A Qualitative Theory of Large Games with Strategic Complementarities*

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Abstract

We study the existence and computation of equilibrium in large games with strategic complementarities. Using monotone operators defined on the space of distributions endowed with the first order stochastic dominance partial order, we prove existence of the greatest and least distributional Nash equilibrium. In particular, we obtain our results under a different set of conditions than those in the existing literature. Moreover, we provide computable monotone distributional equilibrium comparative statics with respect to the parameters of the game. Finally, we apply our results to models of social distance, large stopping games, keeping up with the Joneses, as well as a general class of linear non-atomic games.

Keywords: large games, distributional equilibria, supermodular games, games with strategic complementarities, computation of equilibria, non-aggregative games

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1 Introduction

Beginning with the seminal work of Schmeidler (1973) and Mas-Colell (1984), there has been a great deal of work in the economic literature focusing on games with a continuum of players.[1] In a separate, yet related set of papers, researchers have turned their attention to the question of the existence of equilibrium comparative statics in large games with strategic complementarities (henceforth LGSC) between player actions or "traits" (e.g., see Guesnerie and Jara-Moroni, 2011 or Acemoglu and Jensen, 2010, 2013. This latter strand of work focused primarily on non-atomic, *aggregative* games, in which the payoff of individual players is affected by an aggregate of actions of other players in the game (or, perhaps, a vector of aggregates). In such a game the questions concerning equilibrium existence, its computation, and equilibrium comparative statics, can often be simplified to a great extent.[2]

As we show in this paper, there are classes of large games with a continuum of players, in which the payoff of an individual agent inherently depends on the *entire distribution* of actions and characteristics of other players. In particular, such games cannot be analysed using the standard toolkit, as none of the results in the existing literature provides answers for existence, computation, or comparative statics of equilibria. Finally, the importance of studying large (non-aggregative) games with "traits" or diverse personal characteristics has been highlighted, among others, by Khan, Rath, Yu, and Zhang (2013) and Khan, Rath, Sun, and Yu (2013). In these papers, the authors stress the cardinality of traits as a key factor not only to verify conditions of equilibrium existence, but also to use large games in the study of actual economic problems.

In this paper, we ask a number of questions. First of all, can the methods used to verify the existence of equilibrium in games with strategic complementarities (with a finite number of players; henceforth GSC) be extended to a general class of games with a continuum of players? If this is possible, can one develop conditions for equilibrium comparative statics results and what will be the nature of such equilibrium comparative statics? Finally, can we develop a theory of *computable* equilibrium comparative statics for these games?

[1]For example, see the series of papers by Khan, Rath, Sun, and Yu (2013), Khan, Rath, Yu, and Zhang (2013), and the references within.

[2]Actually, in many aggregative games with strategic complementarities, the existence and equilibrium comparative statics questions can be obtained using the standard tools from the theory of game of strategic complementarities with a *finite* number of players (as in the seminal work of Topkis, 1979, Vives, 1990, and Milgrom and Roberts, 1990). On the other hand, these tools are not applicable in more general classes of large games with strategic complementarities (LGSC). See Balbus, Dziewulski, Reffett, and Woźny (2014) for a discussion.

We begin our analysis in Section 2, where we provide sufficient conditions under which there exists a distributional equilibrium of a LGSC. In addition, we provide an order-theoretic characterization of the set of equilibria. This characterization plays a central role in developing results on equilibrium comparative statics as well as an iterative method for computing the extremal equilibria. By appealing to the notion of order-continuity in our results, we are able to relax the key continuity conditions on payoffs, which are required for equilibrium existence in the existing literature that refers to the standard topological arguments (e.g., continuity conditions in weak topologies).

Moreover, our approach provides *constructive methods* for computing the extremal equilibria by their successive approximation, starting from the extremal elements of the space of all feasible distributions of player actions and "traits". To our knowledge, there are no similar results in the existing literature on computation of extremal equilibria in LGSC. This result proves to be central in identifying sufficient conditions for the existence of *computable equilibrium comparative statics* with respect to parameters of a game.

The central point this paper makes is that, although the tools used for a study of equilibrium in GSC and LGSC are similar in a broad methodological sense, the particulars of the methods are significantly different. This dissimilarity arises due to the inherent infinite dimensional structure of large games. In the framework we discuss, joint strategies of players as well as equilibria are defined in terms of probability distributions. Since, in general, sets of probability distributions are not lattices, the methods applied in the analysis as well as the characterisation of equilibria for LGSC change with respect to GSC. For example, even though it is possible to provide conditions under which the set of distributional equilibria has the greatest and the least element, the set of distributional equilibria is, in general, *not* a complete lattice.[3] At most, we are able to show that whenever the best responses of players are *functions*, the set of distributional equilibria is a chain complete partially ordered set (but not necessarily a complete lattice).[4]

Additionally, the measurability issues induced by the infinite-dimensional specification of LGSC create significant impediments in determining the existence of an equilibrium of the game, let alone its monotone comparative statics. An alternative approach to the measurability issue was recently proposed by Yang and Qi (2013). The authors restricted their attention to equilibria in strategies that are monotone with respect to traits/characteristics. Their key assumption, that the set of traits/characteristics is a

[3] A *complete lattice* is a partially ordered set $(X, \geq)$ such that, for any subset $Y \subseteq X$, the supremum and infimum of Y (with respect to $\geq$) belongs to X.

[4] A *partially ordered set* (henceforth a *poset*) X is chain complete (henceforth CPO), if for an arbitrary chain $C \subset X$, $\sup(C)$ and $\inf(C)$ are each in X. If this completeness condition holds only for countable chains, we say that X is countably chain complete (henceforth CCPO).

chain, allows them to apply the results from the existing literature on monotone games.[5] In our general specification of the game, we do not impose such restrictions on equilibrium strategies. In fact, in Section 3 we apply our methods to economic problems in which strategies of players are non-monotone with respect to players characteristics. As a result, both the techniques used and the results obtained in our paper differ from the analysis performed by Yang and Qi (2013).

The remainder of the paper is organized as follows. In Section 2, we provide our main results on the existence of distributional equilibrium, as well as our results on computable equilibrium comparative statics. We also discuss the relationship between our results, and those in the existing literature. In Section 3, we show how our results can be applied to Akerlof's social distance model, large optimal stopping games, keeping up with the Joneses, as well as a general class of linear non-atomic supermodular games. Finally, in the Appendix we introduce the requisite mathematical terminology used in the paper as well as the auxiliary results.

2 Distributional equilibria in large games

Let Λ be a compact, perfect Hausdorff topological space of player characteristics.[6] Endow Λ with the Borel σ-field $\mathcal{L}$ and a regular probability measure λ vanishing at each singleton.[7] Let $A \subseteq \mathbb{R}^m$ ($m \in \mathbb{N}$) be an action set endowed with the natural product order and Euclidean topology. By $\tilde{A} : \Lambda \to 2^A$ we denote a correspondence that assigns a sets of feasible actions $\tilde{A}(\alpha) \subseteq A$ to each player $\alpha \in \Lambda$.

By $\mathcal{A}$ we denote a family of Borel subsets on A. Let $\mathcal{R}$ denote the set of regular probability measures defined over $\mathcal{L} \otimes \mathcal{A}$, with the marginal distribution on Λ equal to λ.[8]

[5] As we discuss in a related paper (see Balbus, Reffett, and Woźny, 2014b), the results of Yang and Qi require stronger conditions on the traits/characteristics space than those imposed in their paper. See a counterexample in Section 2 of Balbus, Reffett, and Woźny (2014b).

[6] A topological space is *perfect*, if all of its elements are accumulation points. In particular, the space has no isolated points.

[7] Observe that it does not imply that λ is a non-atomic measure (see Lemma 12.18 in Aliprantis and Border, 2006). Moreover, given that Λ is perfect, Theorem 12.21 in Aliprantis and Border (2006) implies that such measure exists. In fact, the assumption that Λ is perfect is crucial. For example, let Λ be a set of ordinals $[0, \omega_1]$ where ω_1 is the first uncountable number. Let $\mathcal{L}$ be the order topology, while measure λ vanishes at each singleton. Then, all closed sets not containing ω_1 are countable. Hence, whenever we take any neighbourhood of ω_1, its complement is λ-null set. Therefore, any neighbourhood of ω_1 has a full measure, while $\lambda(\{\omega_1\}) = 0$. Each probability measure vanishing at a singleton is not regular. In addition, all successors are isolated points.

[8] The measure is *regular*, if $\lambda(X) < \infty$, for any compact set $X \in \mathcal{L}$, and if it is both outer regular and tight. Additionally, note that since Λ is a compact Hausdorff space, any tight measure is at the same time inner regular. Therefore, any regular measure is both inner and outer regular, hence, *normal*.

Endow $\Lambda \times A$ with a partial order $\geq_p$ that satisfies:

$$(\alpha', a') \geq_p (\alpha, a) \;\Rightarrow\; a' \geq a.^{9}$$

Observe that the partial order $\geq_p$ and any action $a \in A$ induce a partial order on Λ, say $\geq_p^a$, defined as follows: $\alpha \geq_p^a \alpha'$ if and only if $(\alpha, a) \geq_p (\alpha', a)$. We accept an additional assumption according to which, for each $a \in A$ and $\alpha \in \Lambda$, a closed interval of the form $\{\alpha' \in \Lambda : \alpha' \geq_p^a \alpha\}$ belongs to $\mathcal{L}$. This assumption is clearly satisfied if all closed intervals on the form $\{(\alpha', a') : (\alpha', a') \geq_p (\alpha, a)\}$ belong to $\mathcal{L} \otimes \mathcal{A}$.

Endow $\mathcal{R}$ with the *first order stochastic dominance* partial order and denote it by $\succeq_P$.[10] By Lemma A.1 (see the Appendix), $(\mathcal{R}, \succeq_P)$ is a partially ordered set.[11] Let

$$\mathcal{D} := \left\{ \tau \in \mathcal{R} : \tau\big(\operatorname{Gr}(\tilde{A})\big) = 1 \right\},$$

where $\operatorname{Gr}(\tilde{A})$ is the graph of correspondence $\tilde{A}$, denote the set of all feasible distributions. Endow $\mathcal{D}$ with the weak topology and consider a payoff function $r : \Lambda \times A \times \mathcal{D} \to \mathbb{R}$. We define a game by $\Gamma := \big((\Lambda, \mathcal{L}, \lambda), A, \tilde{A}, r\big)$.[12]

Definition 1 (Distributional equilibrium). *A* distributional equilibrium *of the game* Γ *is a probability measure* $\tau^* \in \mathcal{D}$ *such that:*

$$\tau^*\Big(\big\{(\alpha, a) : r(\alpha, a, \tau^*) \geq r(\alpha, a', \tau^*), \text{ for all } a' \in \tilde{A}(\alpha)\big\}\Big) = 1.$$

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[9] Clearly, if Λ is an ordered set, this implication is satisfied whenever $\geq_p$ is a product order. However, the implication may also be satisfied if the order is such that $(\alpha', a') \geq_p (\alpha, a)$ iff $(\alpha = \alpha')$ and $(a' \geq a)$. In particular, this allows us to study the case where the set of players has only trivial orders.

[10] That is, we have $\tau' \succeq_P \tau$ if and only if $\int f(\alpha, a) d\tau'(\alpha, a) \geq \int f(\alpha, a) d\tau(\alpha, a)$, for any increasing (with respect to $\geq_p$), bounded, and measurable function $f : \Lambda \times A \to \mathbb{R}_+$.

[11] The set of all isotone (with respect to $\geq_p$), measurable functions $f : \Lambda \times A \to \mathbb{R}$ separates the points in $\Lambda \times A$. Take any (α_1, a_1) and (α_2, a_2). Assume that a_1 is no greater then a_2. Define $f(\alpha, a) = \chi_{\{a' \in A : a \geq a_2\}}(a)$. Clearly, function f is isotone (with respect to $\geq_p$). Finally, it is $\mathcal{L} \otimes \mathcal{A}$-measurable, as $0 = f(\alpha_1, a_1) \neq f(\alpha_2, a_2) = 1$. Let $a_1 = a_2 = a$ and $\alpha_1 \not\geq_p^a \alpha_2$. Then by our assumption on $\geq_p$ all points of the form (α_1, a) and (α_2, a) may be separated by the indicator of $\{\alpha' : \alpha' \geq_p^a \alpha_2\}$.

[12] Unlike Mas-Colell (1984), who characterized players by their payoff functions only, in our specification the space of characteristics is equivalent to the measure space $(\Lambda, \mathcal{L}, \lambda)$. Nevertheless, whenever we define $r(\alpha, a, \tau) := \alpha(a, \tau)$, where $\Lambda := \{\alpha : (a, \tau) \to \alpha(a, \tau) \text{ is a continuous function}\}$, our framework embeds the one proposed by Mas-Colell. Alternatively, we can interpret α as a fixed trait, just like we do in Section 3. See also the work of Khan, Rath, Sun, and Yu (2013); Khan, Rath, Yu, and Zhang (2013), who analyze games with traits.

2.1 The main result

Our initial interest is to determine sufficient conditions for the existence of a distributional equilibrium of a LGSC. We impose the following assumptions:

Assumption 1. *Assume that*

(i) correspondence $\tilde{A}$ is weakly $\mathcal{L}$-measurable, while its values are complete sublattices of $\mathbb{R}^m$;[13]

(ii) for λ-a.e. player $\alpha \in \Lambda$, function r is quasisupermodular over A and has single crossing differences in (a, τ) on $A \times \mathcal{D}$;[14]

(iii) for any $\tau \in \mathcal{D}$, function $(\alpha, a) \to r(\alpha, a, \tau)$ is Carathéodory.[15]

In order to proceed with our main result concerning the existence of a distributional equilibrium, we define the best response correspondence $m : \Lambda \times \mathcal{D} \to 2^A$ by:

$$m(\alpha, \tau) := \operatorname*{argmax}_{a \in \tilde{A}(\alpha)} r(\alpha, a, \tau),$$

Denote its *greatest selection* by $\overline{m} : \Lambda \times \mathcal{D} \to A$, where $\overline{m}(\alpha, \tau) := \bigvee m(\alpha, \tau)$. Similarly, the *least selection* $\underline{m} : \Lambda \times \mathcal{D} \to A$ is defined by $\underline{m}(\alpha, \tau) := \bigwedge m(\alpha, \tau)$.[16] Define the *upper distributional* operator $\overline{T} : \mathcal{D} \to \mathcal{D}$ by:

$$\big(\overline{T}\tau\big)(G) := \lambda\Big(\big\{\alpha \in \Lambda : \big(\alpha, \overline{m}(\alpha, \tau)\big) \in G\big\}\Big), \text{ for any } G \in \mathcal{L} \otimes \mathcal{A}.$$

Similarly, using the least best reply $\underline{m}(\alpha, \tau)$, we define the *lower distributional* operator $\underline{T}$. We proceed with our first lemma that characterizes the monotonicity properties of the two operators.

[13] A sublattice Y of X is *complete* whenever, for any subset $C \subseteq Y$, $\bigvee C := \sup(C)$ and $\bigwedge C := \inf(C)$ belong to S, where the sup/inf operators $\bigvee / \bigwedge$ are defined with respect to the induced order of X. Moreover, since $\tilde{A}(\alpha) \subseteq \mathbb{R}^m$ is a complete sublattice, correspondence $\tilde{A}$ is compact-valued.

[14] Let $(X, \geq_X)$ be a lattice and $(Y, \geq_Y)$ be a partially ordered set. Function $f : X \to \mathbb{R}$ is *quasisupermodular* over X whenever, for any x, x' in X, $f(x) \geq f(x \wedge x')$ implies $f(x \vee x') \geq f(x')$, and $f(x) > f(x \wedge x')$ implies $f(x \vee x') > f(x')$. Function $g : X \times Y \to \mathbb{R}$ has *single crossing differences* in (x, y) on $X \times Y$ whenever, for any $x' \geq_X x$ in X and $y' \geq_Y y$ in Y, $g(x', y) \geq g(x, y)$ implies $g(x', y') \geq g(x, y')$, and $g(x', y) > g(x, y)$ implies $g(x', y') > g(x, y')$.

[15] That is, the function $(\alpha, a) \to r(\alpha, a, \tau)$ is continuous on A and $\mathcal{L}$-measurable.

[16] In the remainder of the paper we use symbols $\bigvee$ and $\bigwedge$ to denote the sup and inf of an underlying set, where both operators are defined with respect to the induced partial order. We shall often refer to these selections collectively as to the "extremal" selections. The extremal elements of a complete sublattice are the greatest greatest and the least elements.

Lemma 1. *Operators $\overline{T}$ and $\underline{T}$ are well-defined and $\succeq_P$-increasing.*

Proof of Lemma 1. We prove the result for operator $\overline{T}$. Using an analogous argument we can show that the lemma is also true for $\underline{T}$.

Claim 1: Operator $\overline{T}$ is well defined. Since r is continuous and quasisupermodular on A, by Milgrom and Shannon (1994, Theorem 4), $m(\alpha, \tau)$ is a complete sublattice of A, for any $\alpha \in \Lambda$ and $\tau \in \mathcal{D}$. In particular, set $m(\alpha, \tau)$ is non-empty and $\overline{m}(\alpha, \tau) := \bigvee m(\alpha, \tau)$ is well-defined, for any $\alpha \in \Lambda$ and $\tau \in \mathcal{D}$.

Next, we show that correspondence m has an increasing measurable selector. Given that function r is Carathéodory, for all $\tau \in \mathcal{D}$, and correspondence $\tilde{A}$ is weakly measurable, by Theorem 18.19 in Aliprantis and Border (2006), correspondence m is $\mathcal{L}$-measurable, for any $\tau \in \mathcal{D}$. As $m(\cdot, \tau)$ maps a measurable space into a metrizable space, it is also weakly measurable (see Aliprantis and Border, 2006, Theorem 18.2).

Let $\pi_i : \mathbb{R}^m \to \mathbb{R}$, be a projection defined by $\pi_i(x_1, \ldots, x_i, \ldots, x_m) := x_i$, for any $(x_1, \ldots, x_m) \in \mathbb{R}^m$. Note that, $\overline{m}(\alpha, \tau) = (\bar{a}_1, \ldots, \bar{a}_m)$, where $\bar{a}_i := \max_{a \in m(\alpha,\tau)} \pi_i(a)$. Since the projection π_i is a continuous function, for any $i = 1, \ldots, m$, while correspondence $m(\cdot, \tau)$ is weakly measurable, by Theorem 18.19 in Aliprantis and Border (2006), function $\overline{m}(\alpha, \tau)$ is λ-measurable. Therefore, by Himmelberg Theorem (see Lemma 18.4 in Aliprantis and Border, 2006), function $\overline{m}(\cdot, \tau)$ is $\mathcal{L}$-measurable, for any $\tau \in \mathcal{D}$.

In order to complete the first part of the proof, we need to show that, for any $\tau \in \mathcal{D}$, measure $(\overline{T}\tau)$ belongs to $\mathcal{D}$. Clearly, $(\overline{T}\tau)(\mathrm{Gr}(\tilde{A})) = 1$. Hence, it suffices to show that $(\overline{T}\tau)$ is normal and regular. Note that, as the marginal distribution of $(\overline{T}\tau)$ over Λ is equal to λ, it is regular. Moreover, since the marginal distribution of $(\overline{T}\tau)$ over metrizable A is a bounded Borel measure, by Theorem 12.5 in Aliprantis and Border (2006), it is regular. Therefore, by Lemma A.2 (see the Appendix), we conclude that measure $(\overline{T}\tau)$ is regular.

Claim 2: Operator $\overline{T}$ is $\succeq_P$-isotone. Since, for λ-a.e. $\alpha \in \Lambda$, function r has single crossing differences in (a, τ) on $A \times \mathcal{D}$, by Theorem 4 in Milgrom and Shannon (1994), correspondence m is isotone in the strong set order on $\mathcal{D}$, for λ-a.e. $\alpha \in \Lambda$. In particular, this implies that function $\overline{m}$ increases on $\mathcal{D}$, for λ-a.e. $\alpha \in \Lambda$. As a result, for λ-a.e. $\alpha \in \Lambda$, we have $\overline{m}(\alpha, \tau') \geq \overline{m}(\alpha, \tau)$, whenever $\tau' \succeq_P \tau$. We now show that $(\overline{T}\tau') \succeq_P (\overline{T}\tau)$.

Let $f : \Lambda \times A \to \mathbb{R}$ be an increasing and measurable function. Then,

$$\begin{aligned}\int_{\Lambda\times A} f(\alpha,a)d\big(\overline{T}\tau'\big)(\alpha,a) &= \int_{\Lambda} f\big(\alpha,\overline{m}(\alpha,\tau')\big)d\lambda(\alpha) \\ &\geq \int_{\Lambda} f\big(\alpha,\overline{m}(\alpha,\tau)\big)d\lambda(\alpha) \qquad (1) \\ &= \int_{\Lambda\times A} f(\alpha,a)d\big(\overline{T}\tau\big)(\alpha,a),\end{aligned}$$

where the first and the last equality follows from the definition of $\overline{T}$, while inequality (1) is implied by monotonicity of f and $\overline{m}(\alpha,\cdot)$. □

Our next result characterises the order structure of the partially ordered set $(\mathcal{D},\succeq_P)$. The following lemma follows directly from Proposition A.1 (see the Appendix).

Lemma 2. *$(\mathcal{D},\succeq_P)$ is a chain complete poset.*

With these two lemmas in place, we are ready to state the main theorem of the paper. Before we state our result, let $\overline{\delta}$ and $\underline{\delta}$ denote the greatest and least elements of set $\mathcal{D}$.

Theorem 1. *Under Assumption 1*

(i) there exists the greatest and the least distributional equilibrium of Γ;

(ii) if $\underline{T} = \overline{T}$ (i.e., if the best response correspondence is a function), then the set of distributional equilibria is a chain complete poset.[17]

Denote the greatest and the least equilibrium of Γ by $\overline{\tau}^*$ and $\underline{\tau}^*$ respectively. If for any countable chain $\{\tau_n\} \subseteq \mathcal{D}$ such that $\tau_n \to \tau$, we have $r(\alpha,a,\tau_n) \to r(\alpha,a,\tau)$, then

(iii) $\overline{\tau}^* = \lim_{n\to\infty} \overline{T}^n(\overline{\delta})$ and $\underline{\tau}^* = \lim_{n\to\infty} \underline{T}^n(\underline{\delta})$.

Proof of Theorem 1. In order to make our argument transparent, we present it via a number of claims.

Claim 1: The greatest fixed point of $\overline{T}$ is a distributional equilibrium of Γ. By Lemma 2, pair $(\mathcal{D},\succeq_P)$ is a chain complete poset. By Lemma 1, $\overline{T}$ is a well-defined, $\succeq_P$-isotone operator that maps $\mathcal{D}$ into itself. Therefore, by Theorem A.1 (see Appendix 3.4), we

[17] Conditions under which the best response correspondence is a function are well-known. For example, whenever correspondence $\tilde{A}$ is convex-valued and the payoff function r is strictly quasi-concave on A.

conclude that the set of fixed points of operator $\overline{T}$ is non-empty and has the greatest element. Denote it by $\overline{\tau}^*$. Clearly $\overline{\tau}^* \in \mathcal{D}$. Moreover, we have:s

$$\overline{\tau}^*\Big(\big\{(\alpha, a) : a \in m(\alpha, \tau^*)\big\}\Big) \geq \overline{\tau}^*\Big(\big\{(\alpha, a) : a = \overline{m}(\alpha, \tau^*)\big\}\Big) = 1,$$

which implies that $\overline{\tau}^*$ is a distributional equilibrium of Γ. Analogously, we can show that the least fixed point of operator $\underline{T}$, denoted by $\underline{\tau}^*$, is a distributional equilibrium of Γ.

Claim 2: $\overline{\tau}^*$ is the greatest distributional equilibrium of Γ*.* Let τ be any equilibrium of the game. Then, by definition of distributional equilibria, we have:

$$1 = \tau\Big(\big\{(\alpha, a) : a \in m(\alpha, \tau)\big\}\Big) \leq \tau\Big(\big\{(\alpha, a) : a \leq \overline{m}(\alpha, \tau)\big\}\Big).$$

Therefore, τ is concentrated over the set $E := \big\{(\alpha, a) : a \leq \overline{m}(\alpha, \tau)\big\}$. Take any increasing function $f : \Lambda \times A \to \mathbb{R}$. Then,

$$\begin{aligned} \int_{\Lambda \times A} f(\alpha, a) d\tau(\alpha, a) &= \int_E f(\alpha, a) d\tau(\alpha, a) \\ &\leq \int_\Lambda f\big(\alpha, \overline{m}(\alpha, \tau)\big) d\lambda(\alpha) \qquad (2) \\ &= \int_{\Lambda \times A} f(\alpha, a) d\big(\overline{T}\tau\big)(\alpha, a), \end{aligned}$$

where (2) follows from the definition of E and the final equation is implied by the definition of $\overline{T}$. Therefore, $(\overline{T}\tau) \succeq_P \tau$. Since $\overline{T}$ is $\succeq_P$-isotone, by Theorem A.2 (see the Appendix), we have that $\overline{\tau}^* \succeq_P \tau$. Hence, $\overline{\tau}^*$ is the greatest distributional equilibrium of Γ. Analogously, we can show that $\underline{\tau}^*$ is the least distributional equilibrium of Γ.

Observe that Claims 1 and 2 support statement (i) of Theorem 1. Statement (ii) is directly implied by Theorem A.1 (see the Appendix). Since the set of fixed points of operator $T = \overline{T} = \underline{T}$ is a chain complete poset, the set of distributional equilibria of Γ is also chain complete. In the remainder of the proof we assume that, for any countable chain $\{\tau_n\} \subseteq \mathcal{D}$ such that $\tau_n \to \tau$, we have $r(\alpha, a, \tau_n, s) \to r(\alpha, a, \tau, s)$. Consider the following claims.

Claim 3: Operator $\overline{T}$ is monotonically inf-preserving. Let $\{\tau_n\} \subseteq \mathcal{D}$ be a monotonically decreasing sequence with infimum τ. By Lemma 2, we have $\tau \in \mathcal{D}$. Moreover,

$$r\big(\alpha, \overline{m}(\alpha, \tau_n), \tau_n\big) \geq r(\alpha, a, \tau_n).$$

Since $\overline{m}(\alpha, \tau_n)$ belongs to $\tilde{A}(\alpha)$ which is compact, the limit of this sequence, denoted by

$\overline{m}^*$, exists and belongs to $\tilde{A}(\alpha)$. By continuity of r, we have $r(\alpha, \overline{m}^*, \tau) \geq r(\alpha, a, \tau)$, for all $a \in \tilde{A}(\alpha)$. Therefore, $\overline{m}^* \in m(\alpha, \tau)$ and

$$\overline{m}^* \leq \overline{m}(\alpha, \tau). \tag{3}$$

On the other hand, since $\tau_n \succeq_P \tau$, for all n, by monotonicity of function $\overline{m}$, we have $\overline{m}(\alpha, \tau) \leq \overline{m}(\alpha, \tau_n)$ and

$$\overline{m}(\alpha, \tau) \leq \lim_{n \to \infty} \overline{m}(\alpha, \tau_n) = \overline{m}^*. \tag{4}$$

Combining (3) and (4), we have $\overline{m}(\alpha, \tau) = \lim_{n \to \infty} \overline{m}(\alpha, \tau_n)$. Therefore, operator $\overline{T}$ is monotonically inf-preserving. Analogously, we may show that operator $\underline{T}$ is monotonically sup-preserving.

Claim 4: $\overline{\tau}^* = \lim_{n \to \infty} \overline{T}^n(\overline{\delta})$. By Claim 3, operator $\overline{T}$ is monotonically inf-preserving. Moreover, it is also $\succeq_P$-isotone. Therefore, by Theorem A.2 we conclude that $\overline{T}^n(\overline{\delta}) \to \overline{\tau}^*$. Analogously, we can show that $\underline{\tau}^* = \lim_{n \to \infty} \underline{T}^n(\underline{\delta})$. □

We next turn to the question of monotone comparative statics of the set of distributional equilibria of a LGSC. In particular, our concern is not only to prove the existence the comparative statics, but also to provide sufficient conditions that would allow for their computation. Given our generalization of Markowsky's Theorem (see Theorem A.4 in the Appendix), we are able to characterize the order structure of the set of fixed points of sup- and inf-preserving maps defined over countably chain complete posets.

Before, we state our next result, we need to introduce some additional notation. Consider a parameterized version of our game $\Gamma(s) := \big((\Lambda, \mathcal{A}, \lambda), A(\cdot, s), \tilde{A}(\cdot, s), r(\cdot, s)\big)$, for each $s \in S$, where the space of parameters $(S, \geq_S)$ is a partially ordered set. We impose the following assumptions of $\Gamma(s)$.

Assumption 2. *For any $s \in S$, suppose that*

(i) for all $s \in S$, correspondence $\tilde{A}(\cdot, s)$ and function $r(\cdot, s)$ satisfy Assumption 1;

(ii) for all $\alpha \in \Lambda$, correspondence $s \rightrightarrows \tilde{A}(\alpha, s)$ is ascending in the strong set order;[18]

(iii) for λ-a.e. player α, function r has single crossing differences in (a, s) on $A \times S$;

(iv) for any countable chain $\{\tau_n\} \subseteq \mathcal{D}$ such that $\tau_n \to \tau$, we have $r(\alpha, a, \tau_n, s) \to r(\alpha, a, \tau, s)$.

[18]Suppose that $(X, \geq_X)$ is a poset, while $(Y, \geq_Y)$ is a lattice. Correspondence $F : X \to 2^Y$ is ascending in the strong set order if, for any $x' \geq_X x$ in X and $y \in F(x)$, $y' \in F(x')$, we have $y \wedge y' \in F(x)$ and $y \vee y' \in F(x')$. Clearly, the above definition implies that correspondence F is sublattice-valued.

Under the above assumption, we are able to provide the following equilibrium monotone comparative statics result.

Corollary 1. *Under Assumption 2, for any $s \in S$, there exist the greatest and the least distributional equilibrium of $\Gamma(s)$, denoted by $\overline{\tau}^*(s)$ and $\underline{\tau}^*(s)$ respectively. Moreover, functions $s \to \overline{\tau}^*(s)$ and $s \to \underline{\tau}^*(s)$ are increasing.*

The above corollary follows directly from Theorems 1 and A.4 (see the Appendix). Therefore, we omit its sproof.

2.2 Remarks and discussion

In this subsection we discuss three issues. First, we compare our results to the related work in the existing literature concerning distributional equilibria in large games. Next, we discuss how the structure of the set of equilibria in LGSC differs from the one in GSC. Finally, we relate our results to the literature on monotone equilibrium comparative statics in games with a continuum of players.

Theorem 1 establishes the existence of distributional equilibria under a different set of assumptions than in Mas-Colell (1984) or in other related papers in the literature that followed. In particular, we do not require for the payoff function r to be weakly continuous with respect to distributions τ on $\mathcal{D}$. Instead, we endow sets $\Lambda \times A$ and $\mathcal{D}$ with a partially ordered structure, impose the quasisupermodularity and single crossing differences conditions on the payoff function, and require that the set of feasible actions of players is a complete sublattice of $\mathbb{R}^m$. In this sense, our approach allows to analyze large games with discontinuous payoffs.[19] See also a discussion in Rath (1996) and Carmona and Podczek (2014).

Second of all, our main theorem establishes the existence of extremal distributional equilibria which, aside from generalizing some of the results for GSC, allows us to develop the order-theoretic characterization of distributional equilibria. Furthermore, under stronger continuity conditions on payoffs, it provides a method of *computing* the equilibrium comparative statics. Additionally, given Theorem 1(ii), in a special case we provide an order-theoretic characterization of the entire set of distributional equilibria. Therefore, this result highlights some aspects of the existence theorems obtained for GSC via various versions of Tarski's Theorem (see Veinott, 1992; Zhou, 1994).

[19]For example, our methods allow to analyse large games of Bertrand competition in which the demand function is discontinuous due to product substitutability. Observe that this class of games cannot be analysed under the assumptions imposed by, e.g., Rath (1996).

However, it is worth mentioning that, unlike in GSC, we do *not* expect the set of distributional equilibria in LGSC to be a complete lattice. In fact, given Theorem 1(ii), we can at most expect that the set is a chain complete poset. The reason for this weaker characterization of equilibrium is very simple. In general, although the action set A is a complete lattice, the set of distributions over A is not.[20] As a result, instead of applying the result of Tarski, we must appeal to an alternative result by Markowsky.[21]

Our approach suggests a direct method for computing particular distributional equilibria of a game as well as their monotone comparative statics (compare with Topkis, 1998, Chapter 4.3). Under sufficient order-continuity conditions imposed on payoffs, in order to compute the extremal distributional equilibria, one needs to calculate the order-limit of a sequence generated by the upper and the lower distributional operator $\overline{T}$ and $\underline{T}$. The sequences are generated by iterating downward (respectively upwards) from the greatest (respectively the least) element of the chain complete poset $(\mathcal{D}, \succeq_P)$. Given that operators $\overline{T}$ and $\underline{T}$ are monotonically inf- and sup-preserving under our assumptions, the order-limits of our iterations are attainable in a *countable* number of steps. Hence, they are computable.[22]

Nevertheless, in order to obtain the computability result, we require that payoffs are weakly continuous on the space of distributions $\mathcal{D}$. Unlike in the original result by Mas-Colell (1984), the weak continuity assumption in not *critical* for existence of an equilibrium; rather, we need this condition for the approximation of the extremal equilibria and their computable comparative statics.

Finally, the only result concerning the existence of equilibrium comparative statics for large games, of which we are aware, is presented by Acemoglu and Jensen (2010). Their approach to equilibrium comparative statics is very similar to ours, as they impose conditions guaranteeing that the joint best response mapping has increasing selections with respect to the parameters of the game (see Definition 3 in their paper). However, there are differences. First of all, as they concentrate on *aggregative games*, where players best respond to the average/mean action of other players, the class of games in which they obtain the result is quite different (and more restrictive) than ours. In particular, our framework includes their class of games, but also allows for more general specifications

[20]For example, consider set of distributions on $A \subseteq \mathbb{R}^m, m \geq 2$, ordered by first order stochastic dominance ordering. In this case, it well-known that the spaces of probability measures on A is not a lattice (see Kamae, Krengel, and O'Brien, 1977); rather, it is only a CPO.

[21]It is worth noting that in Theorem 11 of his paper, Markowsky (1976) characterizes chain complete posets using their fixed point property relative to increasing mappings. In this sense, Markowsky provides a converse to Theorem A.1, just as Davis (1955) provides a converse to Tarski's Theorem.

[22]See the Appendix for a discussion of monotonically sup- and inf-preserving mappings.

of large games. Second of all, in case of a single dimensional action space A, Acemoglu and Jensen manage to show the comparative statics of the extremal (aggregative) equilibria *without* the single crossing property between player actions and aggregates. This constitutes an extension of our results in the case of one dimensional action spaces. However, once the space of actions is multidimensional, Acemoglu and Jensen require that payoffs have increasing differences in the action of a player and the aggregate, which is stronger than the ordinal notion of single crossing differences that we use. Finally, in order to show existence of an aggregate equilibrium Acemoglu and Jensen use the topological fixed point theorem of Kakutani. This makes the issues of equilibrium comparative statics and computability of equilibrium difficult to address. On the other hand, our use of order-theoretical fixed point results allows to address both issues directly.

We conclude this subsection with some additional remarks concerning generality of our results.

Remark 1. In his original paper, Mas-Colell (1984) focused on anonymous games in which the payoff function depended on the marginal distribution over the action space. In particular, the characteristics of agents where not taken into consideration by the players. Hence, the term *anonymous*. In other words, the reward function took the form of $r(\alpha, \tau) := \tilde{r}(\alpha, \tau_A)$, for some function $\tilde{r}$, where τ_A is the marginal distribution of τ over the set of actions A. Note that, our framework embeds anonymous games as a special case. Clearly, using a simple transformation $\tau_A(\cdot) = \tau(\Lambda \times \cdot)$, we can always construct payoffs that depend on τ_A, rather than τ.

Remark 2. After a careful read of the proof of Theorem 1, it is easy to see that the theorem consists of two separate results. The first one states that the greatest fixed point of operator $\overline{T}$ constitutes the greatest distributional equilibrium of Γ. The second one argues that the least fixed point of operator $\underline{T}$ is the least equilibrium of the game. As long as the two operators are well-defined, the two results hold simultaneously. However, if only one of them exists, it is still possible to show that there is either the greatest or the least equilibrium of the game, using the same argument as in the proof of Theorem 1. In particular, this allows us to analyze the class of *superextremal games*, introduced by Shannon (1990), LiCalzi and Veinott (1992), and Veinott (1992), in the large framework. Observe that, whenever we allow for the payoff function to exhibit a weaker form of complementarity then quasisupermodularity and single crossing differences, like join (meet) superextremality and join (meet) upcrossing differences, and allow for $\tilde{A}$ to have join (meet) sublattice values, rather than lattice values, then, by the work of LiCalzi and Veinott (1992), we can guarantee that that the upper (the lower) distributional operator exists. Hence, there is the greatest (the least) equilibrium of the game. Moreover, the monotone comparative statics of the equilibrium, established in Corollary 1, also apply.

Remark 3. In Section 3 that follows, we concentrate on applications which are inherently infinite-dimensional. That is, we discuss examples in which players take into account the whole distribution τ in order to evaluate their payoffs. Nevertheless, there are numerous applications in which agents interact with each other via an aggregate (e.g., see Acemoglu and Jensen, 2010; Guesnerie and Jara-Moroni, 2011). The results presented in this paper are still applicable to this subclass of games. However, they can be strengthen in this particular framework.[23] Suppose that the payoff function $r(\alpha, a, g)$ depends on an aggregate value g, of players characteristics and actions. Suppose that g is an element of some complete lattice G. Moreover, let g be determined by function $h : \mathcal{D} \to G$, which is increasing on $\mathcal{D}$.[24] Define an operator $\overline{\Psi} : G \to G$ by $\overline{\Psi}(g) := h\big(\overline{T}g\big)$, where

$$\big(\overline{T}g\big)(H) = \lambda\Big(\big\{\alpha : \big(\alpha, \overline{m}(\alpha, g)\big) \in H\big\}\Big), \text{ for some } H \in \mathcal{L} \otimes \mathcal{A},$$

while $\overline{m}(\alpha, g) = \bigvee \operatorname{argmax}_{a \in \tilde{A}(\alpha)} r(\alpha, a, g)$. Whenever Assumption 1 is satisfied with r having single crossing differences in (a, g) on $A \times G$, it is easy to show that operator $\overline{\Psi}$ is well-defined and isotone. Since G is a complete lattice, we can apply Tarski's Theorem to show that the operator has the greatest fixed point $\overline{g}^*$. In particular, this implies that there is the greatest distributional equilibrium of the game $\big(\overline{T}\overline{g}^*\big)$. Similarly, we may prove the existence of the least equilibrium.

Remark 4. Following the tradition of Schmeidler (1973), it is possible to define an equilibrium of a LGSC in terms of *strategy profiles*, rather than distributions. In this framework, a strategy profile is a function $f : \Lambda \to A$ that assigns an action $f(\alpha) \in \tilde{A}(\alpha)$ to agent $\alpha \in \Lambda$. Moreover, we find it desirable to restrict our attention to $\mathcal{L}$-measurable functions.

To see how could we adapt our techniques to the above concept of equilibrium, define a set of $\mathcal{L}$-measurable functions $f : \Lambda \to A$ by $\mathcal{F}$. Endow the space with the *pointwise* order.[25] Therefore, the payoff function is $r : \Lambda \times A \times \mathcal{F} \to \mathbb{R}$.

Under similar conditions to those stated in Assumption 1, it is possible to provide an analogous characterization of the set of equilibria, including the existence of the greatest and the least elements of the equilibrium set. However, the method of proving the result is

[23] We thank one anonymous referee of this journal for recommending us this application.

[24] Suppose that $G \subset \mathbb{R}^n$. One example of an aggregator function may be $h = (h_1, \ldots, h_n)$, where $h_i : \mathcal{D} \to \mathbb{R}$ is defined by $h_i(\tau) := \int_{\Lambda \times A} f_i(\alpha, a) d\tau(\alpha, a)$, for some increasing $f_i : \Lambda \times A \to \mathbb{R}$, $i = 1, \ldots, n$.

[25] That is, we say that function f' dominates f in the pointwise order if $f'(\alpha) \geq f(\alpha)$, for all $\alpha \in \Lambda$. Note, this is a different ordering, than the one discussed in our main result.

different. In particular, we are forced to use a different fixed point result in our argument, as the Markowsky's Theorem is no longer applicable.

The reason we need to refer to a different argument is implied by the fact that, unlike the space of distributions, the set of bounded Borel-measurable functions endowed with the pointwise order is *not* a chain complete poset. In fact, the set is a σ-complete lattice, which means that only *countable* subsets of $\mathcal{F}$ have their greatest and least elements contained in $\mathcal{F}$ (see Heikkilä and Reffett, 2006, Example 2.1). Therefore, the set is a *countable chain complete*. Clearly, this follows from the very definition of measurability, which need not be preserved under uncountable operations.

The lack of the strong form of completeness of the joint strategy set forces us to use the Tarski-Kantorovich Theorem (see Theorem A.2 in the Appendix) in order to prove the existence of an equilibrium in the sense of Schmeidler (1973). This on the other hand, requires additional order-continuity assumptions on the payoff function.[26] Recall that in case of distributional equilibria, the continuity conditions were only required for the approximation of the extremal equilibria and the monotone comparative statics, but not for equilibrium existence. Nevertheless, our order-continuity assumptions are weaker than weak continuity assumed in the existing literature (e.g., see Khan, 1986).[27] Moreover, our methods allow for the iterative approximation of the extremal equilibria, as well as the monotone comparative statics of the equilibrium set.

3 Applications

In this section we present several economic applications of the methods discussed in the first part of the paper.

3.1 Social distance model

To motivate the results in our paper, we begin by considering distributional equilibria in a version of the *social distance* model described originally by Akerlof (1997). The model studies the distribution of social ranks/statuses over a large number of heterogeneous individuals. Consider a continuum of agents distributed over a compact interval $X \subset \mathbb{R}$,

[26]Formally, we require that the payoff function preserves the limits of monotone sequences in $\mathcal{F}$. This guarantees that the extremal best response selections are monotonically inf- and sup-preserving – a feature required for the application of Tarski-Kantorovich fixed point theorem.

[27]Since order-continuity has to be verified only with respect to all sub-chains of the original set, it is straightforward to provide an example of payoff functionals that are order-continuous, but not weakly continuous.

where a *location* of an individual agent is denoted by $x \in X$. Let $Y \subset \mathbb{R}$ be the set of all possible positions in the society (e.g., social statuses/ranks), where Y is compact and convex. Each individual is characterized by an *identity* $y \in Y$, which determines the social status/rank to which the agent aspires. We shall refer to the location-identity pair (x, y) as the *characteristic* of an agent.

Suppose that the distribution of characteristics across the population is determined by a probability measure λ, defined over the Borel-field of $X \times Y$. We assume that every agent knows his own characteristic (x, y), as well as the distribution of characteristics across the population. In this game, we study how agents determine their optimal individual choice of social status $a \in Y$, given their location $x \in X$ and identity $y \in Y$.

In this model, the payoff of an individual is determined as follows. First of all, every agent aims to attain a status/rank that is in proximity to his true identity y. Therefore, the agent will suffer a penalty, whenever his social status a does not match to his true identity y. Moreover, the further away the actual status is from the true identity, the more disutility the agent receives.

Second of all, the individual payoff is affected by interactions with other agents in the game. Assume that the players meet at random. Whenever an agent meets another player, he suffers a disutility if his social status a differs from the social status a' of the other individual. Moreover, the disutility increases the greater is the distance between the two statuses. This incorporates a form of peer pressure or conformism to the game.

Let u, $v : \mathbb{R} \to \mathbb{R}$ be a pair of continuous, decreasing functions. In addition, assume that v is concave. Consider an agent characterized by (x, y), who chooses a social status $a \in Y$. Whenever an agent meets another individual with a social status $a' \in Y$, his utility is given by:

$$u(|a - y|) + v(|a - a'|).$$

As both functions u and v are decreasing, the objective of every player is to choose an action as close as possible to their true identity y and the identity a' of the other player. Moreover, given concavity of function v, the further away is the status of the agent from the social rank of the other player, the steeper are the changes in the disutility.

In order to make our notation compact, denote $\Lambda := X \times Y$, with a typical element $\alpha = (x, y)$. Suppose that the frequency of interactions of the agent with other individuals is governed by a probability measure μ_α, defined over the product Borel-field of Λ. Therefore, for any set $U = U_x \times U_y$, where $U_x \subset X$ and $U_y \subset Y$, value $\mu_\alpha(U)$ is the probability of encountering an agent with a characteristic $(x', y') \in U_x \times U_y$. We assume that the measure μ_α depends on $\alpha = (x, y)$, as both the location x of the agent as well as his aspirations

y may determine the frequency of interactions with other members of the population. For example, the greater the distance between the locations of any two agents, the less frequent will be their interaction. Analogously, agents with similar aspirations are more likely to meet.

Let τ be a probability measure defined over the Borel-field of $\Lambda \times Y$. Suppose that the marginal distribution of τ over Λ is λ. Denote the set of all such measures by $\mathcal{D}$. Clearly, τ is a probability distribution of player characteristics and social ranks (α, a). Hence, for any Borel sets $U \subseteq \Lambda$ and $A \subseteq Y$, value $\tau(U \times A)$ denotes the measure of agents with characteristic $\alpha \in \Lambda$ and social rank $a \in A$.

Given the notation, we define the decision problem faced by a typical agent in the game. The objective of a player is to choose his social status $a \in Y$ that maximizes his expected payoff given by

$$r(\alpha, a, \tau) := u(|a-y|) + \int_{\Lambda} \int_{Y} v(|a-a'|) d\tau(a'|\alpha') d\mu_{\alpha}(\alpha'),$$

where $\alpha = (x, y)$ and $\tau(\cdot|\alpha')$ is the distribution of actions of other players in the population conditional on $\alpha' \in Y$. Therefore, the payoff of an agent is the sum of utilities that he receives from individual interactions with other agents. According to the above definition, the social status of an individual cannot be contingent on the social statuses of other agents, but has to be chosen ex-ante before any interaction occurs.[28]

Formally, since the set of feasible actions is common for all agents and constantly equal to Y, Assumption 1(i) is satisfied. By assumption, function v is concave and decreasing. Therefore, Lemma A.4 (see the Appendix) implies that $v(|a-a'|)$ has increasing differences in (a, a'). It is easy to show that this is sufficient for payoff function $r(\alpha, a, \tau)$ to have increasing differences in (a, τ) on $Y \times \mathcal{D}$, whenever the set of distributions $\mathcal{D}$ is ordered with respect to the first order stochastic dominance $\succeq_P$. Hence, Assumption 1(ii) is also satisfied. Finally, given that functions u and v are continuous, and assuming that function $\alpha \to \mu_{\alpha}$ is measurable, Assumption 1(iii) also holds.

By Theorem 1, we conclude there exist the greatest and least distributional equilibria of the game. Furthermore, as under our assumptions $r(\alpha, a, \tau)$ satisfies the order-continuity on $\mathcal{D}$, which allows for the iterative approximation of the extremal equilibria. Moreover, since agents care about the status of other players as well as their own true identity, the extremal equilibria are trivial only in some special cases. That is, in general it is not

[28] One interpretation of the game is as follows. With probability $\mu_{\alpha}(\{\alpha'\})$ player α meets an individual with a characteristic α'. Then, with probability $\tau(\{a'\}|\alpha')$ player α' chooses action a'. This allows agent α to calculate his expected payoff. In this sense, we can think of this as an interim game. However, we do not analyze ex-post "matching" of agents.

the case that, in the greatest (the least) equilibrium of the game, the measure of agents choosing the greatest (the least) possible social status is equal to 1. Therefore, the approximation methods may be very useful in determining and computing the distributional equilibria.

An inherent feature of the above example is that agents need to observe the *entire* distribution of characteristics and actions of other players in order to evaluate their payoffs. In particular, the externality in the game cannot be summarized by an aggregate of actions of players as in Acemoglu and Jensen (2013). Moreover, since the the set of probability distributions is not a lattice, the externality in the game cannot be formulated as a "lattice externality" (see Guesnerie and Jara-Moroni, 2011).

Moreover, note that the space of characteristics is not a chain (i.e., is not totally ordered). In fact, it is crucial that the space of characteristics is a subset of the two dimensional real space, as both the location and the true identity of an agent affect his decision. As it was stressed by Akerlof (1997) or Akerlof and Kranton (2000), both the *neighborhood* effect (the players location) and the *family background effect* (the players true identity) are the two key factors in determining the social interaction and the distribution of equilibrium social distance. However, this means that the space of player characteristics is not a chain. Therefore, the developments of Yang and Qi (2013) cannot be applied in the above framework (see also Balbus, Reffett, and Woźny (2014b)).

3.2 Linear non-atomic supermodular games

The previous example is a special case of a larger class of games that we call *linear non-atomic supermodular games*. Assume that the measure space of player characteristics is denoted by $(\Lambda, \mathcal{L}, \lambda)$. As in Section 2, let the set of all possible actions be A, and the correspondence mapping the characteristics of agents into the set of feasible strategies be denoted by $\tilde{A}$. In addition, we introduce a poset of parameters denoted by $(S, \geq_S)$. In the class of linear nonatomic supermodular games, all the agents in the population interact with each other *individually*. Therefore, the player's ex-post payoff is a sum of utilities from every separate interaction.

More specifically, suppose that the payoff from a single interaction is determined by function $u : \Lambda \times S \times A \times A \to \mathbb{R}$. That is, an agent with characteristic $\alpha \in \Lambda$, given parameter $s \in S$ and action $a \in \tilde{A}(\alpha)$, yields $u(\alpha, s, a, a')$ units of utility from an individual interaction with an agent playing $a' \in A$. As previously assumed, the frequency of interactions with other players will depend on the trait of a given player. Hence, for any $\alpha \in \Lambda$, there is a non-atomic probability measure μ_α defined over $\mathcal{L}$. Hence, for any measurable group $U \in \mathcal{L}$, $\mu_\alpha(U)$ denotes the probability that an agent with characteristic α will meet an individual with a characteristic belonging to U.

Let $\tau \in \mathcal{D}$ be a distribution of characteristics and actions in the population. Given the above description of the game, the ex-post payoff of an agent with $\alpha \in \Lambda$ is

$$r(\alpha, s, \tau, a) := \int_\Lambda \int_A u(\alpha, s, a, a') d\tau(a'|\alpha') d\mu_\alpha(\alpha').$$

Clearly, in order for the payoff to be well defined, we require that function $u(\alpha, s, a, \cdot)$ is $\mathcal{A}$-measurable for any α, s, and a.

Assume that function u is $\mathcal{L}$-measurable, continuous and supermodular with respect to a, and has increasing differences in (a, a'), for any $s \in S$. Moreover, let μ_α be measurable as a function of α. Given Theorem 1, under the above assumptions, any such linear nonatomic supermodular game has the greatest and the least distributional equilibrium, which can be approximated using our iterative method.

Assume that, for any $\alpha \in \Lambda$ and $a'' \geq a$ in $\tilde{A}(\alpha)$, the family of functions $\{\Delta_\alpha(\cdot, a')\}_{a' \in A}$, where $\Delta_\alpha(s, a') := u(\alpha, s, a'', a') - u(\alpha, s, a, a')$, obey the signed-ratio monotonicity (see Quah and Strulovici, 2012). Then, function $r(\alpha, s, \tau, a)$ has single crossing differences in (a, s) on $A \times S$. Therefore, by Corollary 1, the greatest and the least distributional equilibrium of the game *increases* with respect to the deep parameter s.

The key feature of the above class of games is that the payoff function is linear with respect to measure τ. Therefore, it is weakly continuous on the space of probability measures. This implies, that the additional order-continuity assumption imposed prior to Theorem 1(iii) is always satisfied. Hence, no additional assumptions have to be imposed in order for the result to hold.

3.3 Large stopping games

We next turn to an optimal stopping time example. Suppose that a continuum of agents are deciding how long they should each one participate in an investment project that lasts at most T periods. Each period t, each agent takes part in the investment, from which she receives a profit of $\pi(t, m)$, where m is the measure of agents participating in the project at time t. We assume that $\pi(t, m)$ may take both positive and negative values; however, it is increasing in m. In other words, the more agents who participate in the project, the higher are the profits (or, lower are the losses) to every individual agent. Finally, whenever the agent is not participating at the project, her payoff is equal to zero.

In the following analysis we concentrate solely on the case where time is discrete.

Define the set of time indices by $\{1, 2, \ldots, T\}$. Suppose that the time at which the agent joins the project is determined exogenously and that it defines the characteristic of an agent. Hence $\alpha \in \Lambda$, where $\Lambda := \{1, 2, \ldots, T\}$. The distribution of characteristics across agents is determined by some measure λ over Λ. Hence, $\lambda(\alpha)$ is the measure of agents that join the project at time α. Endow Λ with a product order.

Since the time at which agents join the investment is given exogenously, they can only decide when to leave the investment. Assume that agents can leave the project only once (i.e., just like in a standard optimal stopping game). Given this, an action of a player is equivalent to a time index at which the agent decides to leave the investment. Using our notation, the set of all possible actions A is equivalent to the set of time indices Λ.[29] Moreover, the correspondence mapping agents characteristics into the set of feasible actions is defined by $\tilde{A}(\alpha) := \{\alpha, \ldots, T\}$.

Assume that a distribution of characteristics and actions for the population is given by $\tau \in \mathcal{D}$. Therefore, for any $(\alpha, a) \in \Lambda \times A$, $\tau(\{(\alpha, a)\})$ is the measure of agents joining the investment at time α and leaving at time a. Define function $F : \mathcal{D} \times \Lambda \to [0, 1]$ as

$$F(\tau, t) := \tau\Big(\big\{(\alpha, a) : \alpha \leq t \leq a\big\}\Big).$$

In other words, $F(\tau, t)$ is a measure of agents participating in the project at time t. Note that function $F(\tau, \cdot)$ is not a probability distribution nor a cumulative distribution. Clearly, the sum of its values might not be equal to one and it need not be monotone.

Given the notation, we define the payoff of an agent with characteristic $\alpha \in \Lambda$ by

$$r(\alpha, a, \tau) := \sum_{t=\alpha}^{a} \pi\big(t, F(\tau, t)\big),$$

The objective of every agent is to maximize $r(\alpha, \tau, a)$ with respect to $a \in \tilde{A}(\alpha)$.

In order to make sure that the above game satisfies Assumption 1, we first need to show that function $F(\tau, \cdot)$ is pointwise increasing as the measures $\tau \in \mathcal{D}$ shifts upward with respect to the first order stochastic dominance. Consider any τ' and τ in $\mathcal{D}$ such that $\tau' \succeq_P \tau$. By definition of the first order stochastic dominance, this implies that for any $(\alpha, a) \in \Lambda \times \Lambda$, we have

$$\tau'\Big(\big\{(\alpha', a') : (\alpha', a') \leq (\alpha, a)\big\}\Big) \leq \tau\Big(\big\{(\alpha', a') : (\alpha', a') \leq (\alpha, a)\big\}\Big).$$

Since both τ' and τ belong to $\mathcal{D}$, they have the same marginal distribution over the space

[29] We shall differentiate the notation of these two sets in order to avoid confusion.

of characteristics Λ. Therefore, $\tau'(\{(\alpha', a') : a' \leq a\}) \leq \tau(\{(\alpha', a') : a' \leq a\})$. Then,

$$\begin{aligned} F(\tau, t) &:= \tau\Big(\big\{(\alpha, a) : \alpha \leq t \leq a\big\}\Big) \\ &\leq \tau'\Big(\big\{(\alpha, a) : \alpha \leq t \leq a\big\}\Big) \\ &=: F(\tau', t), \end{aligned}$$

for any $t \in \Lambda$. Observe that the above property is sufficient for the payoff function r to have increasing differences in (a, τ) on $A \times \mathcal{D}$. Clearly, since function $\pi(t, \cdot)$ is increasing,

$$\begin{aligned} r(\alpha, a', \tau) - r(\alpha, a, \tau) &= \sum_{t=a}^{a'} \pi\big(t, F(\tau, t)\big) \\ &\leq \sum_{t=a}^{a'} \pi\big(t, F(\tau', t)\big) \\ &= r(\alpha, \tau', a') - r(\alpha, \tau', a), \end{aligned}$$

for any $\alpha \in \Lambda$ and $a' \geq a$ in A. In order to complete the argument, note that, as the space of characteristics and actions is finite, the payoff function is trivially continuous on $\Lambda \times A$. Therefore, the conditions specified in Assumption 1 are satisfied. Moreover, once we assume that function $\pi(t, \cdot)$ is continuous for any $t \in \Lambda$, the payoff function is order-continuous with respect to τ. Hence, by Theorem 1, we conclude that the game has the greatest and the least distributional equilibrium which can be approximated using our iterative methods.

Finally, we are able to determine the comparative statics of the extremal equilibria. Assume that each period the payoff function is parameterized by a deep parameter s belonging to a poset $(S, \geq_S)$. Hence, each period the agent that participates in the investment receives $\pi(t, s, m)$, where m is the measure of players taking part in the project at the given time. Suppose that $\pi(t, \cdot, m)$ is an increasing function for any $t \in \Lambda$ and $m \in [0, 1]$. Clearly, for any α and τ, the payoff function

$$r(\alpha, s, a, \tau) := \sum_{t=\alpha}^{a} \pi\big(t, s, F(\tau, t)\big),$$

has increasing differences in (a, s) on $A \times S$. By Corollary 1, this is sufficient to conclude that the extremal equilibria are increasing functions of s.

3.4 Keeping up with the Joneses

We finish this paper with an application of our results to large economies with peer effects. Consider an economy consisting of a continuum of consumers. Suppose that every consumer in the economy is characterized by their initial wealth $m \geq 0$, and a number $i \in [0,1]$. In our framework, the number i will correspond to the relative social position to which the agent refers to when choosing her consumption level. We shall define the notion formally in the remainder of this subsection.

Assume that the set off all possible values of wealth m is a compact subset M of $\mathbb{R}_+$, where $\max M := \bar{m}$. Let $\Lambda := M \times [0,1]$, and $\alpha = (m,i)$ be one of its elements. The distribution of characteristics is determined by a probability measure λ defined over the product Borel-field $\mathcal{L}$ of λ.

There are two markets in the economy: the *consumption good* market and the *labor* market. Every agent is endowed with m units of the initial wealth, expressed in units of the consumption good, and one unit of time that can be devoted either to labor or leisure. Given the normalized price of consumption $p = 1$ and wage $w > 0$, the budget set is

$$B(m,w) = \big\{(a,n) \in \mathbb{R}_+ \times [0,1] : m + w \geq a + wn\big\},$$

where by (a,n) we denote a pair of consumption a and leisure n. Note that the set of feasible consumption levels in our framework is given by $A := [0, \bar{m}+w]$, which is compact.

Apart from the consumption and leisure, every agent takes into account the relation of her consumption to the consumption of other agents in the economy. Number $i \in [0,1]$, characterizing the agent, denotes the quantile of the distribution of consumption in the population that the agent is treating as his reference point, when choosing her consumption. In other words, the higher is the agent's consumption above the i'th quantile of the distribution, the better. This feature of the model incorporates the *keep up with the Joneses* effect, but in a *heterogenous* manner. That is, every agent might be characterized by a different number i.

Assume that the distribution of characteristics and consumption in the economy is defined by a probability measure τ over $\Lambda \times A$. Let

$$q(\tau,i) := \min\Big\{a' \in A : \tau\Big(\big\{(\alpha,a) \in \Lambda \times A : a \leq a'\big\}\Big) \geq i\Big\}.$$

Hence, $q(\tau,i)$ is the i'th quantile of the distribution of consumption, given τ.

Every consumer is endowed with a pair of utility functions $u : \mathbb{R}_+ \times [0,1] \to \mathbb{R}_+$ and $v : \mathbb{R}_+ \to \mathbb{R}$. We assume that both functions are continuous, increasing, and concave.

Given the distribution of consumption τ, the objective of every agent is to solve

$$\max_{(a,n)\in B(m,w)} u(a,n) + v\big(a - q(\tau,i)\big).$$

Therefore, when choosing his consumption and leisure every agent takes into account their direct effect summarized by function u, as well as the utility that comes from the relative consumption.

Note that in the above framework we model the social status differently than in the example discussed in Section 3.1. Since agents' utility depends positively on the difference between their consumption and a certain quantile of the distribution of consumption, the social status is defined relatively to the mass of agents that consume less than the given player. We consider this to be a good approximation of the consumer choice, when the social class concerns are taken into consideration. Clearly, agents want their consumption to dominate the consumption of a certain fraction of the population.

Given monotonicity of u and v, the budget constraint is always binding. Hence, in any optimal solution (a,n), we have $n = (m + w - a)/w$. This allows us to reduce the number of variables in the consumer optimization problem. Given the condition, the modified payoff of the consumer is

$$r(\alpha, a, \tau) := u\big(a, (m + w - a)/w\big) + v\big(a - q(\tau,i)\big),$$

where $\alpha = (m,i)$, while the set of feasible strategies can be represented by $\tilde{A}(\alpha) = \big[0, (w+m)/w\big]$. Hence, given τ, the consumer objective is to maximize $r(\alpha, a, \tau)$ with respect to $a \in \tilde{A}(\alpha)$.

In order to apply our main results to the above example, we need to show that it satisfies the conditions of Assumption 1. Clearly, set A is a lattice, while correspondence $\tilde{A}$ is continuous and complete lattice valued. By assumptions imposed on functions u and v, function r is continuous in a and measurable with respect to the characteristic α. Therefore, it suffices to show that is has single crossing differences in (a,τ).

First, note that the quantile function $q(\cdot, i)$ is increasing on $(\mathcal{D}, \succeq_P)$ for any i. Take any $a' \geq a$ in $\tilde{A}(\alpha)$ and $\tau' \succeq_P \tau$ in $\mathcal{D}$. Then,

$$\begin{aligned} r(\alpha, a', \tau) - r(\alpha, a, \tau) &= u\big(a', (m+w-a')/w\big) - u\big(a, (m+w-a)/w\big) \\ &\quad + v\big(a' - q(\tau, i)\big) - v\big(a' - q(\tau, i)\big) \\ &\leq u\big(a', (m+w-a')/w\big) - u\big(a, (m+w-a)/w\big) \\ &\quad + v\big(a' - q(\tau', i)\big) - v\big(a' - q(\tau', i)\big) \\ &= r(\alpha, a', \tau') - r(\alpha, a, \tau'), \end{aligned}$$

where the inequality is implied by monotonicity of $q(\cdot, i)$ and Lemma A.3 (see the Appendix). Hence, function $r(\alpha, \tau, a)$ has increasing differences in (a, τ). Therefore, the above example satisfies the conditions stated in Assumption 1 and our results follow.

Appendix

In the following section we present several auxiliary results used in the arguments supporting our main theorems. First, we present a theorem by Markowsky (1976). See Theorem 9 of his paper for the original reference.

Theorem A.1 (Markowsky's Fixed Point Theorem). *Let $F : X \to X$ be increasing, and X be a chain complete poset. Then, the set of fixed points of F is a chain complete poset. Moreover, $\overline{x} := \bigvee \{x : x \leq F(x)\}$ and $\underline{x} := \bigwedge \{x : x \geq F(x)\}$ constitute the greatest and the least fixed point of F respectively.*

For the sake of completeness, we present the proof of the theorem.

Proof of Theorem A.1. Construct a transfinite sequence, where $f(0) := \bigwedge X$, while $f(\alpha) := \bigvee \{F(f(\alpha')) : \alpha' < \alpha\}$, for $\alpha \in [0, \xi + 1]$[30].Consider the following steps.

Step 1: We claim that $f(\alpha)$ is a well-defined chain, isotone with respect to α. For $\alpha = 0$ the proof is straightforward. Assume the thesis is satisfied for some α_0. If $\alpha_0 + 1$ is the successor of α_0 then $\{F(f(\alpha')) : \alpha' < \alpha_0\}$ is a chain. Hence, it has a supremum. Moreover, $F(f(\alpha_0))$ is a bound of this set. Therefore,

$$\begin{aligned} F\big(f(\alpha_0)\big) &= \bigvee \Big\{F\big(f(\alpha')\big) : \alpha' < \alpha_0\Big\} \cup \Big\{F\big(f(\alpha_0)\big)\Big\} \\ &= \bigvee \Big\{F\big(f(\alpha')\big) : \alpha' < \alpha_0 + 1\Big\} \\ &=: f(\alpha_0 + 1). \end{aligned}$$

[30] A similar construction was applied by Echenique in a proof of Tarski's Theorem.

This implies that $\{f(\alpha') : \alpha' < \alpha_0 + 1\}$ is a chain. If α_0 is a limiting point and the thesis is satisfied for all ordinal numbers smaller then α_0, then

$$\Big\{F\big(f(\alpha')\big) : \alpha' < \alpha_0\Big\} = \bigcup_{\alpha<\alpha_0} \Big\{F\big(f(\alpha')\big) : \alpha' < \alpha\Big\}.$$

Clearly, the right hand side is an ascending sequence of chains. Hence, it is a chain. Since X is chain complete, this set possess a supremum $f(\alpha_0)$. By transfinite induction, $f(\alpha)$ is a well defined sequence.

Step 2: We show the existence of the greatest and the least fixed points of F. Since $\xi + 1$ has a lower cardinality than X, there exists α_0 such that $f(\alpha_0) = f(\alpha_0 + 1)$. Hence the set of ordinal numbers $\{\alpha : f(\alpha) = f(\alpha+1)\}$ has the least element α_0. Let $\underline{x} := f(\alpha_0)$. Then $\underline{x} = f(\alpha_0) = f(\alpha_0 + 1) = F(f(\alpha_0)) = F(\underline{x})$. Therefore $\underline{x}$ is a fixed point. Let e be an arbitrary fixed point. We show that, for all α, $f(\alpha) \leq e$. We examine $P_\alpha : f(\alpha) \leq e$ by transfinite induction. For $\alpha = 0$ it is straightforward. Assume that this hypothesis is satisfied for all $\alpha' < \alpha$. Then, $F(f(\alpha')) \leq F(e) = e$. Thus,$f(\alpha) \leq e$. Hence $f(\alpha) \leq e$ for all α. Observe that $\underline{x} = f(\alpha_0) \leq e$, which implies that $\underline{x}$ is the least fixed point. Similarly we show existence of the greatest fixed point, denoted by $\overline{x}$.

Step 3: We claim that $\overline{x} = \bigvee \big\{x : x \leq F(x)\big\}$. Let $E := \big\{x : x \leq F(x)\big\}$. Observe that F maps E into itself. If $x \in E$, then $F(x) \geq x$ and by monotonicity $F\big(F(x)\big) \geq F(x)$. Therefore, $F(x) \in E$. Note that, that all fixed points are elements of E. Moreover, if we take arbitrary element of E such that $x_0 < F(x_0)$, then x_0 may not be the greatest element, since $F(x_0) \in E$. Hence, we claim that only $\bigvee E$ can be the greatest fixed point of F. To show it, we can repeat the construction from Step 1 with $f(0) = x_0$, for an arbitrary $x_0 \in E$. Then, sequence $f(\alpha)$ is isotone, and there is some α_0 such that $f(\alpha_0) = f(\alpha_0 + 1)$, which implies that $f(\alpha_0)$ is a fixed point. Hence $x_0 \leq f(\alpha_0) \leq \overline{x}$. Since $\overline{x} \in E$, it must be that $\overline{x} = \bigvee E$. Similarly we show that $\bigwedge\{x : x \geq F(x)\}$ is the least fixed point.

Step 4: We claim that the set of fixed points is a chain complete poset. Let C be a chain of fixed points, with $c := \bigvee C$. Consider a set of upper bounds of C, namely $Y := \{x : x \geq c\}$. If $x \in Y$ then, for all $e \in C$, we have $x \geq c \geq e$. Therefore, $F(x) \geq F(c) \geq F(e) = e$. By definition of c, this implies that $F(x) \geq c$, and so $F(Y) \subset Y$. Clearly Y is chain complete poset, while $G := F|_Y$ is monotone. Hence, by the previous part, $F|_Y$ has the least fixed point. This is the least upper bound of C in the set of fixed points of F. □

In many cases, we are referring to a constructive version of Markowsky's Theorem.

The Tarski-Kantorovich Fixed Point Theorem is one such theorem. We now provide a generalization of the Tarski-Kantorovich result that additionally provides a fixed-point comparative statics results in the spirit of Veinott's (1992) version of Tarski's theorem. See also Balbus, Reffett, and Woźny (2014a), where it was originally stated. For a monotone sequence $\{x_n\}_{n=0}^{\infty}$, let

$$\bigvee x_n := \sup_{n\in\mathbb{N}} x_n \quad \text{and} \quad \bigwedge x_n := \inf_{n\in\mathbb{N}} x_n.$$

By $F^n(x)$, we denote the n'th orbit (or iteration) of function F from point $x \in X$. That is, $F^n(x) := F \circ F \circ \ldots \circ F(x)$.

Definition A.1. *Function $F : X \to X$ is monotonically sup-preserving (monotonically inf-preserving) if, for any monotone sequence $\{x_n\}_{n=0}^{\infty}$, we have $F\big(\bigvee x_n\big) = \bigvee F(x_n)$ ($F\big(\bigwedge x_n\big) = \bigwedge F(x_n)$). F is monotonically sup/inf-preserving if and only if it is both monotonically sup- and inf-preserving.*

It is worth mentioning that a monotonically sup- or inf-preserving function is necessarily increasing. The Tarski-Kantorovich Theorem (see Dugundji and Granas, 1982, Theorem 4.2) states the following.

Theorem A.2 (Tarski-Kantorovitch Fixed Point Theorem)**.** *Let X be a countably chain complete poset with the greatest and the least element denoted by $\overline{x}$ and $\underline{x}$ respectively. Let Φ denote the set of fixed points of function $F : X \to X$. Then,*

(i) if F is monotonically inf-preserving, $\bigwedge F^n(\overline{x})$ is the greatest fixed point of F;

(ii) if F is monotonically sup-preserving, $\bigvee F^n(\underline{x})$ is the least fixed point of F.

Next, we present two new theorems that can be developed using the Tarski-Kantorovitch Fixed Point Theorem. First, we characterize the set of fixed-point of function F. The second result provides the monotone comparative statics of the set.

Theorem A.3. *Let X be a countably chain complete poset, $F : X \to X$ a monotonically sup/inf-preserving function, and Φ is the set of fixed points of F. Then, set Φ is a non-empty countably chain complete poset with the greatest and the least element denoted by*

$$\bar{\phi} := \bigvee \{x : F(x) \geq x\} \quad \textit{and} \quad \underline{\phi} := \bigwedge \{x : F(x) \leq x\}.$$

Proof of Theorem A.3. By Tarski-Kantorovich Theorem, F has a nonempty set of fixed points. Let $\{e_n\}$ be a countable chain of fixed points, with $\bar{e} := \bigvee\{e_n\}$. Then,

$$F(\bar{e}) = F\left(\bigvee\{e_n\}\right) = \bigvee F(e_n) = \bigvee e_n = \bar{e}.$$

Similarly, we prove the analogous result for $\underline{e} := \bigwedge e_n$. Next, we show that $\bar{\phi} := \bigvee\{x : F(x) \geq x\}$. Let $\overline{x}$ be the greatest element of X. Take an be arbitrary point x such that $x \leq F(x)$. Clearly $x \leq \overline{x}$. Assume that $x \leq F^n(\overline{x})$. Then, $x \leq F(x) \leq F\big(F^n(\overline{x})\big) = F^{n+1}(\overline{x})$. Hence, $x \leq \bar{\phi}$. Since $\bar{\phi} \in \{x : F(x) \geq x\}$, it must be that $\bar{\phi} := \bigvee\{x : F(x) \geq x\}$. We prove the second claim analogously. □

Theorem A.4. *Let X be a countably chain complete poset with the greatest and least element, T be a poset, and $F : X \times T \to X$ be an increasing function such that $F(\cdot, t)$ is monotonically inf-preserving (monotonically sup-preserving) on X, for all $t \in T$. Moreover, denote the greatest (the least) fixed point of $F(\cdot, t)$ by $\bar{\phi}(t)$ ($\underline{\phi}(t)$), for some $t \in T$. Then, function $t \to \bar{\phi}(t)$ ($t \to \underline{\phi}(t)$) is increasing.*

Proof of Theorem A.4. Let $t_1 \leq t_2$. From Theorem A.3 we know that $m_i := \bar{\phi}(t_i) = \bigvee\Gamma_i := \bigvee\{x : F(x, t_i) \leq x\}$. Note that by monotonicity of $F(x, \cdot)$ we obtain $m_1 = F(m_1, t_1) \leq F(m_1, t_2)$. Hence $m_1 \in \Gamma_2$. Since m_2 is the greatest element of Γ_2, this implies that $m_1 \leq m_2$. □

In this part of the section, we prove that a space of regular measures defined over a compact support and endowed with the first order stochastic dominance order is a chain complete poset. Suppose that $(X, \geq)$ is a partially ordered set, where X is a compact space endowed with Hausdorff topology $\mathcal{X}$ and a Borel σ-field $\mathcal{B}$. By $C(X)$ we denote the set of continuous, real-valued functions on X. Let $M(X)$ be a set of isotone, measurable, and bounded real-valued functions over X. We say that measure μ first order stochastically dominates ν, and denote it by $\mu \succeq_P \nu$, whenever

$$\int_X f d\mu \geq \int_X f d\nu, \text{ for all } f \in M(X).$$

Let $\mathcal{R}$ be a set of regular measures over X.

Lemma A.1. *If $M(X)$ separates the points of X, then $(\mathcal{R}, \succeq_P)$ is an ordered space.*

Proof of Lemma A.1. Clearly $\succeq_P$ is reflexive and transitive. We need to show that it is antisymmetric. Assume $\mu \succeq_P \nu$ and $\nu \succeq_P \mu$. Then.

$$\int_X f d\mu = \int_X f d\nu, \text{ for all } f \in M(X). \tag{5}$$

Clearly, set $M(X) - M(X)$ is a Riesz subspace of $C(X)$.[31] Moreover, it contains a constant function. Therefore, by the Stone-Weierstrass Theorem (see Aliprantis and Border, 2006, Theorem 9.12) $M(X) - M(X)$ is uniformly dense on $C(X)$. In particular, condition (5) is satisfied for all $f \in C(X)$.

Take any two closed subsets F and F' of X such that $F \cap F' = \emptyset$, while $\mu(F \cup F')$ and $\nu(F \cup F')$ are arbitrarily close to 1. Clearly, since X is compact and Hausdorff, any closed set is compact. Since the two measures are regular (hence, tight) such subsets exist. Let an arbitrary $\varepsilon > 0$ be such that $\mu(F \cup F') > 1 - \varepsilon$ and $\nu(F \cup F') > 1 - \varepsilon$. As X is compact and Hausdorff, by Urysohn's Lemma (see Aliprantis and Border, 2006, Lemma 2.46), it is possible to construct a continuous function $f : X \to [0,1]$ that satisfies $f(x) = 1$, for $x \in F$, and $f(x) = 0$, for $x \in F'$. Since $\int_X f d\mu = \int_X f d\nu$, we obtain

$$\begin{aligned} \nu(F) &\leq \nu(F) + \int_{F^c \cap F'^c} f d\nu \\ &= \int_X f d\nu \\ &= \int_X f d\mu \\ &= \mu(F) + \int_{F^c \cap F'^c} f d\mu \\ &\leq \mu(F) + \varepsilon. \end{aligned}$$

Analogously, we can show that $\mu(F) \leq \nu(F) + \varepsilon$. Given that ε is arbitrary, this implies that $\mu(F) = \nu(F)$, for any closed (hence, compact) $F \subseteq X$. Since the measures are tight, this implies that $\mu = \nu$. □

Given the above preliminary result, we may proceed with our characterisation of the ordered space of regular measures.

Proposition A.1. *$(\mathcal{R}, \succeq_P)$ is a chain complete poset.*

Proof of Proposition A.1. Let $\{\tau_t\}$ be a decreasing chain. Recall that $M(X)$ is the set of isotone, measurable, and bounded real-valued functions over X. Define $T : M(X) \to \mathbb{R}$,

$$T(f) := \inf_t \left\{ \int_X f(s) d\tau_t(s) \right\},$$

[31] See Aliprantis and Border (2006, p. 646, footnote 5).

and for antitone functions $f \in -M(X)$

$$T(f) = -\inf_t \left\{ \int_X -f(s) d\tau_t(s) \right\} = \sup_t \left\{ \int_X f(s) d\tau_t(s) \right\}.$$

Note that T is a functional on $M(X)$ which preserves the addition and multiplication by a positive scalar. T can be extended to the vector subspace $M(X) - M(X)$ in the natural way. That is, whenever $f = g - h$, for g, $h \in M(X)$, then $T(f) = T(g) - T(h)$, which is well defined. Indeed, if $f \in M(X) - M(X)$, then $f = g - (g - f)$, for $g \in M(X)$ and $f - g \in M(X)$. Hence,

$$\inf_t \int_X g(x) d\tau_t(x) = \lim_t \int_X g(x) d\tau_t(x),$$

as well as

$$\inf_t \int_X g(x) \tau_t(dx) = \lim_t \int_X f(x) - g(x) \tau_t(dx).$$

Since T is isotone,

$$|T(f)| \leq \|f\|_\infty, \tag{6}$$

while the inequality may hold as $T(1) = 1$. By the Banach Extension Theorem, there exists an extension $\hat{T}$ of T for all elements in $C(X)$ that satisfy (6). By The Riesz Representation Theorem (see Aliprantis and Border, 2006, Theorem 14.12) there is a unique regular measure $\underline{\tau}$ such that

$$\hat{T}(f) = \int_X f(s) d\underline{\tau}(s),$$

for all $f \in M(X)$. Moreover, by condition (6), it is a probability measure. Clearly, $\underline{\tau}$ is a lower bound of the chain $\{\tau_t\}$. We need to show it is the greatest lower bound in $\mathcal{R}$.

We prove our claim by contradiction. Suppose there is another measure τ_0 which is a lower bound of $\{\tau_t\}$, but it is not dominated by $\underline{\tau}$. Then, there exists a function $f \in M(X)$ such that $\int_X f d\tau_0 > \int_X f d\underline{\tau}$. As τ_0 is a lower bound of $\{\tau_t\}$, we have

$$T(f) = \inf_t \left\{ \int_X f(s) d\tau_t(s) \right\} \geq \int_X f d\tau_0 > \int_X f d\underline{\tau} = T(f),$$

which yields a contradiction. Hence, $\underline{\tau}$ is a the greatest lower bound of $\{\tau_t\}$. Similarly, we can prove the thesis for increasing chains. $\square$

Next, we present an auxiliary result that is used in the proof of Lemma 1.

Lemma A.2. *Let X and Y be Hausdorff topological spaces endowed with σ-fields $\mathcal{B}_X$ and $\mathcal{B}_Y$ respectively. Let μ be a finite measure on $\mathcal{B}_X \otimes \mathcal{B}_Y$. Whenever the marginals of μ on X and Y are regular measures, then μ is a regular measure.*

Proof of Lemma A.2. By Theorem 12.4 in Aliprantis and Border (2006), it is sufficient to show that measure μ is tight. Define

$$\mathcal{E} := \Big\{V \in \mathcal{B}_X \otimes \mathcal{B}_Y : \mu(V) = \sup\big\{\mu(K) : K \subseteq V,\ K \text{ is compact}\big\}\Big\}.$$

By the standard argument (see the proof of Theorem 12.5 in Aliprantis and Border, 2006), $\mathcal{E}$ is a σ-field. We need to show that $\mathcal{E}$ includes sets of the form $U_x \times U_y$, with $U_x \in \mathcal{B}_X$ and $U_y \in \mathcal{B}_Y$. Since the marginals of μ are tight, hence for given $\varepsilon > 0$ there are compact sets $K_x \subset U_x$ and $K_y \subset Y_y$ such that $\mu_x(U_x \setminus K_x) < \varepsilon/2$ and $\mu_y(U_y \setminus K_y) < \varepsilon/2$, where μ_x and μ_y denote the marginals of μ on X and Y respectively. Then,

$$\mu\big((U_x \times U_y) \setminus (K_x \times K_y)\big) \le \mu\big((U_x \setminus K_x) \times Y\big) + \mu\big(X \times (Y \setminus K_y)\big) < \varepsilon.$$

Hence, $U_x \times U_y \in \mathcal{E}$. □

Finally, we present one result applied in Section 3.1.

Lemma A.3. *Let $f : X \to \mathbb{R}$ be a concave function defined over a convex subset $X \subseteq \mathbb{R}$. Define $Y := \{(x,s) \in \mathbb{R}^2 : (x-s) \in X\}$. Then, function $g : Y \to \mathbb{R}$, defined by $g(x,s) := f(x-s)$, has increasing differences in (x,s) on $X \times X$.*

Proof of Lemma A.3. Since f is concave, for any $x' \ge x$, and $s' \ge s$, we have

$$\frac{f(x'-s') - f(x-s')}{(x'-s') - (x-s')} \ge \frac{f(x'-s) - h(x-s)}{(x'-s) - (x-s)},$$

which implies that $f(x'-s') - f(x-s') \ge f(x'-s) - f(x-s)$. □

Lemma A.4. *Let $f : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}_+$ is convex, be a decreasing and concave function. Let $Y := \{(x,s) \in \mathbb{R}^2 : |x-s| \in X\}$. Then, function $g : Y \to \mathbb{R}$, defined by $g(x,s) := f(|x-s|)$ has increasing differences in (x,s).*

Proof of Lemma A.4. First, we prove function $h : X \to \mathbb{R}$, $h(x) := f(|x|)$, is concave. Take any $x', x \in X$ and $\alpha \in [0,1]$. Then

$$f(|\alpha x + (1-\alpha)x'|) \ge f(\alpha|x| + (1-\alpha)|x'|) \ge \alpha f(|x|) + (1-\alpha) f(|x'|),$$

where the first inequality is implied by the triangle inequality and monotonicity of f, while the second follows from concavity of f. The rest is implied by Lemma A.3. □

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