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Mathematical Economics:
Analysis, Optimization, and Economic Equilibrium

Many arguments in economics take place in the context of a explicit mathematical model of economic equilibrium. Therefore, in the end, economists must be adept at using and applying various branches of mathematics. Mathematics is a very exact discipline; hence, its application to economic arguments allows one to state the theory very precisely. This is an important motivation for studying mathematical economics.

Now, in an abstract sense, all formal economic models can be reduced to parameterized fixed point and/or parameterized optimization problems. In such a setting, the "economics" implied by the models is provided by the mathematical characterization of economic equilibria. Such a characterization often involves allowing the deep parameters of the economy vary, and studying how economic predictions change. Such questions are often referred to in mathematics (and operations research) as "sensitivity analysis", but in economics as "comparative statics/dynamics". In the end, this course is about comparative statics of economic equilibrium. Since the work of Arrow, Debreu, and McKenzie (ADM) in the 1950s, all economic models of significant interest have assumed agents are optimizing in their current situation (i.e., given a parameterization of the economic environment). This situation implies the need for studying parameterized optimization problems at a formal (and abstract) level. Often these problems in such economic models will be presented as constrained optimization problems. The nature of these constraints will determine the mathematical structure of the feasible correspondence facing economic agents, which in turn is critical in characterizing the properties of optimal solutions. In such cases, we will use the theory of the Karash-Kuhn-Tucker (KKT) approach to study the structure of optimal solutions (and value or "marginal" functions). We shall begin with a systematic study of "convex optimization", and the KKT that is associated with it. Such an approach can be seen as a generalization of the standard duality approach embodied in standard Lagrangian methods for convex optimization problems.

Further, in ADM environments, equilibrium is constructed by solving systems of equations. In some such situations, the system of equations have "finite dimensional" solutions, often found in n-dimensional Euclidean space. For example, often such solutions solve equilibrium versions of agent's first order conditions under local or global second order conditions to obtain pricing systems that define economic equilibrium. Further, often solutions to such equations cannot be constructed in "closed-form"; so approximate solution methods are required. We shall then discuss a bit about numerical solutions to systems of equations. Finally, as agent decision problems and economic equilibrium equations often depend on "deep parameters" of the economic environment, economic equilibrium is the solution to a parameterized systems of equations. This situation generates the need to study "fixed point" methods for parameterized systems of equations. So we will spend a some time discussing various methods for solving systems of equations (all in $\mathbf{R}^n$) that often arise in problems of economic equilibrium, as well as obtaining approximate solutions when necessary.

So, the point of this course is to introduce each of you to using mathematics to formalize your economic arguments. I should also mention, this is **not** a pure mathematics course; rather, this is an course in tools for economic theory (and econometric estimation). Most of our work will be done in standard "Euclidean" spaces (finite dimensional). I will not emphasize proofs here, although, I will often sketch how the proof works. (Proofs of all arguments discussed in this course are found in standard reference books. See list below). So the course is not about just mathematics; it is about economics.

Reading Materials:

Texts:

Sundaram, *First Course in Optimization Theory,* Cambridge University Press
Kolmogorov and Fomin, *Introductory Real Analysis,* Dover Press edition.

The books by Sundaram and Kolmogorov and Fomin are both excellent. The Sundaram book samples discussion from every subject in the course outline. It will be one guiding direction for the actual topics we cover in the course. Kolmogorov and Fomin is a great reference book of important aspects of both real and functional analysis. It will also be used a good bit in the class. In all cases, feel free to look for used copies of any of these books also (actually, I strongly suggest you look on websites such as Amazon and Bookfinder for finding cheaper used copies of these books, as well as any of the books below that are mentioned).

Finally, one good book that is related to the subjects covered in this course is

Dean Corbae, Max Stinchcombe, Juraj Zeman, *Introduction to Mathematical Analysis for Economic Theory and Econometrics,* Princeton Press.

In general, this book is advanced, and is not being used as a textbook for the current course. Still, Chapters 1-6 are great references for the material in this course, and I recommend the book.

Other Reference Books, Mathematics

Michael Spivak, *Calculus on Manifolds*, HarperCollins.
Angus E Taylor, *General Theory of Functions and Integration,* Blaisdell Press
Angus E Taylor, *Advanced Calculus,* Blaisdell Press.
T. Apostol, *Mathematical Analysis,* Addison Wesley
T. Apostol, *Calculus, vol 1 and 2,* Blaisdell Press.
Kolmogorov and Fomin, *Introductory Real Analysis*, Dover Press.
J. Marsden and M. Hoffman, *Elementary Classical Analysis,* Freeman Press
J. Munkres, *Topology: A First Course,* Prentice Hall
H. Royden, *Real Analysis,* MacMillan Press
W. Rudin, Principles of Mathematical Analysis, McGraw Hill
S. Saks, *The Theory of the Integral,* Warszawa-Lwow, G.E. Stechert and Co Press

Other Reference Books, Economics

Ross Starr, *General Equilibrium Theory: An Introduction,* Cambridge Press
Donald Katzner, *A Walrasian Vision of the Microeconomy,* Michigan Press.

Andreu MasColell, *The Theory of General Economic Equilibrium,* Cambridge Press
Hal Varian, *Intermediate Microeconomics,* Norton
Hal Varian, *Microeconomics,* (Various editions) Norton
Martin, Osborne, *A Course in Game Theory,* MIT Press
E. Silberberg, *Structure of Economics,* (Various editions), McGraw Hill
B. Binger, *Microeconomics with Calculus,* Addison Wesley
A. Dixit, *Optimization in Economic Analysis,* Oxford Press
G. Jehle and P. Reny, *Advanced Microeconomic Theory,* Addison Wesley

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The preferred way of contacting me is via email.

Grading:

Your initial course grade is based on a three term exams, a final exam, and a course project, each comprising 20% of your initial course grade. The term exams are currently set for September 24, October 22, and November 14**.** The final exam will be scheduled per the ASU final exam schedule on Dec 14 (in the morning). The reason for four (shorter) exams is to prevent there being too much material covered on a single exam. Your term paper will be due at the beginning of your final exam, **no exceptions**.

Per the exams, each will be given in a standard format (i.e., problems and short answer). Your course project will be a 20 page paper that will study an application of mathematics to a specific economic problem. That is, the course project will be a paper where the student presents a formal model of an economic argument, and uses mathematical tools to characterize the structure of that model at hand. I will mention problems and applications during lecture which I suggest students consider working. Students can feel free to find other applications. As mentioned above, this paper is due December 14 at the beginning of your final. You will need to give me hardcopy of this paper at that time (and keep an electronic copy for your records).

Additionally, student performance on homeworks will be given a grade evaluation also. I will typically randomly select what homework problems I want to correct. The HWs will be grades on a 1-3 scale (1=fail, 2=good, 3=excellent). Typically, a homework will be given every 2 weeks. If the average grade you receive on HW is below a 2, I will lower your letter grade by 1 grade level (e.g., A- to B- if you HW grade is a 1). Further, your HW grade will be used to break "ties" if you are close to another letter grade. For example, if your HW grade is near a 3, and you are on the border of A-/B+, I will give you an A-; if not, you get the B+.

Course Topic Outline

This is a set of potential topics for our course. We will study a strict subset of this material (as the material listed below would require much more time than 1 semester). The exact material covered will be determined, in part, by both the preparation of students, as well we feel as a group will be most useful for your future work in graduate school. I will announce the specific topics we shall cover as the semester unfolds w/ a sufficient lead time you will be able to prepare for class. I think this flexibility should be a benefit, not a complication.

1. Motivating Examples

Examples of applications of mathematics in various field of economics and game theory that will be used to motivate the material in the course.

2. Some Results in Mathematical Analysis

2.1. *Basic Real/Functional Analysis* .

The real number system, induction, algebra of sets, Relations, functions, correspondences, inverse/implicit relations, order relations, preorders, application to consumer choice, partially ordered sets and monotone mappings, spaces of interest in economies (e.g, convex sets, linear (vector) spaces, normed spaces, metric spaces, compactness in $\mathbf{R}^n$, basic "function spaces", partially ordered sets, lattices, Veinott's powerdomain, etc.)

2.2. *Differentiability* .

Limits, continuity, Directional derivatives, Dini directionals, smooth functions, inverse/implicit function theorems, higher order derivatives, basic matrix algebra.

3. Parameterized Optimization under Convexity Conditions

3.1. *Convexity and Separation* .

Convex Sets, Half-spaces, Separation theorems, dual representations of convex functions, Minkowski representation theorem for convex sets (and duality), differentiability of convex functions and subgradients, convex cones

3.2. *Maximum Theorems* .

Continuity in $\mathbf{R}^n$, uniform continuity, Lipschitz continuity, Weierstrass theorem, Berge's maximum theorems, definition of the value function, basic structure of optimal solutions.

3.3. *Convex programming* .

Unconstrained convex optimization, constrained convex optimization, classical Lagrangians, constraint qualifications, duality, saddlepoint stability and minimax game interpretation of optimal solutions, Karash-Kuhn-Tucker (KKT) theory for first order characterizations of optimal solutions, envelope theorems, economic interpretation of KKT multipliers.

3.4. ***Some extensions of the theory*** .

Quasiconvex functions, quasiconvex programming, and issues relationships w/ KKT theory and conjugation via surrogate duality.

3.5. ***Smooth comparative Statics of optimal solutions*** .

Implicit function theorems and "smooth" comparative statics of optimal solutions in a parameter (w/ a discussion of Cramer's rule for computation)

4. Lattice Programming*

Veinott's maximum theorems, supermodular/submodular functions, increasing differences (w/ a discussion of ordinal generalizations), supermodular lattice programming (w/ remarks on "ordinal" generalizations), Constrained supermodular programming (i.e, "C_i-supermodular programming), dual formulations of constrained lattice programming problems, preserving supermodular structure to value functions and multistage lattice programming, applications: economic growth, game theory, consumer theory, producer theory.

5. Solving Parameterized equations

5.1. ***Some important Fixed point theorems*** .

Banach's contraction mapping theorem, Brouwer's theorem (w/ reference to Kakutani's theorem), Tarski's theorem, Amann's theorem, each with emphasis on systems of equations in $\mathbf{R}^n$.

5.2. ***Uniqueness of solutions*** .

Geometric conditions for uniqueness, contractive conditions for uniqueness, Janos and Bessaga stability, Converse of the Contraction Mapping theorem.

5.3. ***Fixed point comparative statics*** .

Selections, more on implicit and inverse function theorems, Bonsall-Nadler theorems, Veinott's fixed point comparative statics theorem.

5.4. ***Applications*** .

Equilibrium in a simple exchange economy, game theory, and steady states in economic growth.

6. Basics of Dynamic Programming

6.1. ***Sequential decision problems and Aggregation*** .

Multistage decision theory and aggregation, sequential value functions, sequential constrained optimization problems and Kuhn-Tucker theory, with micro applications.

6.2. ***Dynamic programming*** .

Nonstationary dynamic programming, stationary dynamic programming, principle of optimality, w/ macro applications.