Generalized Envelope Theorems*

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Abstract

We derive a collection of generalized envelope theorems for a broad class of Lipschitz programs with both nonsmooth objectives and constraints applicable to many economic environments where nonconvexities play a key role.

We first provide sufficient conditions for the value function to be Lipschitz and obtain bounds for its upper and lower Directional Dini derivatives of this value function. Next we establish sufficient conditions for the directional differentiabiliy and/or differentiability of the value function, and show how standard smooth envelope theorems are special cases of our results.

We then apply our findings to decision models with discrete choices, to dynamic programming with and without concavity, and to derive new results on the existence and characterization of Markov equilibrium in dynamic economies with nonconvexities.

1 Introduction

The use of envelope theorems to characterize optimal solutions of constrained optimization problems is widespread in microeconomic and macroeconomic theory. In its simplest form, an envelope is basically an equality between the derivative of the value function and the derivative of the objective evaluated at the optimum along a fixed direction ignoring the "indirect" effects due to changes in the optimal solution. In smooth (meaning continuously differentiable) convex programs, envelopes are typically standard derivatives giving precise information concerning the rate of growth of the value function in all directions at a given point, and are therefore an essential tool for comparative statics in static and dynamic models.

While envelopes of convex unconstrained programs are derived directly from the objective of the optimization program, in convex constrained programs they are generally obtained from the Lagrangian, both cases requiring the use of subdifferentials.

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Nonconvexities, however, arise in many problems such as dynamic programming with discrete choices, constrained lattice programming problems, incentive constrained dynamic programs, "bi-level"/Stackelberg games, to name a few. And to make matters more difficult, objectives and/or constraints are not even smooth in many cases. Clearly one then cannot expect envelopes to be simple derivatives as many technical difficulties arise simultaneously: Some constraints may be active, Lagrange multipliers may not be unique, and the absence of traditional derivatives (or gradients) mandates the use of some sort of generalized gradients which, typically, are not singletons.

Nevertheless much progress has been made in the analysis of economic models with nonconvexities, resulting in findings that are spread out in many papers, each targeting specific applications (see, for instance, Amir, Mirman and Perkin [1], Bonnisseau and LeVan [9], Askri and LeVan [5], Milgrom and Segal [32], Ricon-Zapatero and Santos [41] and many others). This paper present a comprehensive and unified way to derive generalized envelope theorems for a large class of nonconvex and/or nonsmooth programs.

Fundamentally, we seek a class of programs constructed with continuous functions (objective and constraints) that are not necessarily convex and/or continuously differentiable, but with properties strong enough for the existence of some generalized derivatives. We show that Lipschitz programs meet this objective and provide a suitable environment for the generalization of existing classical envelope theorems.

An important feature of Lipschitz programs is the preservation of the Lipschitz property by maximization under relatively weak hypothesis, as demonstrated in Section 2. One of these hypothesis is a non-smooth constraint qualification related to the work of Hiriart-Urruty [23] and Auslender [6] and easily checked in many applications. The other, taken from the work of Clarke [11], imposes restrictions on the choice domain and the objective which are shown to be satisfied under some very general conditions. Together these two restrictions are sufficient for the value function to be Lipschitz.

Lipschitz functions have well defined Dini derivatives, and the first result in Section 3 gives lower and upper bounds for these derivatives, a characterization which may prove useful in computational work as well as for establishing the absolute continuity of the value function (an essential step in the proof of its supermodularity in a large class of models, as shown in Section 4). The rest of Section 3 consists in narrowing these bounds to sharpen the characterization of the rates of growth of the value function in specific directions. Gateaux differentiability is the next step since it permits comparative statics in all directions at a specific point.

The addition of Clarke regularity grants more power to Gateaux derivatives at specific points which then behave almost like bounds on the rate of growth of the value function in a neighborhood of that point, and are just one step short of continuous differentiability. Clarke regularity also, under some conditions, is preserved under maximization and can be thought of as generailization of convexity. Because continuously differentiable functions are necessarily Lipschitz while convex functions are upper Clarke regular, classical envelope theorems are just special cases of our more general results.

In Section 4 our results are applied to dynamic programming with and without concavity, decision models with discrete choices, and to a proof of existence of equilibrium in a large class of dynamic models. Finally, some brief but important mathematical definitions and results are gathered in the Appendix.

2 Lipschitz programs

We consider Lipschitz programs of the form:

$$\max_{a \in D(s)} f(a,s) \tag{1}$$

in which $f : A \times S \to \mathbb{R}$ is the objective function, and $D : S \rightrightarrows A$ the feasible correspondence defined as:

$$D(s) = \{a | g_i(a,s) \le 0, \quad i = 1, ..., p \text{ and } h_j(a,s) = 0, \quad j = 1, ..., q\}.$$

where $g_i : A \times S \to \mathbb{R}$, $i = 1, ..., p$ and $h_j : A \times S \to \mathbb{R}$, $j = 1, ..., n$.

The choice set A and the state space (or parameter space) S are both are open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. In contrast to standard "smooth" optimization problems in which constraints and objectives are continuously differentiable (i.e., C^1 or "smooth") all functions f, g_i and h_j are initially only assumed to be Lipschitz[1] at every $(a,s) \in A \times S$. The function $V : S \to \mathbb{R}$, defined as $V(s) = \max_{a \in D(s)} f(a,s)$, is the value function, and the correspondence $A^* : S \rightrightarrows A$, defined as $A^*(s) = \arg\max_{a \in D(s)} f(a,s)$, is the optimal solution correspondence. The classical Lagrangian[2] associated with the above program is:

$$L(a,s,\lambda,\mu) = f(a,s) - \lambda g(a,s) - \mu h(a,s)$$

where λ, and μ are vectors in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively.

2.1 Constraint Qualifications

Recall the definition of a KKT point, which requires the existence of a vector of multipliers satisfying a specific "multiplier rule".

Definition 1 *Given $s \in S$, $a \in D(s)$ is a Karush-Kuhn-Tucker (KKT) point of Program (1) if there exists a vector $(\lambda,\mu) \in \mathbb{R}^p_+ \times \mathbb{R}^q$ such that:*

$$0 \in \partial_a (f - \sum_{i=1}^{p} \lambda_i g_i - \sum_{j=1}^{q} \mu_j h_j)(a,s)$$

and $\lambda_i g_i(a,s) = 0$ for all $i = 1, ..., p$.

Denoting by $K(a,s)$ the closed and convex (but possibly empty) set of vectors (λ,μ) satisfying the above "multiplier rule" at (a,s), the role of constraint qualifications is precisely to guarantee this set is no-empty and bounded. A standard CQ in smooth programs is the Mangasarian-Fromovitz CQ (defined below), which requires the existence of a direction belonging to both the cone of inward directions of active inequality constraints and to the set of tangent directions of equality constraints.

[1] Lipschitz in the sense of "locally Lipschitz" (see Appendix for definitions and some mathematical results).

[2] If A is closed, then the abstract constraint $a \in A$ induces an additional term in the Lagrangian (see, for instance, Clarke [11], Chapter 6).

Definition 2 *The Mangasarian-Fromovitz Constraint Qualification (MFCQ) is satisfied at* $a^*(s) \in A^*(s)$ *if there exists* $y \in \mathbb{R}^n$ *such that:*

$$\begin{aligned} \nabla_a g_i(a^*(s), s) \cdot y &< 0,\ i \in I(a^*(s), s), \\ \nabla_a h_j(a^*(s), s) \cdot y &= 0\ j = 1, ..., q \end{aligned}$$

where $I(a^*(s), s)$ *is the set of indexes of the active inequality constraints (those for which* $g_i(a^*(s), s) = 0$*), and the matrix* $\nabla_a h(a^*(s), s)$ *has full rank.*

Gauvin ([17], Lemma 1) proved that, in smooth programs, MFCQ at $a^*(s)$ is equivalent to the compactness of $K(a^*(s), s)$. Kyparisis [30] sharpened this result under the following slightly less general condition which simply treats active inequality constraints for which multipliers are strictly positive ("binding constraints") as equality constraints.

Definition 3 *The Strict Mangasarian-Fromoviz Constraint Qualification (SMFCQ) is satisfied at* $a^*(s) \in A^*(s)$ *if there exists* $y \in \mathbb{R}^n$ *such that:*

$$\begin{aligned} &\nabla_a g_i(a^*(s), s) \cdot y < 0,\ i \in I_s(a^*(s), s) \\ &\nabla_a g_i(a^*(s), s) \cdot y = 0,\ i \in I_b(a^*(s), s) \\ &\nabla_a h_j(a^*(s), s) \cdot y = 0\ j = 1, ..., q \end{aligned}$$

where $I_s(a^*(s), s) = \{i \in I(a^*(s), s),\ \lambda_i = 0\}$*, and* $I_b(a^*(s), s) = \{i \in I,\ \lambda_i > 0\}$*, and the vectors* $\nabla_a g_i(a^*(s), s)$*,* $i \in I_b(a^*(s), s)$*,* $\nabla_a h_j(a^*(s), s)$*,* $j = 1, ..., q$ *are linearly independent.*

Kyparsis ([30], Proposition 1.1) showed that the SMFCQ is both necessary and sufficient for $K(a^*(s), s)$ to be a singleton in smooth programs.(see also Bonnans and Shapiro [8], Remark 4.49).

No classical gradients generically exist in Lipschitz programs, so we rely on a generalization of the MFCQ (referred to as the "Generalized MFCQ", or GMFCQ) introduced by Hiriart-Urruty [23] and stated in terms of Clarke's generalized gradients. Denoting by $\overline{g}(a^*(s), s)$ the vector of binding inequality constraints at $a^*(s)$, so that $\overline{g} : A \times S \to \mathbb{R}^{\overline{p}}$ (where $\overline{p} = Card(I(a^*(s), s))$), GMFCQ can be stated as follows:

Definition 4 *The Generalized Mangasarian-Fromovitz Constraint Qualification (GMFCQ) is satisfied at* $a^*(s) \in A^*(s)$ *if there exists* $y \in \mathbb{R}^n$ *such that:*

$$\forall (\gamma_a, \upsilon_a) \in \partial_a(\overline{g}, h)(a^*(s), s),\ \gamma_a \cdot y < 0,\ \text{and}\ \upsilon_a \cdot y = 0$$

and $\partial_a h(a^*(s), s)$ *is of maximal rank.*

Hiriart-Urruty ([23],Theorem 4.2) proved that the GMFCQ at some $a^*(s)$ implies the non-emptiness of $K(a^*(s), s)$.

Remark 5 *We note that:*

$$\partial_a \overline{g}(a^*(s), s) \subset \prod_{i \in I(a^*(s), s)} \partial_a g_i(a^*(s), s)$$

so this version of GMFCQ is slightly more general than that of Auslender ([6], Theorem 2.1).

2.2 Lipschitz value functions

The Lipschitz properties of both objective and constraints in Program (1) are, of course, not sufficient for the value function V to even be continuous, and additional restrictions are therefore needed to insure that V is Lipschitz. We adopt the following general hypothesis made in Clarke ([11], Hypothesis 6.5.1, page 241):

Criterion 6 *(Clarke's Hypothesis) $V(s)$ is finite and there exists a compact set Λ and a positive number ε_0 such that for all $s' \in \varepsilon_0 B(s)$ for which $V(s') \geq V(s) - \varepsilon_0$, necessarily $A^*(s') \cap \Lambda \neq \emptyset$.*

Although Clarke's Hypothesis is not expressed in terms of primitive data of the problem, we show that it is satisfied in three important cases. The first is a simple condition mentioned by Clarke, the second is the inf-compactness condition used by Bonnans and Shapiro [8] to derive stability results for differentiable programs, and the third is the uniform compactness condition central to the work of Gauvin and Dubeau [18].[3]

Proposition 7 *Clarke's Hypothesis in the following three cases:*

(i) (growth condition) $\forall r \in \mathbb{R}$, the sets $\{(a', s') \in A \times S,\ f(a', s) \geq r\}$ are compact;

(ii) (inf-compactness) there exists $r \in \mathbb{R}$ and a compact set $\Omega \subset A$ such that for every s' in a neighborhood of s the set $\{a' \in D(s'),\ f(a', s') \geq r\}$ is a nonempty set contained in Ω;

(iii) (uniform compactness) there exists a neighborhood S' of s such that $cl\left[\cup_{s' \in S'} D(s)\right]$ is compact.

Proof. (i). Given any $s \in S$ necessarily there exists some r such that $G(s) = \{a \in A, f(a, s) \geq r\} \cap D(s) \neq \varnothing$; by hypothesis $G(s)$ is compact so $A^*(s) = \arg\max_{D(s)} f(a, s) = \arg\max_{G(s)} f(a, s)$ is non-empty and $V(s)$ is finite. Consider $s' \in \varepsilon_0 B(s)$ such that $V(s') \geq V(s) - \varepsilon_0$. Since $A^*(s')$ is nonempty and included in the compact (by hypothesis) set $\Lambda = \{(a, s') \in A \times S, f(a, s') \geq V(s) - 2\varepsilon_0\}$, Clarke's Hypothesis is satisfied.

(ii). By inf-compactness at s, there exists r such that if $s' \in \varepsilon_0 B(s)$ then $\{a' \in D(s'),\ f(a', s') \geq r\}$ is nonempty and included in the compact set Ω. Letting $\Lambda = \Omega, \forall s' \in \varepsilon_0 B(s)$ necessarily $A^*(s') \subset \Omega$ and $A^*(s') \subset \{a' \in D(s'),\ f(a', s') \geq r\} \subset \Omega$ and $A^*(s')$ is nonempty thus $A^*(s) \cap \Lambda \neq \varnothing$ hence Clarke's Hypothesis is satisfied.

(iii). Uniform compactness of D near s implies the existence of a neighborhood S' of s such that $cl\left[\cup_{s' \in S'} D(s)\right]$ is compact. As a result:

$$\exists \varepsilon_0 > 0,\ s' \in \varepsilon_0 B(s) \Longrightarrow\ D(s') \text{ is compact}$$

since $D(s')$ is a closed subset of the compact $cl\left[\cup_{s' \in S'} D(s)\right]$. This implies that $A^*(s')$ is nonempty (the objective f is continuous on the compact $D(s')$), and Clarke's Hypothesis is satisfied by letting $\Lambda = cl\left[\cup_{s' \in S'} D(s)\right]$. ■

When combined with the GMFCQ, any one of these conditions implies a very powerful result analogous to Berge's maximum theorem on the preservation of continuity under maximization: The value function is Lipschitz and A^* is upper hemicontinuous. The upper hemicontinuity of the optimal correspondence is a very important property, since it implies that as s_n converges to s the maxima of $f(., s_n)$ become arbitrarily close to some of the maxima of $f(., s)$.

[3] A fourth mild compactness condition is given later in this section.

Theorem 8 *If the GMFCQ holds at any* $a^*(s) \in A^*(s)$ *and if Clarke's Hypothesis is satisfied (hence whenever any of the condition of Proposition 7 is satisfied),* V *is Lipschitz at* s *and* A^* *is upper hemicontinuous at* s*. Moreover:*

$$\partial V(s) \subset co\left\{ \bigcup_{a^*(s)\in A^*(s)} \bigcup_{(\lambda,\mu)\in K^*(a_n^*(s),s)} \partial_s(f-\lambda g-\mu h)(a^*(s),s) \right\}$$

Proof. The Lipschitz property (and therefore continuity) of V and the formula for the generalized gradient follow directly from Clarke [11] (Corollary 1, page 242). The upper hemicontinuity of A^* is established separately for the three conditions of Proposition 7.

(i). Given any $s_n \to s$ and any sequence $\{a_n\}$ such that $a_n \in A^*(s_n) \subset D(s_n)$, $V(s_n) = f(a_n, s_n) \to V(s)$ by continuity of V. Given $\varepsilon' > 0$ there exists N such that $\forall n \geq N$, $f(a_n, s_n) \geq V(s) - \varepsilon'$. By the growth condition the sequence $\{a_n, s_n\}_{n\geq N}$ belongs to a compact set, and therefore has a convergent subsequence to $\{a, s\}$. By continuity of f, $f(a_n, s_n) \to f(a, s)$, hence $f(a, s) = V(s)$, and by closeness of D at s, $a \in D(s)$, hence $a \in A^*(s)$.

(ii). Under the inf-compactness condition, there exists r such that the set $A_s = \{a' \in D(s'), f(a', s') \geq r\}$ is nonempty for all $s' \in \delta B(s)$ and is included in a compact set Ω. Thus there exists N such that $\forall n \geq N$, $a_n \in A^*(s_n) \subset A_s \subset \Omega$ and the sequence $\{(a_n, s_n)\}_{n\geq N}$ has a convergent subsequence to (a, s). By continuity of V and f $V(s_n) = f(a_n, s_n) \to f(a, s) = V(s)$ and closeness of D implies the desired result.

(iii). Since V is continuous at s, the map $L : s \to \{a, f(a, s) - V(s) \geq 0\}$ is closed at s. Under the uniform compactness condition, since $A^*(s) = L(s) \cap D(s)$, the correspondence $A^* : s \to A^*(s)$ is the intersection of the closed mapping L with the upper hemicontinuous (since closed and uniformly compact) mapping D. Consider then $s_n \to s$ and any $a_n \in A^*(s_n) = L(s_n) \cap D(s_n)$. Since D is upper hemicontinuous at s , there exists a subsequence of a_n converging to some $a \in D(s)$. Since L is closed at s, the limit a of the subsequence of a_n necessarily belong to $L(s)$. Thus, $a \in A^*(s) = L(s) \cap D(s)$, which proves that A^* is upper hemicontinuous at s. ■

Note again that the continuity of V cannot come from a direct application of Berge's maximum theorem since the feasible correspondence D is not necessarily continuous, even though all constraints are continuous. The correspondence D defined as:

$$D(s) = \{(x, y), x + y \leq s \text{ and } (s - 11)(10 - x) \leq 0\}$$

is not continuous at $s = 11$.

3 Generalized Envelope Theorems

The same conditions sufficient for the preservation of the Lipschitz property under maximization in Theorem (8) are shown below to also be sufficient for the derivation of specific bounds for the Dini derivatives of the value function. In the rest of this section, we impose several stronger conditions on the primitive data (such as concavity, differentiability, Clarke regularity, continuous differentiability) to derive sharper envelope theorems going beyond the simple existence of bounds all the way to C^1 envelopes.

3.1 A Central Result on Stability Bounds

Under the conditions of Theorem 8 the value function is Lipschitz, so its Dini derivatives exist. Our first result states specific bounds for these Dini derivatives, obtained as a consequence of Clarke [11] Corollary 4 (page 243) (see also Tarafdar [45] for an alternative proof independent of Clarke's results), and expressed in terms of the primitive data.

Theorem 9 *If the GMFCQ holds at any $a^*(s) \in A^*(s)$ and under Clarke's Hypothesis, for any $x \in \mathbb{R}^m$:*

$$D^+V(s;x) \leq \max_{a^*(s)\in A^*(s)} \left(\sup_{\lambda \in K(a^*(s),s)} \left(\max_{\theta \in \partial_s (f-\lambda g-\mu h)(a^*(s),s)} \{\theta \cdot x\} \right) \right)$$

and:

$$\max_{a^*(s)\in A^*(s)} \inf_{\lambda \in K(a^*(s),s)} \left(\min_{\theta \in \partial_s (f-\lambda g-\mu h)(a^*(s),s)} \{\theta \cdot x\} \right) \leq D_+V(s;x)$$

Proof. Omitting the equality constraints to simplify the proof, the Lipschitz program (1) becomes:

$$-V(s) = \min -f(a,s) \text{ s.t. } g(a,s) \leq 0$$

and is identical to the "modified program":

$$-V(s) = \min -f(a,a') \text{ s.t.} g(a,a') \leq 0 \text{ and } -a' + s = 0$$

with its associated Lagrangian:[4]

$$L_m((a,a'),s,\lambda,\theta) = -f(a,a') + \lambda g(a,a') + \theta\left[-a' + s\right]$$

The two programs have the same set of solutions, in the sense that $a^*(s) \in A^*(s)$ if and only if $(a^*(s),s) \in A^*_m(s)$, and the same set of multipliers, in the sense that $\lambda \in K(a^*(s),s)$ if and only if $(\lambda,\theta) \in K_m(a^*(s),s)$.

By Theorem 6.1.1 in Clarke [11], there exists $\lambda \geq 0$, and θ such that $\lambda g(a^*(s),s) = 0$ and:

$$0 \in \partial_{(a,a')} L((a^*(s),s),s,\lambda,\theta)$$

which implies the existence of $(\sigma_a + \lambda\gamma_a) \in \partial_a(-f+\lambda g)(a^*(s),s)$ and of $(\sigma_{a'} + \lambda\gamma_{a'}) \in \partial_{a'}(-f+\lambda g)(a^*(s),s)$ such that, for all (u,v) :

$$0 = (\sigma_a + \lambda\gamma_a)u + (\sigma_{a'} + \lambda\gamma_{a'})v - \theta v$$

As a result, necessarily:

$$\sigma_a + \lambda\gamma_a = 0$$

[4]The subscript m is used to identify objects relevant to the "modified program".

and:

$$\theta = \sigma_{a'} + \lambda \gamma_{a'} \in \partial_{a'}(-f + \lambda g)(a^*(s), s)$$

The assumptions of Corollary 4 in Clarke [11] are satisfied (in Clarke's notations, if the GMFCQ holds at each $a^*(s)$ then $M^0(\sum) = \{0\}$) hence:

$$(-V)^+(s; x) \leq \inf_{(a^*(s),s)\in\sum} \sup_{(\lambda,\theta)\in M^1(a,s)} \{\theta \cdot x\} = \inf_{a^*(s)\in A^*(s)} \sup_{\lambda\in K(a^*(s),s)} \sup_{\theta\in\partial_{a'}(-f+\lambda g)(a^*(s),s)} \{\theta \cdot x\}$$

and:

$$(-V)_+(s; x) \geq \inf_{a^*(s)\in A^*(s)} \inf_{\lambda\in K(a^*(s),s)} \inf_{\theta\in\partial_s(-f+\lambda g)(a^*(s),s)} \{\theta \cdot x\}$$

Noticing that $(-V)^+(s; x) = -V_+(s; x)$ and that $(-V)_+(s; x) = -V^+(s; x)$, we obtain:

$$\begin{aligned} V^+(s; x) &\leq -\inf_{a^*(s)\in A^*(s)} \inf_{\lambda\in K(a^*(s),s)} \inf_{\theta\in\partial_s(-f+\lambda g)(a^*(s),s)} \{\theta \cdot x\} \\ &= \sup_{a^*(s)\in A^*(s)} \sup_{\lambda\in K(a^*(s),s)} \sup_{\theta\in\partial_s(f-\lambda g)(a^*(s),s)} \{\theta \cdot x\} \end{aligned}$$

and in a similar manner:

$$V_+(s; x) \geq \sup_{a^*(s)\in A^*(s)} \inf_{\lambda\in K(a^*(s),s)} \inf_{\theta\in\partial_s(f-\lambda g)(a^*(s),s)} \{\theta \cdot x\}$$

which proves the desired result, noting that both sets $\partial_s(f - \lambda g)(a^*(s), s)$ and $A^*(s)$ are compact valued, so inf and sup become min and max, respectively. ■

3.2 Differentiability of the Value Function

3.2.1 De-constraining a program

Getting additional characterization of the rate of growth of the value function requires more than just the Lipschitz structure of both the objective and the constraints. Things would of course be simpler in the absence of constraints, so our first results concern Lipschitz programs satisfying a mild compactness condition which imply that constraints can be locally (at least) ignored and that and Clarke's Hypothesis is automatically satisfied.

Proposition 10 *Suppose that there exist a compact set Λ and a neighborhood $N(s)$ of s such that $\forall s' \in N(s)$, $A^*(s') \subset \Lambda \subset D(s')$, then V is Lipschitz at s. Furthermore, if f is upper Clarke regular at s for each $a^*(s) \in D(s)$, then V is Gateaux differentiable and:*

$$V'(s; x) = \max_{a^*(s)\in A^*(s)} f_s(a^*(s), s; x)$$

Proof. The hypothesis implies that:

$$V(s) = \max_{a\in\Lambda} f(a, s)$$

hence by Berge's theorem of the maximum, V is continuous at s thus finite. Clarke's Hypothesis is satisfied by choosing any $\varepsilon_0 > 0$ such that $\varepsilon_o B(s) \subset N(s)$. Consequently, V is Lipschitz at s by Theorem 8, and:

$$\partial V(s) \subset co\left\{\cup_{a^*(s)\in A^*(s)}\partial_s f(a^*(s), s)\right\}$$

The additional assumption of Clarke upper regularity permits squeezing together upper and lower Dini derivatives of V. Indeed, for all $a^*(s) \in A^*(s)$:

$$\begin{aligned}
\liminf_{t\downarrow 0} \frac{V(s+tx)-V(s)}{t} &= \liminf_{t\downarrow 0} \frac{f(a^*(s+tx), s+tx) - f(a^*(s), s)}{t} \\
&\geq \liminf_{t\downarrow 0} \frac{f(a^*(s), s+tx) - f(a^*(s), s)}{t} \\
&= f_s(a^*(s), s; x)
\end{aligned}$$

the last equality obtained from the Gateaux differentiability of f in s for each $a^*(s)$. In addition:

$$\begin{aligned}
\limsup_{t\downarrow 0} \frac{V(s+tx)-V(s)}{t} &\leq \max_{a^*(s)\in A^*(s)} \max_{\varsigma\in\partial f(a^*(s),s)} \{\varsigma \cdot x\} \\
&= \max_{a^*(s)\in A^*(s)} f^o(a^*(s), s; x) \\
&= \max_{a^*(s)\in A^*(s)} f_s(a^*(s), s; x)
\end{aligned}$$

the last equality from the Clarke upper regularity of f in s for each $a^*(s)$. Upper and lower Dinis therefore coincide, hence V is Gateaux differentiable and:

$$V'(s;x) = \max_{a^*(s)\in A^*(s)} f_s(a^*(s), s; x) = \max_{a^*(s)\in A^*(s)} f_s(a^*(s), s) \cdot x$$

Finally, we also note that this implies:

$$-V'(s;-x) \leq V'(s;x)$$

■

Remark 11 *Since continuously differentiable functions are necessarily upper Clarke regular, V is Gateaux differentiable whenever the compactness condition of Proposition 10 holds and f is continuously differentiable in s for each $a^*(s) \in D(s)$. This is precisely Lemma 3.1 in Amir, Mirman and Perkins [1].*

To generate sharper differentiability properties beyond Gateaux differentiability, one must seek restrictions guaranteeing the preservation of some form of regularity (upper or lower) of the objective under maximization. One can for instance, follow Clarke [11] and impose conditions sufficient for V to inherit the upper Clarke regularity of the objective, as done for instance in Askri and LeVan [5]. We state Clarke's result (specifically, Theorem 2.8.2 without (ii) since all functions are continuous) without proof.

Proposition 12 *Under the compactness condition of Proposition 10 if (i) there exists a neighborhood U of s such that each $s' \longmapsto f(a, s')$ is globally Lipschitz of rank K on U for each $a \in \Lambda$, (ii) $s' \longmapsto f(a, s')$ is upper Clarke regular at s for each $a^*(s) \in D(s)$ and (iii) the correspondence $(s', a') \rightrightarrows \partial_s f(a', s')$ is closed at $(a^*(s), s)$ for each $a^*(s) \in A^*(s)$ then V is upper Clarke regular at s and:*

$$\partial V(s) = co\left\{\cup_{a^*(s)\in A^*(s)} \partial_s f(a^*(s), s)\right\}$$

Alternatively, assuming the choice domain is convex, concavity (which implies lower Clarke regularity) is preserved under maximization, a property we exploit next by combining it with differentiability.

Corollary 13 *Under the compactness condition of Proposition 10, if f is concave in (a, s) and differentiable at s for each a, and if $graphD$ is convex, then V is concave and continuously differentiable, with:*

$$V'(s) = f_s(a^*(s), s)$$

for all $a^*(s) \in A^*(s)$.

Proof. Concave functions are lower Clarke regular, and Clarke regular differentiable functions are in continuously differentiable (see Appendix) so f is trivially upper Clarke regular. Consequently, by Proposition 10 V is Gateaux differentiable, and as noted in the proof of that proposition, $-V'(s; -x) \leq V'(s; x)$. If $graphD$ is convex, then V is concave thus lower Clarke regular, hence for all x:

$$V^{-o}(s; x) = V'(s; x) \leq V^0(s; x) = -V^{-o}(s' - x) = -V'(s; -x)$$

As a result:

$$V'(s; x) = V^{-o}(s; x) = V^0(s; x) = -V'(s; -x)$$

so V is continuously differentiable at s. ■

3.2.2 Constrained programs

Next we turn our attention to general Lipschitz programs with continuously differentiable objective and constraints and for which the SMFCQ guarantees that Clarke gradients are singletons and that the multiplier is unique. In that case the value function can be shown to be at least Gateaux differentiable.

Proposition 14 *Under Clarke's Hypothesis, if the SMFCQ holds at every optimal solution $a^*(s) \in A^*(s)$, and the primitive data is continuously differentiable in s, then V is Gateaux differentiable at s and:*

$$V'(s; x) = \max_{a^*(s)\in A(s)} \{L_s(a^*(s), s, \lambda, \mu) \cdot x\}$$

Proof. Follows from Theorem 8 and Theorem 9 given uniqueness of multipliers. ■

An alternative to the SMFCQ is to assume enough concavity to "squeeze" the lower and upper Dini bounds to obtain Gateaux (directional) envelopes, as done in Milgrom and Segal ([32], Corollary 5). We derive such a result for a less restrictive setting.

Corollary 15 *Under Clarke's hypothesis and if the GMFCQ holds at every $a^*(s) \in A^*(s)$, if the primitive data is continuously differentiable in s, f, and $-g$ concave, h affine in a, the derivatives f_s, $-g_s$, h_s are upper semicontinuous in a, then V is Gateaux differentiable and:*

$$V'(s;x) = \max_{a^*(s)\in A^*(s)} \min_{(\lambda,\mu)\in K(a^*(s),s)} \{L_s(a^*(s), s, \lambda, \mu) \cdot x\}$$

Proof. By Theorem 9:

$$\max_{a^*(s)\in K(a^*(s),s)} \min_{\lambda\in K(a^*(s),s)} L_2(a^*(s), s, \lambda, \mu) \leq D_+V(s;x)$$

Imposing additional conditions on the primitive data helps tighten the upper bound as follows. First, choose a sequence $\{t_n\} \downarrow 0$ such that:

$$\limsup_{t\downarrow 0} \frac{V(s+tx) - V(s)}{t} = \lim_{n\to\infty} \frac{V(s+t_nx) - V(s)}{t_n}$$

Next, consider a sequence $\{a^*(s + t_nx)\}$ with $a^*(s + t_nx) \in A^*(s + t_nx)$ for all $n \in \mathbb{N}$. By Clarke's Hypothesis, for n large enough all $a^*(s + t_nx)$ belong to a compact set, so without loss of generality we assume that $a^*(s + t_nx)$ converges to some a^*. By closeness of D, necessarily in $a^* \in D(s)$; by continuity of V, $V(s) = f(a^*, s)$. As a result $a^* \in A^*(s)$ is a global maxima. By strong duality, the Lagrangian has a global saddle point at (a^*, s, λ, μ) where $(\lambda, \mu) \in K(a^*, s)$. Thus, for any $(\lambda, \mu) \in K(a^*, s)$:

$$\begin{aligned}
&\lim_{n\to\infty} \frac{V(s+t_nx) - V(s)}{t_n} \\
&= \lim_{n\to\infty} \frac{L(a^*(s+t_nx), s+t_nx, \lambda_n, \mu_n) - L(a^*, s, \lambda, \mu)}{t_n}
\end{aligned}$$

where $(\lambda_n, \mu_n) \in K(a^*(s + t_nx), s + t_nx)$. Consequently for any $(\lambda, \mu) \in K(a^*, s)$:

$$\begin{aligned}
&\lim_{n\to\infty} \frac{V(s+t_nx) - V(s)}{t_n} \\
&\leq \lim_{t_n\to 0^+} \frac{L(a^*(s+t_nx), s+t_nx, \lambda, \mu) - L(a^*, s, \lambda, \mu)}{t_n} \\
&\leq \lim_{t_n\to 0^+} \frac{L(a^*(s+t_nx), s+t_nx, \lambda, \mu) - L(a^*(s+t_nx), s, \lambda, \mu)}{t_n} \\
&= L_s(a^*(s+t_nx), s, \lambda, \mu) \cdot x
\end{aligned}$$

The first and the second inequality follows from the fact that $(a_n^*(s + t_nx), s + t_nx, \lambda_n, \mu_n)$ and (a^*, s, λ, μ) are global saddle points of L for any $s + t_nx$ and s respectively.

As a result:

$$\limsup_{t\to 0^+} \frac{V(s+tx)-V(s)}{t}$$
$$\leq \min_{(\lambda,\mu)\in K(a^*(s),s)} L_s(a^*(s+t_n x), s, \lambda, \mu)\cdot x$$

Given that $a_n^*(s) \to a^* \in A^*(s)$ and that $L_s(.)$ is upper semicontinuous in its first argument, the above inequality implies:

$$D^+V(s;x)$$
$$\leq \max_{a^*(s)\in A^*(s)} \min_{(\lambda,\mu)\in K(a^*(s),s)} L_s(a^*(s), s, \lambda, \mu)\cdot x$$
$$\leq D_+V(s;x)$$

Thus $D_+V = D^+V$ and the result follows. ■

Next, in the absence of equality constraints, we show that if the primitive data is jointly concave and continuously differentiable, and if the SMFCQ hold for every optimal solution, then the value function is once continuously differentiable.

Corollary 16 *Under Clarke's hypothesis, if the primitive data is continuously differentiable in (a, s), f, and $-g$ are jointly concave in (a, s), and the SMFCQ holds at all $a^*(s)$ in $A^*(s)$, then V is continuously differentiable and:*

$$V'(s) = L_s(a^*(s), s, \lambda)$$

for any $a^*(s) \in A^*(s)$.

Proof. Under the SMFCQ, and with continuously differentiable primitives, the multiplier is unique and by Corollary 15 V is Gateaux differentiable with:

$$V'(s,x) = \max_{a^*(s)\in A^*(s)} \left\{ \left(f_s(a^*(s), s) - \lambda_{a^*(s)} g_s(a^*(s), s) \right) \cdot x \right\}$$

in which $\left\{\lambda_{a^*(s)}\right\} = K(a^*(s), s)$. In addition:

$$-V'(s;-x) \leq V'(s;x)$$

The concavity of V (inherited from that of f and $-g$), together with its Gateaux differentiability implies it is continuously differentiable hence lower Clarke regular therefore:

$$V'(s;x) = V^{-o}(s;x) \leq V^o(s;x) = -V^{-o}(s;-x) = -V'(s;x)$$

As a result:

$$V^o(s;x) = V^{-o}(s;x)$$

that is:

$$\max_{a^*(s)\in A^*(s)} \left\{ \left(f_s(a^*(s), s) - \lambda_{a^*(s)} g_s(a^*(s), s) \right) \cdot x \right\} = \min_{a^*(s)\in A^*(s)} \left\{ \left(f_s(a^*(s), s) - \lambda_{a^*(s)} g_s(a^*(s), s) \right) \cdot x \right\}$$

hence V is continuously differentiable at s and:

$$V'(s) = f_s(a^*(s), s) - \lambda_{a^*(s)} g_s(a^*(s), s)$$

for any $a^*(s) \in A^*(s)$ and $\left\{\lambda_{a^*(s)}\right\} = K(a^*(s), s)$. ■

We note that Corollary 16 does not require the set of optimal solutions to be singleton. However, any optimal solution along with its associated unique multiplier can be used to calculate the gradient of the value function.

4 Applications and Extensions

4.1 Optimization problems with discrete choice variables

We illustrate the use of non-smooth constraint qualifications, Lagrange multiplier rules, and of our envelope results in a simple consumer program with Lipschitz primitive data and discrete choices. We handle discrete constraints (in our example, to work full time or not to work at all) by rewriting them as equality constraints. The upside of this strategy is that these equality constraints are trivially continuously differentiable; the downside is that they restrict the choice of directions satisfying the MFCQ (since such direction must be in the set of tangent directions of equality constraints).

Consider then a consumer endowed with one unit of time and $e > 0$ units of the consumption good. The consumer chooses leisure $l = 0$ or $l = 1$ and consumption c so as to maximize utility given by $U(c, l)$. The utility function $U : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ is assumed to be continuous, increasing, and Lipschitz at every point in the interior of its domain. In addition, we assume that $U(c, 1) > U(c, 0)$ for all $c \geq 0$. The consumption good is produced by a firm with production function $F : \mathbb{R} \longrightarrow \mathbb{R}$ assumed to be increasing, continuous and Lipschitz at every point.

Consider then the program:

$$V(e) = \max U(c, l)$$

subject to:

$$\begin{aligned} &h(c, l) = l(1 - l) = 0 \\ &g_1(e, (c, l)) = c - F(1 - l) - e \leq 0 \\ &g_2(e, (c, l) = -c \leq 0 \end{aligned}$$

The multiplier rule characterizing KKT points is:

$$0 \in \partial_{(c,l)}\{U - \lambda_1 g_1 - \lambda_2 g_2 - \mu h\}(e, (c^*, l^*))$$

so that:

$$\begin{aligned} \lambda_1 - \lambda_2 &\in \partial_c U(c^*, l^*) \\ \mu\,(1 - 2l^*) &\in \partial_l U(c^*, l^*) - \lambda_1 \partial_l F(1 - l^*) \end{aligned}$$

together with the complementary slackness conditions:

$$\lambda_1(c^* - F(1-l^*) - e) = 0$$
$$\lambda_2 c^* = 0$$

First, we note that Clarke's hypothesis is satisfied because the choice domain $D(e)$ (a closed set) is included in the compact set $[0, F(1)+e] \times [0,1]$, hence the uniform compactness condition of Proposition 7 is trivially satisfied. Second, It is easy to verify that the GMFCQ is satisfied at any solution (c^*, l^*), as:

$$\partial_{(c,l)} h = (0, 1-2l^*)$$
$$\partial_{(c,l)} g_1 = \{1\} \times -\partial_l F(1-l^*)$$
$$\partial_{(c,l)} g_2 = (-1, 0)$$

so that:

$$\left(\partial_{(c,l)} h \cdot y = 0\right) \Longrightarrow (y_2 = 0)$$

Consequently, if $c^* = 0$ (and thus $\lambda_1 = 0$ and $l^* = 1$) then $(y_1, 0)$ with $y_1 > 0$ satisfies the GMFCQ while if $c^* > 0$ (hence $\lambda_2 = 0$) then $(y_1, 0)$ with $y_1 < 0$ satisfies the GMFCQ. The following result is then a direct consequence of proposition ??.

Proposition 17 *V is Lipschitz around any $e > 0$ and:*

$$D^+V(e;x) \leq \max_{(c^*,l^*)\in A^*(e)} \sup_{\lambda_1 \in K((c^*,l^*),e)} \lambda_1$$

and:

$$D_+V(e;x) \geq \max_{(c^*,l^*)\in A^*(e)} \inf_{\lambda_1 \in K((c^*,l^*),e)} \lambda_1 \geq 0$$

where $\lambda_1 = 0$ if $c^* = 0$ and $\lambda_1 \in \partial_c U(c^*, l^*)$ if $c^* > 0$.

We note that the SMFCQ can only be satisfied if none of the inequality constraints are active, i.e., only if $0 < c^* < F(1-l^*) + e$. When that is the case, Proposition 14 implies that V is Gateaux differentiable at any $e > 0$ since the primitive data is trivially continuously differentiable in e.

Alternatively, if U is continuously differentiable in its first argument, then the multiplier is unique since $\lambda_1 = U_c(c^*, l^*) \geq 0$ if $c^* > 0$, and V is Gateaux differentiable with:

$$V'(e;x) = \max_{(c^*,l^*)\in A^*(e)} U_c(c^*, l^*) \text{ if } c^* > 0$$

4.2 Lipschitz Dynamic Programming

We extend the results of Laraki and Sudderth [31] and Hinderer [22] on the preservation of Lipschitz continuity in recursive dynamic programs by weakening their global Lipschitz conditions on the primitive data to just (local) Lipschitzness.

Consider the following dynamic program:

$$V_{n+1}(s) = T(V_n)(s) = \max_{a \in D(s)} \{f(a,s) + \beta V_n(a)\}$$

in which $D(s) = \{a \in A \subset \mathbb{R}^n,\ g(a,s) \leq 0\}$, $s \in S \subset \mathbb{R}^m$ and $V_0 = 0$, with its corresponding Lagrangian:

$$L_{n+1}(a,s) = f(a,s) + \beta V_n(a) - \lambda g(a,s)$$

and solution set $A^*_{n+1}(s) = \arg\max_{a \in D(s)} f(a,s) + \beta V_n(a)$. Both functions f and g are only assumed to be Lipschitz in (a,s), and $0 < \beta < 1$.

Given $V_0 = 0$, the following result is a direct consequence of a repeated application of Theorem 8.

Proposition 18 *If the GMFCQ is satisfied for all $a^*_{n+1}(s) \in A^*_{n+1}(s)$ for each n, then if Clarke's Hypothesis is satisfied (or under any of the conditions in Proposition 7 the sequence $\{V_n\}$ is a sequence of Lipschitz functions, with Clarke gradients satisfying:*

$$\partial V_{n+1}(s) \subset co \left\{ \bigcup_{a^*_{n+1}(s) \in A^*_{n+1}(s)} \ \bigcup_{\lambda \in K^*(a^*(s),s)} \partial_s (f - \lambda g)(a^*_{n+1}(s), s) \right\}$$

Remark 19 *Of course, whether or not the GMFCQ and Clarke's hypothesis are satisfied will depend on the specific problem considered. The GMFCQ is easier to satisfy the fewer the number of active constraints (it is trivially satisfied when all constraints are inactive). Clarke's hypothesis is automatically satisfied if the choice domain is uniformely bounded (as in bounded growth models).*

It is well known that the sequence $\{V_n\}$ of Lipschitz functions converges uniformly to the unique continuous function V satisfying $V = T(V)$. Unfortunately, uniform limits of sequences of locally Lipschitz functions are not necessarily locally Lipschitz, since the Weistrass Approximation Theorem asserts that any continuous functions, Lipschitz or not, may be uniformly approximated by polynomials (which are Lipschitz).

Nevertheless, it is possible to prove that V is Lipschitz (and more) under certain conditions. One can, for instance, demand that a global Lipschitz condition be satisfied, as in Laraki and Sudderth [31] and Hinderer [22]. Alternatively, one can work impose sufficient structure on the primitive data to guarantee that all optimal solutions are located on a compact domain on which the objective is then globally Lipschitz, as in Askri and LeVan [5], in effect "de-constraining" the program.

Concavity, a feature of many economic models, is an important property since it is preserved under pointwise limits, and since concave functions are Lipschitz on the interior of their domain. We exploit the implications of concavity in dynamic programs next. requirement may be sufficient to guarantee that V^* is locally Lipschitz as discussed next.

4.2.1 Concave Dynamic Programming

Much more (than just Lipschitzness) can be revealed concerning the differentiability properties of the value function in the presence of concavity, providing the primitive data is also assumed to be differentiable with respect to s. The presence of multiple multipliers is of course a hindrance, but that too can be set aside if one assumes that the SMFCQ holds.

Proposition 20 *Assuming that (i) f and g are Lipschitz and concave in (a, s) as well as continuously differentiable in s, (ii) the derivatives f_s and g_s are upper semicontinuous in a, (iii) the MFCQ is satisfied at every optimal solution, and (iv) Clarke's Hypothesis is satisfied, then the Lipschitz value function V is concave and Gateaux differentiable with:*

$$V'(s,x) = \max_{a^*(s)\in A^*(s)} \min_{\lambda\in K(a^*(s),s)} (F_s(a^*(s),s) - \lambda g_s(a^*(s),s)) \cdot x$$

If, in addition, the SMFCQ is satisfied at every optimal solution, then V is continuously differentiable and:

$$V'(s) = f_s(a^*(s),s) - \lambda_{a^*(s)} g_s(a^*(s),s)$$

for any $a^*(s) \in A^*(s)$ and $\lambda_{a^*(s)} = K(a^*(s),s)$.

Proof. V is concave hence Lipschitz and satisfies the Lipschitz program:

$$V(s) = \max_{a\in D(s)} \{f(a,s) + \beta V(a)\}$$

Under the MFCQ a direct application of Corollary 15 to this program implies that V is Gateaux differentiable with:

$$V'(s,x) = \max_{a^*(s)\in A^*(s)} \min_{\lambda\in K(a^*(s),s)} (f_s(a^*(s),s) - \lambda g_s(a^*(s),s)) \cdot x$$

but the existence of multiple Lagrange multipliers generically prevents V from being continuously differentiable.

However, assuming next that the SMFCQ is satisfied at each optimal solution, the multiplier set is a singleton and by Corollary 16:

$$V'(s) = f_s(a^*(s),s) - \lambda_{a^*(s)} g_s(a^*(s),s)$$

for any $a^*(s) \in A^*(s)$ and $\lambda_{a^*(s)}$. ■

Our result on the continuous differentiability generalizes Benveniste and Scheinkman [7] by allowing the inequality constraints to be active at the optimal solution. The cost is a stronger constraint qualification (SMFCQ), although weaker than the LICQ in Rincon-Zapatero and Santos [41].

4.2.2 Differentiability of the Pareto Frontier

Consider the model of an exchange economy with commitment frictions of Kocherlakota [28] and Koeppl [29] (see also Rincon-Zapatero and Santos [41]) in which two infinitely lived agents receive a stochastic endowment in each period which they mutually share under limited commitment. As in Koeppl, the endowment for agent $i = 1, 2$ in period t is (ω_s^1, ω_s^2) which is determined by the realization of θ_t. The stochastic process $\theta = \{\theta_1, \theta_2, ...\}$ is a sequence of iid random variables, each having finite support $\Theta = \{1, 2, ..., S\}$, and . The probability that θ_t equals s is denoted by $\pi_s = \Pr\{\theta_t = s \in \Theta\}$.

We will assume the following, in which we relax Koeppl's assumptions of strict monotonicity, strict concavity and C^2 utility function. We extend the result in Rincon-Zapatero and Santos [41] by providing the form of Gateaux envelopes at points of nondifferentiability of the value function.

Assumption 4.2.2: The utility function $u : \mathbb{R}_+ \to \mathbb{R}$ *is increasing, concave, continuously differentiable with* $\lim_{c \to 0^+} u'(c) = \infty$*, and* $0 < \beta < 1$*, and for each* U_0*, the feasible set is uniformly compact.*

We characterize the incentive feasible allocations (see Koeppl [29] for details):

$$V(U_0) = \max_{\{c_s, u_s\}_{s=1}^S} \sum_{s=1}^S \pi_s \left[u(\overline{\omega}_s - c_s) + \beta V(U_s)\right]$$

subject to

$$\begin{aligned}
& U_0 - \sum_{s=1}^S \pi_s \left[u(c_s) + \beta U_s\right] \leq 0 \\
& u(\omega_s^1) + \beta U_{aut} - u(c_s) - \beta U_s \leq 0 \\
& u(\overline{\omega}_s - \omega_s^1) + \beta U_{aut} - u(\overline{\omega}_s - c_s) - \beta V(U_s) \leq 0 \\
& U_s \in [U_{aut}, U_{\max}]
\end{aligned}$$

The set of optimal solutions is denoted $Y^*(U_0)$, and a typical element of this set is $\{c_s^*, U_s^*\}_{s=1}^S$; the KKT multiplier vector takes the form $(\lambda_1, \{\lambda_{2s}\}, \{\lambda_{3s}\}, \{\lambda_{4s}\}, \{\lambda_{5s}\}) \in K(\{c_s^*, U_s^*\})$. Given assumption *4.2.2*, and the joint concavity of the objective and the constraints in $(\{c_s, u_s\}, U_0)$, the hypothesis of Proposition 20 are met. Thus, by Proposition 20, if the GMFCQ is satisfied for every optimal solution $(c^*(U_0), U^*(U_0)) \in Y^*(U_0)$, then V is concave with Gateaux dirivative:

$$V'(U_0; x) = \max_{(c^*, U^*) \in Y^*(U_0)} \min_{\lambda \in K(c_s^*, U_s^*)} \{-\lambda_1 \cdot x\}$$

Further, if the SMFCQ is satisfied for every optimal solution $(c^*(U_0), U^*(U_0)) \in Y^*(U_0)$, then V is concave and C^1 with derivative:

$$\begin{aligned}
\nabla V(U_0) &= -\lambda_1 \\
&= -\frac{u'(\overline{\omega}_s - c_s)}{u'(c_s)}
\end{aligned}$$

for any $(\lambda_1, \{\lambda_{2s}\}, \{\lambda_{3s}\}, \{\lambda_{4s}\}, \{\lambda_{5s}\}) \in K(\{c_s^*, U_s^*\})$.

4.3 Computing Markov equilibrium in growth models with nonsmooth nonconvex technologies

Recent work on optimal growth models with nonsmooth and nonconvex technologies have largely set aside the issue of existence of recursive and/or sequential equilibrium (see, for instance, Kamihigashi and Roy [26][27]), an issue we propose to address in this section by combining the envelope results of this paper with the lattice programming methodology of Mirman, Morand, and Reffett [34].

In these models, nonconvexities typically arise when a consumer's decisions depends on the aggregate or per capita state K in addition to its own individual state k. Our strategy is to impose sufficient conditions on the primitive data such that our search for a recursive equilibrium can be restricted to a large class of monotone functions. It is therefore related to the work of Hopenhayn and Prescott [25], extended in Morand, Reffett, and Tarafdar [37], on the existence of monotone controls on standard stochastic optimal growth models.

To simplify we work here in a deterministic setup, but, just as in Hopenhayn and Prescott [25] and central in Mirman, Morand and Reffett [34], we use a key result from lattice programming relating the supermodularity of a program's objective to the monotonicity (in (k, K)) of optimal decisions. It is in the proof of the supermodularity of the value function that our generalized envelope results play a critical role in models where the primitive data is not continuously differentiable.

4.3.1 Model and definition of recursive equilibrium

We consider a class of models with a continuum of identical infinitely-lived households/firms, each household entering period $t = \{0, 1, 2, ...\}$ with an individual stock of capital k_t and supplying inelastically one unit of time to firms. Common in the literature (e.g., Coleman [12], Greenwood and Huffman [19]), and consistent with recent work (e.g., [44] [26] [27]), we adopt a "reduced-form" production function $F(k, n, K, N)$ where k and n are, respectively, the firm's capital and labor inputs. Since $n = N = 1$ we use the notation $f(k, K, z) = F(k, 1, K, 1, z)$ and make the following standard assumptions.

Assumption *(i) There exists* $\hat{k} > 0$ *such that* $F(\hat{k}, 1, \hat{k}, 1) = \hat{k}$ *and* $F(k, 1, k, 1) < k$ *, for all* $k > \hat{k}$, so we denote by $\mathbb{K}$ *the interval* $[0, \widehat{k}]$. Function $F : \mathbb{K} \times [0, 1] \times \mathbb{K} \times [0, 1] \to \mathbb{R}$ *is continuous, increasing, concave in its first two arguments, and exhibits constant returns to scale in* (k, n).

Assumption *(ii)* $u : \mathbb{K} \mapsto \mathbb{R}$ *is increasing continuous, concave, and satisfies* $u(0) = 0$.

We replace the usual Inada condition $lim_{c\to 0} u'(c) \to \infty$ by the following assumption:

Assumption (iii). For all $M > 0$*, there exists* $x_0 \in \mathbb{K}$*,* $x_0 > 0$ *such that* $\xi > M$ *for all* $\xi \in \partial u(x_0)$*.*

As in Hopenhayn and Prescott, we also need a curvature condition requiring that the degree of complementarity between private and aggregate per capita capital stocks be high relative to the curvature of the utility function. Noting that the function $u(f(k, .) - y)$ is Lipschitz at any point K satisfying $f(k, K) - y > 0$, and therefore almost everywhere differentiable at such points, we state this assumption as follows:

Assumption (iv) At points where $u(f(k, .) - y)$ *is differentiable, the function* $u'(f(k, K) - y)f_1(k, K)$ *is increasing in* K*.*

A consumer seeks to maximize utility given by:

$$E_0\left\{\sum_{i=0}^{\infty}\beta^i u(c_i)\right\}$$

given initial state $k_0 = K_0 > 0$. Period t choices of consumption and investment must satisfy:

$$c_t + k_{t+1} \leq f(k_t, K_t)$$

and the consumer is constrained to use a law of motion h to recursively compute the sequence $\{K_t\}$ of future per capita capital stocks as $K_{t+1} = h(K_t)$.

We require h to belong to the set $\boldsymbol{B}$ defined as:

$$\boldsymbol{B} = \{h : \mathbb{K} \to \mathbb{K}, 0 \leq h(k) \leq f(k,k), h \text{ usc and increasing}\}$$

and note that $(\boldsymbol{B}, \leq)$ is a complete lattice, where $\leq$ is the pointwise partial (see for instance Davey and Priestley [15]), and that the set of subsets of $\boldsymbol{B}$ endowed with the induced set order $\leq_a$ is also a complete lattice.

For a given $h \in \mathbf{B}$, by a standard argument there exists a unique value function V satisfying:

$$V(k,K) = TV(k,K) = \sup_{y\in\Gamma(k,K)}\{u(f(k,K)-y) + V(y,h(K))\},$$

where $\Gamma(k,K) = \{y \in \mathbb{K}, 0 \leq y \leq f(k,K)\}$ is non-empty, compact and convex, and the continuity of f, implies that Γ is a continuous correspondence. Denote by Y^* the set of solutions to the above program, that is:

$$Y^*(k,K;h) = \arg\sup_{y\in\Gamma(k,K)}\{u(f(k,K)-y) + v(y,h(K))\}$$

and by $\vee Y^*$ and $\wedge Y^*$ its greatest and least selection, respectively.

Interpreting a selection from the optimal correspondence Y^* as a "best response" to the law of motion h, we define a recursive equilibrium as a law of motion that is best response to itself, that is:

Definition 21 *A recursive equilibrium is a element h^* in $\boldsymbol{B}$ such that $h^*(k) \in Y^*(k,k;h^*)$ for all $k \in \mathbb{K}$.*

4.3.2 Existence of Recursive equilibrium

Because recursive equilibria are precisely the fixed point of mapping A defined as:

$$h \in \boldsymbol{B} \rightrightarrows Ah = \{h' \in \boldsymbol{B}, \forall k \in \mathbb{K}\ h'(k) \in Y^*(k,k;h')\},$$

our existence proof relies on the order-preserving properties of this mapping, as well as the related operator $\overline{A}$ defined as:

$$h \in \boldsymbol{B} \to \overline{A}h = \vee Y^*(k,k;h)$$

Both operators share the following property:

Lemma 22 *Both* $A : (\boldsymbol{B}, \leq) \to (2^{\boldsymbol{B}} \backslash \varnothing, \leq_a)$ *and* $\overline{A} : (\boldsymbol{B}, \leq) \to (\boldsymbol{B}, \leq)$ *are isotone mappings.*

Proof. The proof follows precisely the argument in Mirman, Morand and Reffett [33] (Lemma 7) except for the proof of supermodularity of V. We set aside this (important) detail for now and turn to the main result of this section. ■

Proposition 23 *The set of recursive equilibria is a non-empty complete lattice. The sequence* $\{\overline{A}^n f\}$ *pointwise converges to the greatest recursive equilibrium.*

Proof. See Mirman, Morand and Reffett [34] (Theorems 6 and 9). ■

4.3.3 Proof of supermodularity of the value function

We finally turn to the proof of supermodularity of V, for which our generalized envelope results are needed because the traditional lattice theoretic argument (Theorem 2.7.6 in Topkis [46]) on the preservation of supermodularity under maximization is not applicabel (or limited simply to Leontieff production functions, as in Hopenhayn and Prescott [25]). Neither does the argument in Mirman, Morand and Reffett [34] since it requires the primitive data to be at least continuously differentiable.

Proof. Given $V_0 = 0$ we prove by induction that each element of the sequence $\{V_n = T^{(n)}V_0\}$ is supermodular, so that V inherits that property as the pointwise limit of that sequence. Fix $h \in \boldsymbol{B}$ and assume then that V_n is Lipschitz and supermodular, and consider the Lipschitz program:

$$V_{n+1}(k, K) = \max_{0 \leq y \leq f(k,K)} \{u(f(k, K) - y) + V_n(y, h(K))\}$$

Given that h is increasing, $\beta V_n(y, h(K))$ has increasing differences in $(y; K)$ (on $\mathbb{R}^2$ supermodularity and increasing differences are equivalent properties) while $u(f(k, K) - y)$ has increasing differences in $(y; (k, K))$ since u is concave and f is increasing. Consequently, the objective in the above program has increasing differences in $(y; (k, K))$, while the choice correspondence $[0, f(k, K)]$ is strong set order ascending, hence the optimal choice set Y^*_{n+1} is strong set order ascending and $\vee Y^*_{n+1}$ and $\wedge Y^*_{n+1}$ are both isotone selections in (k, K) by Theorems 2.8.1 and 2.8.3 in Topkis [46]. Note that by the same argument both $f - \vee Y^*_{n+1}$ and $f - \wedge Y^*_{n+1}$ are also isotone selections in (k, K).

Inada conditions imply interiority of solutions (so that all multipliers are 0), so it follows from Theorem 9 that V_{n+1} is Lipschitz and that:

$$\max_{y^*(k,K) \in Y^*_{n+1}(k,K)} \left(\min_{\theta \in \partial(u(f(k,K) - y^*(k,K)))} \theta \cdot x \right) \leq D_+ V_{n+1}(k, K; x)$$

and:

$$D^+ V_{n+1}(k, K; x) \leq \max_{y^*(k,K) \in Y^*_{n+1}(k,K)} \left(\max_{\theta \in \partial(u(f(k,K) - y^*(k,K))} \theta \cdot x \right)$$

in which the Dini derivatives are with respect to the first variable of V.

The concavity of u implies that if $c' > c$ then $\forall(\theta, \theta') \in \partial u(c) \times \partial u(c')$ necessarily $0 \leq \theta' \leq \theta$. As a result, for any $x > 0$:

$$\max_{y^*(k,K) \in Y^*_{n+1}(k,K)} \left(\max_{\theta \in \partial(u(f(k,K)-y^*(k,K))} \theta \cdot x \right) \leq \max_{\theta \in \partial(u(f(k,K)-\wedge Y^*_{n+1}(k,K))} \theta \cdot x$$

and also, given any $\overline{k} \in \mathbb{K}$, for all $\widehat{k} \geq k \geq \overline{k}$:

$$\max_{\theta \in \partial(u(f(k,K)-\wedge Y^*_{n+1}(k,K))} \theta \cdot x \leq \max_{\theta \in \partial(u(f(\overline{k},K)-\wedge Y^*_{n+1}(\overline{k},K))} \theta \cdot x$$

Thus $\forall \widehat{k} \geq k \geq \overline{k}$ and $\forall x > 0$:

$$\begin{aligned} 0 &\leq D_+ V_{n+1}(k, K; x) \\ &\leq D^+ V_{n+1}(k, K; x) \leq \max_{\theta \in \partial(u(f(\overline{k},K)-\wedge Y^*_{n+1}(\overline{k},K))} \theta \cdot x \end{aligned}$$

which proves that the Dini derivatives of V_{n+1} are uniformly bounded above on any interval $[\overline{k}, \widehat{k}]$. A symmetric argument holds for the direction $x < 0$, thus proving that on any interval $[\overline{k}, \widehat{k}]$ both Dinis are uniformly bounded. This implies that imply $k \to V_{n+1}(k, K)$ is absolutely continuous on $[\overline{k}, \widehat{k}]$ for any $0 < \overline{k} < \widehat{k}$.

This absolute continuity together with the properties that V_{n+1} is increasing and continuous in it first argument, imply that V_{n+1} is absolutely continuous on $\mathbb{K} = [0, \widehat{k}]$ (see Problem 37 in Royden [43]). By the fundamental theorem of integral calculus (Theorem 10, Chapter 6 in Royden [43]), $k \to V_{n+1}(k, K)$ is therefore almost everywhere differentiable and, for all $k \in \mathbb{K}$:

$$V_{n+1}(k, K) = \int_0^k V'_{n+1}(s, K) ds \tag{2}$$

At the points where $k \to V_{n+1}(k, K)$ is differentiable, by definition both Dinis must coincide, hence:

$$V'_{n+1}(s, K) = u'(f(s, K) - \wedge Y^*_{n+1}(s, K)) f_1(s, K)$$

Note that for any $K' > K$, where the derivative exists:

$$\begin{aligned} V'_{n+1}(s, K) &= u'(f(s, K) - \wedge Y^*_{n+1}(s, K)) f_1(s, K) \\ &\leq u'(f(s, K') - \wedge Y^*_{n+1}(s, K)) f_1(s, K') \\ &\leq u'(f(s, K') - \wedge Y^*_{n+1}(s, K')) f_1(s, K') \\ &= V'_{n+1}(s, K') \end{aligned}$$

the first inequality resulting from the curvature assumption (Assumption (iv)), and the second from the isotonicity of $\wedge Y^*_{n+1}$ previously established. In light of (2) this proves the desired supermodularity of V_{n+1}. By induction, the supermodularity property is true for all n, and the sequence of functions $\{V_n\}_{n=0}^{\infty}$ is a collection of supermodular functions in (k, K). Its pointwise limit, precisely V, must therefore inherit that property. ∎

5 Appendix: Mathematical Tools

5.1 Derivatives and subgradients

Given an open set $\Omega \subset \mathbb{R}^n$, the function $f : \Omega \to \mathbb{R}^m$ is said to be Lipschitz at $x \in \Omega$ if $\exists k > 0$ and $\exists \delta > \delta$ such that:

$$\forall x', x'' \in \delta B(x), \ \ |f(x'') - f(x')| \leq k \, |x'' - x'| \, ,$$

where $B(x)$ is the open ball of radius 1 centered on x. If the modulus can be chosen independently of x on an open subset of Ω, f is said to be globally Lipschitz on that subset.

If f is Lipschitz at x then the upper and lower Dini derivatives, respectively defined as the functions:

$$d \longmapsto D^+ f(x; d) = \lim\sup_{t \to 0^+} \frac{f(x+td) - f(x)}{t} \quad \text{and:}$$
$$d \longmapsto D_+ f(x; d) = \lim\inf_{t \to 0^+} \frac{f(x+td) - f(x)}{t}$$

always exist.

Function f is said to be (a) Gateaux (directionally) differentiable at x if both Dini derivatives coincide for all d, in which case the Gateaux derivative is:

$$f'(x; d) = \lim_{t \to 0^+} \frac{f(x+td) - f(x)}{t},$$

and (b) differentiable at x if it is Gateaux differentiable at x and if $f'(x; d) = \nabla f(x) \cdot d$. Note that the function $x \to |x|$ is directionally differentiable but not differentiable at 0. However, by Rademacher's theorem, if f is Lipschitz at all point of an open set $\Theta \subset \Omega$, then it is almost everywhere differentiable on Θ. Finally, if the function $x \longrightarrow \nabla f(.)$ is continuous at x, then f is said to be continuously differentiable at x.

Lipschitz functions also have the property that the upper and lower Clarke derivatives, respectively defined as the functions:

$$d \longmapsto f^o(x; d) = \lim\sup_{\substack{y \to x \\ t \to 0^+}} \frac{f(y+td) - f(y)}{t} \quad \text{and:}$$
$$d \longmapsto f^{-o}(x; d) = \lim\inf_{\substack{y \to x \\ t \to 0^-}} \frac{f(y+td) - f(y)}{t}$$

also always exist. Note that:

$$f^{-o}(x; d) \leq -f^{-o}(x; -d) = f^o(x; d) \tag{3}$$

If f is Lipschitz and Gateaux differentiable at x, then clearly:

$$f^{-o}(x; d) \leq f'(x; d) \leq f^o(x; d)$$

Gateaux differentiable Lipschitz functions (at some x) are said to be upper Clarke regular at x if $f^o(x;d) = f'(x;d)$ (and lower Clarke regular at x if. $f^{-o}(x;d) = f'(x;d)$). Function f is said to be strictly differentiability at x if both upper and lower Clarke derivatives coincide. In finite dimensional spaces, strict differentiability and continuous differentiability are equivalent.

Finally, the Clarke gradient of a Lipschitz function f at x is the nonempty compact convex set:

$$\partial f(x) = \overline{co}\{\lim \nabla f(x_i) : x_i \to x, x_i \notin \Theta, x_i \notin \Omega_f\}$$

where $\overline{co}$ denotes the convex hull[5], Θ is any set of Lebesgue measure zero in the domain, and Ω_f is a set of points at which f fails to be differentiable. Clarke [11] (Proposition 2.1.5) shows that $x \rightrightarrows \partial f(.)$ is an upper hemicontinuous correspondence. Clarke [11] (Proposition 2.1.2) shows that:

$$f^o(x;d) = \max_{\zeta \in \partial f(x)} \{\zeta.d\}$$

hence $f^o(x;d)$ is a convex function of d.

It is important to note the following important properties of convex functions $f : \Omega \to \mathbb{R}^m$:

Lemma 24 *Suppose $f : \Omega \to \mathbb{R}^m$ is convex. Then:*

(i) f is Lipschitz at every $x \in \Omega$.

(ii) f is upper Clarke regular at every $x \in \Omega$.

(iii) the Clarke gradient of f at x coincides with the subgradient of convex analysis, i.e. the set of $p \in M_{m \times n}$ satisfying $\forall d, p \cdot d \leq f(x_0 + d) - f(x_0)$.

(iv). If f is also differentiable at x, then f is continuously differentiable at x.

Proof. (i), (ii) and (iii) are well-known. We prove now that differentiability together with upper (or lower) Clarke regularity implies continuous differentiability. Upper Clarke regularity and differentiability imply that:

$$f^o(x;d) = f'(x,d) = \nabla f(x).d$$

hence, using (3) above:

$$\begin{aligned} f^{-o}(x;d) &= -f^o(x;-d) = -\nabla f(x).(-d) \\ &= \nabla f(x).d = f^o(x;d) \end{aligned}$$

which proves that f is strictly differentiable (thus continuously differentiable) at x. A similar argument clearly applies if f is lower (rather than upper) Clarke regular. ■

[5] In the formula, either co or $\overline{co}$ will do since we work with finite dimensional spaces.

5.2 Properties of Correspondences

We work in metric spaces, so we can state topological properties of correspondences exclusively in terms of sequences.

Definition 25 *Given $A \subset R^n$ and $S \subset R^m$, a non-empty valued correspondence $D : S \twoheadrightarrow A$ is:*

(i) lower hemicontinuous at s if for every $a \in D(s)$ and every sequence $s_n \rightarrow s$ there exists a sequence $\{a_n\}$ such that $a_n \rightarrow a$ and $a_n \in D(s_n)$.

(ii) upper hemicontinuous at s if for every sequence $s_n \rightarrow s$ and every sequence $\{a_n\}$ such that $a_n \in D(s_n)$ there exists a convergent subsequence of $\{a_n\}$ whose limit point a is in $D(s)$.

(iii) closed at s if $s_n \rightarrow s$, $a_n \in D(s_n)$ and $a_n \rightarrow a$ implies that $a \in D(s)$ (In particular, this implies that $D(s)$ is a closed set).

(iv) open at s if for any sequence $s_n \rightarrow s$ and any $a \in D(s)$, there exists a sequence $\{a_n\}$ and a number N such that $a_n \rightarrow a$ and $a_n \in D(s_n)$ for all $n \geq N$.

Note that $D(s) = \{a \in A,\ g_i(a, s) \leq 0,\ i = 1, ..., p\}$, in which the g_i are locally Lipschitz (and thus continuous), is necessarily closed at s. The same property holds true in the presence of locally Lipschitz equality constraints.

Another property of correspondences which is critical in our analysis is that of uniform compactness.

Definition 26 *A non-empty valued correspondence D is said to be uniformly compact near s if there exists a neighborhood S' of s such that $cl\left[\cup_{s' \in S'} D(s)\right]$ is compact.*

We note the result in Hogan [24] that if D is uniformly compact near s, then D is closed at s if and only if $D(s)$ is a compact set and D is upper hemicontinuous at s. When D is defined by a system of continuous equality and inequality constraints, uniform compactness near s thus implies compactness and upperhemicontinuity at s. In fact, for any s' sufficiently close to s, since $D(s')$ is a closed subset of $cl\left[\cup_{s' \in S'} D(s)\right]$ it is therefore compact.

Finally, we will need the following property of hemicontinuous correspondences (and thus of Clarke gradients).

Proposition 27 *If D is an upper hemicontinuous correspondence, then for every compact neighborhood K of x, the set:*

$$\bigcup_{z \in K} D(z)$$

is compact.

Proof. Consider a sequence $\{y_n\}$ in $\bigcup_{z \in K} D(z)$ so that $y_n \in D(z_n)$ for some z_n in K. The sequence $\{z_n\}$ is the compact K, so there exists a subsequence of $\{z_{\varphi(n)}\}$ of $\{z_n\}$ converging to some $z' \in K$. By upper hemicontinuity of D at z', there exists a subsequence of $\{y_{\varphi(n)}\}$ converging to some $y \in D(z')$. This proves that the initial sequence $\{y_n\}$ has a convergent subsequence, and therefore that the set $\bigcup_{x \in K} D(x)$ is compact. ■

5.3 Posets, Lattices, Supermodularity and Lattice Programming

A *partially ordered set* (or Poset) is a set X ordered with a reflexive, transitive, and antisymmetric relation. If any two elements of X are comparable, X is referred to as a complete partially ordered set, or chain. An upper (resp. lower) bound of $B \subset X$ is an element x^u (resp. x^l) in B such that $\forall x \in B$, $x \leq x^u$ (resp. $x^l \leq x$). A *lattice* is a set X ordered with a reflexive, transitive, and antisymmetric relation $\geq$ such that any two elements x and x' in X have a least upper bound in X, denoted $x \wedge x'$, and a greatest lower bound in X, denoted $x \vee x'$. The product of an arbitrary collection of lattices equipped with the product (coordinatewise) order is a lattice. $B \subset X$ is a *sublattice* of X if it contains the sup and the inf (with respect to X) of any pair of points in B.

Let $(X, \geq_X)$ and $(Y, \geq_Y)$ be Posets. A mapping $f : X \to Y$ is *isotone* (or *increasing*) on X if $f(x') \geq_Y f(x)$, when $x' \geq_X x$, for $x, x' \in X$. A correspondence (or multifunction) $F : X \to 2^Y$ is *ascending* in the set relation on 2^Y denoted by $\geq_S$ if $F(x') \geq_S F(x)$, when $x' \geq_X x$. A particular set relation of interest is Veinott's strong set order (See Veinott [47], Chapter 4). Let $L(Y) = \{A | A \subset Y,\ A \text{ a nonempty sublattice}\}$ be ordered with the *Strong Set Order* $\geq_a$: if $A_1, A_2 \in L(Y)$, we say $A_1 \geq_a A_2$ if $\forall (a, b) \in A_1 \times A_1$, $a \wedge b \in A_2$ and $a \vee b \in A_1$.

Let X be a lattice. A function $f : X \to R$ is *supermodular (resp., strictly supermodular)* in x if $\forall (x, y) \in X^2$, $f(x \vee y) + f(x \wedge y) \geq$(resp., $>$) $f(x) + f(y)$. An important property of the class of supermodular functions is they are closed under pointwise limits. (Topkis, [46], Lemma 2.6.1). Consider a partially ordered set $\Psi = X_1 \times P$ (with order $\geq$), and $B \subset X_1 \times P$. The function $f : B \longrightarrow R$ has *increasing differences* in (x_1, p) if for all $p_1, p_2 \in P$, $p_1 \leq p_2 \Longrightarrow f(x, p_2) - f(x, p_1)$ is non-decreasing in $x \in B_{p_1}$, where B_p is the p section of B. If this difference is strictly increasing in x then f has *strictly increasing differences* on B.

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