An Application of Kleene's Fixed Point Theorem to Dynamic Programming: A Note*

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Abstract

In this note, we show that the least fixed point of the Bellman operator in a certain set can be computed by value iteration whether or not the fixed point is the value function. As an application, we show one of the main results of Kamihigashi (2014, "Elementary results on solutions to the Bellman equation of dynamic programming: existence, uniqueness, and convergence," *Economic Theory* 56, 251–273) with a simpler proof.

Keywords: Dynamic programming, Bellman equation, value function, fixed point.

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1 Introduction

Dynamic programming is one of the most important tools in studying dynamic economic models. Recently, under a minimal set of conditions, Kamihigashi (2014a) showed that the value function of a stationary dynamic programming problem can be computed by value iteration.

In this note, we show that the least fixed point of the value iteration algorithm, or the Bellman operator, in a certain set can be computed whether or not the fixed point is the value function. Although it is insufficient for computing the value function itself, this result is significant in that it separates the issue of convergence, or computability, from the existence and uniqueness of a fixed point of the Bellman operator.

We prove the above result by applying what is known as "Kleene's fixed point theorem," which has rarely been used directly in economics.[1] We present this fixed point theorem in the next section. In Section 3, we introduce some definitions and notations. In Section 4, we prove our main result. In Section 5, we illustrate the usefulness of our main result by showing one of the main results of Kamihigashi (2014a) with a simpler proof.

2 Kleene's Fixed Point Theorem

In this section, we present "Kleene's fixed point theorem" (e.g., Baranga, 1991). We begin with some mathematical terminology. Let $(P, \preceq)$ be a partially ordered set; i.e., $\preceq$ is a reflexive, antisymmetric, and transitive binary relation on P. An *upper bound* of a set $Q \subset P$ is an element $p \in P$ satisfying $q \preceq p$ for all $q \in Q$. The *least* element of $Q \subset P$ is an element $p \in Q$ satisfying $p \preceq q$ for all $q \in Q$; note that the least element, if it exists, is uniquely defined since $\preceq$ is antisymmetric.[2] The *supremum* of $Q \subset P$, denoted as $\sup Q$, is the least upper bound of Q.

A sequence $\{p_n\}_{n\in\mathbb{N}}$ is called *increasing* if $p_n \preceq p_{n+1}$ for all $n \in \mathbb{N}$. We say that $(P, \preceq)$ is ω-*complete* if every increasing sequence in P has a supremum in P. A function $f : P \to P$ is called ω-*continuous* if for every increasing sequence $\{p_n\}$ in P having a supremum, we have $f(\sup_{n\in\mathbb{N}} p_n) = \sup_{n\in\mathbb{N}} f(p_n)$. An ω-continuous function f is necessarily increasing in the sense that $f(p) \preceq f(q)$ whenever $p \preceq q$.

[1] One of the exceptions is Vassilakis (1992).

[2] That is, if $p \preceq q$ and $q \preceq p$, then $p = q$.

Theorem 1. *Let $(P, \preceq)$ be an ω-complete partially ordered set. Let $f : P \to P$ be ω-continuous. Suppose that there exists $\underline{p} \in P$ such that $\underline{p} \preceq f(\underline{p})$. Then $\underline{p}^* \equiv \sup_{n\in\mathbb{N}} f^n(\underline{p})$ is the least fixed point of f in $\{p \in P : \underline{p} \preceq p\}$.*

Proof. See Stoltenberg-Hansen et al. (1994, p. 24). □

Baranga (1991) calls the above result "Kleene's Fixed Point Theorem," while Sangiorgi (2009, p.35) states that "it is indeed unclear who should be credited for the theorem."[3] According to Jachymski (2000, p. 249), Theorem 1 is equivalent to the Tarski-Kantorovitch fixed point theorem (e.g., Granas and Dugundji, 2003, Theorem 1.2, p. 26).[4]

3 Dynamic Programming

Our setup is identical to that of Kamihigashi (2014a). Here we briefly introduce definitions and notations necessary for presenting our results.

Let X be a set, and Γ be a nonempty-valued correspondence from X to X. Let D be the graph of Γ:

$$D = \{(x, y) \in X \times X : y \in \Gamma(x)\}. \tag{1}$$

Let $u : D \to [-\infty, \infty)$. Let Π and $\Pi(x_0)$ denote the set of feasible paths and that of feasible paths from x_0, respectively:

$$\Pi = \{\{x_t\}_{t=0}^{\infty} \in X^{\infty} : \forall t \in \mathbb{Z}_+, x_{t+1} \in \Gamma(x_t)\}, \tag{2}$$

$$\Pi(x_0) = \{\{x_t\}_{t=1}^{\infty} \in X^{\infty} : \{x_t\}_{t=0}^{\infty} \in \Pi\}, \qquad x_0 \in X. \tag{3}$$

Let $\beta \geq 0$. The value function $v^* : X \to \overline{\mathbb{R}}$ is defined by

$$v^*(x_0) = \sup_{\{x_t\}_{t=1}^{\infty} \in \Pi(x_0)} \mathop{\mathrm{L}}_{T\uparrow\infty} \sum_{t=0}^{\infty} \beta^t u(x_t, x_{t+1}), \qquad x_0 \in X, \tag{4}$$

where $\mathrm{L} \in \{\underline{\lim}, \overline{\lim}\}$ with $\underline{\lim} = \liminf$ and $\overline{\lim} = \limsup$. We define

$$\Pi^0 = \left\{ \{x_t\}_{t=0}^{\infty} \in \Pi : \mathop{\mathrm{L}}_{T\uparrow\infty} \sum_{t=0}^{\infty} \beta^t u(x_t, x_{t+1}) > -\infty \right\}, \tag{5}$$

[3] To be precise, Sangiorgi (2009) refers to a special case of Theorem 1.

[4] The Tarski-Kantorovitch fixed point theorem has recently been used rather extensively in economics; see Mirman, Morand, and Reffett (2008), Van Zandt (2010), Balbus, Reffett, and Wozny (2013, 2014), and McGovern et al. (2013) among others.

$$\Pi^0(x_0) = \{\{x_t\}_{t=1}^\infty \in \Pi(x_0) : \{x_t\}_{t=0}^\infty \in \Pi^0\}, \qquad x_0 \in X. \tag{6}$$

Following the convention that $\sup \emptyset = -\infty$, we see that

$$\forall x_0 \in X, \quad v^*(x_0) = \sup_{\{x_t\}_{t=1}^\infty \in \Pi^0(x_0)} \operatorname*{L}_{T\uparrow\infty} \sum_{t=0}^{\infty} \beta^t u(x_t, x_{t+1}). \tag{7}$$

Let V be the set of functions from X to $[-\infty, \infty)$. The Bellman operator B on V is defined by

$$(Bv)(x) = \sup_{y\in\Gamma(x)} \{u(x, y) + \beta v(y)\}, \qquad x \in X, v \in V. \tag{8}$$

A fixed point of B is a function $v \in V$ such that $Bv = v$.

The partial order $\leq$ on V is defined in the usual way:

$$v \leq w \quad \iff \quad \forall x \in X,\ v(x) \leq w(x). \tag{9}$$

Given $v, w \in V$ with $v \leq w$, we define the order interval $[v, w]$ by

$$[v, w] = \{f \in V : v \leq f \leq w\}. \tag{10}$$

4 The Main Result

We are ready to prove the main result of this note as a consequence of Kleene's fixed point theorem.

Theorem 2. *Suppose that there exist $\underline{v}, \overline{v} \in V$ such that*

$$\underline{v} \leq \overline{v}, \tag{11}$$

$$B\underline{v} \geq \underline{v}, \tag{12}$$

$$B\overline{v} \leq \overline{v}. \tag{13}$$

Then the following conclusions hold:

(a) $\underline{v}^* \equiv \sup_{n\in\mathbb{N}}(B^n\underline{v})$ *is the least fixed point of B in $[\underline{v}, \overline{v}]$.*

(b) *The increasing sequence $\{B^n\underline{v}\}_{n\in\mathbb{N}}$ converges to $\underline{v}^*$ pointwise.*

Proof. By (11)–(13), B maps $[\underline{v}, \overline{v}]$ into itself. We apply Kleene's fixed point theorem to $B : [\underline{v}, \overline{v}] \to [\underline{v}, \overline{v}]$. For this purpose, it suffices to verify that $[\underline{v}, \overline{v}]$ is ω-complete and that B is ω-continuous on $[\underline{v}, \overline{v}]$.

To see that $[\underline{v}, \overline{v}]$ is ω-complete, let $\{v_n\}_{n\in\mathbb{N}}$ be an increasing sequence in $[\underline{v}, \overline{v}]$. Then the pointwise supremum of $\{v_n\}$ is equal to $\sup_{n\in\mathbb{N}} v_n$, the least upper bound of $\{v_n\}$. Since $\underline{v} \leq v_n \leq \overline{v}$ for all $n \in \mathbb{N}$, we have $\sup_{n\in\mathbb{N}} v_n \in [\underline{v}, \overline{v}]$. It follows that $[\underline{v}, \overline{v}]$ is ω-complete.

To see that B is ω-continuous on $[\underline{v}, \overline{v}]$, let $\{v_n\}_{n\in\mathbb{N}}$ be an increasing sequence in $[\underline{v}, \overline{v}]$ again. Let $x \in X$. We have

$$[B(\sup_{n\in\mathbb{N}} v_n)](x) = \sup_{y\in\Gamma(x)} \{u(x,y) + \beta(\sup_{n\in\mathbb{N}} v_n)(y)\} \tag{14}$$

$$= \sup_{y\in\Gamma(x)} \sup_{n\in\mathbb{N}} \{u(x,y) + \beta v_n(y)\} \tag{15}$$

$$= \sup_{n\in\mathbb{N}} \sup_{y\in\Gamma(x)} \{u(x,y) + \beta v_n(y)\} \tag{16}$$

$$= [\sup_{n\in\mathbb{N}} (B v_n)](x), \tag{17}$$

where (15) holds since $u(x,y)$ is independent of n, and (16) follows by interchanging the two suprema using Kamihigashi (2008, Lemma 1). Since x is arbitrary, it follows that $B \sup_{n\in\mathbb{N}} v_n = \sup_{n\in\mathbb{N}} B v_n$. That is, B is ω-continuous on $[\underline{v}, \overline{v}]$.

Now by Kleene's fixed point theorem, conclusion (a) follows. To see conclusion (b), note from (12) that $\underline{v} \leq B\underline{v} \leq B^2\underline{v} \leq \cdots$. Thus $\{B^n\underline{v}\}_{n\in\mathbb{N}}$ is increasing, and for each $x \in X$, we have $(B^n\underline{v})(x) \uparrow \underline{v}^*(x)$ as $n \uparrow \infty$ by the definition of $\underline{v}^*$; i.e., $\{B^n\underline{v}\}$ converges to $\underline{v}^*$ pointwise. □

Theorem 2 allows one to compute the least fixed point of B in $[\underline{v}, \overline{v}]$ by iterating B from $\underline{v}$. Of course, this fixed point may not be the value function v^* without additional conditions, but it is exactly this point that makes Theorem 2 significant: convergence to the least fixed point is valid whether or not it is the value function.

5 Applications

In this section, we illustrate the usefulness of Theorem 2 by showing how to use it to prove one of the main results of Kamihigashi (2014a), which replaces $\underline{v}^*$ with the value function v^* in the conclusions of Theorem 2.

The next lemma slightly generalizes an argument used in the proof of Theorem 4.3 in Stokey and Lucas (1989), providing a sufficient condition for any fixed point v of B with $\underline{v} \leq v$ to satisfy $v^* \leq v$.[5]

Lemma 1. *Let $\underline{v} \in V$ be such that*

$$\forall \{x_t\}_{t=0}^{\infty} \in \Pi^0, \qquad \varliminf_{t \uparrow \infty} \beta^t \underline{v}(x_t) \geq 0. \tag{18}$$

Let $v \in V$ be a fixed point of B with $\underline{v} \leq v$. Then $v^* \leq v$.

Proof. See the Appendix. □

The following result is immediate from Theorem 2 and Lemma 1.

Lemma 2 (Kamihigashi, 2014a, Lemma 6.3)**.** *Let $\underline{v}, \overline{v} \in V$ satisfy (11)–(13) and (18). Then $v^* \leq \underline{v}^*$, where $\underline{v}^*$ is defined in Theorem 2.*

This result is proved in Kamihigashi (2014a) using the additional result that finite iterations of B correspond to finite-horizon approximations of the infinite-horizon problem (4) (Kamihigashi, 2014a, Lemma 6.2). Our proof of Lemma 2 shows that this approximation result is unnecessary once Theorem 2 and Lemma 1 are available. Let us now use Theorem 2 and Lemma 2 to prove one of the main results of Kamihigashi (2014a).[6]

Theorem 3 (Kamihigashi, 2014a, Theorem 2.2)**.** *Suppose that $v^* \in V$. Suppose further that there exists $\underline{v} \in V$ satisfying (12), (18), and*

$$\underline{v} \leq v^*. \tag{19}$$

Then the following conclusions hold:

(a) *v^* is the least fixed point of B in $[\underline{v}, \overline{v}]$.*

(b) *The increasing sequence $\{B^n \underline{v}\}_{n=1}^{\infty}$ converges to v^* pointwise.*

Proof. Since v^* is a fixed point of B (see Kamihigashi, 2014a, Lemma 2.1), (11)–(13) hold with $\overline{v} = v^*$. Define $\underline{v}^*$ as in Theorem 2. Then $\underline{v}^* \leq \overline{v} = v^*$. The reserve inequality holds by Lemma 2; thus $\underline{v}^* = v^*$. Now both conclusions follow from Theorem 2. □

As another application of Theorem 2, it can be used along with Lemma 6.1 in Kamihigashi (2014a) to show Theorem 2.1 in Kamihigashi (2014a).

[5] Although the lemma is not explicitly shown in Kamihigashi (2014a), it is recognized and mentioned in Kamihigashi (2014a, 2014b).

[6] In Kamihigashi (2014a, Theorem 2.2), v^* is shown to be the least fixed point of B in a larger set. The same set can be used here with an additional argument.

A Proof of Lemma 1

We show that $v^*(x_0) \leq v(x_0)$ for any $x_0 \in X$. Let $x_0 \in X$. If $\Pi^0(x_0) = \emptyset$, then $v^*(x_0) = -\infty \leq v(x_0)$. Suppose that $\Pi^0(x_0) \neq \emptyset$. Let $\{x_t\}_{t=1}^{\infty} \in \Pi^0(x_0)$. We have

$$
\begin{aligned}
v(x_0) &\geq u(x_0, x_1) + \beta v(x_1) && (20)\\
&\geq u(x_0, x_1) + \beta u(x_1, x_2) + \beta^2 v(x_2) && (21)\\
&\ \ \vdots && (22)\\
&\geq \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) + \beta^T v(x_T) && (23)\\
&\geq \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) + \beta^T \underline{v}(x_T). && (24)
\end{aligned}
$$

Applying $\mathrm{L}_{T\uparrow\infty}$ to the rightmost side, we have

$$
\begin{aligned}
v(x_0) &\geq \mathop{\mathrm{L}}_{T\uparrow\infty} \left\{ \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) + \beta^T \underline{v}(x_T) \right\} && (25)\\
&\geq \mathop{\mathrm{L}}_{T\uparrow\infty} \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) + \varliminf_{T\to\infty} \beta^T \underline{v}(x_T) && (26)\\
&\geq \mathop{\mathrm{L}}_{T\uparrow\infty} \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}), && (27)
\end{aligned}
$$

where (26) follows from the properties of $\varliminf$ and $\varlimsup$,[7] and (27) holds by (18). Since (25)–(27) hold for any $\{x_t\}_{t=1}^{\infty} \in \Pi^0(x_0)$, applying $\sup_{\{x_t\}_{t=1}^{\infty} \in \Pi^0(x_0)}$ to (27) and recalling (7), we obtain $v(x_0) \geq v^*(x_0)$.

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[7] We have $\varliminf(a_t + b_t) \geq \varliminf a_t + \varliminf b_t$ and $\varlimsup(a_t + b_t) \geq \varlimsup a_t + \varliminf b_t$ for any sequences $\{a_t\}$ and $\{b_t\}$ in $[-\infty, \infty)$ whenever both sides are well-defined (e.g., Michel, 1990, p. 706).

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