothing$ is a chain subcomplete correspondence that is ascending in the Veinott's weak set relation. Then, (a) $G(t)$ admits an isotone selection. If, in addition, we assume $G(t)$ is meet- (respectively, join-) sublattice-valued for each $t \in T$, then (b) the isotone selection is $\wedge G(t)$ (respectively, $\vee G(t)$).*

Veinott proves more versions of the above isotone selection theorem assuming stronger hypotheses than (a), e.g., $G(t)$ quasi-sublatticed valued for each $t \in T$, but with weaker hypotheses than assumed for result (b). We present two different set of sufficient conditions for the existence of WCS from Smithson[77].

Proposition 1.3.12. *(Smithson[77], Theorem 1.7): Let X be a partially ordered set, T a set, and let $G(t) : T \to 2^X$ be a nonempty correspondence that is ascending in Smithson's weak set relation (C1) (respectively, (C2)) in (x, t). If, in addition, $\sup G(t) \in G(t)$ (respectively, $\inf G(t) \in G(t)$) for all $t \in T$, the there is an isotone selection, namely $g(t) = \sup G(t)$ (respectively, $g(t) = \inf G(t)$).*

We now define Range Majorizing Condition (RMC) and Range Intersection Property (RIP) that are required for stating the second isotone selection theorem. We say a correspondence $F(x)$ satisfies *Range Majorizing Condition* if for $C = \{z|x_1 \leq z \leq x_2\}$, $x_1 \leq x_2$,when $F(x) \cap C \neq \varnothing$, $\sup(F(x) \cap C) \in F(x) \cap C$.Further, if for any $x_1 \leq x_2$, $y(x_1) \in F(x_1)$ and $y(x_2) \in F(x_2)$ such that $y(x_1) \leq y(x_2)$, and for all $x \in [x_1, x_2], F(x) \cap [y(x_1), y(x_2)] \neq \varnothing$ then we say that $F(x)$ has the *Range Intersection Property (RIP).*

Proposition 1.3.13. *(Smithson[77], Theorem 1.9) Let X be a partially ordered set which contains an element $x^u \in X$ that is a least upper bound in X. If for each $t \in T$, $G(t) : T \to 2^Y$ is nonempty and satisfies conditions MCSC, RIP, and RMC, then $G(t)$ admits an isotone selection.*

We make a final remark on Proposition1.3.12 and Proposition1.3.13. The proofs rely heavily on an application of the Axiom of Choice (namely, the Zorn's lemma). In principle, this can be a serious problem for developing constructive methods that address the question of approximating monotone selections. Recently, alternative methods are developed for the results in Smithson [77] that do not rely upon the Axiom of Choice (see, Jachymski[44], Theorem 2.21). Also, Heikkilä and Reffett[40] develop chain methods for computing particular selections that are not based on either the Axiom Schema Replacement or the Axiom of Choice. In addition, they provide new WCS theorems on the fixed point correspondence in Problem 2. These extensions are important if one wants to avoid the non-constructive nature of the monotone selection results based on applications of the Axiom of Choice.

1.4 An Economy with Classical Technology

We generalize Brock and Mirman[15] to allow for more general "distorted classical" stochastic technologies. In these economies, time is discrete and indexed by $t \in T = \{0, 1, 2, ...\}$. There is a continuum of *ex ante* and *ex post* identical infinitely-lived households. The only form of uninsured risk is an aggregate production function shock. Aggregate production in each state is assumed to be constant returns to scale in private returns. Therefore, the value of all firms is zero in equilibrium. Each period households are endowed with a unit of time which is supplied inelastically in competitive markets. For simplicity, we assume uncertainty comes in the form of a finite state, first-order Markov process denoted by $\theta_t \in \Theta$, with stationary transition probabilities $\chi(\theta, \theta')$.[8] Let the set $\mathbf{K} \subset \mathbf{R}_+$ contain all feasible values for the aggregate endogenous state variable K, i.e., the capital to labor ratio, and define the product space $\mathbf{S} : \mathbf{K} \times \Theta$. Since the household also enters each period with an individual level of the endogenous state variable k, the individual capital to labor ratio, we denote the state of a household by the vector $s = (k, S) \in \mathbf{K} \times \mathbf{S}$.

The preferences are represented by a period utility index $u(c)$, where $c \in \mathbf{K} \subset \mathbf{R}_+$ is period consumption. Letting $\theta^i = (\theta_1, ..., \theta_i)$ denote the history of the shocks until period i, a household's lifetime preference is defined over infinite sequences indexed by date and history $\mathbf{c} = (c_{\theta^i})$ and is,

$$U(\mathbf{c}) = E_0 \left\{ \sum_{i=0}^{\infty} \beta^i u(c_i) \right\},$$

where E_0 is the mathematical expectation with respect to the probability structure of the shocks over the infinite horizon. We impose the following assumption on preferences:

Assumption - P1 :*The utility function $u : \mathbf{K} \mapsto \mathbf{R}$ is bounded, twice continuously differentiable, strictly increasing, strictly concave. In addition, marginal utility, $u'(c)$ satisfies the standard Inada conditions:*

$$\lim_{c \to o} u'(c) = \infty \text{ and } \lim_{c \to \infty} u'(c) = 0.$$

We assume that the output available to the household in the current period can be represented by the function $F(k, 1, K, 1, \theta, t) = f(k, K, \theta; t)$, where t is a parameter that is possibly infinite dimensional (e.g., a continuous mapping that represents distortions and influences technology). We assume that this

[8] The reader should keep two things in mind while reading the results reported in this paper: (i) the results are valid for the case of deterministic nonoptimal growth by setting the shocks to a constant in all states; and, (ii) stochastic optimal growth is obtained as a special case by setting all equilibrium distortions to zero.

production function is evaluated at equilibrium employment levels with $n = N = 1$ and make the following assumptions on technology:

Assumption - T1 The production function $F(k, n, K, N, \theta, t)$ is such that:
(i) $F(k, n, K, N, \theta; t)$ *is constant returns to scale in* (k, n) *for each* (K, N, θ, t) *such that* $F(0, 1, K, 1, \theta, t) = f(0, K, \theta; t) = 0$ *for all* $K \in \mathbf{K}, \theta \in \Theta$ *and* $t \in \mathbb{T}$.
(ii) $f\ (k, K, \theta, t)$ *is twice continuously differentiable, strictly increasing in* (k, K) *and strictly concave in its first argument.*
(iii) $f_1(K, K, \theta; t)$ *is weakly decreasing (i.e., non-increasing) in* K; there exists a $k_0 > 0$ such that $f(k_0, k_0, \theta; t) - k_0 > 0$, and $\beta \int f_1(k_0, k_0, \theta'; t)\chi(\theta, d\theta') \leq 1$ for all (θ, t).
(iv) There exist $\hat{k}(\theta) > 0$ *such that* $f(\hat{k}(\theta), k(\theta), \theta; t) = \hat{k}(\theta)$ *and* $f(k, k, \theta; t) < k$ *for all* $k > \hat{k}(\theta)$ *and for all* $\theta \in \Theta$.

The restrictions on the primitives in Assumptions P1 and T1 are standard. (See, e.g., Coleman [19] for discussion). As we consider some baseline comparative statics issues, we consider the economy studied in Coleman [19]. In this setting, there is a state contingent capital income tax; in addition, we allow for nonconvexities in production in social returns. The distorted reduced-form technology f can be written as follows:

$$f(k, K, \theta; t) = (1 - t_1(K, \theta))g(k, K, \theta) + t_2(K, \theta),$$

where g is also a reduced-from distorted classical production function, the parameters $t_1(K, \theta) : \mathbb{S} \to [0, 1]$ and $t_2(K, \theta)$ can be interpreted as the state-contingent tax and a lump sum transfer, respectively. If we define the standard lexicographic order relation on the set of parameter vectors $t \in \mathbb{T}$ as $t'(K, \theta) \succeq t(K, \theta)$ if either $t_1^{'}(K, \theta) > t_1(K, \theta)$ for all $S \in (K, \theta) \in \mathbf{K} \times \Theta = \mathbf{S}$, or $t_1^{'}(K, \theta) = t_1(K, \theta)$ and $t_2^{'}(K, \theta) \geq t_2(K, \theta)$, then $f(k, K, \theta; t)$ is increasing in t.

We make the following assumption on the nature of distortion:

Assumption - D1: *The functions* $t_1(K, \theta)$ *and* $t_2(K, \theta)$ *are Lipschitz continuous on* $\mathbf{K} \times \Theta = \mathbf{S}$.

In developing our existence arguments, we fix $t \in \mathbf{T}$.(and, for the moment suppress notation). For any given $t \in \mathbf{T}$, define the household's feasible correspondence to be $\Gamma(k, K, \theta)$ where Γ defines the set of actions (c, k') that satisfy the standard budget constraint:

$$c + k' = f(k, K, \theta); \text{ and } c, k' \geq 0.$$

Under Assumption T1, $\Gamma(k, K, \theta)$ is a "well-behaved" nonempty correspondence for each $s = (k, K, \theta) \in \mathbf{K} \times \mathbf{S}$. In particular, as f is continuous and isotone, we conclude that Γ is a non-empty, compact and convex-valued, continuous correspondence for each state s that is ascending in (k, K, θ) for each t in the set inclusion order on $\mathbf{2^{KxK}}$ along an equilibrium restriction where $k = K$ and a balanced budget for the government.

Let $\mathbf{C}(\mathbf{S})$ denote the space of continuous functions $h(S)$:$\mathbf{S}\to\mathbf{K}$ equipped with the standard uniform norm topology (i.e., $\| h \|= \sup_{S\in\mathbb{S}}|h(S)|$) and pointwise Euclidean partial order where $\mathbf{S}$ is a compactum, and let $\mathbf{C}^{+}(\mathbf{S})$ be its cone. To construct the household's decision problem, consider that aggregate capital-labor ratio evolves according to:

$$K' = h(K,\theta) \in \mathbf{C}^{+}(\mathbf{S}), 0 \leq h \leq f,$$

where for any given t, $h(S) : \mathbf{S}\to\mathbf{K}$ is continuous in both its arguments, increasing in K for each θ. The household solves the following dynamic program:

$$J(s) = \sup_{(c,k')\in\Gamma(s;t)} \{u(c) + \beta \int_{\Theta} J(s')\chi(\theta, d\theta')\}. \tag{1.1}$$

Standard arguments prove the existence of a $J \in \mathbb{V}$ that satisfies this functional equation, where $\mathbb{V}$ is a space of bounded, continuous, real valued functions with the sup norm (see, for instance, Stokey, Lucas and Prescott[78]). In addition, under assumptions P1-T1, following the argument in Mirman and Zilcha[61] (lemma 1) J is differentiable in k.

We define an recursive equilibrium as follows:

Definition: *A (recursive) competitive equilibrium for this economy consists of a parameter vector* (t_1, t_2), *a value function for the household* $J(s)$, *and the associated individual decisions* c *and* k' *such that: (i)* $J(s)$ *satisfies the household's Bellman equation (1.1), and* c, k' *solve the optimization problem in the Bellman's equation given* t; *(ii) all markets clear: i.e.,* $k' = h(S) = K'$ *and (iii) the government budget balances.*

1.4.1 The Existence of MEDPs

The second welfare theorem does not apply in this economy. Therefore, the social planning approaches to characterizing MEDPs do not suffice. We adopt an alternative strategy, the so-called "Euler equation approach". To facilitate our construction, we consider a stronger version of Amann's theorem in Proposition 1.3.7, Section 1.3. This result is proved in Morand and Reffett [62] and considers Amann's theorem for isotone transformations of equicontinuous fixed point spaces.[9]

Proposition 1.4.1. *Let* E *be an equicontinuous fixed point space of continuous functions, each defined on a compact set* X, *equipped with the* sup *continuous uniform topology and the pointwise partial Euclidean order. Let* $[\overline{y}, \widehat{y}]$ *be a closed*

[9] Let $\mathbf{B}(X)$ be a bounded subset of the space of continuous functions $\mathbf{C}(X)$ and $E \subset \mathbf{B}(X)$. We say that E is *equicontinuous at a point* $x_0 \in X$ if, for any $\varepsilon > 0$, there is a $\delta > 0$ such that any x in the δ-neighborhood of x_0 we have $\| h(x) - h(x_0) \| \leq \epsilon$ for every $h \in E$ (it is important that δ is independent of h). We say that E is *equicontinuous* if it is equicontinuous at every point of E.

order interval in E. Suppose that $A : [\overline{y}, \widehat{y}] \to [\overline{y}, \widehat{y}]$ is an isotone, continuous map. Then A has a maximal fixed point $\widehat{x}$ and $\widehat{x} = \lim_{n\to\infty} A^n\widehat{y}$, and the sequence $\{A^n\widehat{y}\}_{n=0}^{\infty}$ is decreasing.

Proof: See Morand and Reffett[62], Proposition 2.■

To construct existence of recursive equilibrium, we define a candidate non-linear operator A whose fixed points coincide with a MEDP. The Euler equation associated with the optimal policy function in Bellman's equation (*1.1*) along an equilibrium trajectory where $k = K$ (appealing to the Mirman-Zilcha envelope condition) generates the following necessary and sufficient condition for a recursive competitive equilibrium: the existence of a function $c^*(K, K, \theta) = c^*(K, \theta)$ such that,

$$u'(c^*(K,\theta)) = \beta \int_{\Theta} u'[c^*(F(K,\theta) - c^*(K,\theta), \theta')] r(F(K,\theta) - c^*(K,\theta), \theta')\chi(\theta, d\theta'). \tag{1.2}$$

Here, $F(K, \theta) = f(K, K, \theta; t)$ and $r(K, \theta) = f_1(K, K, \theta; t)$ for notational simplicity.

Definition: $\mathbf{H}^0$ *is the set of consumption functions h such that:*(i) $h : \mathbf{S}\to\mathbf{K}$;(ii) $0 \leq h(K,\theta) \leq F(K,\theta)$ *for all* $(K,\theta) \in \mathbf{S}$;(iii) $0 \leq h(K',\theta) - h(K,\theta) \leq F(K',\theta) - F(K,\theta)$ *for all* $K' \geq K, (K, K') \in \mathbf{K}\times\mathbf{K}$ *and all* θ.

Equip $\mathbf{H}^0$ with the standard sup uniform metric topology; and adopt the Euclidean partial order $\geq$ induced by the cone structure of $\mathbf{C}^+(\mathbf{S})$. That is, $h' \geq h$ if and only if $h'(K,\theta) \geq h(K,\theta)$ for all $(K,\theta) \in \mathbf{S}$. The following lemma summarizes some important properties of the space $\mathbf{H}^0$.

Lemma 1.4.1. *Under assumption T1, (i)* $\mathbf{H}^0$ *is a closed, convex, equicontinuous order interval of continuous function (e.g., a convex, compact, order interval); (ii)* $\mathbf{H}^0$ *is a complete lattice.*

Proof: (i) See Coleman[19] Proposition 3. (ii) See Morand and Reffett[62], Lemma 1.■

To construct a recursive equilibrium, we define a nonlinear operator Ah based on an equilibrium version of the Euler equation. Consider any $h \in \mathbf{H}^0$, $h > 0$, and any (K, θ),

Definition: The operator $Ah(K,\theta) = \{y | y : \text{for } h > 0, u'(y) = \beta \int_{\Theta} u'(h(F-y,\theta'),\theta')r(F-y,\theta')\chi(\theta,d\theta'); \text{ if } h = 0 \text{ in any } (K,\theta), \text{ we set } Ah(K,\theta) = 0\}$.

The following lemma lists a few key properties of the operator A:

Lemma 1.4.2. *Under Assumptions P1 ,T1, and D1:(i) For any $h \in \mathbf{H}^0$, and any (k, θ), there exists a unique $Ah(k, \theta)$;(ii) A maps $\mathbf{H}^0$ into itself (e.g., is a transformation of $\mathbf{H}^0$);(iii) A is continuous on $\mathbf{H}^0$; (iv) there exists a maximal fixed point $h^* \in \mathbf{H}^0$ and the sequence $\{A^n F\}$ converges uniformly to h^*; and, (v) the maximal fixed point is strictly positive.*

Proof: The proofs of (i)-(iii) are in Coleman[19] (Proposition 4). Claim (iv) follows directly from Proposition 1.4.1. Claim (v) follows from a standard dynamic programming argument that is presented in the main theorem in Greenwood and Huffman[34] p 615.■

It is important to note that neither (i), (ii), nor (iii), rely on compactness of the state-space, and are therefore valid under Assumptions P1 and T1 only. We can now state our existence result for MEDPs.

Proposition 1.4.2. *Under Assumptions P1, T1 and D1, there exists a recursive equilibrium.*

Proof: Follows from Lemma 1.4.1 and 1.4.2.■

1.4.2 The Uniqueness of MEDPs

We next consider the uniqueness of MEDPs. Let C^+ be the cone of a real Banach space C, and consider a transformation $A : C^+ \to C^+$. We say an operator $A : C^+ \to C^+$ is *$e-$concave* if there exists non-zero $e \in C^+$, such that (i) for an arbitrary non-zero $c \in C^+$ the inequalities $\alpha e \leq Ac \leq \beta e$, where α and β are positive, are valid and (ii) for every $c \in C^+$ such that $\alpha_1(c)e \leq c \leq \beta_1(c)e$ with $(\alpha_1(c), \beta_1(c)) \gg 0$, and there is a number $\eta(c,t) > 0$ such that $A(tc) \geq (1+\eta)tAc$ for any $t \in (0,1)$. An operator is said to be *pseudo-concave* on C^+ if for all $t \in (0,1), c \in C^+, c > 0, Atc >> tAc$. Let C^+ be a solid cone, the operator $A : C^+ \to C^+$ is *strongly sublinear* if $Atc >> tAc$ for all non-zero $c \in C^+$ and $0 < t < 1$. (See Guo and Lakshmikantham[35], Definition 2.2.2). If A is isotone and strongly sublinear, it is well-known A is e-concave. If an operator that is strongly sublinear on the interior of a solid cone, then it is pseudo-concave. Notice that pseudo-concavity is a weaker condition than $e-$concavity; uniqueness of strictly positive fixed point therefore typically requires stronger conditions on the operator and/or cone. Let $\mathbf{P}$ be a solid cone. Two such related conditions for the operator A to have unique strictly positive fixed points are that of cone compression and k_0-monotonicity. The former is used to guarantee existence of positive fixed points. The latter is used for uniqueness relative to the cone.

An operator $Ah : \mathbf{P} \to \mathbf{P}$ is a *cone compression* on the normal cone $\mathbf{P}$ if their exists a pair of numbers $R, r > 0$ such that

$$Ah \nleqq h, \text{for } h \in \mathbf{P}, \|h\| < r, h \neq 0;$$
$$Ah \ngeqq h, \text{for } h \in \mathbf{P}, \|h\| > R.$$

Let $H \subset \mathbf{C}^{+}(\mathbf{S})$ be an compact, order interval where $\mathbf{C}^{+}(\mathbf{S})$ is the space of positive continuous functions on the compact set $\mathbf{S} = \mathbf{K} \times \Theta$. We say an operator A is k_0-*monotone* on H if it is (i) isotone on H, and (ii) if for any strictly positive fixed point h_1, there exists a $k_0 > 0$, $0 \leq k_1 \leq k_0$ and $h_2 \in H$ such that $h_2 \leq h_1$, for all $k \geq k_1$, and $h_1(k,\theta) \geq Ah_2(k,\theta)$ all $k \geq k_1$, for all θ. Notice if A is k_0-monotone, A is a cone compression.

In our argument, we construct new sufficient conditions for existence of a uniquely strictly interior fixed point. We first construct the operator $\widehat{A}$ as in Coleman[21], but we prove additional properties of this operator that are useful for our argument that are not in Coleman. We then show that the operator is strongly sublinear on its interior and a cone compression (which implies the existence of a strictly positive fixed point. We then show the operator is additionally k_0-monotone, which implies he has a unique strictly positive fixed point.

We first define the set of functions $\mathbf{M}$ as follows

Definition: $\mathbf{M} = \{\ m : \mathbf{R}_{+} \times \Theta \rightarrow \mathbb{R} |$(i) m is continuous, (ii) for all $(K,\theta) \in \mathbf{R}_{+} \times \Theta$, $0 \leq m(K,\theta) \leq F(K,\theta)$ and (iii) for any $K = 0$, $m(K,\theta) = 0\}$

Endow $\mathbf{M}$ with the standard partial pointwise order and the C^0uniform topology. We note that $\mathbf{H}^0$ and $\mathbf{M}$ can be directly related to each other by a simple mapping. For $m \in \mathbf{M}$, consider the function $\Psi(m(K,\theta))$ implicitly defined by,

$$u'[\Psi(m(K,\theta))] = \frac{1}{m(K,\theta)}, \text{ for } m > 0, \text{ 0 elsewhere.}$$

Clearly, Ψ is continuous, increasing, $\lim_{m\rightarrow 0} \Psi(m) = 0$, and $\lim_{m\rightarrow F(K,\theta)} \Psi(m) = F(K,\theta)$.Using the function Ψ, for any $m > 0$, we denote the solution (for y) to the following equation by $\widehat{A}m(K,\theta)$,

$$\widehat{Z}(m,y,K,\theta) = \frac{1}{y} - \beta E_{\theta}[\frac{H(F(K,\theta) - \Psi(y),\theta')}{m(F(K,\theta) - \Psi(y),\theta')}] = 0,$$

and set $\widehat{A}m = 0$ when $m = 0$. Since $\widehat{Z}(m,y,K,\theta)$ is strictly decreasing and continuous in y and $\lim_{y\rightarrow 0} \widehat{Z}(m,y,K,\theta) = \infty$ and $\lim_{y\rightarrow F(K,\theta)} \widehat{Z}(m,y,K,\theta) = -\infty$, for each $m(K,\theta) > 0$,with $K > 0$, and $\theta \in \Theta$, there exists a unique $\widehat{A}m(K,\theta)$.

It is easy to show that to each fixed point of the operator A corresponds a fixed point of the operator $\widehat{A}$. Indeed, consider x such that $Ax = x$ and define $y = \frac{1}{u'(x)}$ (or, equivalently $\Psi(y) = x$). It is also easy to verify that $Am \subset \mathbf{M}$ and is monotone on $\mathbf{M}$. By definition, for all (K,θ), x satisfies,

$$u'(x(K,\theta)) = \beta E_\theta\{H(F(K,\theta) - x(K,\theta),\theta') \times u'(x(F(K,\theta) - x(K,\theta),\theta'))\}.$$

Substituting the definition of y into this expression yields

$$\frac{1}{y} = \beta E_\theta\{\frac{H(F(K,\theta) - \Psi(y(K,\theta)),\theta')}{y(F(K,\theta) - \Psi(y(K,\theta),\theta'))}\},$$

which shows that y is a fixed point of $\widehat{A}$.

We are now prepared to prove our uniqueness result:

Proposition 1.4.3. *Under Assumptions P1, T1, D1, (i) The operator $\widehat{A}$ is strongly sublinear; (ii) $\hat{A}$ has at most one strictly positive fixed point; and, (iii) there exists a unique recursive equilibrium in* $\mathbf{H}^0$.

Proof: (i). First note both $\mathbf{H}^0$ and $\mathbf{M}$ are order intervals in solid cones of continuous functions defined on a compact set. Therefore since $\widehat{Z}$ is strictly decreasing in its second argument, a sufficient condition for strong sublinearity of $\hat{A}m$ on the interior of $\mathbf{M}$ is:

$$\widehat{Z}(tm, t\widehat{A}m, K, \theta) > \widehat{Z}(tm, \widehat{A}tm, K, \theta) = 0. \tag{1.3}$$

By definition,

$$\widehat{Z}(tm, t\widehat{A}m, K, \theta) = \frac{1}{t\widehat{A}m} - \beta E_\theta\{\frac{H(F(K,\theta) - \Psi(t\widehat{A}m(K,\theta)),\theta')}{tm(F(K,\theta) - \Psi(t\widehat{A}m(K,\theta)),\theta')}\},$$

so that,

$$t\widehat{Z}(tm, t\widehat{A}m, K, \theta) = \frac{1}{\widehat{A}m} - \beta E_\theta\{\frac{H(F(K,\theta) - \Psi(t\widehat{A}m(K,\theta)),\theta')}{m(F(K,\theta) - \Psi(t\widehat{A}m(K,\theta)),\theta')}\}.$$

Since Ψ is increasing and $H(K',\theta')/m(K',\theta')$ is decreasing in K',

$$\frac{1}{\widehat{A}m} - \beta E_\theta\{\frac{H(F(K,\theta) - \Psi(t\widehat{A}m(K,\theta)),\theta')}{m(F(K,\theta) - \Psi(t\widehat{A}m(K,\theta)),\theta')}\}$$

$$> \frac{1}{\widehat{A}m} - \beta E_\theta\{\frac{H(F(K,\theta) - \Psi(\widehat{A}m(K,\theta)),\theta')}{m(F(K,\theta) - \Psi(\widehat{A}m(K,\theta)),\theta')}\} = 0,$$

and $\widehat{Z}(tm, t\widehat{A}m, K, \theta) > 0$ so it must be the case that $\hat{A}tm > t\ \hat{A}m$.Therefore, $\hat{A}m$ is strongly sublinear on the interior of its domain.

(ii) As $\hat{A}m$ is strongly sublinear on the interior of its domain, we conclude $\hat{A}m$ is pseudo concave. Given the presence of the Inada condition on technology, a standard argument in Coleman shows $\hat{A}m$ is K_o-monotone. Further by the main theorem in Coleman , we conclude that $\hat{A}m$ has at most a single strictly

positive fixed point. The last question pertains to existence then of strictly positive fixed points. Note that given the definition of $\hat{A}m$, whenever $m > 0$, necessarily $\hat{A}m > 0$; $\hat{A}m < m$. We therefore have $\hat{A}$ a *cone compression* on the interior of its domain. Then Krasnosel'skii and Zabreiko ([**?**], Theorem 46.4), $\hat{A}m$ has a strictly positive fixed point. Therefore by the proposition immediately above, $\hat{A}m$ actually has a unique strictly positive fixed point. Finally, (noting the relationship between the orbits of $\hat{A}$ and A discussed earlier in this section) as we have a unique strictly positive fixed point for $\hat{A}$, namely $m^* > 0$, we must have a unique fixed point for A , say $h^* > 0$. Since $h^* > 0$ implies strictly positive consumption, it is a MEDP.
(iii) As the $\hat{A}m$ has a unique strictly positive fixed point in M, by the definition of $\hat{A}m$ and the fact that $\hat{A}[M]$ is isomorphic to $A[\mathbb{H}^0]$, we conclude there is a strictly positive fixed point $h^* \in \mathbf{H}^0$ with consumption positive in all states $K > 0$, each θ. By a standard argument (e.g., see Le Van and Vailakis[53], Section 5) , interiority of consumption and investment (along with the fact $h^* \in \mathbf{H}^0$) is sufficient in this case to support prices in $l^1_+ \backslash \{0\}$.■

1.4.3 Monotone Comparison Theorems using Euler Equation Methods

In this section, we construct monotone comparison theorems using Euler equation methods. The monotonicity of the mapping A in lemma 1.4.2 can be exploited to derive strong comparative statics (SCS) results on the space of deep parameters $t \in T$ using the selection theorems in Section 1.3. The set of equilibrium is a non-empty complete lattice, so, in the absence of the uniqueness result, comparative statics analysis requires defining orders on both the set of parameters and on the set of equilibrium. We show that the set of equilibrium is ascending in the strong set order of Veinott in t, consequently, we conclude that the minimal and maximal fixed points are also monotonic in t.

Proposition 1.4.4. *Suppose that the assumptions of lemma 1.4.2 and Proposition 1.4.2 are satisfied for each mapping A_t belonging to the set $\{A_t : \mathbf{H}^0 \to \mathbf{H}^0,\ t \in T\}$, where $(T, \geq_T)$ is a poset, and $G(t)$ is the fixed point correspondence of A_t. If A_t is isotone in t, that is if $t' \geq_T t$ implies that, for all x in X, $A_{t'}x \geq A_t x$, then $G(t)$ is ascending in Veinott's strong set order $\geq_s$ on $2^{\mathbf{H}^0}$ and the minimal and maximal fixed points (respectively, $\wedge G(t)$ and $\vee G(t)$) of A_t are isotone mappings into $\mathbf{H}^0$on T.*

Proof: The claims follow from the proof in Morand and Reffett[62], Theorem 2, noting that $G(t)$ is isotone in Veinott's strong set order, a direct implication of Proposition 1.3.10.■

For an application of this result, consider a perturbation in the discount rate β. Since the right side of the Euler equation in (1.2) is increasing in β, as a consequence, the root $y^*(K,\theta,h,t) = A_{t=\beta}(c)$ that defined the operator, is increasing in $\beta \in (0,1) = T$, where T is endowed with the dual order $\geq_T$ on the real line (i.e., $\beta' \geq_T \beta$ if $\beta' \leq \beta$). By Proposition 1.4.4, the maximal and minimal fixed points increase with t (i.e., decrease with β). By Proposition 1.4.3, the set of MEDPs increase in the pointwise strong set order $\geq_{ss}$ and there is a unique isotone selection.

For another application, consider the tax rate $t \in T$, where T is the set of continuous functions $t(K,z) \in [0,1]$ that are monotone in K. Endow T with the standard pointwise Euclidean order for a space of functions, i.e., $t' \geq_T t$ if $t'(K,z) \geq t(K,z)$ for all (K,z). Then $A_{t'}c \geq A_t c$ in the order defined on E and the equilibrium set (the set of fixed points of the operator A_t) is isotone in t the strong set order. Again, by Proposition 1.4.3, we can obtain a unique isotone selection on T from the set of MEDPs.

1.5 An Economy with Nonclassical Technology

We now allow for more general versions of bounded nonconvex production technologies, linear preferences, Markov technology shocks and a role for public policy. By "distorted nonclassical" production technologies, we mean two cases: the reduced-form production function $f(k,K,\theta)$ is such that (i) $f_1(k,K,\theta)$ is not decreasing in k when $k = K$, and/or (ii) f is not necessarily constant returns to scale in private inputs. In (ii), that there is an issue with interpreting exit and entry conditions in the industry within the equilibrium model but we ignore the industry dynamics. Uncertainty (and much of the model) is as before. Preferences and technologies are denoted as before, except we now have weaker assumptions:

Assumption - P2: *The utility index $u(c) \in \mathbf{U}$ where $\mathbf{U}$ consists of all $u(c) : \mathbf{K} \mapsto R$ that are bounded, continuous, strictly increasing, and either strictly concave on $\mathbf{K}$ or linear on $\mathbf{K}$.*

Assumption - T2: *The aggregate production functions $f \in F$, where F consists of isotone functions $f(k,K,\theta)$:$\mathbf{K} \times \mathbf{K} \times \Theta$, each space ordered with pointwise Euclidean order, f is continuous in k and there exists $\hat{k}(\theta) > 0$ such that $f(\hat{k}(\theta),\theta) + (1-\beta)\hat{k}(\theta) = \hat{k}(\theta)$ and $f(k,\theta) < k$ for all $k > \hat{k}(\theta)$ for all $\theta \in \Theta$; and f is twice differentiable its arguments.*[10]

We impose a joint restriction on the curvature of $u(c)$ relative to the complementarity of the equilibrium distortion in $f(k,K,\theta)$. This restriction is used *only* for our methods when $f \in F$ such that $f(K,K,\theta)$ is not concave in K.(See

[10] We also refer to an isotone function as a monotone function.

section 1.6.2 for further discussion of this point, and how this restriction can be eliminated in the case $f(K, K, \theta)$ is concave.)

Assumption - PT1: *The utility index $u \in U$ and the aggregate production technology $f \in F$ are such that $u'(\gamma(K))f_1(k, K, \theta)$ is isotone in K for each function $\gamma(K)$ where $\gamma(K)$ satisfies $0 \le \gamma(K') - \gamma(K) \le f(k, K', \theta) - f(k, K, \theta)$* for $K' \ge K$.[11]

We need a regularity property on the stochastic process of shocks.

Assumption - M1: *The transition matrix $\chi \in \Xi$ is an irreducible Markov process that satisfies the standard Feller property.*

When discussing the long-run properties of a Markovian equilibrium (and equilibrium comparative statics on limiting distributions), it is useful to restrict attention to a subset of economies where we can prove Markovian dynamics are jointly monotone in (K, θ). Therefore, we note the following additional assumptions:

Assumption - PT2: *The class U and F have $u'(\gamma(\theta))f_1(k, K, \theta)$ are isotone in θ for each $\gamma(\theta)$ such that $0 \le \gamma(\theta') - \gamma(\theta) \le f(k, K, \theta') - f(k, K, \theta)$.*[12]

Assumption - M2: *The measure $\chi \in \Xi$ is stochastically increasing (or equivalently, totally positive of order 2).*[13]

The case of optimal growth under uncertainty is embedded in above assumptions. Our results are more general than those obtained for the optimal growth model with Markov shocks in Hopenhayn and Prescott[43]. Although they claim a more general result, a careful reading of their proofs reveals that Hopenhayn and Prescott can *only* claim sufficient conditions for monotone controls in the optimal growth model with Markov shocks when production functions are the fixed-coefficient, Leontief-type.[14] Note that, we can dispense with assumption M1 or M2 for the optimal growth case. Also, if the class of

[11] If one is willing to adopt the slightly stronger complementarity condition related to the one mentioned in Hopenhayn and Prescott [43] (i.e., $u''(c)f_1f_2 + u'(c)f_{12} \ge 0$), we can allow $u(c)$ in assumption P2 to be concave (but not necessarily linear).

[12] This assumption includes the case for Markov shocks mentioned (but not studied) in Hopenhayn and Prescott [43] for the optimal growth model.

[13] See Topkis [83] for a definition of stochastically increasing.

[14] The problem with application of a key theorem in Topkis [83] (Theorem 2.7.6) also arises in Amir [5]. In this paper, if one follows the proofs, one realizes that the author can *only* claim the existence of monotone controls in the nonclassical optimal multisector growth model when production functions are either (i) Leontief or (ii) defined on domains where the inputs are chained. Our approach using generalized envelopes can be applied in the multisector growth model to obtain more general sufficient conditions for monotone controls in multisector models than found in Amir's work.

shocks $\chi \in \Xi$ consists of a collection of independent and identically distributed random variables, then we obtain joint monotonicity for decentralized Markovian equilibrium under weaker conditions. We can completely dispense with Assumption PT2, and we still obtain joint monotonicity of the decentralized MEDPs. For the optimal growth case, we only require $u \in U$ concave, and $f(K,\theta)$ monotone in (K,θ).

1.5.1 The Parameter Space and Household Decision Problems

Consider the existence of MEDPs under the assumptions P2, T2, PT1 and M1. We begin by defining the fixed point space we use to compute MEDPs.

Definition: $\mathbf{C}_1 = \{h |\ 0 \leq h(K,\theta) \leq f(K,K,\theta)\ \forall\ (K,\theta);\ h(K',\theta) - h(K,\theta) \geq 0\ if\ K^{'} \geq K\}$.

Here $h \in \mathbf{C}_1 \subset \mathbf{B(S)}$, $\mathbf{S}$ is a compact partially ordered topological space with the pointwise Euclidean order (and the usual topology on $\mathbf{R}^n$). $\mathbf{B(S)}$ is the set of bounded functions $\mathbf{S^K}$ endowed with the standard pointwise Euclidean order and C^0 uniform topology, and $\mathbf{C}_1$ consists of all positive functions that are isotone in K, and socially feasible, monotone in K.

Assume that households take as given the recursion h on per-capita aggregate capital stock K, which is used to compute future returns on capital (and, therefore, factor prices),

$$K' = h(K,\theta) \in \mathbf{C}_1, 0 \leq h \leq f.$$

If we make additionally assume PT2 and M2, we obtain stronger characterizations of Markovian equilibrium. For that situation, consider the space,

Definition: $\mathbf{C}_2 = \{\ h | h(K,\theta) \in \mathbf{C}_1$ that are jointly isotone in $(K,\theta)\}$.

Clearly $\mathbf{C}_2(\mathbf{S})$ is a closed sublattice of $\mathbf{C}_1 \subset \mathbf{B(S)}$. The spaces $\mathbf{C}_1$ and $\mathbf{C}_2$ are used to find Markovian equilibrium for economies without and with assumptions PT2 and M2, respectively. We next prove a lemma that is useful in constructing a Markovian equilibrium.

Lemma 1.5.1. *Both* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *are convex and subcomplete in* $\mathbf{B(S)}$.

Proof: See Mirman, Morand and Reffett[59], lemma 1. ■

Therefore, $\mathbf{C}_1$ (respectively, $\mathbf{C}_2$) is a natural place to pose the existence of MEDP question.

We now characterize the best response mapping of households facing an aggregate environment $h \in \mathbf{C}_1$, under the assumptions P2, T2, PT1 and M1. Consider a household entering the period in state $p = (p_c, \theta) \in P = \mathbf{K} \times \mathbf{K} \times \Theta$, $p_c = (k, K) \in \mathbf{K} \times \mathbf{K}$, facing an aggregate economy with aggregate dynamics (and prices) summarized by the function $h \in \mathbf{C}_1$. Let consumption and investment be given as $a = (c, y) \in A \subset \mathbf{K} \times \mathbf{K}$. The value function for the household is a function $v^*(p, h)$ that is a solution of the functional equation:

$$v^*(p; h) = \sup_{a \in \Gamma(p)} \{u(c) + \beta \int_{\Theta} v^*(y, h(K, \theta), \theta'; h)\chi(\theta, d\theta')\}, \tag{1.4}$$

where the feasible correspondence $\Gamma(p) = \{a | c + y \ \leq f(p), \, c, y \geq 0\}$. In order to study the existence of a v^* that satisfies the above functional equation, consider the operator B^C:

$$\begin{aligned} B^C v(p; h) &= \sup_{a \in \Gamma(p)} \{u(c) + \\ &\qquad \beta \int_{\Theta} v(y, h(K, \theta), \theta'; h)\chi(\theta, d\theta')\}. \end{aligned}$$

Here the operator B^C is defined on the space $V_c = \{v(p; h) : \mathbf{P} \times \mathbf{C}_1 \rightarrow \mathbf{R}$, v bounded in (k, K, θ, h), isotone in p for each h, continuous in k for each $(K, \theta, h)\}$. Equip V_c with the standard C^0 topology (and the associated uniform metric) and the pointwise Euclidean partial order. V_c is a complete metric space. Lemma 1.5.2 provides a set of results characterizing the unique function v^* that satisfies (1.4):

Lemma 1.5.2. *Under assumptions P2, T2, PT1 and M1, (i) $B^C v \subset V_c$;(ii) there exists a unique $v^* \in V_c$ that satisfies the Bellman equation (1.4); and, (iii) the fixed point v^* is strictly increasing in p for each $h \in \mathbf{C}_1$.*

Proof: A standard argument. See Stokey, Lucas, and Prescott[78].■

We now use lattice programming to further characterize the value function.[15] Define the optimal solution associated with $v^*(p; h)$ by $a^*(p, h)$,

$$\begin{aligned} a^*(p; h) &= \{\arg \sup_{c, y \in \Gamma(p)} \{u(c) + \\ &\qquad \beta \int_{\Theta} v(y, h(K, \theta), \theta'; h)\chi(\theta, d\theta')\}\}. \end{aligned} \tag{1.5}$$

To characterize the optimal solution $a^*(p, h) \subset 2^A$, we define a set of partial orders over choices of consumption c and investment y. The class of partial orders is referred to as "direct value" orders and was pioneered in the work

[15] We assume familiarity in this section with the basic terminology of lattice programming (supermodular functions etc.). Important references are LiCalzi and Veinott [54], Veinott [85], and Topkis [83].

of Antoniadou[7]. To fix ideas, consider the simple two good version of the consumer decision problem. Assume that the relative price is one. Define a collection of direct value orders for unit price for $a = (c, y) \in A \subset \mathbf{K} \times \mathbf{K}$ (denoted by $\geq_{vi}$, where $i \in I$, an index set) as follows: $a, a' \in A$, we say $a' \geq_{vi} a$ if and only if $c^{'} + y' \geq_e c + y$ *and* $a' \geq_{Li} a$. Here $\geq_e$ is referred to as the value quasi-order on A, and $\geq_{Li}$ is the standard lexicographic order defined using the index set $I = \{c, y\}$on $A \subseteq R^2_+$. We use this collection of valuation lattices $(A, \geq_{vi})$ to model the action space for the stochastic growth model $A \subseteq R^2_+$. When indexing the lexicographic order in the valuation order by c, we refer to the resulting lattice, on the commodity space $(A, \geq_{vc})$, as the *consumption value lattice*. We also make reference to the *investment value lattice* when indexing the lexicographic order in the valuation order by investment $(A, \geq_{vy})$. Antoniadou [7] shows that the space $(A, \geq_{vi})$ is (i) a partially ordered set for each $i \in I = \{c, y\}$, and (ii) $\geq_{vi}$ induces a lattice structure on A for each $i = c, y$. Define, $\Gamma(p) = \{a| \ c + y \leq m, c, y \geq 0, m = f(p)\} \subseteq A$ when $(A, \geq_{vi})$ $i = 1, 2$.Under assumptions P2, T2, PT1 and M1, and each index $i = c, y$, the feasible correspondence $\Gamma(p)$ is (i) an isotone mapping $P \to 2^A$ in the strong set order $\geq_a$ endowed with either of the partial orders $i = c, y$; and (ii) it is a nonempty, continuous, compact, convex, and complete sublattice for each $p \in P$.

We turn next to a characterization of supermodular functions on the collection $(A, \geq_{vi})$. In the next lemma, we characterize additively separable supermodular objectives on the direct value lattices $(A, \geq_{vi})$. Let $U(x, y)$: $A \to R$ on the lattice $(A, \geq_{vi})$.

Lemma 1.5.3. *Assume $U(x, y) = u_1(x) + u_2(y)$, where each $u_i(.)$ is isotone for $i = 1, 2$. Then (i) $U(x, y)$ is supermodular (strictly supermodular) on the x valuation lattice $(A, \geq_{vx})$ if and only if $u_2(y)$ concave (strictly concave), (ii) $U(x, y)$ is supermodular (strictly supermodular) on the collection $(A, \geq_{vI})$ for $I = x, y$ if and only if both $u_1(x)$ and $u_2(y)$ are concave (strictly concave).*

Proof: See Mirman, Morand and Reffett[59], Lemma 4. ■

Now, we consider sufficient conditions for monotone controls $a^*(p, h)$ from (1.5) to be isotone in the Euclidean order on A. The parameters of interest are $p_c = (k, K)$ and $h \in \mathbf{C}_1$. A major obstacle to studying the dynamic complementaries in (1.5) is characterizing sufficient conditions for preserving supermodularity under maximization. One set of sufficient conditions for preserving supermodularity under maximization on arbitrary projections to the parameter space is found in Topkis[83] (Theorem 2.7.6). This set of sufficient conditions cannot be applied in growth models with multidimensional parameter spaces as they require the graph of the feasible correspondence to be sublattice valued in the powersets of $A \times P$; a condition not available in growth models unless the production function is Leontief. We, therefore, do not follow this line of argument. We develop results on generalized envelope conditions found in the literature on nonsmooth analysis. See Clarke[17] (chapter 2) and Rockafellar and Wets[71]. This approach is used in Askri and Le Van[8] who study envelope theorems in the multisector optimal growth model with nonclassical technologies. Unfortunately, however, their results only apply to economies for which the optimal solutions are strictly interior. In our framework, their methods cannot be directly applied. We extend Askri and Le Van[8] results to economies without boundary restrictions, such as Inada conditions. Our method is based on Gauvin and Dubeau[33].

Let $p \in P$. Note that P is a convex sublattice. Consider the subspace of value functions $V(p) \subset V_c$ consisting of the $v(k, K, \theta, h) \in V_c$ with the following additional restrictions:

(i) $v(p)$ is supermodular in $p_c = (k, K) \in P_c$ for each θ;

(ii) $v(k, K, \theta, h)$ Lipschitz in k with the Lipschitz constant,

$$L = \sup_{c,k,K,\theta,h} |\{u'(c)f_1(k, K, \theta), u'(0)f_1(k, K, \theta) + \varepsilon|,$$

where $\varepsilon = \beta \int u'(f(k, h(K, \theta), \theta'))f_1(k, h(K, \theta), \theta')\chi(\theta, d\theta') - u'(0)$. The subset V is a closed subset of the complete metric space of functions V_c.Also, recall that supermodularity is closed under pointwise limits (see Topkis[83], lemma 2.6.1). We have the following monotonicity result,

Proposition 1.5.1. *Let us assume P2, T2, PT1 and M1 and let $v \in V(p)$. Then (i) the optimal solution $a^*(h; p)$: $\mathbf{C}_1 \rightarrow 2^A$ is ascending in h in the strong set order $\geq_a$ on the investment valuation lattice $(A, \geq_{vy})$; and, (ii) the maximal and minimal selections for investment $a_y^u(h; p) = \max_y a^*(h; p)$ and $a_y^l = \min_y a^*(h; p)$ are isotone functions from $\mathbf{C}_1 \rightarrow A$.*

Proof: See Mirman, Morand and Reffett[59], Theorem 5. ■

Notice that monotonicity on the investment lattice $(A. \geq_{vy})$ implies that investment monotonicity on the Euclidean lattice $(A, \geq_E)$. Proposition 1.5.1 implies that the extremal selections of the best response map are monotone on the space $\mathbf{C}_1$ for each (k, K, θ).Corollary 1.5.1 shows that the extremal selections form self maps to the space $\mathbf{C}_1$:

Corollary 1.5.1. *Assume P2, T2, PT1 and M1, let $v^* \in V$ in equation* **??** *and for each $\theta \in \Theta$; then for $h \in \mathbf{C}_1$ (i) the optimal solution $a^*(p_c, \theta; h)$ is ascending from $P_c \rightarrow 2^A$ in the strong set order $\geq_a$ on the investment valuation lattice $(A, \geq_{vy})$; and, (ii) the minimal and maximal selections for investment $a_y^u(p_c, \theta; h) = \max_y a^*(p_c, \theta; h)$ and $a_y^l = \min_y a^*(p_c, \theta; h)$ are isotone functions from $P_c \rightarrow A$.Under additional assumptions PT2 and M2, and for $h \in \mathbf{C}_2$,(iii) the optimal solution $a^*(p; h)$ is ascending from P to 2^A in the strong set order $\geq_a$ on the investment valuation lattice $(A, \geq_{vy})$; and, (iv) the minimal and maximal functions for investment $a_y^u(p) = \max_y a^*(p)$ and $a_y^l = \min_y a^*(p)$ are isotone functions from $P \rightarrow A$.*

In the proof of Proposition 1.5.1 and Corollary 1.5.1 in Mirman, Morand and Reffett[59], they also prove a new envelope theorem that generalizes the result in Mirman and Zilcha [61], Amir, Mirman and Perkins[6], and Askri and Le Van[8]. With this envelope, it is straightforward to check that the right side of (1.4) at a solution v^* has all the requisite complementary structure to obtain isotone increasing controls in Veinott's strong set order $\geq_s$ (namely, the requisite increasing differences between the controls and the parameters). Given that this new generalized envelope is of independent interest, we present the argument for its existence.

We need to define some terms. A correspondence $\Gamma(p)$ is said to *uniformly compact near* p if there is a neighborhood $N(p)$ of p such that the closure of $\cup_{p' \in N(p)} \Gamma(p')$ is compact. Given the continuity of f in p for economies $\Delta \in E$, one can prove that the feasible correspondence on (1.4), $\Gamma(p)$, is uniformly compact near p. Rewrite the constraints in (1.4), more generally, as $\Gamma(p) = \{a | g(a,p) \leq 0\}$ where $g(a,p)$ is the set of implicit constraints defined in (1.4). We say a pair $(a,p) \in gr\Gamma(p)$ satisfies the *Mangasarian-Fromowitz regularity conditions (* or, are *MF-regular)* if there exists a direction $r \in R^2$ such that the Jacobian $\nabla_a g(a,p) r < 0, g(a,p) = 0$.[16] Here $gr\Gamma(p)$ is the graph of $\Gamma(p)$. In our problem, the constraints are additively separable with constant gradients in the controls, for any pair $(a,p) \in A \times P$; therefore each point (a,p) is MF regular. Therefore, any optimal solution $(a^*(p,h),p) \in gr\Gamma(p)$ is a MF-regular point. Further, because these coefficients do not change as a function of a, we also note that we have a stronger constraint qualification present, namely that the basis elements $\nabla_a g(a^*(p,h),p)$ are linearly independent. Therefore, our problem also satisfies the so-called "linear independence" *(LI)* constraint qualification discussed in Gauvin and Dubeau[33].

Next, note a few properties of Bellman operator B^c. Let $v \in V$. We know that the feasible correspondence $\Gamma(p) : P \to 2^A$ is a continuous, strong set order ascending correspondence in $p = (k, K, z)$ for each $h \in \mathbf{C}_1$. Further, for each $p, \Gamma(p)$ is nonempty, compact, convex and subcomplete in $(A, \geq_{vi})$ for $i = c, y$. As $u(c)$ is Lipschitz (as its C^1 with bounded gradient on any eighborhood of $\mathbf{K}$ that is strictly interior), and the sum of two Lipschitz functions is Lipschitz, we conclude that the objective is Lipschitz in (c,y) for each (p,h). By a standard application of Berge's maximum theorem[11] p.116), the value function $B^C v$ is continuous in k, and the optimal solutions $a^*(p,h)$ form a nonempty, compact-valued correspondence for each (p,h). Noting the continuity of the objective, a^* is also upper hemi-continuous correspondence in k. As the order on P pointwise Euclidean, when P is endowed with the standard metric/topology, P is a Banach lattice with a continuous lattice structure. Also note that $(A, \geq_{vi})$ $i = c, y$, A has a continuous lattice structure, and A=$\mathbf{K} \times \mathbf{K}$ is Hausdorff. Therefore, by Debreu[26], the optimal solutions $a^*(p) : P \to 2^A$

[16] As all constraints are inequalities, we are writing that MF regularity constraint qualifications for a problem with only inequality constraints, i.e, we do not require for all binding constrants, say $h(a,p) = 0$, to satisfy that the direction r is othrogonal to $\nabla_a h(a,p)$ where h is the collection of all the equality constraints.

is are upper-measurable. (See also Hopenhayn and Prescott[43] for discussion of upper-measurability).

We next prove that the value function is locally Lipschitz under our assumptions in this section. This result is used to prove under our assumptions, $B^c v \in V$.

Proposition 1.5.2. *The Bellman operator,* $B^c : P \times \mathbf{C}_1 \to R$, *is locally Lipschitz near* $k > 0$, *for each* (K, z, h) *and* $v^*(k, K, \theta, h)$ *is Clarke differentiable in its first argument for each* (K, θ, h).

Proof: We have two cases.

Case 1: The optimal solutions $a^*(p,h)$ are strictly interior in $A = \mathbf{K} \times \mathbf{K}$; i.e., for all $a(p,h) \in a^*(p,h), a(p,h) \in int(R^2_+)$

By a result in Amir, Mirman, and Perkins[6] (lemma 3.3) left and right Dini derivatives exist in k for each (K, z, h) and are bounded. By Rockafellar[70] (Proposition 5), $B^c v$ is therefore locally Lipschitz with a upper estimate of the Lipschitz modulus of $L_v(p,h) = \sup_{p,k>0}\{B^{c+}v, B^{c-}v\} \leq L$ where for example B^{c+} is the right Dini at (p,h), $k > 0$.

Case 2: The optimal solutions $a^*(p,h)$ is such that there is an $a(p,h) \in a^*(p,h)$ not interior.

Using a standard Lagrangian approach, the operator $B^c v$ is given as follows: for $h \in \mathbf{C}_1$,

$$\begin{aligned} B^c v &= \sup_{a,\lambda,\varphi_c,\varphi_y} L(a,p,h) \\ &= \sup_{a,\lambda,\varphi_c,\varphi_y} u(c) + \beta \int v(y, h(K,\theta), \theta')\chi(\theta, d\theta') \\ &\quad +\lambda(f - c - y) + \varphi_c c + \varphi_y y \end{aligned} \tag{1.6}$$

where $\lambda, \varphi_c, \varphi_y$ are the multipliers associated with the respective constraints that define $\Gamma(p) = \{a | c + y \leq f(k,K,z), c \geq 0, y \geq 0\}$. As (i) each element of $(a^*(p,h), p)$ is *MF-regular* such that it also satisfies the condition *(LI)* and (ii) the primitive data of the problem is Lipschitz, by corollary 4.4 in Gauvin and Dubeau[33], $B^c v$ has bounded right and left Dini derivatives in k with $B^{c+}v_k(k,K,z,h) = \max_{a \in a^*(p)} \nabla^+_k L(a,p,h) \leq L$, and $B^{c-}v_k(k,K,z,h) = \max_{a \in a^*(p)} \nabla^-_k L(a,p,h) \leq L$ for $k > 0, p \in P$. Then by Gauvin and Dubeau[33] (Theorem 5.1), $B^c v$ is locally Lipschitz in $k > 0$, $p \in P, h \in \mathbf{C}_1$ (see also Rockafellar[70], Proposition 5).■

This generalized Clarke envelope is a critical step: the economies that satisfy assumptions P2, T2, PT1 and M1, the value function $v^*(k,K,\theta,h)$ has increasing differences in $(k; K, h)$ for each θ. If, in addition, we assume PT2 and M2, then we obtain v^* also having increasing differences in $(k; \theta)$.

1.5.2 The Existence of MEDPs

We prove the existence of a complete lattice of Markovian equilibrium. Noting the dependence of best responses on the environment (in the next section we conduct monotone comparative statics on the space of environments), we denote a correspondence,

$$Th(K,\theta) =$$
$$\{a(K,K,\theta;h)|a \text{ any monotone selection for investment in } a_y^* \text{ in (1.5)}\}$$

We state some useful properties of the correspondence Th. In particular, we focus on the sublattice structure of its range:

Lemma 1.5.4. *Under assumptions P2, T2, PT1 and M1, $Th \subset \mathbf{C}_1, Th$ is ascending on $\mathbf{C}_1$ in the strong set order $\geq_a$ to 2^{C_1} and is complete-latticed valued; with additional assumptions PT2 and M2, $Th : \mathbf{C}_2 \to 2^{\mathbf{C}_2}$ is ascending in the strong set order $\geq_a$ and T is complete lattice valued.*

Recalling the Veinott-Zhou version of Tarski's theorem in Proposition 1.3.2, we obtain our first result (the proof follows directly from Lemma 1.5.4 and Proposition 1.3.2),

Proposition 1.5.3. *Under the assumptions P2, T2, PT1 and M1, the set of fixed points φ_T^* is a nonempty complete lattice in $\mathbf{C}_1$; with additional conditions PT2 and M2, the set of fixed points φ_T^* is a nonempty complete lattice in $\mathbf{C}_2$.*

1.5.3 Monotone Comparison Theorems via Lattice Programming Methods

We first point out straightforward monotone comparison results with respect to changes in the discount rate and shock process. Consider ordered perturbations of the discount rate β and/or uncertainty $\chi \in \Xi$ (where the ordered perturbation of measure χ take place in a setting of first order stochastic dominance). Using variations of existing arguments (e.g., Amir, Mirman and Perkins[6] (Theorem 5.1) and Hopenhayn and Prescott[43] (corollary 7) for perturbations in β and χ, respectively), we obtain a Veinott strong set order monotone comparative statics result in the pointwise Euclidean order from the extremal selections of agent investment decisions for investment $a_y^*(p,h;\beta,\chi)$, under assumptions P2, T2, PT1 and M1. Then by the Veinott-Topkis SCS theorem, we obtain Veinott strong-set order fixed point correspondence comparison with the operator Th by $\varphi_T^*(\beta,\chi)$ and have the SCS via Proposition 1.3.10, Section 1.3. We conclude that the fixed point correspondence $\varphi_T^*(\beta,\chi)$ exhibits strong set order comparative statics, i.e., $\varphi_T^* : (0,1) \times \Xi \to 2^{\mathbf{C}_1}$ is a strong set order increasing correspondence.

To study monotone comparative statics with respect to the space of reduced-form distorted production functions, our argument requires the development of a set of partial orders that is suitable for ordering the envelope conditions for agents' decisions. This partial order involves "gradient monotonicity" conditions. Infinite dimensional single crossing properties relative to a space of payoff functions for a collection of parameterized dynamic programs have been studied by Lovejoy[55]. Consider the order on the space of technologies F : $f' \geq_F f$ when $u(f'(k,K,z)) - u(f(k,K,z))$ is increasing in k, for each (K,z) with $f' - f = 0$, when $k = 0, (k,K,z) \in P$, and P is compact.[17] Observe the following: (a) $(F, \geq_F)$ is a partially ordered space (antisymmetry follows given f vanishes at zero), (b) $f' \geq_F f$ implies $f'(p) \geq f(p)$ for all p in the pointwise Euclidean order, and, (c) $f' \geq_F f$ implies that the gradients, $\partial_k f'(p) \geq \partial_k f$ (p), are pointwise ordered in the Euclidean order.

Proposition 1.5.4 provides some monotone comparison results for MEDPs (and stationary Markovian equilibrium). We have examples of SCS and WCS; namely SCS on the set of MEDPs, and WCS on the set of invariant distributions. We note that so far, the existence of *measurable* MEDPs has not been addressed. We need to address this question prior to discussing the structure of Markov operators used to study invariant distributions. Let $(\mathbf{K} \times \mathbf{Z}, \mathcal{K} \times \mathcal{Z})$ be a Borel measurable space where $\mathcal{K} \times \mathcal{Z}$ is the set of Borel subsets of $\mathbf{K} \times \mathbf{Z}$. Let $\mathcal{C}_1$ (respectively, $\mathcal{C}_2$) be the the space of $h \in \mathbf{C}_1$ (respectively, $h \in \mathbf{C}_2$) such that $h(K,z)$ is jointly measurable. By a standard result (e.g., Halmos [36], Section 20, Theorem A), if X_c is any countable subset of $\mathbf{C}_1$ (respectively, $\mathbf{C}_2$), then $\vee X_c$ and $\wedge X_c$ are in $\mathbf{C}_1$ (respectively, $\mathbf{C}_2$). Therefore, we conclude that $\mathcal{C}_1$ (respectively $\mathcal{C}_2$) are $\sigma-$complete lattices. Using the optimal solutions in (1.5) that solve the agents dynamic programs in (1.4), define operators based upon the extremal selectors; namely, $A^u h = \sup_h a^*(p;h)$ (respectively $A^l h = \inf_h a^*(p,h)$). We remark these are both well-defined isotone and measurable operators on $\mathcal{C}_1$ (respectively $\mathcal{C}_2$) where isotonicity follows from Proposition 1.5.1 and measurability follows from Hopenhayn and Prescott ([43], Proposition 2) (when restricting the domain of each extremal operator to $h \in \mathcal{C}_1$ and $h \in \mathcal{C}_2$, respectively). We also note that by an argument in Mirman, Morand and Reffett [59], $A^u h$ and $A^l h$ are both order continuous operators on sequences in $\mathcal{C}_1$ (respectively, $\mathcal{C}_2$). Then by Proposition 1.3.6, successive approximations on $A^u(f)$ (respectively, $A^l(0)$ converges in order to the maximal (respectively, minimal) fixed point of Th in (1.6) (when Th is restricted to $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively). We can use these extremal fixed points (which are appropriately measurable) to conduct monotone comparative dynamics.

As a prerequisite to stating our comparison results, we define a few terms that are useful in characterizing the order theoretic properties of the random

[17] Note that the partial order defined with respect this difference is increasing in each component of p. We fix (K,z), and emphasize the role of k in our discussion below.
Also, similar orders can be developed to obtain monotone controls in consumption, relative to the space of production function by developing the obvious dual argument using the dual order relative to capital.

dynamical systems. Let $\mathbf{M}(\mathbf{K}\times\mathbf{Z})$ be the space of finite measures on $\mathbf{K}\times\mathbf{Z}$, endow $\mathbf{M}$ with the stochastic dominance partial order, that is $\lambda' \geq_M \lambda$ if for every monotone, measurable, nonnegative, and bounded function $f : \mathbf{K}\times\mathbf{Z}\to R_+$, $\int f\lambda'(dk\times d\gamma) \geq \int f\lambda(dk\times d\gamma)$. Hopenhayn and Prescott[43] (Proposition 3) show that when this order is restricted to the space of monotone, measurable, bounded, and nonnegative functions, $(\mathbf{M}, \geq_M)$ is a partially order set under the stochastic dominance order $\geq_M$. When viewed from a topological perspective, Dudley[29] (Proposition 11.3.2) provides a metric under which $\mathbf{M}$ is a compact metric space. Let $(\mathbf{K}\times\mathbf{Z},\mathcal{B}(\mathbf{K})\times\mathcal{B}(\mathbf{Z}))$ be measurable spaces where $\mathcal{B}(.)$ denotes the Borel measurable subsets. Consider the adjoint operator $J(\lambda;h) : \mathbf{M}(\mathbf{K}\times\mathbf{Z})\times\mathcal{C}_2 \to \mathbf{M}(\mathbf{K}\times\mathbf{Z})$ defined as,

$$J(\lambda;h)(A\times B) = \int I_A(h(k,z))\chi(z,B)\lambda(dk\times dz), \tag{1.7}$$

where I_A is the indicator function for a measurable set $A\in\mathcal{B}(\mathbf{K})$, $B\in\mathcal{B}(\mathbf{Z})$**.** For each $h\in\mathcal{C}_2$, define the fixed point correspondence for the operator J $(\lambda;h)$ to be $\Psi_J^*(h) = \{\lambda\in\mathbf{M},\lambda = J(\lambda,h)\}$. Define $\lambda_m(h) = \min\Psi_J^*(h)$, and let $\varphi_J^*(f)$ be the set of invariant distributions associated with the set of Markovian equilibrium $\varphi_T^*(f)$, for any production function $f\in F$.We have,

Proposition 1.5.4. *Assume P2, T2, PT1 and M1, let $f\in(F,\geq_F)$. Then (i) the correspondence of Markovian equilibrium, $\varphi_T^*(f) : F\to 2^{C_1}$ is ascending in the strong set order $\geq_a$ .Further, with additional assumptions PT2 and M2, (ii) the set of equilibrium invariant distributions $\varphi_J^*(f) : F\to 2^M$ is ascending in both (C1) and (C2) of Smithson's-weak set relation $\geq_{as}$and therefore admits a monotone selection on F; and (iii) the dynamics exhibit monotone comparative dynamics in the Smithson-weak set relations (C1) and (C2).*

Note that standard arguments can be used to prove the existence of an invariant distribution for a Markovian equilibrium in $\varphi_T^*(f)$. The main contribution of Proposition 1.5.4 concerns comparative dynamics results on the space of equilibrium correspondence. The problem of ruling out limiting distributions that do not have ergodic sets on a strictly positive support is nontrivial. We leave further characterization of a stationary Markovian equilibrium for future work. Note that, isotone selections in $\varphi_J^*(f)$ exist as one can easily check the conditions of Smithson's weak isotone selection theorem discussed in Section 1.3, Proposition 1.3.12.

1.6 An Economy with Elastic Labor Supply

We revisit the model with classical technology (Section 1.4) and allow for elastic labor supply. This model is formulated as in Datta, Mirman and Reffett[22]. As in the previous sections, we consider a continuum of household/firms populating the economy. Uncertainty and market structure are also similar to that in Sections 1.4 and 1.5 but the household cares about leisure. For each period and state, preferences are represented by a period utility index $u(c_i, l_i), (c_i, l_i) \in R_+ \times [0,1]$. Letting $\theta^i = (\theta_1, ..., \theta_i)$ denote the history of the shocks until period i, the households lifetime preferences are additively separable and defined over infinite sequences indexed by dates and histories,

$$U(\mathbf{c}, \mathbf{l}) = E_0 \left\{ \sum_{i=0}^{\infty} \beta^i u(c_i, l_i) \right\}.$$

Here E_0 is the expectation with respect to the probability structure of future histories of the shocks θ^i given the transition matrix χ. The period utility function $u : R \times [0,1] \mapsto R$, satisfies,

Assumption - P3: *The period utility index $u(c, l)$ is such that:*

(i) $u(c, l)$ is continuously differentiable, strictly increasing, and strictly concave in (c, l).

(ii) The partial derivatives $u_c(c, l)$ and $u_l(c, l)$ satisfy the Inada conditions:

$$lim_{c \to 0}\ u_c(c, l) = \infty,\ lim_{c \to \infty} u_c(c, l) = 0,\ lim_{l \to 0} u_l(c, l) = \infty.$$

(iii) The second partials satisfy,

$$\frac{u_{cc}}{u_c} \leq \frac{u_{lc}}{u_l}, \frac{u_{ll}}{u_l} \leq \frac{u_{cl}}{u_c}.$$

The assumptions on period utility are standard. (See Datta et al[22] and Le Van and Vailakis[53] for discussion of this assumption). Note that condition P3(iii) can be thought of as "normality". It also means that the marginal rate of substitution $\frac{u_l}{u_c}$ is non-decreasing in c and $\frac{u_l}{u_c}$ is non-increasing in l. And this is slightly stronger than quasi-concavity of the period utility function (we assume it to be strictly concave) because it implies,

$$u_c^2 u_{ll} + u_l^2 u_{cc} \leq 2 u_c u_l u_{cl},$$

which is a necessary condition for quasi-concavity. This condition is automatically satisfied if $u_{cc}(c, l) < 0$, $u_{ll}(c, l) < 0$ and $u_{cl}(c, l) \geq 0$. If the cross-partial is negative, the condition restricts its magnitude.

Each household is endowed with a unit of time, and enters into a period with an individual stock of capital k. We assume a decentralization where firms do not face dynamic decision problems. Households own the firms as well as both the factors of production, and they rent these factors of production in competitive markets. In addition, to allow for externalities in the production process, as in previous sections, we assume that the production technologies of the firms to depend on per capita aggregates. Assume that technology satisfies,

Assumption - T3: *The production function $f : \mathbf{K} \times [0,1] \times \mathbf{K} \times [0,1] \times \Theta \to R$ satisfies,*

(i) $f(0, 0, K, N, \theta) = 0$ for all $(K, N, \theta) \in \mathbf{K} \times [0,1] \times \Theta$,

(ii) $f(k,n,K,N,\theta)$ is continuous, increasing, differentiable; in addition, it is concave and homogeneous of degree one in (k,n).

(iii) $f(k,n,K,N,\theta)$ also satisfies the standard Inada conditions in (k,n) for all $(K,N,\theta) \in \mathbf{K} \times [0,1] \times \Theta$; i.e.,

$$\begin{aligned} \lim_{k \to 0} f_k(k,n,K,N,\theta) &= \infty, \\ \lim_{n \to 0} f_n(k,n,K,N,\theta) &= \infty, \\ \lim_{k \to \infty} f_k(k,n,K,N,\theta) &= 0. \end{aligned}$$

(iv) There exists a $\hat{k}(\theta) > 0$, such that $f(\hat{k}(\theta),1,\hat{k}(\theta),1,\theta) + (1-\beta)\hat{k}(\theta) = \hat{k}(\theta)$ and $f(k,1,k,1,\theta) < k$ for all $k > \hat{k}(\theta)$, for all $\theta \in \Theta$.

Assumption T3 is standard in the stochastic growth literature (see Brock and Mirman[15]). With the initial stock k_0, we can define $\bar{k} = \max\{k_0, \sup_\theta \hat{k}(\theta)\}$ and the state space for the capital stock and output can be defined on the compact set $\mathbf{K} \subseteq [0,\bar{k}]$. Let $\mathbf{K}_+$ denote the set of strictly positive values for k.

1.6.1 The Household Decision and Equilibrium

Imagine a consumer faced with a choice problem of a single good and leisure in the first stage. The objective is to maximize the difference between the level of utility and the expenditure to obtain that level of utility (see Topkis[83]). Normalizing on the price of consumption goods, consumers take the price of leisure $w(K,\theta)$, the level of per capita consumption C, and the per capita leisure level $L(C,K,\theta)$, as given. Here $C \in \mathbf{K}$, $w : \mathbf{K} \to R_{++}$, $L : \mathbf{K} \times \mathbf{S} \to [0,1]$, and L is a continuously once-differentiable function, and as in previous sections, $\mathbf{S} := \mathbf{K} \times \Theta$.Given w, the household solves,

$$\upsilon(C,K,L,\theta)) = \sup_{l \in [0,1]} \frac{u(C,l)}{u_c(C,L)} - wl,$$

for each $(C,K,L,\theta) \in \mathbf{K}^2 \times [0,1] \times \Theta$. Given the assumption P3, standard arguments using the Theorem of the Maximum, establish that the value function υ is well-defined and continuous (e.g., see Berge[11], p.115). Further, by the strict concavity of period utility in P3, the optimal policy correspondence associated with υ is a singleton. The necessary condition for this first-stage maximization problem is,

$$\frac{u_l(C, l^*(C,K,\theta))}{u_c(C,L)} = w(K,\theta).$$

To finish our description of the first stage, we need to determine equilibrium factor prices as functions of the aggregate state variable. We do this from the representative firm's static production problem. Assume that firms maximize profits under perfect competition, i.e., the firms maximize profits subject to given factor prices, say $\bar{r}(K,\theta)$ and $\bar{w}(K,\theta)$, the rental rate for capital and the wage rate, respectively. The factor prices are continuous functions of the aggregate state variable. The representative firm's maximum profit is,

$$\Pi(\bar{r},\bar{w},K,N,\theta) = sup_{k,n} f(k,n,K,N,\theta) - \bar{r}k - \bar{w}n$$

where anticipating the standard definition of competitive equilibrium, we set $k = K$ and $n = N(S)$, for $S \in \mathbf{S}$.

In the second-stage, the household solves a dynamic capital accumulation problem. To describe this problem, we parameterize the aggregate economy facing a typical decision maker. Define to be the space of bounded, continuous functions with domain $\mathbf{S}$ and range $\mathbf{R}_+$. To parameterize the household's decision problem, we first describe the aggregate economy.

If the aggregate per capita capital stock is K, then households assume a continuous function for per capita labor supply $0 \leq N(S) \leq 1$, and a recursion of the capital stock K' is given by,

$$K' = h(S);\ h \in \mathbf{C}^+(\mathbf{S}), 0 \leq h \leq f(K, 1-N(S), \theta)$$

where $\mathbf{C}^+(\mathbf{S})$ is as before the space of positive continuous functions on $\mathbf{S}$ with the uniform topology. Using the solution to the household's first stage decision problem (and, imposing equilibrium on the labor market), define the per capita aggregate labor supply $N(S) = 1 - l^*(C,K,\theta)$. Then the aggregate economy consists of functions $\Omega = (w,r,h,C,N)$ from a space of functions with suitable restrictions needed to parameterize the household's decision problem in the second-stage. Assume that the policy-induced equilibrium distortions have the following standard form,

$$r = [1-\pi_k(S)]\bar{r},\ w = [1-\pi_n(S)]\bar{w},$$

where $\pi = [\pi_k, \pi_n]$ is a continuous mapping $\mathbf{S} \rightarrow [0,1) \times [0,1)$. We assume regularity conditions on the distorted prices,

Assumption - D2 *:The vector of distortions $\pi = [\pi_k, \pi_n]$ is such that the distorted wage $w = (1-\pi_n(K,\theta))\bar{w}$ and the distorted rental rate $r = (1-\pi_n(K,\theta))\bar{r}$ satisfy,*

(i) $w : \mathbf{K} \times \Theta \rightarrow \mathbf{R}_+$ is continuous, at least once-differentiable and (weakly) increasing in K,

(ii) $r : \mathbf{K}_+ \times \Theta \rightarrow \mathbf{R}_+$ is continuous and decreasing in K such that,

$$\lim_{K\rightarrow 0} r(K,\theta) \rightarrow \infty.$$

In other words, we assume that the distorted wage and rental rates behave as the non-distorted rates $\bar{w}$, $\bar{r}$ or the marginal products of labor and capital, respectively. Assumptions D2(i) and P3(iii) imply that leisure increases with higher consumption and decreases with larger capital accumulation.

Next define the lump-sum transfer to each agent, $d(S) = \pi_k K + \pi_n N(K,\theta)$. Then household's total income is $y(s) = rk + wN + \Pi + d(s)$ where s is the individual household's state, $s = (k, S) = (k, K, \theta)$ and Π is profit. Note that under assumptions P3, T3 and D2, $y(s)$ is a continuous function. We next define the household's feasible correspondence, $\Psi(s)$, which consists of the set (c, k') $\in \mathbf{R}^2_+$ that satisfy,

$$c + wl^*(C, K, \theta) + k' = y,$$

given $(k, K, \theta) \gg 0$. Notice that $\Psi(s)$ is well behaved. In particular since Π is continuous, Ψ is a non-empty, compact and convex-valued, continuous correspondence.

Next, we state the second stage decision problem for the household. At the beginning of any period the aggregate state for the economy is given by $S \in \mathbf{S}$. Each household enters the period with their individual capital stock $k \in \mathbf{K}$, so their individual state is $s \in \mathbf{K}\times\mathbf{S}$. Then the households dynamic decision problem is summarized by the Bellman equation,

$$v(s) = sup_{(c,k')\in\Psi(s)} u(c, l^*(C, K, \theta)) + \beta \int_{\Theta} v(s')\chi(\theta, d\theta') \qquad (1.8)$$

Standard arguments show the existence a $v \in \mathbb{V}$ that satisfies this functional equation, where $\mathbb{V}$ is again the space of bounded, continuous functions with the uniform norm. In addition, since u is strictly concave in c, standard arguments also establish that v is strictly concave in its first argument, k. Once again, from Mirman and Zilcha[61], the strict concavity of v also implies that the envelope theorem applies and the solution v to the Bellman equation is once differentiable in k.

We are now prepared to define equilibrium.

Definition: *A (recursive) competitive equilibrium for this economy consists of sequences functions $r, w, d,$ and κ; a value function for the household $v(s) \in \mathbb{V}$ and the associated individual decisions $c^*(s)$ and $n^*(s)$ such that (i) given r, w, d and κ , $v(s)$ satisfies the household's Bellman equation (1.8); (ii) $c^*(s)$ solves the right-hand side optimization in the Bellman's equation, $l^*(s) = 1 - n^*(s)$ solves the first-stage utility maximization; (iii) all markets clear: i.e., $k' = h(S) = K'$, $n^*(s) = N(S)$, $c^*(s) = C(S)$ and the government budget constraint holds, i.e., $d = \pi_k k + \pi_n n^*$*

1.6.2 The Existence of Equilibrium

Before we state the existence problem, we define a number of functions. In equilibrium, $c(s) = C(S)$, $k = K$, $n = N(S)$,then $y(s) = F(K, \theta) = f(K, 1 - l^*(C(S), K, \theta), \theta) + (1 - \beta)K$. The next period capital stock, in equilibrium, is given as $K' = y - C$. Also, for later reference, define $\hat{l}(S)$ as the solution to,

$$\frac{u_l(f(K, 1 - \hat{l}(S), \theta), \hat{l}(S))}{u_c(f(K, 1 - \hat{l}(S), \theta), \hat{l}(S))} = (1 - \pi_n(S))f_n(K, 1 - \hat{l}(S), \theta).$$

Notice that $\hat{l}$ is the amount of leisure that is compatible with no household investment in the first-stage utility maximization. At any (aggregate) state

S,the maximum possible amount of consumption occurs if $c = f$ and, i.e., if there is no investment. In general, the amount of consumption is less than f and leisure, which is positively related to consumption, is therefore less than $\hat{l}(S)$. That is, for a given state S, $1-\hat{l}(S)$ is the lower bound for the amount of labor supplied. In addition, $\hat{l}(S)$ is differentiable with respect to K, by the implicit function theorem, since the marginal utilities, technology and the distorted wage is differentiable in K. Moreover, for the special case, $u_{cl} \geq 0$, $\hat{l}(S)$ is increasing in K. $\hat{l}(S)$ is also increasing in K, for the case $u_{cl} < 0$, if

$$u_{ll} - f_n u_{cl} < 0, u_{cl} - f_n u_{cc} > 0.$$

The Euler equation, associated with the right side of the Bellman equation (1.7) above, can be rewritten as,

$$u_c(c, l^*(c, K, \theta)) = \beta \int_{\Theta} u_c(c(K', \theta'), l^*(c', K', \theta'))r(K', \theta')\chi(\theta, d\theta'). \tag{1.9}$$

Here the $'$ notation refers to next period value of the particular variable. Given a candidate function $c(S)$, we rewrite the Euler equation (1.9) in equilibrium as,

$$\begin{aligned} u_c(c, l^*(c, K, \theta)) \quad &= \quad \beta \int_{\Theta} u_c(c(F_c - c, \theta'), l^*(c(F_c - c, \theta'), K', \theta')) \cdot \\ & \qquad r(F_c - c, \theta')\chi(\theta, d\theta'), \end{aligned} \tag{1.10}$$

where $F_c = f(K, 1 - l^*(c(K, \theta), K, \theta), \theta) + (1 - \beta)K$. We can use equation (1.10) to define a nonlinear operator that yields a strictly positive fixed point in the space of consumption functions. This fixed point is an equilibrium for the economy.

Define $F^u(S) = F^u(K, \theta) = f(K, 1 - \hat{l}(K, \theta), \theta) + (1 - \beta)K$ and consider the following space of functions,

Definition: $\mathbf{H}_l = \{h : \mathbf{S} \rightarrow \mathbf{K}$, h continuous, $h(S) \in [0, F^u(S)]$ and h such that $u_c(h(S), l^*(h(S), S))$ is decreasing in h, $u_c(h(S), l^*(h(S), S))$ is decreasing in K.$\}$

Equip $\mathbf{H}_l$ with the sup norm. Note that the assumption the marginal utility of consumption is decreasing in h means that the space $\mathbf{H}_l$ differs from the space of consumption functions studied in Coleman[20]. It is easily verified that for the preferences considered in that paper, the restriction u_c decreasing in h is implied. However, since the class of preferences studied in this paper is larger than that studied in Coleman, additional restriction is necessary on the space of consumption functions.

Define the extended real valued mapping $Z : \mathbf{H}_l \times \mathbf{Y} \times \mathbf{K} \times \mathbf{Z} \rightarrow \bar{\mathbf{R}}$ where $\mathbf{Y} \subset \mathbf{R}_+$, as

$$Z(h,\zeta,K,\theta) = \Psi_1(\zeta,K,\theta) - \Psi_2(h,\zeta,K,\theta), \tag{1.11}$$
$$\Psi_1 = u_c(\zeta, l^*(\zeta,K,\theta)), \tag{1.12}$$

$$\Psi_2 = \beta \int_\Theta u_c(h(F_\zeta - \zeta,\theta'), l^*(h(F_\zeta - \zeta,\theta'), F_\zeta - \zeta,\theta')) r(F_\zeta - \zeta,\theta')\chi(\theta, d\theta'). \tag{1.13}$$

Here $F_\zeta = f(K, 1 - l^*(\zeta,K,\theta) + (1-\beta)K$. Then define the nonlinear operator $A : \mathbf{H}_l \rightarrow \mathbf{H}'$ as follows:

$$Ah(K,\theta) = \{\zeta \text{ such that } Z(h,\zeta,K,\theta) = 0, h > 0; Ah(K,\theta) = 0 \text{ elsewhere}\} \tag{1.14}$$

where $\mathbf{H}'$ at this point is an appropriate Banach space.

We discuss some properties of the operator A as defined by equations (1.11) - (1.14).

Proposition 1.6.1. *Under Assumptions P3, T3 and D2, for any $h \in \mathbf{H}_l$, there exists a unique $Ah = \tilde{h}$ such that $Z(h, \tilde{h}, K, \theta) = 0$, for any (K,θ).*

Proof: Datta, Mirman and Reffett[22], Proposition 1. ■

Proposition 1.6.1 implies that for all states, the operator Ah is well defined and under the continuity assumptions on preferences, technologies, and distorted prices, continuity of Ah is obvious. To study the fixed points of A, we first establish that A is a transformation of $\mathbf{H}_l$: i.e., $A : \mathbf{H}_l \rightarrow \mathbf{H}_l$. It will be convenient to assume

Assumption - P4: *The cross-partial of the utility function is non-negative, that is, $u_{cl} \geq 0$.*

Greenwood and Huffman [34] only consider the case where $u_{cl} = 0$. Coleman [20] allows for $u_{cl} \geq 0$ and also some cases where $u_{cl} < 0$. However, he considers a restricted homothetic class of preferences and, in addition, imposes more restrictions (jointly on utility, production functions and distortions) to study the case of negative cross partials of u. The same case of negative cross-partials of u can be handled in our setting also. At this stage, we are unable to capture more general cases of negative cross partials of u than Coleman [20], therefore, we focus only on the $u_{cl} \geq 0$ case. And, we have the following:

Proposition 1.6.2. *Under assumptions P3, P4, T3 and D2, $Ah \subset \mathbf{H}_l$.*

Proof: Datta, Mirman and Reffett[22], Theorem 1.■

Notice that $\mathbf{H}_l$ is a non-empty, convex subset of a space of continuous, bounded real-valued functions but it not equicontinuous, and is therefore not relatively compact.[18] Since the space of all continuous functions on a compactum X, denoted by $\mathbf{C(S)}$, with the sup-norm metric is a Banach lattice,

[18] A set is relatively compact if its closure is compact.

$\mathbf{H}_l$ is a sublattice in $\mathbf{C}(\mathbf{S})$. Now, a closed subset of continuous, bounded real-valued functions (on a compact domain) equipped with sup-norm metric is compact if and only if it is equicontinuous. The theorem of Arzela and Ascoli (see Dieudonne [27], p.136-137) says that a set of equicontinuous, pointwise compact subset of the continuous functions is relatively compact.

Define the following subset of $\mathbf{H}_l$,

Definition: $\bar{\mathbf{H}} = \{h \in \mathbf{H}_l$ such that $0 \leq | \ h(K_2,\theta) - h(K_1,\theta) \ | \leq | F(K_2, l^*(h(K_2,\theta), K_2, \theta) - F(K_1, l^*(h(K_1,\theta), K_1, \theta) \ |$, for all $K_2 \geq K_1.\}$

A standard argument shows that the space of consumption functions $\bar{\mathbf{H}} \subset \mathbf{H}_l$ is a closed, pointwise compact, and equicontinuous set of functions. Then by a standard application of Arzela-Ascoli, $\bar{\mathbf{H}}$ is a compact, convex, order interval in $\mathbf{H}_l$. Notice that the restriction on consumption in the space $\bar{\mathbf{H}}$ that distinguishes it from $\mathbf{H}_l$ implies that the investment function $K' = F_h - h$ is an increasing functions of the current capital stock K which follows because F_h is increasing in K (since l^* is decreasing in K, the marginal products of capital and labor are positive).

We note some important properties of the operator A and the space $\bar{\mathbf{H}}$,

Proposition 1.6.3. *Under assumptions P3, P4, T3 and D2,* $\bar{\mathbf{H}}$ *is a complete lattice and* A *is a transformation on* $\bar{\mathbf{H}}$*, i.e.,* $Ah \subset \bar{\mathbf{H}}$.

Proof: See Datta, Mirman, and Reffett[22], Lemma 1 and Theorem 2.■

To apply a lattice-theoretic fixed point theorem, we need to verify isotonicity,

Proposition 1.6.4. *Under assumption P3, P4, T3 and D2,* A *is isotone on* $\mathbf{H}_l$.

Proof: Datta et al[22] ,Theorem 3.■

We now restrict the mapping A to the subspace $\bar{\mathbf{H}}$ (which is well-defined since A is continuous, $\bar{\mathbf{H}}$ is compact, order subinterval in $\mathbf{H}_l$ and apply a version of Amann's theorem,

Proposition 1.6.5. *Under assumptions P3, P4, T3 and D2 , the set of fixed points of* $A : \bar{\mathbf{H}} \to \bar{\mathbf{H}}$ *has a maximal fixed point* $Ah^* \in \bar{\mathbf{H}}$ *such that* $lim_{n\to\infty} A^n F \to Ah^* = h^*$*, uniformly..*

Proof: Apply Proposition 1.4.1; see also Datta et al[22], Proposition 2.■

1.6.3 The Uniqueness of Equilibrium

As in the case of classical production with inelastic labor supply, we apply our new approach to existence of strictly positive fixed points once again. First, define a function $f^u(K,\theta) = f(K, 1-\hat{l}(K,\theta),\theta)$ and consider the set of functions $\mathbf{M}$ for the inverse of marginal utility in equilibrium,

Definition: $\mathbf{M}_l = \{m(K,\theta) \mid m : \mathbf{K}\times\Theta \to \mathbf{K}$ is continuous; $0 \le m(K,\theta) \le \frac{1}{u_c(f^u(K,\theta),\hat{l}(K,\theta))}$ for $K>0$; $m(K,\theta)=0$ for $K=0$; and $\frac{r(K',\theta)}{m(K',\theta)} \le \frac{r(K,\theta)}{m(K,\theta)}$ for $K' \ge K\}$

By assumption D2, $r(K,\theta)$ is continuous and $\mathbf{K}$ is a compact set, therefore, r is uniformly continuous. As in Section 1.4, one can verify that $\mathbf{M}_l$ is a closed, equicontinuous, pointwise compact subset of the space of continuous functions on a compact topological space, namely $\mathbf{C}^+(\mathbf{S})$. $\mathbf{M}_l$ is, therefore, compact.

We now define a suitable operator on the space $\mathbf{M}_l$ and find a unique strictly positive fixed point of this operator (to prove the uniqueness of recursive equilibrium in $\bar{\mathbf{H}}$). As before, define the function $H(m,K,\theta)$ for each $m \in \mathbf{M}_l$ implicitly as follows (the following lemma makes sure that this definition is meaningful),

$$\begin{aligned} u_c(H(m(K,\theta),K,\theta), l(H(m(K,\theta),K,\theta),K,\theta)) &= \frac{1}{m(K,\theta)}, m>0; \\ \text{and } \; H(m,K,\theta) &= 0, \; m=0. \end{aligned}$$

Note that, $H(m(K,\theta),K,\theta) = h(K,\theta)$, pointwise. The proof of uniqueness takes place in three lemmata.

Lemma 1.6.1. *Assume P3, P4, T3 and D2. Then the mapping $H(m,K,\theta)$ is well-defined for each m, K and θ.*

Proof: Datta, Mirman and Reffett[22], Lemma 2.■

To characterize $H(m,K,\theta)$, take $m' \ge m$ in the pointwise partial order on $\mathbf{M}_l$. Define $h_2 = H(m',K,\theta)$ and $h_1 = H(m,K,\theta)$. Notice when $m' \ge m$, we have $h_2 \ge h_1$. We can now show that $f(k, 1-l(H(m,K,\theta),K,\theta)) - H(m,K,\theta)$ is decreasing in m by the definition of $H(m,K,\theta)$. Define

$$\begin{aligned} &\Delta(h, f_h - h, \theta) = \\ &\beta \int u_c(h(f_h - h,\theta'), l(h(f_h-h,\theta'), f_h - h,\theta'))r(f_h-h,\theta')\chi(\theta,d\theta')) \end{aligned}$$

Then for $m' \ge m$, we have the following inequality

$$\begin{aligned} u_c(Ah_1, l(Ah_1,K,\theta) &= \Delta(h_1, f_{Ah_1} - Ah_1, \theta) \\ &\ge \Delta(h_2, f_{Ah_1} - Ah_1, \theta) \end{aligned}$$

Therefore, for such a perturbation of h, the mapping Z used in the definition of Ah is now nonnegative. Therefore, the first term in the definition of Z must decrease and the second term must increase in a solution Ah_2. The latter implies $f_{Ah_2} - Ah_2 \leq f_{Ah_1} - Ah_1$. Consequently, by the definition of $H(m, K, \theta)$, $f(K, 1 - l(H(m, K, \theta), K, \theta)) - H(m, K, \theta) = f_{H(m)} - H(m)$ must be decreasing in m.

Now, define the mapping

$$\hat{Z}(m, \tilde{m}, K, \theta) = \frac{1}{\tilde{m}} - \beta \int_{\Theta} \frac{r(f_{\tilde{m}} - H(\tilde{m}, K, \theta), \theta')}{m(f_{\tilde{m}} - H(\tilde{m}, K, \theta), \theta')} \chi(\theta, d\theta'),$$

where $f_m - H(m, K, \theta) = f(K, 1 - l(H(m, K, \theta), K, \theta), \theta) - H(m, K, \theta)$ and we are ready to define the operator,

$$\hat{A}(m) = \{\tilde{m} \in \mathbf{M}_l \mid \hat{Z}(m, \tilde{m}, K, \theta) = 0, \text{for } m > 0; \text{and, } 0 \text{ elsewhere}\}.$$

Defining the standard partial order on $\mathbf{M}_l$, that is, $m' \geq m$, $m', m \in \mathbf{M}_l$ if and only if $m'(K, \theta) \geq m(K, \theta)$ for all (K, θ).

Finally, if $m'(K, \theta) > m(K, \theta)$, $m, m' \in \mathbf{M}_l$, the mapping H must be such that $u_c(H(m, K, \theta), l(H(m, K, \theta), K, \theta))$ is decreasing in m for each (K, θ). Since $h \in \bar{\mathbf{H}}$, $u_c(c, l(c, K, \theta)$ is decreasing in c, and there exists $h, h' \in \bar{\mathbf{H}}$ such that $h' = H(\frac{1}{u_c(h', l(h', K, \theta))}, K, \theta) = H(m', K, \theta)$ and $h = H(\frac{1}{u_c(h, l(h, K, \theta))}, K, \theta) = H(m, K, \theta)$.

If the operator $\hat{A}m$ is well defined, we are able to relate orbits of the operator $\hat{A}^n m_0 \in \mathbf{M}_l$ to those of the operator $A^n h_0 \in \bar{\mathbf{H}}$ by the following construction. Consider some $h_0 \in \bar{\mathbf{H}}$. For such an h_0, there exists an $m_0 = \frac{1}{u_c(h_0, l(h_0, K, \theta))} \in \mathbf{M}_l$ such that $H(\frac{1}{u_c(h_0, l(h_0, K, \theta))}) = h_0$. By definition,

$$\begin{aligned} \hat{Z}(m_0, \hat{A}m_0, K, \theta) &= \hat{Z}(H(\frac{1}{u_c(h_0, l(h_0, K, \theta))}, K, \theta), \\ \hat{A}H(\frac{1}{u_c(Ah_0, l(Ah_0, K, \theta))}), K, \theta) &= Z(h_0, Ah_0, K, \theta). \end{aligned}$$

Therefore, $h_1 = Ah_0 = H(\frac{1}{u_c(Ah_0, l(Ah_0, K, \theta))}) = H(\hat{A}m_0)$. A similar argument establishes $A^n h_0 = H(\hat{A}^n m_0)$, $n = 1, 2, ...$ We next show that the operator $\hat{A}m$ is well defined.

Lemma 1.6.2. *Under assumptions P3, P4, T3 and D2, the operator $\hat{A}$ is a well-defined transformation on $\mathbf{M}_l$.*

Proof: Datta et al[22], Lemma 3.■

We now provide the last step of our argument.

Lemma 1.6.3. *Under assumptions P3, P4, T3 and D2, if $\hat{A}$ has a strictly positive fixed point then it is unique.*

Proof: Since $\hat{Z}$ is increasing in m, and decreasing in $\tilde{m} = \hat{A}m$, $\hat{A}m_1 \geq \hat{A}m_2$ for $m_1 \geq m_2$. A sufficient condition for strong sublinearity is,

$$\hat{Z}(tm, t\hat{A}m, K, \theta) > \hat{Z}(tm, \hat{A}tm, K, \theta).$$

This inequality follows since $m \in \mathbf{M}_l$, and r decreasing in K. Thus,

$$\hat{Z}(tm, t\hat{A}m, K, \theta) = \frac{1}{\tilde{m}} - \beta \int_{\Theta} \frac{r(f_{\tilde{m}} - H(t\tilde{m}), \theta')}{m(f_{\tilde{m}} - H(t\tilde{m}), \theta')} \chi(\theta, d\theta') > 0,$$

and $\hat{Z}(tm, \hat{A}tm, K, \theta) = 0$. Notice also that by examining the definition of $\hat{A}m$, given the Inada condition, $\hat{A}m$ is K_0-monotone. Therefore, by the same argument in the classical production with inelastic labor supply case, via the extension of a uniqueness theorem in Krasnosel'skii and Zabreiko [**?**] found in Coleman [19], if $\hat{A}$ has a strictly positive fixed point, it is unique in $\mathbf{M}_l$ (and, therefore, in $\bar{\mathbf{H}}$). ■

Finally, we prove the existence a strictly positive fixed point.

Proposition 1.6.6. *Under assumptions P3, P4, T3 and D2, there is a unique strictly positive MEDP.*

Proof: Note that, as $\mathbf{M}_l$ is an order interval in a solid cone of continuous functions, and $\hat{A}m$ is strongly sublinear on its interior. Also, given the definition of $\hat{A}m$, whenever $m > 0$, necessarily $\hat{A}m > 0$; particular, k_0-monotonicity implies there is a point $m_0 >> 0$ that maps up. Therefore again, we have $\hat{A}$ a cone compression on the order interval $\mathbf{M}_l$. By Krasnosel'skii and Zabreiko ([**?**], Theorem 46.4), we conclude $\hat{A}m$ has a strictly positive fixed point. Further as $\hat{A}m$ is additionally K_0-monotone, therefore actually has a unique strictly positive fixed point. Finally, again exploiting the relationship between the orbits of $\hat{A}$ and A discussed earlier in this section before the beginning of this proof, as we have a unique strictly positive fixed point for $\hat{A}$ in $\mathbf{M}_l$, namely $m^* > 0$, we have a unique fixed point for A , say $h^* > 0$ in $\bar{\mathbf{H}}$.Since $h^* > 0$ implies strictly positive consumption, it is a MEDP. ■

Again, we note that $h^* > 0$ is crucial for characterizing prices in $l^1_+ \backslash \{0\}$ (e.g., see Le Van and Vailakis[53])

1.7 Concluding Remarks

In this chapter, we survey a new and emerging approach to recursive competitive equilibrium theory that is commonly referred to as isotone recursive methods and we focus on economies with homogenous agents. These methods allow one to unify results on the existence, characterization and computation of MEDPs and the SME for a large class of economies commonly encountered in applied dynamic macroeconomics. Datta, Mirman, Morand and Reffett[23] develop isotone recursive methods to study MEDPs in the stochastic Ramsey models of Becker and Zilcha [10] with heterogeneous agents. They find sufficient conditions for MEDPs to be isotone and Lipschitz continuous and for MEDPs that are just Lipschitz continuous. Another application of isotone recursive methods to the case of heterogeneous agent models is in overlapping generation models. These models form the basis of much work in lifecycle theory on social security. Erikson, Morand and Reffett[31] apply the isotone recursive approach to a class of two period stochastic lifecycle-overlapping generations models with social security, production nonconvexities and public policy (fiscal or monetary). Primarily, they consider the case of i. i. d. shocks but provide some preliminary results with Markov shock. This paper (along with others mentioned below) indicate an important direction for isotone recursive methods in future research; namely, the study of Stationary Markovian equilibrium (SME). In this survey, this question is not addressed. In the existing literature, an SME is often considered to be an invariant distribution (e.g., Hopenhayn and Prescott [43]).[19] In many cases the existence of SME can be established with applications of the fixed point theory for complete partially ordered sets as discussed in section 1.3 though the applications might not be as simple as in the case of continuous MEDPs. For example, consider the existence of a stationary Markov equilibrium for the case of nonconvex production technologies (in addition to Erikson, Morand and Reffett [31], see Hopenhayn and Prescott[43] and Mirman, Morand and Reffett[59]). In general, the extremal MEDPs are only semicontinuous;[20] often, Propositions 1.3.6 and 1.3.7 cannot be applied (as operators are not necessarily order-continuous on and/or appropriately topologically continuous on their respective domains). However, the existence of an extremal limiting distribution can be guaranteed by applying Proposition 1.3.3 or 1.3.5 (along with Proposition 1.3.4). The computational issues for numerical solutions to approximate an SME can be addressed using Propositions 1.3.8 and 1.3.9. Proposition 1.3.8 provides a collection of generalized iterative procedures (that are not necessarily successive approximations). Proposition 1.3.9 provides sufficient conditions for the existence of an underlying set for iterations that is cofinal (because the underlying space of probability measures on a compact Polish space is a compact metric space). Therefore, the existence of monotone iterative methods on a countable indexation is obtained via Heikkilä and Lakshmikantham's[39] generalized iterative procedures. Heikkilä and Salonen[41][42] and Heikkilä[37] provide extensive discussions on

[19] In other work, an SME is considered to be an ergodic distribution with a nongenerate support. Our remarks apply to this case also.

[20] Erikson, Morand, and Reffett[31] and Mirman, Morand, and Reffett[59] provide sufficient conditions that distinguish the cases of the existence of continuous MEDPs and the existence of semicontinuous MEDPs.

implementation of such theoretical construction. As for comparative statics of an SME for economies with semicontinous MEDPs: the space of probability measures defined on a compact Polish space is not necessarily a lattice (e.g., consider probability measures defined over a support that $S \subset R_+^N$ for $N > 1$), therefore, the SCS and WCS monotone selection theorems of Veinott in Propositions 1.3.10 and 1.3.11 do not apply. The space of probability measures on a compact subset of a Polish space is a CPO (as it is a compact metric space). One can apply the WCS conditions in Propositions 1.3.12 and 1.3.13 to obtain an isotone selection between the space of economies and the set of SME (see Mirman, Morand and Reffett[59] section 1.3 for a discussion). We feel that similar new and interesting applications of recent work in order theoretic fixed point theory will become paramount in future work that seeks to study MEDPs and SME.

Potentially the most important extension of isotone recursive methods is the so-called "mixed-monotone" recursive methods first presented systematically in Reffett[69], and subsequently applied in Mirman, Reffett and Stachurski[60] to Bewley models with a single asset. The mixed-monotone method build upon the mixed-monotone fixed point theory (also known as "coupled" fixed point theory) that has been developed in the literature on discontinuous differential equations. These methods appear powerful, and deliver MEDPs *on the natural state space of current states* even in situations where MEDPs are not unique. Discussions of mixed monotone fixed point theory are found in Amann[4], Heikkilä and Lakshmikantham[39] and Reffett[68], to name a few. The discovery of mixed-monotone recursive methods appears to be a giant step forward in developing methods based on constructive fixed point theory that can be applied in a wide-array of economic situations. One no longer needs to have isotone operators (nor fixed point spaces) where underlying constructions are based on isotonicity. One problem with this method is that one requires sufficient topological structure relative to the fixed point space for antitone transformations to possess the fixed point property. Preliminary work in a series of recent papers by Reffett[67][68][69], Datta and Reffett [**?**], and Mirman, Reffett and Stachurski [60] indicate that for many interesting economies, such "mixed monotone" fixed point methods are available. For example, these methods provide successive approximation algorithms for computing Bewley models of the sort studied in Aiyagari[2], Krusell and Smith[51], and Miao[57]. In addition, isotone recursive methods are a special case of mixed monotone recursive methods and can be studied in a "single" step using an isotone operators instead of multi-steps for mixed-monotone operators. Mixed monotone recursive methods unify the existing approaches to characterize MEDPs and the SME by allowing researchers to obtain more general results that relate monotone iterative computational procedures to actual fixed point constructions. As numerical methods described in standard monographs (e.g., Krasnosel'skii et al[48]) can build on explicit operators to obtain error estimates of Santos and Vigo[75] and Santos[74]. In principle, one might be able to obtain a complete set of iterative methods for studying numerically, the quantitative properties of the SME in a large class of macroeconomic models to a specified degree of accuracy, which seems to be the goal of quantitative macroeconomics (e.g., real business cycle studies). Indeed, qualitative methods can provide an essential, first step in obtaining a useful (and, mathematically credible) quantitative theory of macroeconomic fluctuations and long-run growth.

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