A Nonsmooth Approach to Envelope Theorems*

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Abstract

We develop a nonsmooth approach to envelope theorems that applies across a broad class of parameterized constrained nonlinear optimization problems that arise typically in economic applications, including those with occasionally binding constraints. Our methods emphasize the role of the Strict Mangasarian-Fromowitz Constraint Qualification (SMFCQ), and include envelope theorems for both the convex and nonconvex case, allow for noninterior solutions, as well as equality and inequality constraints. We give new conditions when the value functions are directionally differentiable, as well as once-continuously differentiable (C^1).We apply our results to stochastic growth models with Markov shocks, as well as constrained lattice programming problems.

1 Introduction

Since the work of Viner [31] and Samuelson [27], the envelope theorem has become a standard tool in economic analysis. In its "classical" form the envelope theorem is simply a continuous derivative of a value function in a parameter. Sufficient conditions for its existence at first required a great

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deal of mathematical structure, including convexity, interiority, as well as the continuous differentiability of objectives and constraints (e.g., Samuelson [27], Rockafellar [24], Mirman and Zilcha [20], Benveniste and Scheinkman [3]).

In subsequent work, some of these assumptions have been relaxed. For instance, versions of Danskin's Theorem in Clarke [6] and Milgrom and Segal [19] relax conditions on the structure of the choice sets in unconstrained problems. Some of these results have been recently extended to problems with interior solutions and integer decisions in very general programming settings in Clausen and Strub [8]. Finally, and more along the lines of this paper, a recent contribution on the classical envelope theorem, Rincon-Zapatero and Santos [23] extends the classical C^1 envelope theorem to infinite horizon stochastic dynamic programming settings with inequality constraints in the presence of noninterior optimal solutions. Per this latter result (as well as Milgrom and Segal's results for the cases with inequality constraints), the programs are *convex*, and it is not clear to extend the result to economic models with nonconvexities and constraints. Additionally, in the optimization literature, a lot of progress has been made on stability bounds for nonconvex problems in Banach spaces (see, for instance, the monographs of Clarke [7] and Bonnans and Shapiro [4]), although the focus has generally not been on simple and practical sufficient conditions for exact directional derivatives.

In this paper we combine the recent results of the optimization literature with some conditions easily verifiable in finite dimensional problems to provide generalized on envelope theorems applicable to economic models with nonconvexities. When considering the question of the envelope theorem in the nonconvex case, at least three important issues arise. First, as continuously differentiable envelope theorems are not expected for the nonconvex case, one would like to propose an appropriate alternative notion of a "generalized" envelope that fits most applications and is a "substitute" for the classical envelope. Second, the proposed approach must work in settings with both equality and inequality constraints, and when optimal solutions are not necessarily interior. Third, when the Slater condition is not appropriate, new constraint qualifications allowing for simple (and, if possible, exact) calculations of the generalized envelopes or for constructing differential bounds of the value function (the latter is often all that is needed in applications) need to be identified.

Methodologically, we take a "nonsmooth" approach closely related to the work of Gauvin and Tolle [13], Gauvin and Dubeau [12], and Auslender [2], among other, which seem most useful in optimization problems with nonconvexities yet reduce to the classical envelope theorems in convex cases. We consider Lipschitz programs in finite dimensional spaces, in which objective functions are locally Lipschitz but not necessarily differentiable, although we assume smooth constraints.[1] We show how progressively stronger conditions on primitive data lead to progressively sharper characterizations of the differentiable properties of the value function. In the end, we give conditions under which value functions admit differential bounds, are Clarke differentiable, directionality differentiable, and once-continuously differentiable (C^1). Our sharpest results focus on constraint systems satisfying the Strict Mangasarian-Fromowitz Constraint Qualification, a refinement of the Mangasarian-Fromowitz Constraint Qualification equivalent to the uniqueness of the Karash-Kuhn-Tucker multiplier in our setup.

The remainder of the paper is laid out as follows. In the next section, we describe the benchmark class of optimization problems we consider. In Section three we present our main results. In particular, we provide results on differential bounds for the value function, directional differentiability, and C^1 differentiability. Applications of some of these results make up the last section of the paper.

2 Lipschitz Programs

Given a space A of actions (or controls), a parameter space S, and an objective function $f : A \times S \to \mathbb{R}$, we consider following the *Parameterized Lipschitz Optimization Program*:

$$V(s) = \max_{a \in D(s)} f(a, s) \tag{2.0.1}$$

where the feasible correspondence is:

$$D(s) = \{a | g^i(a, s) \leq 0, \quad i = 1, ..., p, \ h^j(a, s) = 0, \quad j = 1, ..., q\}$$

and the optimal solution correspondence is:

$$A^*(s) = \arg \max_{a \in D(s)} f(a, s)$$

We will maintain some baseline assumptions throughout the paper:

Assumption 1:

(a) A and S *are open convex subsets of* $\mathbb{R}^n$ and $\mathbb{R}^m$*, respectively,*

[1] Problems with nonsmooth constraints as well as "mixed-integer programming" problems are studied in Morand, Reffett, and Tarafdar [21].

(b) the constraints g^i, $i = 1, .., p$ and $h^j = 0$, $j = 1, .., q$ are jointly C^1, and $n \geq q$,

(c) the objective function $f : A \times S \to \mathbb{R}$ is locally Lipschitz in (a, s),

The Lagrangian duality scheme we use is standard, albeit we study its use in a nonconvex context. Of course, the cost of relaxing the convexity assumptions is immediate, as we will not, in general, have optimal solutions that satisfy strong duality (e.g., zero duality gap, saddlepoint stability, etc.) and constructing a necessary and sufficient first order theory for optimal solutions will be in general impossible. This, however, does not prevent us from providing mild conditions under which sharp characterizations of simple nonsmooth envelope theorems for (2.0.1) can be read from linearizations of the Lagrangian dual at its optimum.

Let the dual variables (or *Karash-Kuhn-Tucker (KKT) vector of multipliers*) be denoted by $(\lambda, \mu) \in \mathbb{R}^p_+ \times \mathbb{R}^q$. We conjugate the problem (2.0.1) with a classical Lagrangian duality scheme as follows:

$$L(a, \lambda, \mu; s) = \begin{cases} f(a, s) - \lambda g(a, s) - \mu h(a, s) & \text{if } a \in A, (\lambda, \mu) \in \mathbb{R}^p_+ \times \mathbb{R}^q \\ -\infty, & \text{if } a \notin A, (\lambda, \mu) \in \mathbb{R}^p_+ \times \mathbb{R}^q \\ \infty, & \text{otherwise} \end{cases}$$

where

$$g(a, s) = [g^1(a, s), ..., g^p(a, s)]; \ h(a, s) = [h^1(a, s), ..., h^q(a, s)]$$

A point $a \in D(s)$ is a *Karush-Kuhn-Tucker (KKT) point* if there exist $(\lambda, \mu) \in \mathbb{R}^p_+ \times \mathbb{R}^q$ such that:

$$0 \in \partial_a f(a, s) - (\lambda \nabla_a g + \mu \nabla_a h)(a, s)$$

and

$$\lambda g(a, s) = 0$$

We denote by $K(a, s)$ the set of KKT points associated with a feasible point $a \in D(s)$.

In constrained optimization, constraint qualifications are needed to guarantee that optimal solutions are KKT points, and that the feasible region around such points does not vanish under local perturbations of the parameters. The strongest of these constraints is the Linear Independence Constraint Qualification (LICQ), which has been key in the work of Gauvin and Dubeau [12] and the focus of Rincon-Zapatero and Santos [23], for example.

Definition 1 *A feasible point $a \in D(s)$ satisfies the LICQ if the following vectors are linearly independent,*

$$\nabla_a g^i(a,s),\ i \in I, \quad \nabla_a h^j(a,s),\ j = 1, ..., q$$

where $I = \{i : g^i(a,s) = 0\}$

LICQ is rather strong and plays no role in our argument. Rather, we focus a weaker constraint, the Mangasarian-Fromowitz Constraint Qualification (MFCQ), which is equivalent to Robinson's constraint qualification in finite dimension and with a finite number of constraints.

Definition 2 *A feasible point $a \in D(s)$ satisfies the MFCQ if:*
(i) the following vectors are linearly independent,

$$\nabla_a h^j(a,s),\ j = 1, ..., q$$

(ii) there exists $y \in \mathbb{R}^n$ such that,

$$\nabla_a g^i(a,s) y < 0,\ i \in I;\ \nabla_a h^j(a,s) y = 0, \quad j = 1, ..., q$$

where the set of active constraints are denoted by $I = \{i : g^i(a,s) = 0\}$.

MFCQ is not sufficient to guarantee the uniqueness of the KKT multipliers, and this will make obtaining directional derivatives and sharp characterizations of the generalized gradients very difficult. However, MFCQ is sufficient for asserting the non-emptiness of the set of multipliers (as well as their compactness; e.g., Gauvin and Tolle ([13] Corollary 2.8), as stated in the following proposition, a standard Lagrange multiplier rule.

Proposition 3 *Under assumption 1 and if MFCQ holds at $a^*(s) \in A^*(s)$, then there exist $\lambda \in \mathbb{R}^p_+$ and $\mu \in \mathbb{R}^q$ such that:*

(a) $\lambda^i g^i(a^*(s), s) = 0$
(b) $0 \in \ \partial_a f(a^*(s), s) - (\lambda \nabla_a g + \mu \nabla_a h)(a^*(s), s)$

We note that (c) above implies that:

$$f_a^{-o}(a^*(s), s; x) - (\lambda \nabla_a g - \mu \nabla_a h)(a^*(s), s) \cdot x \leq 0$$

for each direction $x \in \mathbb{R}^n$.

To obtain uniqueness of KKT multiplier for each optimal solution we will impose a slightly stronger condition than MFCQ (in the sense that it implies MFCQ). Although technically not exactly a constraint qualification (since it assumes the existence of a multiplier), this refinement of the MFCQ has been identified in the literature as the Strict Mangasarian-Fromowitz Constraint Qualification (SMFCQ) or simply the Strict Constraint Qualification (Bonnans and Shapiro [4], Definition 4.46).

Definition 4 *A feasible point $a \in D(s)$ together with a multiplier $(\lambda, \mu) \in K(a, s)$ satisfy SMFCQ if:*

(i) the following vectors are linearly independent,

$$\nabla_a g^i(a,s),\ i \in I_b;\ \ \nabla_a h^j(a,s),\ j = 1, ..., q$$

(ii) there exist $y \in \mathbb{R}^n$ such that,

$$\nabla_a g^i(a,s)y < 0,\ i \in I_s;\ \nabla_a g^i(a,s)y = 0,\ i \in I_b$$
$$\nabla_a h^j(a,s)y = 0, \quad j = 1, ..., q$$

where the set of binding and saturated constraints are denoted by $I_b = \{i \in I : \lambda_i > 0\}$, $I_s = \{i \in I : \lambda_i = 0\}$ respectively.

The equivalence between uniqueness of KKT multiplier and SMFCQ for smooth programs is demonstrated in Kyparisis ([18], Proposition 1.1). We use this result in many of our arguments below. We state Kyparisis below without proof noting, however, that with smooth constraints the strict differentiability of the objective is sufficient to prove Kyparisis' result (see also Bonnans and Shapiro [4] Remark 4.49).

Proposition 5 *Assume that f is differentiable in a, and that assumption 1 holds. Then SMFCQ holds for $(a, (\lambda, \mu))$ iff $K(a, s)$ is a singleton.*

3 Differentiability of the Value Function

This section presents successively stronger characterizations of the envelopes (of course, under successively stronger sufficient conditions).

3.1 Stability Bounds

We first discuss an important set of results on the existence of bounds for the Dini derivatives of the value function for general Lipschitz programs.

The most general results have been obtained in Clarke ([7], Theorem 6.5.2 and Corollary 4. Also, see Bonnans and Shapiro ([4], Theorem 4.26 for a similar version of Clarke's result). One issue though with Clarke's results is that some of the hypotheses of the theorem are not stated on primitves, and involve properties of the value function (e.g., see Hypothesis 6.5.1, p. 241). Therefore, we now give sufficient conditions for Clarke's result appealing to easily checked sufficient conditions in Proposition 6.[2] The conditions involve the MFCQ and standard uniform compactness conditions.[3] Although less general than Clarke's conditions, our sufficient conditions hold in many economic applications.

Proposition 6 *Under assumption 1, if $D(s)$ is nonempty and uniformly compact near s and MFCQ holds at the optimal solution $a^*(s) \in A^*(s)$, then, in any direction $x \in \mathbb{R}^m$, we have:*

$$\liminf_{t \to 0^+} \frac{V(s+tx) - V(s)}{t} \geq \sup_{a^*(s) \in A^*(s)} \inf_{(\lambda,\mu) \in K(a^*(s),s)} \left\{ L_s^{-o}(a^*(s), s, \lambda, \mu; x) \right\}$$

$$\limsup_{t \to 0^+} \frac{V(s+tx) - V(s)}{t} \leq \sup_{a^*(s) \in A^*(s)} \sup_{(\lambda,\mu) \in K(a^*(s),s)} \left\{ L_s^{o}(a^*(s), s, \lambda, \mu; x) \right\}$$

where:

$$L_s^{-o}(a^*(s), s, \lambda, \mu; x) = \min_{\varsigma_s \in \partial_s f(a^*(s),s)} [(\varsigma_s - (\lambda \nabla_s g + \mu \nabla_s h)(a^*(s), s)) \cdot x]$$
$$L_s^{o}(a^*(s), s, \lambda, \mu; x) = \max_{\varsigma_s \in \partial_s f(a^*(s),s)} [(\varsigma_s - (\lambda \nabla_s g + \mu \nabla_s h)(a^*(s), s)) \cdot x]$$

Proof. As suggested by Rockafellar ([25], p. 29), [4] the original program can be re-written as:

$$\begin{aligned} V(s) &= \max f(a, a') \\ g(a, a') &\leq 0 \\ h(a, a') &= 0 \\ -a' + s &= 0 \end{aligned}$$

[2] In Bonnans and Shapiro, the key condition is Robinson's constraint qualification together with an additional hypothesis similar to that of Clarke. Also, in our context, the condition on the absence of nontrivial abnormal multipliers is equivalent to MFCQ. See Rockafellar ([26], p. 215).

[3] We note, in our context, uniform compactness is equivalent to the inf-compactness assumption in often used in the literature (e.g., Bonnans and Shapiro, [4] Proposition 4.1.2).

[4] See also Clarke [7] and Bonisseau and LeVan [5].

with associated Lagrangian:[5]

$$f(a,a') - \lambda g(a,a') - \mu h(a,a') - \theta(-a'+s)$$

■

The assumption of uniform compactness of D near s implies the existence of a neighborhood S' of s such that $cl\left[\cup_{s'\in S'} D(s)\right]$ is compact. As a result:

$$\exists\delta > 0,\ \left|s'-s\right| < \delta \implies D(s') \text{ is compact}$$

(since $D(s')$ is closed and included in the compact $cl\left[\cup_{s'\in S'} D(s)\right]$), which implies that $A^*(s')$ is nonempty, and Hypothesis 6.5.1 in Clarke is therefore satisfied (by setting $\varepsilon_0 = \delta$ and $\Omega = cl\left[\cup_{s'\in S'} D(s)\right]$). Our result is then a straightforward application of Corollary 4 in Clarke [7], as the MFCQ for the original program is equivalent to MFCQ for the above program, is equivalent to the nonexistence of abnormal multipliers. Thus:

$$\begin{aligned}&\sup_{a^*(s)\in A^*(s)} \inf_{\theta\in K(a^*(s),s)} \theta\cdot x\\ &\leq DV_+(s,x) \leq DV^+(s,x)\\ &\sup_{a^*(s)\in A^*(s)} \sup_{\theta\in K(a^*(s),s)} \theta\cdot x\end{aligned}$$

where

$$\inf_{\theta\in K(a^*(s),s)} \theta\cdot x = \inf_{(\lambda,\mu)\in K(a^*(s),s)} \min_{\varsigma_s\in\partial_s f(a^*(s),s)} \left[(\varsigma_s - (\lambda\nabla_s g + \mu\nabla_s h)(a^*(s),s))\cdot x\right]$$

and:

$$\sup_{\theta\in K(a^*(s),s)} \theta\cdot x = \sup_{(\lambda,\mu)\in K(a^*(s),s)} \max_{\varsigma_s\in\partial_s f(a^*(s),s)} \left[(\varsigma_s - (\lambda\nabla_s g + \mu\nabla_s h)(a^*(s),s))\cdot x\right]$$

With these differential bounds in place, we are now ready to state sufficient conditions under which simple "min-max" directional envelopes exist for all class of Lipschitz programming problems.

3.2 Directional Differentiability

Additional assumptions are needed for the existence of directionality differentiable envelopes. The problem is that under only the MFCQ, neither

[5] This is precisely the problem studied in Clarke [7] Chapter 6, for the finite dimensional case.

$\partial_s f(a^*(s), s)$ nor $K(a^*(s), s)$ are singletons for each optimal solution, which implies in generals exact directional calculations are not available.[6] In our result, the assumption of strict differentiability of f in s takes care of the Clarke gradient, and SMFCQ insures uniqueness of multipliers.

Theorem 7 *Under assumption 1, if $D(s)$ is nonempty and uniformly compact near s, if f is smooth in s and if SMFCQ holds for every optimal solution $a^*(s) \in A^*(s)$, then for any direction $x \in \mathbb{R}^m$, the directional envelope is given by:*

$$V'(s, x) = \max_{a^*(s) \in A^*(s)} \{ (\nabla_s f - \lambda \nabla_s g - \mu \nabla_s h) (a^*(s), s) \cdot x \}$$

Proof. The return function f is smooth in s, hence:

$$\nabla_s L(a^*(s), s, \lambda, \mu; x) = (\nabla_s f - \lambda \nabla_s g - \mu \nabla_s h) (a^*(s), s)$$

By Proposition 5, $K(a^*(s), s)$ is a singleton thus by Proposition (6):

$$\begin{aligned} &\sup_{a^*(s) \in A^*(s)} (\nabla_s f - \lambda \nabla_s g - \mu \nabla_s h) (a^*(s), s) \cdot x \\ &= D_+ V(s; x) = D^+ V(s; x) \\ &= \sup_{a^*(s) \in A^*(s)} (\nabla_s f - \lambda \nabla_s g - \mu \nabla_s h) (a^*(s), s) \cdot x \end{aligned}$$

Noting that the supremum is attained (Gauvin and Dubeau [12], Corollary 4.3):

$$D_+ V(s; x) = D^+ V(s; x) = \max_{a^*(s) \in A^*(s)} \{ (\nabla_s f - \lambda \nabla_s g - \mu \nabla_s h) (a^*(s), s) \cdot x \}$$

■

This is a very important result. The key condition used in the argument is the SMFCQ, which in our context is equivalent to uniqueness of the KKT multipliers (in finite dimensional settings), which in turn, allows us to write simple forms for exact directional derivatives (as opposed to Dini's bounds as in the previous section).

It also bears mentioning that Rincon-Zapatero and Santos [23] emphasize the importance of having conditions under which the KKT points are *globally* unique in their proof of C^1 differentiability of the value function (e.g., see [23], Theorem 3.2). Certainly, the LICQ condition they use is *one* such condition (along with enough smoothness in the primitive data), but SMFCQ should be considered an important alternative to LICQ. We give examples in the next section that help highlight this point.

[6] In the earlier draft of this paper, we give examples.

3.3 Unconstrained programs

To unify our approach with some existing results for the case of unconstrained programs (or programs with only interior solutions), we consider the existence of directional envelopes without smoothness conditions in the parameter s. Our next result is related to Clarke's version of Danskin's Theorem (Clarke [6] Theorem 2.1), where Clarke regularity suffices.

Corollary 8 *Under assumption 1, if f is Clarke regular in s, then for every optimal solution $a^*(s) \in A^*(s)$ and any direction $x \in \mathbb{R}^m$, the directional envelope is given by:*

$$V^{'}(s;x) = \max_{a^*(s) \in A^*(s)} f_s'(a^*(s), s; x)$$

Proof. We improve the lower bound in Proposition 6,

$$\begin{aligned}
&\liminf_{t \to 0^+} \frac{V(s+tx) - V(s)}{t} \\
&= \liminf_{t \to 0^+} \frac{f(a^*(s+tx), s+tx) - f(a^*(s), s)}{t} \\
&\geq \liminf_{t \to 0^+} \frac{f(a^*(s), s+tx) - f(a^*(s), s)}{t} \\
&= f_s'(a^*(s), s; x)
\end{aligned}$$

the last equality following from f being directionality differentiable. By Clarke regularity, the upper bound in Proposition 6 satisfies:

$$\begin{aligned}
&\limsup_{t \to 0^+} \frac{V(s+tx) - V(s)}{t} \\
&\leq \max_{\varsigma \in \partial f_s(a^*(s), s)} \{\varsigma \cdot x\} = f^o(a^*(s), s; x) = f_s'(a^*(s), s; x)
\end{aligned}$$

Upper and lower bound coincide for any direction $x \in \mathbb{R}^m$, hence:

$$V'(s;x) = \max_{a^*(s) \in A^*(s)} \left\{ f_s'(a^*(s), s; x) \right\}$$

■

A few remarks on this result. First, if we in addition assume the collection $f_s(a, s)$ forms an equidifferentiable collection, then we arrive at a multidimensional version of Milgrom and Segal's result (e.g., [19], Theorem 3). Their result was, in turn, a generalization of the classic smooth envelope theorem (e.g., Mirman and Zilcha ([20], Lemma 1). Second, applications of this result can be made for infinite horizon dynamic programs as has been done for related nonsmooth envelopes based upon Clarke's version of Danskin's theorem (e.g, see Askri and LeVan [1]), Indeed, we give an application to monotone controls in stochastic growth models in the next section of the paper.

3.4 Clarke Gradient of the Value Function

The existence of bounds for the rate of growth of the value function in s implies the value function is locally Lipschitz (see Tarafdar [28], chapter 2). We now extend the result in Theorem 7 to obtain the Clarke gradient of the value function when solutions are not necessarily interior.

Proposition 9 *Under assumption 1, if $D(s)$ is nonempty and uniformly compact near s and MFCQ holds at every optimal solution $a^*(s) \in A^*(s)$, the Clarke gradient of the value function satisfies:*

$$\partial V(s) \subset co\left\{\cup_{a^*(s)\in A^*(s)} \cup_{(\lambda,\mu)\in K(a^*(s),s)} (\partial_s f - \lambda\nabla_s g - \mu\nabla_s h)(a^*(s), s)\right\}$$

If, in addition, f is smooth and SMFCQ holds at every optimal solution, then:

$$\partial V(s) = co\left\{\cup_{a^*(s)\in A^*(s)}(\nabla_s f - \lambda\nabla_s g - \mu\nabla_s h)(a^*(s), s)\right\}.$$

Proof. The first inclusion is demonstrated in Clarke ([7], Corollary 1, p 242). Then, given the differentiability and uniqueness of multipliers under SMFCQ in Theorem 7, we have

$$\partial V(s) \subset co\left\{\cup_{a^*(s)\in A^*(s)}(\nabla f_s - \lambda\nabla_s g - \mu\nabla_s h)(a^*(s), s)\right\} \qquad (3.4.1)$$

Consider any η, such that:

$$\eta \in \cup_{a^*(s)\in A^*(s)} \left\{\nabla_s f(a^*(s), s) - \lambda\nabla_s g(a^*(s), s) - \mu\nabla_s h(a^*(s), s)\right\} \quad (3.4.2)$$

By construction, for all x :

$$\eta{\cdot}x \leq \max_{a^*(s)\in A^*(s)} \left\{(\nabla f_s(a^*(s), s) - \lambda\nabla_s g(a^*(s), s) - \mu\nabla_s h(a^*(s), s)) \cdot x\right\} = V'(s; x)$$

the last equality following from Theorem 7. Since $V^{'}(s;x) \leq V^o(s;x)$ necessarily:

$$\eta \in \partial V(s)$$

and $\partial V(s)$ contains any convex combination of such η, that is:

$$\partial V(s) \supset co\left\{\cup_{a^*(s)\in A^*(s)}(\nabla f_s - \lambda\nabla_s g - \mu\nabla_s h)(a^*(s), s)\right\}.$$

■

3.5 C^1 Differentiability of the Value Function

Finally, in setting with constraints and noninterior optimal solutions, perhaps the most general result in the literature concerning the existence of a classical C^1 envelope theorem appears in a recent paper by Rincon-Zapatero and Santos ([23], Theorems 3.1 and 3.2).[7] A corollary of our main result on directional differentiability in Theorem 7 relates our results to C^1 differentiability.

Corollary 10 *Under assumption 1, if $D(s)$ is nonempty and uniformly compact near s, if f is smooth and strictly quasi-concave in a, g quasi-convex in a, h affine in a, and if SMFCQ holds for every $a^*(s) \in A^*(s)$,then V is strictly differentiable (i.e., C^1) at s with:*

$$\nabla V(s) = \left(\nabla_s f - \lambda \nabla_s g - \mu \nabla_s h\right)\left(a^*(s), s\right)$$

Proof. By the strict quasi-concavity of f, the quasi-convexity of g, and the assumption that h is affine, $A^*(s)$ is a singleton, and SMFCQ guarantees uniqueness of the multiplier. Then, a result in Clarke [7] (Corollary 2, p. 242) proves the result. ■

The difference between Theorem 10 and the envelope obtained in Rincon-Zapatero and Santos' work is meaningful. One can give simple economic examples (e.g., in consumer theory) where the results of Rincon-Zapatero and Santos [23] do not apply as LICQ is not satisfied, yet the value function is actually C^1. In these cases, the SMFCQ does hold. We give such an example in a moment. Similarly, one can also construct examples of where the SMFCQ fails, MFCQ holds, and the value function is not C^1. (e.g., Tarafdar [28] for such examples, or an earlier draft of this paper.) So the importance of SMFCQ seems very clear in obtaining sharp envelope theorems in the finite dimensional parameter case (either directional, and C^1)

4 Examples and Applications

In this section, we present an important example, and two applications of our results to lattice programming. We begin with an example that show that LICQ is not sufficient for C^1 envelopes.

[7] Mirman and Zilcha [20], Benveniste and Scheinkman [3] provide conditions for the existence of C^1 envelopes for effectively unconstrained problems in dynamic programming framework. Milgrom and Segal [19] too provide similar condition for unconstrained static programming problems.

4.1 A Consumer Problem: SMFCQ vs. LICQ

In the following example, we assume the objective is C^1 and strictly quasi-concave, LICQ fails, SMFCQ holds, and the value function is C^1 (thus illustrating the importance of Theorem 10 in the preceding section).

Consider $u : \mathbb{R}^2_+ \to \mathbb{R}$ given by $u(a_1, a_2) = a_1 \cdot a_2^2$ and prices are as follows:

$$p_1 = \begin{cases} 1 \text{ if } a_1 \leq 5 \\ 2 \text{ if } a_1 > 5 \end{cases}$$

$$p_2 = \begin{cases} 1 \text{ if } a_2 \leq 5 \\ 2 \text{ if } a_2 > 5 \end{cases}$$

while income m belongs to the interval $[8, 12]$. The budget correspondence is:

$$a_1 + a_2 - m \leq 0 \tag{4.1.1}$$

$$2a_1 - 5 + a_2 - m \leq 0 \tag{4.1.2}$$

$$a_1 + 2a_2 - 5 - m \leq 0 \tag{4.1.3}$$

$$2a_1 - 5 + 2a_2 - 5 - m \leq 0 \tag{4.1.4}$$

Since it is convex and the objective is strictly quasi-concave, the optimal solution is unique.

We denote the KKT multipliers corresponding to the constraint j again by λ_j. The optimal solutions associated with the Lagrangian of this problem can be verified to be:

$$(a^*(m), \lambda^*(m)) = \begin{cases} (m-5, 5, 100-10m, 0, 10m-75, 0)\,, \; m < 10 \\ \left(\frac{m+5}{3}, \frac{m+5}{3}, 0, 0\frac{(m+5)^2}{9}, 0\right), \quad m = 10 \\ \left(5, \frac{m}{2}, 0, 0, \frac{20m-m^2}{4}, \frac{m^2-10m}{4}\right), \quad m > 10 \end{cases}$$

For $m < 10$, constraints (4.1.1) and (4.1.3) are active, and LICQ is satisfied. Dually at $m > 10$, constraints (4.1.3) and (4.1.4) are active, and LICQ is satisfied. Interestingly, at $m = 10$, all constraints are active, yet only constraint (4.1.3) is binding ($\lambda_3 > 0$), while all other constraints are saturated. The vector $r = [-2, 1]^T$ satisfies the conditions for SMFCQ; yet at $m = 10$, LICQ is clearly violated. Hence, Rincon-Zapatero and Santos main theorem ([23], Theorem 3.1) does not apply; yet, we still have unique multipliers under Theorem (7). The value function for this problem is given by,

$$V(m) = \begin{cases} (m-5)25, \; m < 10 \\ \frac{(m+5)^3}{27}, \quad m = 10 \\ \frac{5m^2}{4}, \quad m > 10 \end{cases}$$

is continuously differentiable with derivatives given as follows:

$$V'(m) = \begin{cases} 25, & m < 10 \\ \frac{(m+5)^2}{9}, & m = 10 \\ \frac{5m}{2}, & m > 10 \end{cases}$$

4.2 Monotone Controls in Stochastic Growth Models with Markov Shocks

We next use our envelope results to weaken a very strong condition of Hopenhayn and Prescott [16] for obtaining monotone controls in stochastic growth models with Markov shocks. The dynamic programming argument in Hopenhayn and Prescott ([16], Proposition 2) requires the graph of feasible correspondence to be a sublattice, a type of "strict cardinal complementarities" condition, which, in practice, excludes all but Leontief production functions.

We show below how to preserve supermodularity under maximization without the sublattice condition under standard Inada conditions by appealing to repeated applications of Corollary 8.

We impose precisely the same assumptions as Hopenhayn and Prescott ([16], Section 6, Part B), except that we do *not* restrict attention to production functions with sublatticed graph correspondence. Uncertainty is a random production shock $z \in \mathbf{Z} = [z_{\min}, z_{\max}] \subset \mathbb{R}_{++}$ following a Markov process with transition function Q.

Assumption 4.2.1: The utility function $u : K \to \mathbb{R}_+$ *is strictly increasing, concave, smooth, and* $u'(0) = +\infty$ *with* $u(0) = 0$*;* $0 < \beta < 1$.

Assumption 4.2.2: The production function $f : K \times Z \to \mathbb{R}_+$ is *increasing, smooth, supermodular and* $f(0, z) = 0$*. In addition, there exists* $k_{\max} > 0$ *such that* $\forall k > k_{\max}$*,* $f(k, z_{\max}) < k$*, and* $K = [0, k_{\max}]$.

Assumption 4.2.3: Q is stochastically increasing.

By a standard argument, there exists a unique continuous bounded value function V^* satisfying the Bellman equation:

$$V^*(k, z) = \max_{0 \le y \le f(k,z)} \left\{ u(f(k, z) - y) + \beta \int V^*(y, z') Q(z, dz') \right\} \quad (4.2.1)$$

obtained as the pointwise limit of the sequence $\{V_n\}_{n=1}^{\infty}$ defined by:

$$V_{n+1}(k, z) = \max_{0 \le y \le f(k,z)} \left\{ u(f(k, z) - y) + \beta \int V_n(y, z') Q(z, dz') \right\} \quad (4.2.2)$$

and given $V_0 = 0$. By Berge's Theorem of the Maximum, the optimal correspondence $Y_n^*(k,z)$ is non-empty, compact-valued and upper hemicontinuous in k. Denote $y_n^*(k,z) = \vee Y_n^*(k,z)$ the maximal optimal solution.

The proof of supermodularity of V^* goes as follows. First, using our envelope results, we show that each $V_n(.,z)$ is Lipschitz on $[\theta, k_{\max}]$ with θ arbitrarily close to 0. Combined the right continuity of $V_n(.,z)$ at 0 this implies that each $V_n(.,z)$ is absolutely continuous on $[0, k_{\max}]$ and therefore equal to its indefinite integral. Assumptions 4.1.1 - 4.1.4 imply that this indefinite integral is increasing in z, hence each V_n is supermodular, a property inherited by the pointwise limit V^*.

Lemma 11 *Under assumptions 4.1.1 - 4.1.3, for any* $0 < \theta < k_{\max}$ *any* z, $V_n(.,z)$ *is globally Lipschitz on* $[\theta, k_{\max}]$.

Proof. For any z in Z $V_n(.,z)$ is increasing and therefore almost everywhere differentiable, and Corollary (8) implies that where it is differentiable:

$$V_n'(k,z) = \max_{y \in Y_n^*(k,z)} \{u'(f(k,z) - y) f_1(k,z)\}.$$

and by strict concavity of u:

$$V_n'(k,z) = u'(f(k,z) - y_n^*(k,z)) f_1(k,z)$$

Under the Inada conditions, a standard argument implies that $f(k,z) - y_n^*(k,z)$ is bounded away from 0 on any $[\theta, k_{\max}]$ where $\theta > 0$, and $V_{p+1}'(.,z)$ is therefore bounded on $[\theta, k_{\max}]$, and thus globally Lipschitz on $[\theta, k_{\max}]$. ■

We now use this lemma to prove another lemma on the absolute continuity of the value function.

Proposition 12 *Under assumptions 4.1.1-4.1.3, for all* $n \geq 1$, V_n *is equal to its indefinite integral, that is:*

$$V_n(k,z) = \int_0^k V_n'(x,z) dx$$

where:

$$V_n'(k,z) = u'(f(k,z) - y_n^*(k,z)) f_1(k,z)$$

and $y_n^*(k,z) = \vee Y_n^*(k,z)$

Proof. Consider any sequence $\{\theta_i\} \downarrow 0$. For every i large enough:

$$\int_0^k V_n'(x,z)dx = \int_0^{\theta_i} V_n'(x,z)dx + \int_{\theta_i}^k V_n'(x,z)dx$$

V_n is Lipschitz on $[\theta_i, k]$, so by Dem'yanov and Rubinov ([10], Lemma 1.2 p. 77):

$$\int_{\theta_i}^k V_n'(x,z)dx = V_n(k,z) - V_n(\theta_i, z)$$

Also, $V_n(.,z)$ is an increasing function, so by Royden ([**?**]) (Theorem 3, p. 100):

$$\int_0^{\theta_i} V_n'(x,z)dx \leq V_n(\theta_i, z) - V_n(0,z)$$

so by continuity of $V_n(.,z)$ at 0:

$$\lim_{i\to\infty} \left| \int_0^{\theta_i} V_n'(x,z)dx \right| = 0$$

As a result:

$$\int_0^k V_n'(x,z)dx = V_n(k,z) - V_n(0,z)$$

■

We now show the value functions for the finite horizon problem are each supermodular with increasing optimal solutions (and top and bottom elements increasing selections).

Proposition 13 *Under assumptions 4.1.1-4.1.4, for all $n \geq 1$, V_n is supermodular, $Y_n^*((k_t, z_t)$ is strong set order ascending, and $\vee Y_n^*$ and $\wedge Y_n^*$ are increasing in (k_t, z_t).*

Proof. Assume that V_{n-1} is supermodular and recall that:

$$V_n(k,z) = \max_{0\leq k' \leq f(k,z)} \left\{ u(f(k,z) - k') + \beta E V_{n-1}(k', z') \right\}.$$

By supermodularity of V_{n-1} in (k', z'), $\beta EV_{n-1}(k', z')$ has increasing differences in (k', z). Also, $u(f(k, z) - k')$ has increasing differences in (k', z) since $u'' \leq 0$ and $f_2 \geq 0$. Under assumption 4.1.4, the objective above has increasing differences in (k, z), so by Topkis ([29], Theorem 2.8.3), Y^*_{n+1} is strong set order ascending, and $\vee Y^*_{n+1}$ and $\wedge Y^*_{n+1}$ are increasing in (k, z). Recalling that:

$$V_n'(k, z) = u'(f(k, z) - y^*_n(k, z))f_1(k, z)$$

the curvature assumption 4.1.4 implies that:

$$-u''(f(k, z) - y^*_n(k, z))f_1(k, z) + u'(f(k, z) - y^*_n(k, z))f_{12}(k, z) \geq 0$$

which, together with $y^*_n(k, z) = \vee Y^*_n$ increasing in z implies in turn that $V_n'(k, .)$ is an increasing function of z. Thus, for $k' > k$ and $z' > z$ and any $n \geq 1$, by Proposition (12) above:

$$V_n(k', z') - V_n(k, z') = \int_k^{k'} V_n'(x, z')dx$$

$$\geq$$

$$\int_k^{k'} V_n'(x, z)dx = V_n(k, z) - V_n(k, z)$$

i.e., V_n is a supermodular function in (k, z). Given that the proposition is true for $n = 0$, a recursive argument proves the desired result. ■

Finally appealing to a standard limiting argument, we have our result.

Theorem 14 *Under assumptions 4.1.1-4.1.4, the value function V is supermodular, $Y^*(k, z)$ is strong set order ascending, and the extremal optimal investment policies $\vee Y^*$ and $\wedge Y^*$ are increasing in (k, z).*

Proof. V is the pointwise limit of a sequence of supermodular functions, and therefore also supermodular, and the properties of the optimal investment correspondence Y^* follows from a standard result in Topkis (e.g., [29],Theorem 2.8.3). ■

4.3 Minimax Lattice Programming

We develop a minimax lattice programming method combining nonsmooth envelope theorems and duality to generate *strong* monotone comparative statics in a class of constrained optimization problems. Such sharp characterizations of monotone selections (i.e., least and greatest selections as in a standard lattice programming problem) cannot be obtained using the flexible set order methods of Quah, which only guarantee the existence of a monotone selection, unless all controls are flexible set ordered (see Quah ([22])).

Consider, for simplicity, a standard 2 good consumer problem, although all proofs and results apply directly to a finite number of goods. The commodity space is a nonempty sublattice C of $\mathbb{E}$ ($\mathbb{E}$ is $\mathbf{R}_+^2$ endowed with the componentwise partial order), and $s \in S = [0, \infty)$ is income. Good 2 is the numeraire, and $0 < p < \infty$ is the relative price of good 1. Let $\mathcal{L}(\mathbf{R}_+)$ be the set of all nonempty sublattices of $\mathbf{R}_+$, which we endow with Veinott's strong set order.[8] Assume Assumptions 1, in addition to the following complementarity conditions assumed in Quah([22]) for the cardinal case:

Assumption 4.2.1: The function $u(c)$ supermodular on C, concave in c_2 for each c_1, and strictly increasing in all arguments.

We now demonstrate the following comparative statics result under Assumption 1 and 4.2.1: the consumer demand correspondence for good c_1 is ascending in income $s \in S$ in the *strong set order* to $\mathcal{L}(\mathbf{R}_+)$ in income. To see this, given $p > 0$ and $s \in S$, the consumer solves the *primal* problem:

$$V(s) = \max_{c \in D(s)} u(c) \tag{4.3.1}$$

with:

$$D(s) = \{c | pc_1 + c_2 \leq s, c \in C\}$$

and optimal solutions are denoted:

$$C^*(s) = \arg \max_{c \in D(s)} u(c) \tag{4.3.2}$$

Given $s > 0$ and $c_1 \in [0, s/p)$ we define the *aggregated primal* problem as follows:

$$V^{ap}(c_1, s) = \max_{c_2 \in D(c_1, s)} u(c_2, c_1) \tag{4.3.3}$$

with:

$$D(c_1, s) = \{c \in C | c_2 \leq s - pc_1 = \widehat{s}\} \tag{4.3.4}$$

[8] See appendix for all lattice programming definitions used in this section.

and optimal solutions are denoted:

$$C_2^{ap*}(c_1, s) = \arg \max_{c_2 \in D(c_1,s)} u(c_2, c_1)$$

Clearly, for each $p > 0$:

$$V(s) = \max_{c_1 \in [0,s/p]} V^{ap}(c_1, s) \tag{4.3.5}$$

We show below that under Assumption 4.2.1, Program (4.3.5) is a *standard* (cardinal) lattice programming in good 1, since the objective $V^{ap}(c_1, s)$ is supermodular in (c_1, s) and the feasible correspondence in program (4.3.5) is strong set order ascending in s. Our main result thus follows directly from Topkis ([29], Theorem 2.8.2).

Theorem 15 *Under Assumption 4.2.1, (1.a)* $C_1^* : S \to \mathcal{L}(\mathbf{R}_+)$ *is ascending in Veinott strong set order, (1.b)* $\vee C_1^*(s)$ *and* $\wedge C_2^*(s)$ *are isotone selections.*

The proof of Theorem 15 takes place in two steps, Lemmas 16 and 17, with the proof of supermodularity of V^{ap} relying on Theorem 7. To see this, first note that for Problem 4.3.3 (i) the correspondence $D(c_1, s)$ is uniformly compact in s, and (ii) the SMFCQ is satisfied. The conditions of Theorem 7 are met, and $V^{ap}(c_1, s)$ is directionally differentiable. To compute and characterize the directional envelopes in Theorem 7 for $V^{ap}(c_1, s)$, following a standard construction (e.g., Rockafellar ([24], p. 382), for $c_1 \in [0, s)$, $s > 0$, define the Lagrangian

$$\begin{aligned} L(c_2, \lambda; c_1, s) &= u(c_2; c_1) - \lambda(c_1 + pc_2 - s) \text{ if } c_2 \geq 0, \phi \in \Omega \\ &= +\infty \text{ if } c_2 \geq 0, \phi \notin \Omega \\ &= -\infty \text{ if } c_2 < 0, \ \phi \in \Omega \\ &= \text{ any value in } [-\infty, \infty] \text{ if } c_{-i} \notin C_{-i}, \phi \notin \Omega \end{aligned} \tag{4.3.6}$$

The dual problem of (4.3.3) is the following: for $c_1 \in [0, s/p)$, $s > 0$,

$$V^{ad}(c_1, s) = \inf_{\lambda \geq 0} \sup_{c_2 \geq 0} L(c_2, \lambda; c_1, s) \tag{4.3.7}$$

This is is a standard convex problem under Assumption 4.2.1. In particular, the (nonempty) saddlepoints of (4.3.7) are elements of their effective domains (see Rockafellar ([24], Theorem 37.6 and its corollaries)). Then, by Theorem 7, the directional envelope is:

$$V_{c_1}^{ad}(c_1, s; x) = \max_{c_2^* \in C_2^{ad*}} \{u_1(c_1; c_2^*(\lambda^*(c_1, s), c_1)) - \lambda^*(c_1, s)\} \cdot x \tag{4.3.8}$$

where C_2^{ad*} are optimal dual solutions.

Next, to characterize the properties of this envelope, we use a minimax lattice programming argument to do monotone comparative statics on the saddlepoints. The dual objective is V^L defined as:

$$V^L(\lambda, c_1, s) = \sup_{c_2} L(c_2; \lambda, c_1, s) \tag{4.3.9}$$

with optimal solutions:

$$C_2^{ad*}(\lambda, c_1, s) = \arg\sup_{c_2 \geq 0} L(c_2; \lambda, c_1, s) \tag{4.3.10}$$

and Problem (4.3.7) is simply:

$$V^{ad}(c_1, s) = \inf_{\lambda \geq 0} V^L(\lambda; c_1, s) \tag{4.3.11}$$

with optimal solutions $\Lambda^*(c_1, s)$:

$$\Lambda^*(c_1, s) = \arg_\lambda \inf V^L(\lambda; c_1, s) \tag{4.3.12}$$

Lemmas 16 and 17 below characterize the optimal solutions in (4.3.10) and (4.3.12), respectively.

Lemma 16 *For $p > 0$, under Assumptions 1 and 4.2.1, $C_2^{ad*} : \Omega \times [0, s) \times S \to \mathcal{L}(\mathbf{R}_+)$ is (i) descending in $\lambda \in \Omega$ with all selections antitone, each $(c_1, s) \in [0, s) \times S$; and (ii) ascending in c_1 with all selections isotone, each $(\lambda, s) \in \Omega \times S$.*

Proof. For fixed $(\lambda, c_1, s) \in \Omega \times [0, s) \times S$, consider the parameterized Lagrangian $L_{(\lambda, c_1, s)}$: $\mathbf{R}_+ \mapsto \mathbb{R}_-$. As (i) $(\mathbb{R}_-, *)$ is a posemigroup (with identity element) for $* = +$ (as $*$ is properly increasing), and (ii) the indicator function for any sublattice in $\mathbf{R}$ is super$*$ with $*$=+ properly increasing in this posemigroup (see Veinott [30], Chapter 6, p. 18), then under assumption 4.2.1, the Lagrangian is super$*$ in c_i for $* = +$ and $L_{(\lambda, c_1, s)}(c_2)$ is therefore supermodular in c_2 for each (λ, c_1, s)).Furthermore, it has strict decreasing differences in $(c_2; \lambda)$ for each (c_1, s), and increasing differences in $(c_2; c_1)$ for each (λ, s) (and is a valuation in $(c_2; s)$ for each (λ, c_1)). Therefore, noting assumption 1 (per nonemptiness of the optimal solutions and Berge's theorem), by Topkis' theorem (Topkis [29], Theorem 2.8.3), the correspondence C_2^{ad*} is strong set order descending and subchained in λ for each c_1, and strong set order ascending and subchained in c_1 for each λ. Because of strict decreasing (respectively, increasing) differences, every selection of C_2^{ad*} is (a) antitone in λ for each c_i, and (b) isotone in c_1 for each λ (and independent of s). ■

Lemma 17 *Under Assumptions 1 and 4.2.1, in the Problem (4.3.9), the KKT multipliers* $\Lambda^*(c_1, s)$ *are (a) strong set order ascending in* c_1*, each* $s \in S$ *with* $\wedge\Lambda^*(c_1, s)$ *and* $\vee\Lambda^*(c_1, s)$ *isotone selections in* c_1*, each* $s \in [0, s]$*; (b) strong set order descending in* s*, each* $c_1 \in [0, s]$ *with* $\wedge\Lambda^*(c_1, s)$ *and* $\vee\Lambda^*(c_1, s)$ *antitone selections in* s*, each* $c_1 \in [0, s]$*. Finally, (c) the aggregated primal* $V^{ad}(c_1, s)$ *has increasing differences (and, hence, is supermodular) in* (c_1, s)*.*

Proof. The dual value function $V^L(\lambda, c_1, s)$ is convex in λ for each (c_1, s). It is also submodular in λ (noting that for the posemigroup $(\mathbb{R}_+, *)$ the indicator function of a sublattice is sub$*$ with $*$ properly increasing). By the convexity and submodularity of V^L in λ, the effective domain for V^L is a convex sublattice. By Grinold's version of Danskin's theorem (Grinold [15], Lemma, p. 186), the right directional envelope $V^L_\lambda(\lambda, c_1, s)$ in direction $e^+ > 0$ is:

$$V^L_\lambda(\lambda, c_1, s; e^+) = \min_{c_2 \in C_2^{ad*}} (s - c_1 - pc_2(\lambda, c_1, s)) \cdot e^+$$

and the left-directional in direction $e^- < 0$ is

$$V^L_\lambda(\lambda, c_1, s; e^-) = \max_{c_2 \in C_2^{ad*}} (s - c_1 - pc_2(\lambda, c_1, s)) \cdot e^-$$

As C_2^{ad*} is ascending in c_1 for each s, and independent of s, each c_1, and noting the definition of the (partial) directional derivative $V^L_\lambda(\lambda, c_1, s)$ in any direction e, we conclude $V^L(\lambda, c_1, s)$ has (a) decreasing differences between (λ, c_1), each s, and (b) increasing differences between (λ, s), each c_1. Therefore, (a) and (b) follows directly from Topkis's theorem ([29], Theorem 2.8.3).

To prove (c), by Theorem 7, we have for any direction x

$$\begin{aligned}
V^{ap}_{c_1}(c_1, s; x) &= V^{ad}_{c_1}(c_1, s; x) \\
&= \max_{c_2 \in C_2^{ad*}} \partial_{c_1} L(c_2^d(c_1, s), \lambda^*(c_1, s), c_1, s) \cdot x \\
&= \max_{c_2 \in C_2^{ad*}} \{u_1(c_1; c_2^*(\lambda^*(c_1, s), c_1)) - \lambda^*(c_1, s)\} \cdot x \\
&= \{u_1(c_1; \vee C_2^{ad*}(\wedge\lambda^*(c_1, s), c_1)) - \wedge\Lambda^*(c_1, s)\} \cdot x
\end{aligned}$$

where the last line follows from the fact that $C_2^{ad*}(\lambda, c_1)$ (resp, $\Lambda^*(c_1, s)$) are each subchained-valued with $\vee C_2^{ad*}$ and its associated unique KKT multiplier $\wedge\Lambda^*(c_i, s)$) under SMFCQ and partial concavity of $u(c)$ in c_2, with the third line indicating these solution achieve the max following from $u(c)$ having increasing differences. As $\wedge\Lambda^*(c_1, s))$ is decreasing in s (by the partial concavity of the L in c_2 in the aggregated dual problem), and $\vee C_2^{ad*}(\lambda, c_1)$ is also decreasing in λ (but the submodularity of L in (c_2, λ), but increasing in c_1 (by the supermodularity of L in c), we have $V_{c_1}^{ad}(c_1, s; x)$ increasing in s. Therefore, $V^{ap}(c_1, s)$ has increasing differences in (c_1, s); hence, it is supermodular. ■

Finally, we note that this argument also provides conditions for downward sloping demands. To see this, let $\hat{s} = s - pc_1$, and substitute this definition into the Lagrangian in (4.3.6). Then, using Theorem 7 in Lemma 17, part (c) relative to the parameter p, we have:

$$\begin{aligned} V_p^{ap}(c_1, \hat{s}; x) &= V_p^{ad}(c_1, s - pc_1; x) \\ &= \min_{\lambda^* \in \Lambda^*(c_1, s - pc_1)} \partial_p L(c_2^{ad*}(c_1, s - pc_1), \lambda^*(c_1, s - pc_1), c_1, s - pc_1) \cdot x \\ &= \min_{\lambda^* \in \Lambda^*(c_1, s - pc_1)} \{-p\lambda^*(c_1, s - pc_1)\} \cdot x \\ &= -p \cdot \vee\Lambda^*(c_1, s - pc_1) \cdot x \end{aligned}$$

which is decreasing in c_1 (as the maximal selection $\vee\Lambda^*$ obtains the minimum in the definition of the directional, and $\vee\Lambda^*$ is falling in s and hence increasing in p, and $\vee\Lambda^*$ is increasng in c_1, both by Lemma 17) part (a) and (b)). As the feasible correspondence $\{c_1 | c_1 \in [0, s/p]\}$ strong set order ascending in p when prices are given its *dual* partial order on $\mathbf{R}$ (i.e., higher p implies "lower" p), by Topkis' theorem, $C_1^{ad^*}(s, p)$ is strong set order descending in p, with $\wedge C_1^{ad^*}(s, p)$ and $\vee C_1^{ad^*}(s, p)$ decreasing selections, each s, as desired.

We note that the optimal solutions in this example are not necessarily interior, so the results of Clarke [6] and Milgrom and Segal [19] do not apply. Also, under Assumption 4.2.1 the value functions are not concave in c_1, so the results of Gol'stein [14] and Milgrom and Segal ([19], Theorem 5) do not apply when computing the single crossing property.

5 APPENDIX: Lattice Programming Terminology

5.1 Lattices and Supermodularity

A *partially ordered set* (or Poset) is a set X ordered with a reflexive, transitive, and antisymmetric relation. If any two elements of X are comparable,

X is referred to as a complete partially ordered set, or chain. An upper (resp. lower) bound of $B \subset X$ is an element x^u (resp. x^l) in B such that $\forall x \in B$, $x \leq x^u$ (resp. $x^l \leq x$). A *lattice* is a set X ordered with a reflexive, transitive, and antisymmetric relation $\geq$ such that any two elements x and x' in X have a least upper bound in X, denoted $x \wedge x'$, and a greatest lower bound in X, denoted $x \vee x'$. The product of an arbitrary collection of lattices equipped with the product (coordinatewise) order is a lattice. $B \subset X$ is a *sublattice* of X if it contains the sup and the inf (with respect to X) of any pair of points in B.

Let $(X, \geq_X)$ and $(Y, \geq_Y)$ be Posets. A mapping $f : X \to Y$ is *isotone* (or *increasing*) on X if $f(x') \geq_Y f(x)$, when $x' \geq_X x$, for $x, x' \in X$. A correspondence (or multifunction) $F : X \to 2^Y$ is *ascending* in the set relation on 2^Y denoted by $\geq_S$ if $F(x') \geq_S F(x)$, when $x' \geq_X x$. A particular set relation of interest is Veinott's strong set order (See Veinott [30], Chapter 4). Let $L(Y) = \{A | A \subset Y,\ A \text{ a nonempty sublattice}\}$ be ordered with the *Strong Set Order* $\geq_a$: if $A_1, A_2 \in L(Y)$, we say $A_1 \geq_a A_2$ if $\forall (a, b) \in A_1 \times A_1$, $a \wedge b \in A_2$ and $a \vee b \in A_1$.

Let X be a lattice. A function $f : X \to R$ is *supermodular (resp., strictly supermodular)* in x if $\forall (x, y) \in X^2$, $f(x \vee y) + f(x \wedge y) \geq$(resp., $>$) $f(x) + f(y)$. An important property of the class of supermodular functions is they are closed under pointwise limits. (Topkis, [29], Lemma 2.6.1). Consider a partially ordered set $\Psi = X_1 \times P$ (with order $\geq$), and $B \subset X_1 \times P$. The function $f : B \longrightarrow R$ has *increasing differences* in (x_1, p) if for all $p_1, p_2 \in P$, $p_1 \leq p_2 \Longrightarrow f(x, p_2) - f(x, p_1)$ is non-decreasing in $x \in B_{p_1}$, where B_p is the p section of B. If this difference is strictly increasing in x then f has *strictly increasing differences* on B.

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