On uniqueness of time-consistent Markov policies for quasi-hyperbolic consumers under uncertainty*

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Abstract

In give a set of sufficient conditions for uniqueness of a time-consistent Markov stationary consumption policy for a quasi-hyperbolic household. To the best of our knowledge our uniqueness result is the first presented in the literature for general settings, i.e. under standard assumptions on preferences, as well as a new condition on a transition probabilities. This paper advocates a method to overcome some predicaments of the Harris and Laibson (2001) model and also extends our recent existence result (see Balbus, Reffett, and Woźny, 2014b). We also present few natural followers of optimal policy uniqueness: computation algorithm, monotone comparative statics results and finish with discussion of few extensions to more general setups.

1 Introduction

The problem of dynamic inconsistency first introduced in Strotz (1956), and further developed in Phelps and Pollak (1968) or Peleg and Yaari (1973), has played an increasingly

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important role in many fields in economics. The classical toolkit for analyzing such "time" consistency problems was first proposed by Strotz (1956), and emphasized the language of recursive decision theory. However, as observed by many researchers in subsequent discussions (e.g., Peleg and Yaari (1973) and Bernheim and Ray (1986)), such optimal dynamically (or time) consistent plans need not exist in a class of Markovian strategies, let alone be unique or even simple to characterize and/or compute.

One key reason for this fact lies in the almost inherit presence of discontinuities in intertemporal preferences (or dynamic equilibrium) that arises naturally in such problems in recursive decision theory. The source of the lack of continuity is generated by the lack of commitment between the current "versions" of the decisionmaker, and all her continuation "selves". For example, from a decision theoretic perspective, when a "current" decision maker is indifferent between some alternatives in the future, that same decision maker can still strictly prefer such an alternative in advance and be willing to commit, yets lacks access to a reasonable "commitment device" that would impose discipline on her future "selves" when tomorrow arrives. Due to this discontinuity, the optimal level of "commitment" may be nonexistent (see, for example, Caplin and Leahy (2006)), and the dynamic maximization problem can be poorly defined.

As a way of circumventing these problems, Peleg and Yaari (1973) proposed a dynamic game interpretation of the time-consistency problem. More specifically, in this view of the problem, one envisions the decisionmaker playing a dynamic game between one's current self, and each of her future "selves", with the solution concept in the game being a subgame-perfect Nash equilibrium (SPNE, henceforth). Still the SPNE of the appropriate game need not be a MSNE of a decision problem. This latter fact is due, in part, to the dynamic decision theoretic approach being proposed itself, where future ties are broken in favor of a current self, and that observation is not necessarily true for a SPNE of a dynamic game. Additionally, the set of SPNE may be large, and most importantly, hence, an optimal SPNE (corresponding to an optimal time-consistent policy) may simply do not exist. This latter issue has become the central concern of a now broad literature that has emerged since the pioneering work of Peleg and Yaari (1973). Moreover, even if the question of existence of SPNE is resolved, the equilibrium existence or uniqueness in the particular class of functions, namely Stationary Markov Nash Equilibria (henceforth, SMNE) is still not guaranteed (see Bernheim and Ray (1986) and Leininger (1986)). The works of Kocherlakota (1996) and Maskin and Tirole (2001) provide an extensive set of motivations for why one might be interested in concentrating on SMNE, as opposed to SPNE. Finally, the recent emphasis on numerical simulations in applied work studying models with time consistency issues provides additional reasons for being interested in

SMNE.

In this paper, we develop a classical Strotz (1956) recursive approach to studying equilibrium in the Phelps and Pollak (1968) game theoretic representation of the problem, with an emphasis on developing constructive methods for characterizing SMNE (as well as computing them). That is, we seek conditions under which simple stable iterative numerical algorithms can be developed that both (i) characterize the existence of SMNE from a theoretical perspective, as well (ii) provide explicit and accurate algorithms for computing them. From the perspective of the existence question, our paper is very closely related to the important papers of Bernheim and Ray (1986) or Harris and Laibson (2001), where the authors add noise of invariant support in an effort to develop conditions that guarantee the existence of an time-consistent policy of locally bounded variation and/or Lipschitz for sufficiently small amount of hyperbolic discount factor. But what is critical in understanding the difference between the approaches in this literature, and those in the present paper, is that the methods we propose do not rely on so-called "generalized Euler equation" methods (as, for example, in Harris and Laibson). In the generalized Euler equation approach, the question of existence and characterization is closely related to a generalized Euler inequality method that appeals to the calculus of bounded variation for characterizing the structure of SMNE. In our paper, we propose a very different approach, one based upon a value function method (in the spirit of "promised utility methods"), but defined in spaces of *functions,* as opposed to spaces of correspondences in the promised utility literature. Our methods link the stochastic game studied in Harris and Laibson (2001), with a recursive or value function methods suggested by Strotz (1956) (and further developed by Caplin and Leahy (2006)), and therefore nicely complement all of these existing literatures.

More specifically, under standard assumptions on preferences and certain geometric condition on a transition probability, using our value iteration approach, we are able to show existence and uniqueness of time-consistent policies, provide characterizations of their (Lipschitz) continuity and monotonicity properties, and, therefore, extend the Harris and Laibson (2001) results to *all* discount factors. Finally, and equally as important, as we obtain equilibrium uniqueness, existence of an optimal, time-consistent decision policy is established. The fact that our methods emphasize *both* the computation of values and pure strategies is of central importance in our work as it links the game-theoretic analysis with that of recursive decision theory.

We then turn to the question of computation of SMNE, as well as equilibrium comparative statics on the deep parameters of the game. We are able to construct an simple approximation scheme computing SMNE values in the sup norm, as well as conduct monotone comparative statics on the extremal time-consistent SMNE policies with respect to the model parameters. The comparative statics and approximation results are important for applied research in the field. For example, in Sorger (2004), he proposes settings under which any twice continuously differentiable function can be supported as a policy of a time consistent hyperbolic consumer. This result can be subsequently linked to a Gong, Smith, and Zou (2007) text, showing that a hyperbolic discounting is not observationally equivalent to exponential discounting. That is, it is always possible to calibrate an exponential model so that it predicts the same level of consumption as a hyperbolic model. However, the two models have *radically different* comparative statics. Hence, our approach allows us to sort out the exact nature of this question, and provide theoretical monotone comparative statics results based on the equilibrium set of the stochastic game itself. Such a result can clarify empirical questions that are asked by applied researchers.

The rest of the paper is organized as follows. In section 2 we discuss in details the related literature. In section 3 we specify model, its assumptions and state our main theorems. Finally section 5 concludes by discussing how our results can be potentially used for a study of different approaches or axiomatizations of time-consistent problems.

2 Related methods

It is important to remember that equilibrium non-existence and its multiplicity, related to the class of games we study, have constituted a significant challenge for applied economists who sought to study models where such dynamic consistency failures play a key role. They have been equally as challenging for researchers that seek to identify tractable numerical approaches to computing SMNE in these (and related) dynamic games (e.g., see the discussion in Krusell and Smith (2003) or Judd (2004)). On the one hand, Krusell, Kuruscu, and Smith (2002) propose a generalized Euler equation method for a version of a hyperbolic discounting consumer and obtain explicit solution for logarithmic utility and Cobb-Douglas production examples. But this is an example. On the other hand, in Judd (2004), he uses generalized Euler equation approach to analyze smooth time-consistent policies and proposes a perturbation method for calculating them. The problem here is providing conditions under which at any point in the state space the generalized Euler equations represent a *sufficient* first order theory for an agents value function in the equilibrium of the game.[1] Concentrating on non-smooth policies, Krusell and Smith (2003)

[1]We should elaborate on this point. In a generalized Euler equation method, on an open set of any point in the state space, we can always construct an local linearization (in a space of functions) that might be valid as a linear approximation to the function that satisfies the functional equation near that

define a step function equilibrium and show its existence and resulting indeterminacy of steady state capital levels. Further, in a deterministic setting general existence result of optimal policies under quasi-geometric discounting can be provided using techniques proposed by Goldman (1980) for finite horizon economies, by Harris (1985) for infinite horizon or by Feinberg and Shwartz (1995) in the generalized discounting setting.

Summarizing, from a technical point of view, tools used to show existence and characterize Markovian policies are wide and motivated by specific applications or problems under study. Still the general framework for studying (analytically and numerically) of (possibly nonsmooth) SMNE is missing. To circumvent some of these mentioned predicaments in a unified setup authors also added noise to the decision problems or relevant dynamic games. Specifically, in a *(recursive) decision approach*, by adding noise (making payoff discountinities negligible) Caplin and Leahy (2006) prove existence of recursively optimal plan for a finite horizon decision problem and general utility functions. Similarly Bernheim and Ray (1986) show that by adding enough noise to the dynamic game (to smooth discontinuities away) existence of SMNE is guaranteed. Such *stochastic game approach* was later developed by Harris and Laibson (2001) who characterize the set of smooth SMNE by (generalized) first order conditions. Finally Balbus and Nowak (2008) show conditions for SMNE existence in an infinite horizon, hyperbolic discounting stochastic game with many players in each generation[2].

It is worth mentioning that authors have also analyzed optimal time-consistent policies on enlarged state space. For infinite horizon decision problems Kydland and Prescott (1980, henceforth KP) notice that the state space of an appropriately defined value function must incorporate some pseudo-state variables like Lagrange multipliers for the problem (of finding optimal policies) to be recursive. KP method is linked to the Abreu, Pearce, and Stacchetti (1990, henceforth APS) type arguments. Specifically by adding appropriate noise to the time-consistency game characterization of all sequential equilibria using APS methods can be offered. This approach is undertaken by Bernheim, Ray, and Yeltekin (1999). They analyze our problem using APS type arguments. Specifically they consider a set of (bounded) values for (sequential) subgame perfect equilibria in a Phelps and Pollak (1968) self-game and analyze all subsets of such values. Later they construct a monotone (under set inclusion) operator on this set and numerically analyze its largest fixed point. Using this method they show existence of a sequential time-consistent policy

point; the problem is showing the Euler equation is necessary and sufficient on that open set. For the later claim to be true, you must know the value function in the equilibrium of the game is concave. In our method, such a local expansion will be valid; but then, we do not need the generalized Euler equation to compute the models equilibrium.

[2]See also Nowak (2010) for related results.

and use it to analyze self-control in the context of a low asset trap.

Finally let us note that quasi-hyperbolic discounting problem is linked to a problem of altruism towards successive generations (see Saez-Marti and Weibull (2005) for formal results). This link can be also seen through a technical perspective, where the stochastic games methods (see Balbus, Reffett, and Woźny (2014a)) can be applied for both quasi-hyperbolic discounting and intergenerational altruism (see Balbus, Reffett, and Woźny (2013)) models.

3 Consumption-savings model and main results

In the environment we study, we envision an individual decisionmaker to be a sequence of "selves" indexed in discrete time $t = 0, 1, \ldots$. For a given state $x_t \in S$ (where $S = [0, \overline{S}]$ or $S = [0, \infty)$), the "self t" chooses a consumption $c_t \in [0, s_t]$, and leaves $s_t - c_t$ as an investment for future "selves". As in effect, we rule out borrowing; also we interpret the asset as a productive one, and refer to it as capital.[3] These choices, together with current state s_t, determine a transition probability $Q(ds_{t+1}|s_t - c_t, s_t)$ of a next period state.

Self t preferences are represented by a utility function given by:

$$u(c_t) + \beta E_t \sum_{i=t+1}^{\infty} \delta^{i-t} u(c_i), \tag{1}$$

where $1 \geq \beta > 0$ and $1 > \delta \geq 0$, u is a instantaneous utility function and expectations E_t are taken with respect to a realization of a random variable s_i drawn each period from a transition distribution Q (see Ionescu-Tulcea theorem).

3.1 Generalized Bellman operator

Under some continuity assumptions on u and Q (to be specified later), we can define a Markovian equilibrium pure strategy to be an $h \in \mathcal{H}$ where $\mathcal{H} = \{h : S \to S | 0 \leq h(s) \leq s$ bounded and Borel measurable$\}$ that is time-consistent for the quasi-hyperbolic consumer. That is, if h satisfies the following functional equation:

$$h(s) \in \arg\max_{c \in [0,s]} u(c) + \beta\delta \int_S V_h(s') Q(ds'|s - c, s), \tag{2}$$

[3] We can introduce assets and borrowing into the model also, but at the expense of having to decentralize equilibrium under some meaningful price system. With stochastic production, this can be a difficult problem. We leave this question for future work.

where $V_h : S \to \mathbb{R}$ is a continuation value function for the household of "future" selves following a stationary policy h from tomorrow on. The value in the Markovian equilibrium for the future selves, therefore, must solve the following additional functional equation in the continuation given as follows:

$$V_h(s) = u(h(s)) + \delta \int_S V_h(s')Q(ds'|s - h(s), s). \tag{3}$$

Therefore, if we define the value function for the self t to be:

$$W_h(s) := u(h(s)) + \beta\delta \int_S V_h(s')Q(ds'|s - h(s), s),$$

one obtains the relation

$$V_h(s) = \frac{1}{\beta}W_h(s) - \frac{1-\beta}{\beta}u(h(s)). \tag{4}$$

Equation (4) is our *generalized Bellman equation*. It shows a condition that any Markovian value must satisfy to solve our original maximization problem. Element $\frac{1-\beta}{\beta}u(h(s))$ is our adjustment to account for changing preferences. For $\beta = 1$ case Equation (4) reduces to the standard Bellman equation. Based on equation (4), we can define an operator whose fixed points, say V^*, correspond to values for some *time-consistent* Markov policies.

3.2 Assumptions

To study this problem, we need to make some assumptions on the primitive data of the game:

Assumption 1 *Let us assume:*

- $u : S \to \mathbb{R}_+$ *is continuous, increasing and strictly concave with* $u(0) = 0$ *and* $u(\cdot) \leq \overline{u}$*,*
- *for any* $s, i \in S$ $Q(\cdot|i, s) = p(\cdot|i, s) + (1 - p(\cdot|i, s))\delta_0(\cdot)$*, where* δ_0 *is a delta Dirac measure concentrated at point* 0*, while* $p(\cdot|i, s)$ *is some measure such that* $p(Z|i, s) < 1$ *and* $p(Z|i, 0) \equiv 0$ *for all* $i \in S, Z \subset S$*.*
- *for any integrable* v *assume* $(i, s) \to \int_S v(s')p(ds'|i, s)$ *is continuous, increasing and convex with* i *and measurable with* s*.*

First, our assumptions on preferences are completely standard. Here, we only mention the imposition of strict concavity of a utility function in these assumptions allows us to restrict attention to single valued best replies in the definition 2, and hence we can study the fixed points of a single valued operator whose fixed points generate corresponding equilibrium values and corresponding policies in the game. It bears mentioning that a careful reading of the proof of our main existence theorem below (e.g., Theorem 1) indicates this assumption can be weakened, as we can also work with increasing selections from a best response correspondence (not necessarily unique valued best replies).

Second, our assumption on a transition probability requires a few remarks. Q has the specific form in our conditions above. We should mention that although this is a powerful technical assumption, the conditions are satisfied in many applications (e.g., see the discussion in Chassang (2010) for a particular example of this exact structure). Additionally, as we assume positive returns (i.e., $u(\cdot) \geq 0$), our assumptions above assure that the expected continuation value is monotone in its arguments. This structure is common in the literature. For example, a stronger version of this assumption was introduced by Amir (1996), used in a series of papers by Nowak (2003, 2006, 2007), as well as studied extensively in the context of games of strategic complementarities with public information in Balbus, Reffett, and Woźny (2014a). We refer the reader to our two related papers (see Balbus, Reffett, and Woźny (2013, 2014b)) for a detailed discussion of the nature of these assumptions.

Remark 1 *Observe that we do not require that p is a probability measure. A typical example of p is: $p(\cdot|i,s) = \sum_{j=1}^{J} g_j(i,s)\eta_j(\cdot|s)$, where $\eta_j(\cdot|s)$ are measures on $\mathcal{S}$ and $g_j : S \times S \to [0,1]$ are functions with $\sum_{j=1}^{J} g_j(\cdot) \leq 1$. However there are many examples of p that cannot be expressed by a linear combination of stochastic kernels, and still satisfy our assumptions.*

Finally, our motivation and approach is closely linked to that of Harris and Laibson (2001), so it is useful to compare our assumptions, and theirs. First, we do not impose twice continuous differentiability nor strict monotonicity of a utility function as Harris and Laibson do. Second, as opposed to Harris and Laibson, although we bound our utility between 0 and $\overline{u}$, we do not bound its risk aversion. This latter assumption is critical for any generalized implicit function approach (or calculus of bounded variation). Third, as far as transition probability is concerned, our model generates more sources of noise than Harris and Laibson (2001), as in our case, not only is labor income random, but wealth (or capital) is also draw from Q. Finally, we do not require that Q has a density, let alone imposing conditions on its degree of smoothness.

3.3 MSNE uniqueness

Let $\mathcal{V}$ be a space of bounded (by 0 and $\frac{\overline{u}}{1-\delta}$), Borel measurable, real valued functions on S, with $V(0) = 0$ equipped with a pointwise partial order and a sup-norm. For a given value $V \in \mathcal{V}$ construct an operator T by:

$$TV(s) = \frac{1}{\beta}AV(s) - \frac{1-\beta}{\beta}u(BV(s)), \tag{5}$$

where the pair of operators A and B defined on space $\mathcal{V}$ are given by:

$$AV(s) = \max_{c\in[0,s]} \left\{ u(c) + \beta\delta \int_S V(y)Q(ds'|s-c,s) \right\}, \tag{6}$$

$$BV(s) = \arg\max_{c\in[0,s]} \left\{ u(c) + \beta\delta \int_S V(y)Q(ds'|s-c,s) \right\}. \tag{7}$$

Notice, in the above, we have defined the operator B to map between candidates for equilibrium values $\mathcal{V}$ to spaces of pure strategy best replies $\mathcal{H}$. So in effect, we have a pair of operator equation we need to solve to construct equilibrium values $V^* \in \mathcal{V}$.

Surely T maps $\mathcal{V}$ into itself. Further, for any fixed point V^* of an operator T, this value function corresponds to a stationary, time-consistent Markov policy $h^* = BV^* \in \mathcal{H}$. Equip the space of pure strategies $\mathcal{H}$ with a pointwise partial order. In this case, we obtain our main result:

Theorem 1 (Uniqueness of MSNE) *Let assumption 1 hold. Then, then there is a unique value v^* and corresponding unique time-consistent Markov policy MSNE h^*. Moreover, for any $v \in \mathcal{V}$ we have*

$$\lim_{t\to\infty} ||Tv^t - v^*|| = 0.$$

Proof: We show T is a contraction mapping using Blackwell sufficient conditions.

A is increasing by definition. To see monotonicity of B, consider a function

$$G(c, s, V) = u(c) + \beta\delta \int_S V(s')p(ds'|s-c,s).$$

Then for any $V \in \mathcal{V}$ and $s \in S$, the function $G(\cdot, s, V)$ is supermodular.

Moreover, $(c, V) \to \int_S V(s')p(ds'|s-c,s)$ has decreasing differences. To see this fact, observe we have the following inequalities:

$$\begin{aligned} 0 &\leq \int_S V_2(s')p(ds'|s-c_2,s) - \int_S V_2(s')p(ds'|s-c_1,s), \\ &= \int_S V_2(s')[p(ds'|s-c_2,s) - p(ds'|s-c_1,s)], \\ &\leq \int_S V_1(s')[p(ds'|s-c_2,s) - p(ds'|s-c_1,s)], \end{aligned}$$

where $V_2 \geq V_1$ and $c_2 \geq c_1$. Therefore, for any $s \in S$, the function $(c, V) \to G(c, s, V)$ has decreasing differences on $[0, s] \times \mathcal{V}$. Since $[0, s]$ is a lattice and $\mathcal{V}$ is poset, we obtain by Topkis (1978) theorem that the (unique) best reply $BR(V)(s) = \arg\max_{c\in[0,s]} G(c, s, V)$ is decreasing on $\mathcal{V}$. Since A is increasing and B decreasing, by definition of T, we conclude that T is increasing.

We now show T possesses a discounting property. For this reason let $V \in \mathcal{V}$ be given, fix $z \in \mathbb{R}_+$ and observe:

$$\begin{aligned} &T(V+z)(s) = \frac{1}{\beta}A(V+z)(s) - \frac{1-\beta}{\beta}u(B(V+z)(s)) = \\ &\frac{1}{\beta}A(V+z)(s) - \frac{1-\beta}{\beta}u(BV(s)) = \\ &\frac{1}{\beta}AV(s) + \frac{1}{\beta}\beta\delta z - \frac{1-\beta}{\beta}u(BV(s)) = \\ &TV(s) + \delta z, \end{aligned}$$

where the first equality follows as the argmax B does not depend on a constant, while the second as A has discounting property. Hence, if $\delta < 1$ then T possesses discounting property. As a result T satisfies Blackwell (1965) sufficient conditions for a contraction. The proof finishes by application of Banach fixed point principle for a complete metric space $\mathcal{V}$. ■

Theorem 1 is the central result of our paper. First, it guarantees existence of time-consistent pure strategy equilibrium value v^* and policy h^*. Second, it asserts that such value and policy is unique, where the uniqueness result holds within a class of bounded measurable value functions. This in turn implies: sequences generated by operator T are converging in the sup norm to v^*.

Such strong characterization of time-consistent policies is obtained due to two central assumption: concentrating on Markovian policies and the mixing assumption imposed on Q. Without any of these two our results would be substantially weaker. Operator T is a Bellman type operator and expresses the time-consistency problem recursively for Markovian policies. However, generally (if assumption 1 is not satisfied) T does not poses useful properties of similar operators applied in the study of optimal economies[4]. Finally to mention, although under assumption 1 T is a contraction, useful properties concerning equilibrium h^* characterization do not follow from standard arguments used in Stokey, Lucas, and Prescott (1989). For this reason we present the next two results further characterizing (in terms of smoothness and monotonicity) the actual time consistent equilibrium policy functions.

Theorem 2 (Monotonicity of policies) *Assume 1, and consider a time-consistent policy h^*. If $p(\cdot|i,s)$ is constant with s, for any i, then the optimal time-consistent policy h^* is increasing and Lipschitz with modulus 1.*

Proof: Let $h^* = BV^*$ for some $V^* = TV^*$. Consider the function

$$G(c,s,V^*) = u(c) + \beta\delta \int_S V^*(s')p(ds'|s-c).$$

Observe G is supermodular in c on a lattice $[0,s]$, and the feasible action set $[0,s]$ is increasing in the Veinott's strong set order. Moreover, by concavity of $i \to \int_S V^*(s')p(ds'|i)$ we conclude G has increasing differences with (c,s). By Topkis (1978) theorem argument maximizing h^* is increasing with s on S.

Similarly, if i denotes "investment", we also can rewrite this problem as:

$$H(i,s,V^*) = u(s-i) + \beta\delta \int_S V^*(s')p(ds'|i),$$

where H is supermodular with the choice variable i on a lattice $[0,s]$, and, again, the set $[0,s]$ is increasing in the Veinott's strong set order. Again, by concavity of u, we conclude that H has increasing differences with (i,s). Therefore, again, by Topkis (1978) theorem, the optimal solution i^* is increasing with s on S.

Clearly $i^*(s) = s - h^*(s)$. Finally as both h^* and i^* are increasing on S hence h^* and i^* are Lipschitz with modulus 1. ■

[4]It suffices to change $\delta-$Dirac measure with some other nontrivial one in assumption 1 and equilibrium uniqueness results would not hold. In such case one could Markov-equilibrium existence using topological arguments but with no hope of uniqueness. Also equilibrium computation would become substantially complicated.

Notice the results in the above theorem are important, as they extend the result reported in Harris and Laibson (2001) on Lipschitz continuity of equilibrium to a broader scope of quasi-hyperbolic discount factors.

We next turn to the question of continuous (but not necessarily Lipschitz continuous) time consistent policies. For this, we impose the following Feller type property on the noise.

Assumption 2 *$s \to p(\cdot|i,s)$ is strongly stochastically continuous (i.e. the function $s \to \eta_f(s) := \int_S f(s')p(ds'|i,s)$ is continuous for any $f \in \mathcal{V}$) and any i.*

We now prove a theorem characterizing the continuity structure of time consistent equilibrium policies.

Theorem 3 (Continuity of policies) *Let 1 and 2 hold. Then, the time-consistent policy h^* is continuous.*

Proof: Let $V_{h^*} \in \mathcal{V}$ be equilibrium payoff under time-consistent policy h^*. Then, by Assumption 1, the mapping

$$s \to \eta^j_{h^*}(s) := \int_S V_{h^*}(s')p(ds'|i,s),$$

is continuous. Notice, the function

$$F_{h^*}(c,s) := u(c) + \beta\delta\eta_{h^*}(s),$$

is also continuous and strictly concave with respect to c for fixed $s > 0$. Let $s_n \to s_0$. Since $h^*(s) = \arg\max_{c\in[0,s]} F_{h^*}(c,s)$, we have

$$F_{h^*}(h^*(s_n), s_n) \geq F_{h^*}(c, s_n).$$

Without loss of generality, suppose $h^*(s_n) \to s_0$. By the continuity of F_{h^*}, we have

$$F_{h^*}(c_0, s_0) \geq F_{h^*}(c, x_0).$$

By the strict concavity of $F_{h^*}(\cdot, x)$ and definition of h^*, we obtain $c_0 = h^*(s_0) = \lim_{n\to\infty} h^*(s_n)$. ■

3.4 Monotone comparative statics

Finally, motivated by the indeterminacy result in Gong, Smith, and Zou (2007) (as well as concerns about the possible econometric estimation of our stochastic game), we now consider a parameterized version of our optimization problem in the previous section of the paper. For a partially ordered set Θ , with $\theta \in \Theta$ a typical element, define the unique MSNE as h^*_θ. We make the following assumption.

Assumption 3 *Let us assume:*

- $u : S \times \Theta \to \mathbb{R}$, $c \to u(c, \theta)$ *is continuous, increasing and strictly concave on* S *with* $(\forall \theta \in \Theta)\, u(0, \theta) = 0$. *Also* u *has increasing differences with* (c, θ) *and* $\theta \to u(c, \theta)$ *is decreasing.*
- *For any* $s, i \in S$ *and* $\theta \in \Theta$ *let* $Q(\cdot|i, s, \theta) = (1 - p(\cdot|i, \theta))\delta_0(\cdot) + p(\cdot|i, \theta)$.
- $(\forall v \in V)$,$(\forall \theta \in \Theta)$ *function* $i \to \int_S v(s')p(ds'|i, \theta)$ *is continuous, increasing and concave with* i. *Also for each increasing* v *we have* $(i, \theta) \to \int_S v(s')p(ds'|i, \theta)$ *has decreasing differences with* (i, θ) *and* $\theta \to \int_S v(s')p(ds'|i, \theta)$ *is decreasing on* Θ.
- *where* δ_0 *is a delta Dirac measure concentrated at point* 0, *while* $p(\cdot|i, \theta)$ *is some measure such that* $p(Z|i, \theta) < 1$ *and* $p(Z|0, \theta) \equiv 0$ *for all* $i \in S, Z \subset S$. *is stochastically decreasing with* θ.

We Assumption 3 in place, we can now prove our main result on monotone comparative statics for extremal time consistent equilibrium policies.

Theorem 4 (Monotone comparative statics) *Let Assumption 3 be satisfied. Then mapping* $\theta \to h^*_\theta$ *is increasing on* Θ.

Proof: By theorem 1, for any $\theta \in \Theta$, there exist a unique h^*_θ. Moreover by theorem 2 function h^*_θ, as well as $s \to s - h^*_\theta(s)$ are increasing functions on S. As a result, for each θ and each V we have that

$$s \to u(h^*_\theta(s), \theta) + \delta \int_S V(s')p(ds'|s - h^*_\theta(s), \theta).$$

is monotone, hence V^*_θ is monotone es well.

Now, for increasing $V \in \mathcal{V}$, consider a function

$$G(c, s, \theta, V) = u(c, \theta) + \beta\delta \int_S V(s')p(ds'|s - c, \theta),$$

and observe that G is decreasing with θ, and has increasing differences with (c, θ). Clearly, $A_\theta V(s) = \max_{c\in[0,s]} G(c, s, \theta, V)$ is decreasing with θ. Similarly, by Topkis (1978) theorem, $B_\theta V(s)$ is increasing with θ (where $B_\theta V(s) = \arg\max_{c\in[0,s]} G(c, s, \theta, V)$). Consequently, we have

$$\theta \to T_\theta V(s) = \frac{1}{\beta} A_\theta V(s) - \frac{1-\beta}{\beta} u(B_\theta V(s)),$$

is decreasing on Θ. We therefore conclude that v^*_θ is decreasing with θ. Consequently, $\theta \to G(c, x, \theta, w^*_\theta)$ is decreasing and $(c, \theta) \to G(c, x, \theta, w^*_\theta)$ has increasing differences with (c, θ). Then, by Topkis (1978) theorem, $\underline{h}^*_\theta$ is increasing with θ. ■

4 Two extensions

4.1 A more general model

A careful examination of the methods used in the proof of the main theorem suggest possible generalization of the uniqueness result to a more general multi-dimensional action and state spaces. Let $S = [0, \overline{S}] \subset \mathbb{R}^n$ or $S = [0, \infty) \subset \mathbb{R}^n$, and action set a compact and complete lattice $A \subset \mathbb{R}^m$. By $A(s) \subset A$ denote a set of actions available at state s such that $s \to A(s)$ is measurable. Let $A(s)$ be compact and a complete lattice, for any s.

Assumption 4 *Let*

- $u : S \times A \to \mathbb{R}_+$ *is continuous, increasing and supermodular, strictly concave with* a*, and measurable with* s*, with* $u(0, a) = 0$ *and* $u(\cdot) \le \overline{u}$*,*
- *for any* $s \in S, a \in A(s)$ *let* $Q(\cdot|a, s) = p(\cdot|a, s) + (1 - p(\cdot|a, s))\delta_0(\cdot)$*, where* δ_0 *is a delta Dirac measure concentrated at point* 0*, while* $p(\cdot|a, s)$ *is some measure such that* $p(Z|a, s) < 1$ *and* $p(Z|a, 0) \equiv 0$ *for all* $a \in A, Z \subset S$*.*
- *for any integrable* v *assume* $(a, s) \to \int_S v(s')p(ds'|a, s)$ *is continuous, supermodular, decreasing and concave with* a *as well as measurable with* s*.*

Theorem 5 (Uniqueness of MSNE: general model) *Let assumption 5 hold. Then, then there is a unique value* v^* *and corresponding unique time-consistent Markov policy MSNE* h^**. Moreover, for any* $v \in \mathcal{V}$ *we have*

$$\lim_{t\to\infty} ||Tv^t - v^*|| = 0.$$

4.2 Keeping up with the Joneses

The next generalization considers an infinite horizon stochastic economy with n quasi-hyperbolic discounting households. We consider a 'keeping up with the Joneses' motive where each households is subject to negative consumption externalities of the others consumptions, as well as strategic complementarities incentivising to consume more if others consumer more. Our aim is to design a profile of equilibrium policies where, players best response to each others policies. Formally we model this economy as a stochastic game and focus on the MSNE, i.e. when all players use and know that others use Markov stationary strategies. Here we briefly mention how a framework of a stochastic game and our functional equations can be use to study existence of MSNE of this game. Clearly due to the underline game one cannot hope for MSNE uniqueness without imposing strong conditions, but we rather aim to show how the uniqueness result obtained for every single player can help in determining. It is also an interesting application of mixed-monotone operators in studying stochastic games (see Balbus, Reffett, and Woźny (2013) for the first application of mixed-monotone operators to the study of OLG games).

To analyze MSNE, for any profile of continuation values $(v_1, v_2 \ldots, v_n) \in \mathcal{V}^n$, state $s \in S$, consider an auxiliary game between n-player each with payoffs given by:

$$u_i(s, a_i, a_{-i}) + \beta\delta \int_S v_i(s') Q(ds'|s, a_i, a_{-i}).$$

By $a^*(s, v)$ denote a set of NE of the stage game for a profile of corresponding value functions $v = (v_1, \ldots, v_n)$ and state s. Define an operator $\overline{B}(v)(s) := \overline{a}^*(s, v)$ and $\underline{B}(v)(s) := \underline{a}^*(s, v)$ where $\overline{a}^*(s, v)$ and $\underline{a}^*(s, v)$ denote the greatest and least Nash equilibrium of the auxiliary game. The corresponding operator A is given by:

$$\overline{A}_i v(s) = \left\{ u_i(s, \overline{a}^*(s, v)) + \beta\delta \int_S v_i(s') Q(ds'|s, \overline{a}^*(s, v)) \right\},$$

and similarly we define $\underline{A}_i v(s)$. To continue we need some assumptions:

Assumption 5 *Let for all i*

- $u_i : S \times A \to \mathbb{R}_+$ *be continuous on a, supermodular, increasing with a_i, decreasing with a_{-i} and measurable with s, with $u_i(0, a) = 0$ and $u_i(\cdot) \leq \overline{u}$,*
- *for any $s \in S, a \in A(s)$ let $Q(\cdot|a, s) = p(\cdot|a, s) + (1 - p(\cdot|a, s))\delta_0(\cdot)$, where δ_0 is a delta Dirac measure concentrated at point 0, while $p(\cdot|a, s)$ is some measure such that $p(Z|a, s) < 1$ and $p(Z|a, 0) \equiv 0$ for all $a \in A, Z \subset S$.*

- *for any integrable v assume $(a, s) \to \int_S v(s')p(ds'|a_i, a_{-i}, s)$ is continuous, supermodular, decreasing with a, measurable with s.*

Lemma 1 *For any $(v_1, v_2, \ldots, v_n) \in \mathcal{V}^n$ the auxiliary game possesses the greatest $\overline{a}^*(s, v)$ and the least $\underline{a}^*(s, v)$ Nash equilibrium. Both are decreasing with v.*

Lemma 2 *For any $(v_1, v_2, \ldots, v_n) \in \mathcal{V}^n$ $s \to \overline{a}^*(s, v)$ and $s \to \underline{a}^*(s, v)$ are measurable.*

Now for $v, \tilde{v} \in \mathcal{V}^n$ define an operator:

$$\overline{T}_i(v, \tilde{v})(s) = \frac{1}{\beta}\overline{A}_i v(s) - \frac{1-\beta}{\beta} u_i(s, \overline{a}_i^*(s, v), \overline{a}_{-i}^*(s, \tilde{v})),$$

and a vector $\overline{T}(v) = (\overline{T}_1(v, \tilde{v}), \overline{T}_2(v, \tilde{v}) \ldots, \overline{T}_n(v, \tilde{v}))$. Similarly we define $\underline{T}$.

Lemma 3 *$\overline{T}$ is a mixed monotone operator, i.e. $(v, \tilde{v}) \to \overline{T}(v, \tilde{v})$ is increasing with v and decreasing with $\tilde{v}$. Similarly $\underline{T}$.*

Lemma 4 *$v \to \overline{T}(v, \tilde{v})$ has a unique fixed point. Similarly $\underline{T}$.*

Lemma 5 *$\overline{T}(v, \tilde{v})$ is continuous in ... topology on... Similarly $\underline{T}$.*

...Here we can try to use our JMathE techniques to show continuity and existence...

Theorem 6 *$\overline{T}$ has a non-empty and anti-chained set of fixed points. Similarly $\underline{T}$.*

The above theorem shows that that there exists MSNE. The antichained result means that if we have two different MSNE values, say v^*, w^* it *cannot be* that $(\forall s \in S) \quad v^*(s) \geq w^*(s)$. As the values are unordered generally there *is no optimal* of MSNE. This indicates possibility of *sunspot* MSNE (sequential equilibria), where depending on the current state s different continuation MSNE is chosen.

5 Conclusions

Their are numerous reasons why developing constructive methods in the study of (sequential or Markov perfect) equilibrium in models with dynamic strategic interaction can be important. The first reason steps from the increasingly important role that numerical methods have played in the characterization of dynamic equilibrium over the last few decades (e.g., given the recent interest in many literatures that seek to fit the equilibrium of these models to the data via either calibration or estimation techniques). Indeed, one of the important contributions that the generalized Euler equation methods (of Harris and Laibson (2001)) have opened, is the possibility of developing a large catalog of tractable methods for characterizing the structure of Markov perfect equilibrium in dynamic games. In this spirit, and in the context of a stochastic game representation of the problem of modelling consumers with hyperbolic preferences, we are able to make significant advances on these types of methods. In particular, our methods allow us to compute *both* time-consistent optimal policies and associated value functions, with neither of these equilibrium objects being computed using the local approximation arguments implicitly given in a generalized Euler equation method of Harris and Laibson (2001). Therefore, in an important sense, our work can be viewed as a direct extension of theirs.

Also, more generally, our methods show that stochastic games can provide an very useful setting for studying dynamic problems with strategic interactions between "generations" of agents. In such problems, when constructive methods are available, these methods prove useful also for understanding not only issues related to the approximation/estimation of equilibrium (in a parameter), but they also provide the possibility of exploring numerous important theoretical questions that are not so readily addressed with pure topological methods (e.g., questions concerning equilibrium comparative statics, and understanding the theoretical differences between various models under consideration).

Per this latter theoretical issue, for example, our Theorem 1 asserts existence of the greatest value and hence the optimal time-consistent policy. That is, we show there exists an optimal, time-consistent policy which can be interpreted inducing a infinite sequence of policies stemmed from one undertaken by a "planner", where under a continuation of the state variable, this sequence of future planned policies would actually be implemented by the continuation "doers" in the subsequent periods. Such a situation has a natural interpretation as a "sustainable" policy for a decision maker with hyperbolic preference, and this idea can be integrated nicely into models of optimal policy design with global pollution (e.g., see the work of Karp and Tsur (2011)). Actually, our results on the structure of optimal time-consistent consumption decisions have direct implications for the recent work on welfare analysis and public policy in general (e.g., see the recent work of Nakajima (2010)).

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