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A Survey of Mathematical Methods in Economics:
Optimization, Equilibrium, and Comparative Statics

Many arguments in economics take place in the context of an explicit mathematical model of equilibrium. Sometimes, the equilibrium concept is competitive equilibrium. In this case, sometimes one can compute an equilibrium by first solving a "social planning" problem, and then using fixed point methods and a "duality" argument to decentralize allocations under a suitable price system (e.g., Negishi methods under convexity assumptions, etc.) In other cases, Negishi methods are not available, so one must first solve parametric optimization problems for each economic agent, and then solve systems of market clearing equations (in prices) via a fixed point methods to verify existence of equilibrium. Finally, in some economic models, equilibrium is not competitive, but corresponds with a "Nash equilibrium" in a game. For such problems, again, equilibrium is constructed using fixed point methods (i.e., fixed points of "best response" maps). In all of these cases (and numerous others in economics), optimization techniques and fixed point theory are used extensively.

Once the equilibrium is constructed, the next natural question to raise concerns the characterization of equilibrium (and/or optimal solutions). In particular, one might be interested in how the *set* of equilibrium and/or optimal solutions vary in the "deep parameters" of the economies being studied. This is a "comparative statics" question, which is even interesting in the context of standard parameterized optimization problems. This characterization question raises a number of new interesting mathematical questions. It should be noted that sometimes solutions cannot be computed in " closed form". In such cases, the question of *accurate* approximation of equilibrium and/or optimal solutions becomes a new issue of interest.

The central concern of this course is to introduce students to some mathematical methods that prove useful in the study each of these questions. The course is intended to complement your work in your core theory courses (as well as provide a bit of math useful for econometrics). We will begin the course with a an introduction to typical mathematical questions that arise in economic theory and econometrics to highlight the complications involved in writing down well-defined economic problems. We then turn to the main body of the course, which takes place in the follow steps:

(i) *survey some useful results from mathematical analysis*: we will discuss some mathematical concepts from both real and functional analysis (as well as inductive logic) that prove useful in many arguments in economics (as well as will be used later in the course);

(ii) *qualitative optimization via lattice programming*: lattice programming is a branch of "qualitative optimization" methods that are often used to obtain characterizations of optimal solutions in economic choice problems. In lattice programming, one often does not typically appeal to duality arguments and/or implicit function theorems to study how optimal solutions vary in deep parameters (as is done, for example, in mathematical programs that are sufficiently "convex"). Rather, using the underlying order structure of the problem, along with the complementarity structure of objective functions and constraint correspondence, comparative statics results are deduced. The presence of constraint sets do introduce serious limitations of standard lattice programming methods. That is, for some important cases (e.g., income effects in the consumer's problem, monotone controls in growth models, etc.), it is difficult to apply lattice programing using traditional methods. For these problem, we will study monotone comparative statics using minimax lattice programming via Lagrangian dual methods. We will also discuss order theoretic fixed point theory, and provide applications of this theory to both macroeconomics and game theory.

(iii) *optimization problems under convexity conditions*: we first study conditions where (a) solutions exist; (b) those solutions have desirable structural properties; and (c) those solutions can be computed using an appropriate first order theory via a duality argument. Per question (a), we will study topological "maximum theorems". Per questions (b) and (c), we will develop "Lagrangian dual methods" for characterizing and computing optimal solutions, as well as computing envelop theorems for value functions. In developing the "Lagrangian dual" approach to the constrained optimization, we will include an extensive discussions of minimax theory, saddlepoint stability, with a brief introduction to Nash games in finite dimensions. The discussion in (c) will also require an detailed discuss of suitable "constraint qualifications" which play a key role in dual methods for computing optimal solutions to primal optimization problems. Finally, for local (metric) comparative statics for particular optimal solutions, we will appeal to appropriate versions of the implicit function theorem (IFT). As the IFT is a local construction, we will also discuss conditions for global versions of the implicit function theorem;

(iv) *survey parameterized fixed point theory*: a "parameterized" fixed point problem is a whose solutions, in general, depend on a set of deep parameters. These problems arise often in economics. For example, equilibrium existence problems in Arrow-Debreu-McKenzie models and/or Nash games are often parameterized fixed point problems. Additionally, constructing a recursive equilibrium amounts to solving a is a parameterized fixed point problem. We will study existence, uniqueness, and computation questions for parameterized fixed point problems. Per the question of existence, we will consider both topological and order theoretic methods. Uniqueness conditions discussed will be both geometric and "contractive" conditions. Via the converse of the contraction mapping principle, we will show that in an abstract sense, they are equivalent as global stability of iterative structure in an *arbitrary* space will be shown as equivalent to a contraction in a complete metric space (e.g., via Bessaga's theorem). We finally raise the question of continuous and/or monotone comparative statics for fixed point in deep parameters (i.e., the existence of continuous and/or monotone selection from the set of solutions in a deep parameter), as well as issues concerning the computation of equilibrium comparative statics.

(v) *study functional equations and dynamic programming*: in many economic models (e.g., growth models in macroeconomics), decision problems have a natural recursive structure. To study these problems, we introduce the idea of "recursive aggregation" (e.g., dynamic programming). We will study both nonstationary and stationary dynamic programming for both the deterministic programs (emphasizing the principle of optimality), and discuss (time permitting) stochastic dynamic programming;

Course Materials:

Texts and Main References:

E. Ok, 2007. *Real Analysis with Economic Applications.* Princeton.
Kolmogorov and Fomin, 1975 (reprint of 1970 edition) *Introductory Real Analysis,* Dover.
R.T. Rockafellar, 1970, *Convex Analysis,* Princeton (available for free download on Rockafellar's webpage)

It is difficult to find a single math econ book for this course that is strong in all areas we will study. Ok's book is an excellent reference for many topics in the course pertaining to real and functional analysis. Unfortunately, it is a bit narrow in scope. Kolmogorov and Fomin is a fantastic reference book for analysis (both real and functional analysis). Finally, Rockafellar's monograph *Convex Analysis* is classic reference on finite dimensional convex analysis. For further reference material, also my extended list of reference material at the end of this course outline.

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You can always contact me by via email. That one works great. It's the way to go in my humble opinion. My office phone is 480-965-7006. I check my voicemail at the office religiously, I would say once every couple of years or so. I am not a big cell-phone person either. So its best to contact me is via email. Also, Min Chen will be my teaching assistant, and he will be running the review session that is associated w/ the course. Also, turn in all your homeworks to him. Also, you can always contact he per issues related to the course.

Grading:

The initial course grade is based on student performance on two term exams, and a final exam, each comprising one-third of your term grade. The term exams are currently scheduled on the following dates: October 2 and November 6. The final exam will be on December 9 during finals week. The exams will be standard format (problems, True-False-Uncertain w/ argument, etc.), etc.).

Additionally, student performance on homeworks will be given a grade evaluation also. On HW's, I will give a grade on a 1-3 scale (1=fail/weak, 2=acceptable, 3=excellent). Per grading, for example, I will randomly select which homework problems I want to grade. Typically, there will be a homework assigned every 3 lectures. If the average grade you receive on HW is below a 2, I will lower your initial letter grade by 1 full letter grade (e.g., A- to B- if you HW grade at < 2). Further, your HW grade will be able to help your grade also to break "ties" if you are close to another letter grade (i.e., if your HW grade is near a 3, and you are on the border of A-/B+, I will give you an A-).

Course Outline

This is a set of topics for the course. I am sure we will end up studying a strict subset of this material. We will, therefore, need to be flexible. I place a "*" next to sections of the outline I might not have time to do, but hope to at least discuss at some level of specificity.

1. Mathematical Questions Arising in Economics

Motivation for the course, economic models as mathematical objects, initial formalizations of some important economic questions, examples from micro, macro, approximation/econometrics

2. Preliminaries in Mathematical Analysis

Systems of powersets, relations, binary operations, order relations, closure systems and Moore families, preordered sets, partially ordered sets, lattices, ordering powersets (e.g., set inclusion, weak-induced set order, Veinott's Powerdomain), monotone mappings, semigroups/groups/rings/fields, po-semigroups, well-orderings, the axiom of choice and a few of its equivalents (e.g., Zermelo's well ordering lemma, Zorn's lemma, Kuratowski's maximal element theorem, etc), cardinal and ordinal numbers, induction/transfinite induction and iterative methods, important linear spaces in economics (e.g., $\mathbf{R}^n$, linear (vector) spaces/subspaces, inner product spaces, normed spaces, Banach spaces)

3. Parametric Optimization

3.1. *Parameteric Continuity and Maximum Theorems* .

Metric spaces, metric/norm topologies (and their relationship with topological spaces), convergence structures in metric spaces (including Dini's theorem), compactness in metric spaces, continuity and semicontinuity of functions and correspondences in metric spaces.Level-bounded functions, versions of the maximum theorems of Berge, Hogan, and Rockafellar in metric spaces, remarks on generalizations of maximum theorems to topological spaces, definition of value function, basic structure of optimal solutions and value functions.

3.2. *Lattice Programming* .

Preseving parameteric monotonicity of value functions, super$*$ functions from a lattice to a chain, preserving super$*$ structure under sup/inf operations to value functions, Veinott's order theoretic maximum theorems, supermodular lattice programming and ordered optimal solutions (cardinal conditions), superextremal and quasisupermodular lattice programming (ordinal conditions)*, necessary conditions*, constrained supermodular lattice programming via Lagrangian duals* (e.g., minimax lattice programming), multistage lattice programming, application: income effects in consumer's problem*

3.3. **Convexity, Duality, and Separation in Finite Dimensions.** Linearity and Dual spaces, Convex sets in $\mathbf{R}^n$, convex geometry in finite dimensions (e.g., cones, convex hulls, affine sets, hyperplanes, polyhedral sets, extreme points, extended-real valued convex functions, extended-real valued quasi-convex functions, dual/epigraphical representations of convex functions and Fenchel conjugation, differentiability properties of convex functions, relationship w/ Hahn-Banach theorem/separation*, separating hyperplane theorems, supporting hyperplanes, remarks on extensions of these results to infinite dimensions*.

3.4. ***Convex Optimization and Duality in Finite Dimensions*** .

Maximum theorems under convexity, convexity preserving operations, saddlefunctions, Quadratic forms and smooth characterizations of convex functions and saddlefunctions, Farkas' lemma/theorem of alternative, linear programming, unconstrained convex optimization, constrained convex optimization/classical Lagrangian duality, saddlepoint stability and minimax (including discussion Nash equilibrium, and the fixed point theorems of Brouwer and Kakutani), Karash-Kuhn-Tucker (KKT) theory and Lagrange multiplier rules, various versions of the envelope theorem via duality, pricing via duality, local comparative statics via implicit function theorem, global implicit function theorems, an infinite dimensional KKT theorem (w/ applications to growth)*, complications with extensions of duality to Quasiconvex programming*.

4. Parameterized Fixed Point Theory*

4.1. ***Existence, Uniqueness, and Computation of Fixed Points*** .

This "section" of the course is not really a distinct section; rather, the material here will be covered within the lectures on optimization, game theory, lattice programming, and dynamic programming. Still, I find it useful to layout in one section the structure of the material in parameterized fixed point theory that is used in economic applications. It also bears mentioning that I will post copies of Herbert Amann's lecture notes on topological and order theoretic fixed point theory, as well as Bonsall's lectures on fixed point theory, and those lecture notes will contain all the results discussed below (excepting monotone fixed point comparative statics, which I will discuss elsewhere). Fixed point theory (and the comparative statics of fixed point sets) is a very important topic, and is at the heart of the following questions: (a) existence of Arrow-Debreu equilibrium, (b) pure strategy equilibrium in superextremal/supermodular games), (c) dynamic programming in recursive equilibrium problems. The results needed for these problems, and covered in the lectures during the course, are the following results:

Existence: (i) Brouwer and Kakutani's theorem, (ii) Banach's contraction mapping theorem, (iii) Schauder's theorem, (iv) Tarski-Davis Theorem, (v) Veinott's generalization to ascending correspondences in complete lattices, (vi) Abian-Brown's theorem, (vii) Markowsky's theorem, (viii) Smithson's generalization to ascending correspondences in Posets.

Uniqueness: contractive conditions for uniqueness, Janos/ Bessaga stability, "converses" of the contraction Mapping theorem, geometric conditions for uniqueness*.

COMPUTATION: Iterative methods, (order) continuity and approximation, (i) Tarski-Kantorovich's theorem, (ii) Amann's theorem, computability and continuity

4.2. ***Fixed Point Comparative Statics**** .

Local stability of fixed points via implicit and inverse function theorems, Bonsall-Nadler theorems and continuous fixed point comparative statics, Veinott monotone fixed point comparative statics theorems (w/ applications to superextremal games), Vias-Boas fixed point comparative statics theorems**.

5. Functional Equations and Dynamic Programming

5.1. ***Functional Equations in Economics*** .

Parameterized functional equations, simple examples of some functional equations arising in economics (e.g., recursive equilibrium problems in OLG models, recursive equilibrium in growth models where the second welfare theorem fails)

5.2. ***Sequential decision problems, Separability, and Aggregation*** .

Multistage decision theory and aggregation, sequential value functions, sequential constrained optimization problems (SP) in infinite dimensions and related Kuhn-Tucker theory with macro applications.

5.3. ***Discounted Dynamic Programming*** .

The Bellman equation viewed as a functional equation (FE), relating solutions to (SP) and (FE) and the principle of optimality, nonstationary dynamic programming w/ bounded returns, stationary dynamic programming w/ bounded returns, remarks on extensions to the unbounded returns case, stochastic dynamic programming with countable Markov shocks, more on systems of powersets (e.g., $\pi-$systems, $\lambda-$systems, $\sigma-$fields, Borel algebras), set functions, measures, measurable sets, measurable functions, Lebesque integration, monotone convergence theorem, Lebesque dominated convergence theorem, Fubini's theorem, L^p spaces, extending the arguments to the stochastic case with countable shock spaces, remarks on complications w/ measurability/uncountable shock spaces and unbounded returns.

6. References

This is a list of books in mathematics and economics that I have found very useful over the years. They are categorized somewhat by level of difficulty, and listed in no particular order within that categorization.

6.1. ***Some Nice Fundamental References in Mathematics*** .

Michael Spivak. 1965. *Calculus on Manifolds*, Harper Collins.

T. Apostol. 1957. *Mathematical Analysis,* Addison Wesley

T. Apostol. 1967, 1969. *Calculus, vol 1 and 2,* Blaisdell Press.

J. Marsden and M. Hoffman. 1993. *Elementary Classical Analysis,* Freeman Press

J. Munkres. 2000. *Topology: A First Course,* Prentice Hall

H. Royden. 1968. *Real Analysis,* MacMillan Press

W. Rudin. 1976. Principles of Mathematical Analysis, McGraw Hill

S. Saks. 1940. *The Theory of the Integral,* Warszawa-Lwow, G.E. Stechert and Co Press

A. Dontchev and R.T. Rockafellar. 2009. *Implicit Functions and Solution Mappings.* Springer.

6.2. ***Some Classic References in Economics*** .

J. Von Neuman and O. Morgenstern. 1944. *Theory of Games and Economic Behavior.* Princeton

P. Samuelson, 1947. *Foundations of Economic Analysis.* Harvard.

D. Blackwell, 1954. *The Theory of Games and Statistical Decisions.* Wiley.

L. Savage. 1954. *Foundations of Statistics.* Wiley/Chapman Hall

P. Samuelson and R. Dorfman. 1958. *Linear Programming and Economic Analysis.* MacGraw-Hill

G. Debreu. 1959. *The Theory of Value.* Cowles Foundation.

D. Gale. 1960. *Theory of Linear Economic Models.* MacGraw-Hill.

K. Arrow. 1963. *Social Choice and Individual Values.* Wiley.

H. Nikado. 1969. *Convex Structure and Economic Theory.* Academic Press.

K. Arrow and F. Hahn. 1971. *General Competitive Analysis.* North Holland.

W. Hildenbrad, 1974. *The Core and Equilibria in a Large Economy.* Princeton

A MasColell. 1985. *The Theory of General Economic Equilibrium,* Cambridge

G. Debreu. 1986. *Mathematical Economics: Twenty Papers of Gerard Debreu.*

N. Stokey, R. Lucas, Jr., E. Prescott. 1989. *Recursive Methods in Economic Dynamics.* Harvard.

C. Aliprantis., D. Brown, and O. Burkinshaw. 1989. *Existence and Optimality of Competitive Equilibrium.* New York: Springer.

R. Becker and J. Boyd. 1997. *Capital Theory, Equilibrium Analysis, and Recursive Utility.* Blackwell.

J. C. Moore. 1999. *Mathematical Methods for Economic Theory, vol 1,2.* Springer.

L. McKenzie. 2002. *Classical General Equilibrium Analysis.* MIT.

C. Aliprantis and K. Border. 2006. *Infinite Dimensional Analysis: Hitchhikers Guide.* 3rd edition. Springer.

J. Stachurski. 2009. *Economic Dynamics: Theory and Computation.* MIT.

6.3. ***Advanced References in Mathematics*** .

When listing reference books, I will not only list books useful directly for this course, but also mention titles useful for further study in economics. This means the books listed will be written at different levels of difficulty. So, if you examine a book recommended, and see it is too difficult, just move on. Also, a few remarks.

First, the Bourbaki series in Mathematics is fantastic, but difficult. I do recommend it as all the books in the series are great. I especially recommend the volumes (Chapters) on measure theory, set theory, topological vector spaces, and general topology.

Second, Van Nostrand Press published a fantastic series of books on Mathematics in the 1950s and 1960s. One series is an "undergrad" series, and are nice introductions to various branches of mathematics. The second series is a graduate series, and is wonderful, including classic monographs by J. Kelley, Gillman/Jerison, Halmos (various books, all great), Jacobson, Loeve, among others. I highly recommend this latter series as a starting point for technical references in your graduate work.

Anyway, I now list many references on topics in mathematics that I think are useful in various areas of economics, again in no particular order...well, excepting the first book on the list, for which I am forever indebted to Amanda Friedenberg for generously donating her copy of this *gem* to my library!

E. Klein and A. Thompson. 1984. *Theory of Correspondences: Including Applications to Mathematical Economics.* Canadian Math Society Monograph Series.

C. Castaing and M. Valadier. 1977. *Convex Analysis and Measurable Multifunctions.* Springer.

K. R. Parathasarthy. 1967. *Probability Measures in Metric Spaces.* Academic Press.

P. Billingsley, 1968. *Convergence of Probability Measures.* Wiley.

I. Ekeland and R. Témam. 1976. *Convex Analysis and Variational Problems.* North-Holland.

Davey and Priestly, 2002 *Introduction to Lattices and Order*, Cambridge Press.

G. Birkhoff, 1967 *Lattice Theory,* AMS.

R. T. Rockafellar and R. Wets. 1997. *Variational Analysis.* Springer.

P. Halmos, 1951. *Measure Theory,* Van Nostrand.

P. Halmos, 1960. *Naive Set Theory,* Van Nostrand.

P. Halmos. 1948. *Finite Dimensional Vector Spaces.* Van Nostrand.

A. Grothendieck. 1973. *Topological Vector Spaces.* Gordon Breach.

K. Kuratowski. 1966, 1968. *Topology, vol.1,2.* Academic Press.

L. Gillman and M. Jerison. 1960. *Rings of Continuous Functions.* Van Nostrand.

M.E. Monroe. 1953. *Measure and Integration,* Addison Wesley Press.

D Smart. 1974. *Fixed Point Theorems,* Cambridge Press.

E Cheney. 1982. *Introduction to Approximation Theory,* McGraw Hill.

R. Dudley. 1989. *Real Analysis and Probability,* Wadsworth.

J. Dugundji and A. Granas. 1982. *Fixed Point Theorems,* Warszawa-Lwow.

J. Dieudonne, 1960. *Foundations of Modern Analysis,* Academic Press.

R. Engelking, 1990. *General Topology,* Sigma Series in Pure Mathematics, Springer

N Jacobson, 1960. *Lectures on Abstract Algebra,* vol 1-3, Van Nostrand.

D. Luenberger. 1969. *Optimization by Vector Spaces Methods.* Wiley.

P. A. Meyer, 1966. *Probability and Potentials,* Blaisdell.

J. Kelley, 1955. *General Topology,* Van Nostrand.

J. Kelley et.al., 1963. *Linear Topological Spaces.* Van Nostrand.

S. Willard, 1970. *General Topology,* Addison Wesley.

C. Berge, 1963. *Topological Spaces,* MacMillan.

T. Jech. 1978. *Set Theory.* Academic Press.

G Choquet, 1976. *Lectures on Analysis,* vol 1-3, W. A. Benjamin.

G. Choquet. 1966. *Topology.* Academic Press.

R. T. Rockafeller, 1970. *Convex Analysis,* Princeton.

R. Abraham, 1967. *Foundations of Mechanics,* W.A. Benjamin (1967 edition only).

R. Abraham and J. Robbin, 1967. *Transversal Mappings and Flows,* W.A. Benjamin.

M. Krasnolselskii, 1964. *Positive Solutions to Operator Equations,* Noordhoff.

M. Krasnolselskii (et al). 1972. *Approximate Solutions of Operator Equations,* Noordhoff.

S. Carl and S. Heikkilä, 2011. *Fixed Point Theory in Ordered Sets and Applications.* Springer.

P. Billingsley, 1983. *Probability and Measure,* Wiley.

S Meyn, and R. Tweedie, 1993. *Markov Chains and Stochastic Stability,* Springer.

M. Powell. 1981. *Approximation Theory and Methods.* Cambridge.

E. Lehmann, 1959. *Testing Statistical Hypotheses,* Wiley.

E. Lehmann, 1983. *The Theory of Point Estimation,* Wiley.

D. Topkis. 1998. *Supermodularity and Complementarity,* Princeton.

C.R. Rao. 1973. *Linear Statistical Inference and its Applications.* Wiley.

J. Neveu, 1965.*Mathematical Foundations of the Calculus of Probability.* Holden-Day.

P. Hall and C. Heyde. 1980. *Martingale Limit theory and its Applications.* Academic Press.

S. Karlin. 1968. *Total Positivity.* Stanford Press.

N. Dunford and J. Schwartz. 1957. *Linear Operators: Part 1, General Theory.* Wiley Interscience.

S. Mac Lane. 1971. *Categories for Working Mathematicians.* Springer.

H. Rubin and J. Rubin. 1963, *Equivalents of the Axiom of Choice,* Springer.

H. Rubin and J. Rubin. 1985. *Equivalents of the Axiom of Choice, II.* North-Holland.

6.4. ***Unpublished Notes*** .

W. Fenchel. 1951. *Convex Cones, Sets, and Functions.* Princeton University.

F. Bonsall. 1962. *Lectures on Some Fixed Point Theorems of Functional Analysis.* Tata Institute.

L. Schwartz. 1964. *Functional Analysis.* Courant Institute Lecture Notes.

R. Dudley. 1976. *Probabilities and Metrics.* Lecture Note Series #45, Aarhus.

H. Amann. 1977. *Order Structures and Fixed Points.*SAFA 2, ATTI del 2∘ Seminario di Analisi Funzionale e Applicazioni. MS.

_________. 1977. *Lectures on Some Fixed Point Theorems.*

W. A. Kirk. 1990. *Fixed Point Theory: A Brief Survey.* Notas de Matematicas, 108.

A. McLennan, 1989. *Selected Topics in the Theory of Fixed Points.* University of Minnesota.

Veinott, A. 1992. *Lattice Programming: Qualitative Optimization and Equilibrium.* MS. Stanford.