Lagrange Multipliers in Convex Programs with Applications to Classical and Nonoptimal Stochastic One-Sector Growth Models*

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Abstract

In convex programs defined on infinite dimensional L_∞ Banach spaces, we present sufficient conditions for the existence of Lagrange multipliers in L_1, thus permitting their interpretation as traditional price systems and generalizing the work of LeVan and Saglam (2004) and Dechert (1982) obtained for programs in l^∞. We emphasize the role of Mackey continuity, give examples in which multipliers fail to be price systems, and show that a simple restriction on the feasible domain in the form of a "strict constraint qualification" guarantees the uniqueness of the Lagrange multiplier in Gateaux differentiable programs.

Through a straightforward application of these findings we prove existence of competitive equilibrium in standard stochastic one sector growth models. We also combine our results with the "monotone method" developed by Coleman (1991) to prove existence of competitive equilibrium in a large class of distorted dynamic representative agent economies.

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1 Introduction

Convex constrained optimization programs are a fundamental tool of economic theory, in particular through their connections with various concepts of equilibrium in static and dynamic economies in which the commodity space is modeled as a Banach space. Economists are therefore very interested by general results concerning existence, characterization and uniqueness of solutions to such programs, and for sufficient conditions for the value function to inherit properties of the objective and/or constraints (such as monotonicity, continuity, and differentiability). The analysis of these programs traditionally begins by deriving sufficient conditions for the existence of Lagrange multipliers, a first step especially critical for economists seeking to interpret multipliers as equilibrium commodity and/or asset prices.[1]

Proofs of existence of Lagrange multipliers, which are positive continuous linear functionals, rely on a straightforward application of a geometric version of the Hahn-Banach Separation Theorem together with a Constraint Qualification (CQ). In finite dimensional space ($\mathbb{R}^n$) Lagrange multipliers are immediately interpretable as a price systems (the valuation function is a dot product). In non-reflexive (thus infinite dimensional) Banach spaces, however, many continuous linear functionals do not have a simple representation qualifying them to be interpreted as price systems. As shown by the literature on commodity prices in infinite dimensional spaces, and in particular the seminal works of Bewley (1972), Lucas and Prescott (1972), and Peleg and Yaari (1970), additional restrictions beyond convexity and a CQ must be imposed to insure that a valuation system can be represented by a price system associated with a dot product representation.

Recognizing that the tools and results developed in finite dimensional spaces do not conclusively extend to infinite dimensional Banach spaces, this paper examines the issue of existence and characterization of Lagrange multipliers in convex constrained programs of the form:

$$\begin{gathered}\min F(x)\\ \phi(x) \leq 0,\; x \in X\end{gathered}$$

where $F : X \to \mathbb{R}$ is a proper convex function, $\phi : X \to X$ a convex constraint operator, and X a non-reflexive Banach space. Denoting by X' the set of norm continuous linear functionals on X (the "norm dual" of X) recall that a Lagrange multiplier is an element $\Lambda^* \geq 0$ of X' satisfying:[2]

$$\langle \Lambda^*, \phi(x^*) \rangle = 0$$

and:

$$F(x) + \langle \Lambda^*, \phi(x) \rangle \geq F(x^*)$$

for all x.

In convex programs for which $X = l^\infty$ (the set of bounded real-valued sequences) LeVan and Saglam (2004) and Dechert (1982) have demonstrated that the Asymptotic Non-Anticipativity of the objective and individual constraints (a condition satisfied under Mackey continuity), together with the Asymptotic Insensitivity of the constraints, are sufficient for Lagrange mul-

[1] Lagrange multipliers are also essential for representing equilibria by functional variational inequalities (see Jofre, Rockafellar, and Wets (2007))

[2] For any $\Lambda \in X'$ and $y \in X$, we write $\Lambda(y) = \langle \Lambda, y \rangle$, and $\Lambda \geq 0$ if and only if $\forall y \geq 0,\ \langle \Lambda, y \rangle \geq 0$.

tipliers to be in l^1_+ and thus to be interpreted as a price system. These findings have led to simple proofs of existence of equilibrium prices in optimal deterministic growth models with time separable or recursive utility.

This paper focuses on the more general case $X = L_\infty$, the Banach space of essentially bounded functions defined on an appropriate measure space. First, we establish in Section 2 general sufficient conditions for a Lagrange multiplier to exists, and for it to be a price system, with a special emphasis on the weak* and Mackey topologies. Such emphasis is, of course, no surprise: The Mackey topology is the finest topology on L_∞ for which the set of continuous linear functionals is precisely L_1, and the weak* and Mackey topologies share the same convex closed sets and therefore the same set of lower semicontinuous convex functions. Norm bounded weak* closed sets are weak* compact by Alaoglu's theorem, and the compactness of the feasible domain together with the lower semicontinuity are then sufficient for solutions to exist. Mackey continuity is closely tied to the concepts of myopicity and impatience (see, for instance Araujo, Novinski and Pascoa (2011) as well as Brown and Lewis (1981)).

In many applications the objective and the constraints often take specific forms for which some of the sufficient conditions of Section 2 are easily or even trivially satisfied. In Section 3 we examine and illustrate two such cases, the first when the objective is an expected value, and the second when the constraint system is Insensitive and Non-Anticipatory. We also give examples of simple environments in which price systems cannot exist and present new findings on uniqueness of price systems in Gateaux differentiable programs providing a version of Robinson's CQ holds.

In Section 4 we apply our findings to prove existence of competitive equilibrium in stochastic versions of the classical one sector growth model in LeVan and Saglam (2004) and Dechert (1982). We then greatly extend this result by also proving existence of a competitive equilibrium in a large class of distorted dynamic economies through a combination of our results with the "monotone method" of Coleman (1991) and Morand and Reffett (2003).

Section 5 provides some basic mathematical results from convex and infinite dimensional analysis, although more details can be found in Barbu and Precupanu (2012) and Aliprantis and Border (1999) for instance.

2 Convex Stochastic Programs

Given a set S, a σ-algebra $\mathcal{F}$ of subsets of S, a non-atomic probability measure P defined on $(S, \mathcal{F})$, and $L_\infty(S, \mathcal{F}, P) = L_\infty$ the Banach space of essentially bounded P-measurable functions, we consider in this paper the convex programs:

$$\begin{gathered} \min F(x) \\ \phi(s, x) \leq 0 \\ x \in L_\infty \end{gathered} \tag{1}$$

with the following restrictions:

(a) The objective $x \in L_\infty \mapsto F(x) \in \mathbb{R} \cup \{+\infty\}$ is convex and bounded on a neighborhood of an interior point of its domain (it is thus norm continuous in the interior of its effective domain).

(b) For any $s \in S$ the constraint function $x \in L_\infty \mapsto \phi(s,x) \in \mathbb{R} \cup \{+\infty\}$ is convex. Note that constraint value $\phi(s,x) \leq 0$ depends on the whole range of decisions $x(s'),\ s' \in S$ (and not exclusively on $x(s)$).

(c) F is assumed to be bounded on a neighborhood of an interior point of its effective domain, and is therefore norm continuous in the interior of its effective domain. Recall that the effective domains of the objective and the constraints are, respectively, the sets $C = \{x \in L_\infty,\ F(x) < +\infty\}$ and $\Gamma = dom(\phi) = \{x \in L_\infty,\ \|\phi(x)\| < +\infty\}$.

(d) Slater's Constraint Qualification (CQ) holds, that is:

Condition 1. $\exists x' \in C$ and $\exists \varepsilon > 0$ such that, for all $s \in S$:

$$\phi(s,x') \leq -\varepsilon$$

2.1 Existence of a Lagrange multiplier

Given a solution x^* to Program (1), the proof of existence of a Lagrange multiplier associated with x^* (a positive element of the norm dual of L_∞ which we call a "valuation" in reference to Debreu (1954)) relies on a geometric version of the Hahn-Banach separation theorem exploiting the convexity of the program and the non-emptiness of the the (norm) interior of L_∞^+ combined with Slater's CQ.

Theorem 1. *Under Condition 1, any x^* solution to the convex program (1) has an associated valuation.*

Proof. We follow the proof of Theorem 1 in LeVan and Saglam (2004). Given any solution x*, consider the set:

$$Z = \{(\rho, z) \in \mathbb{R} \times \mathrm{L}_\infty; \exists x \in \mathrm{L}_\infty, \rho > F(x) - F(x^*),\ z(s) > \phi(s,x), a.e.\}$$

Clearly $Z \cap (\mathbb{R}_- \times \mathrm{L}_\infty^-) = \emptyset$, Z is non-empty and convex, and L_∞^- has non-empty interior. By the Hahn-Banach separation Theorem there exists (c, Λ) in $\mathbb{R} \times \mathrm{L}_\infty' \setminus \{0\}$ (a Fritz-Jones multiplier) such that $\forall (\rho, z) \in Z$ and $\forall (\rho', z') \in \mathbb{R}_- \times \mathrm{L}_\infty^-$:

$$c\rho' + \langle \Lambda, z' \rangle \leq c\rho + \langle \Lambda, z \rangle$$

In particular, $c\rho + \Lambda z \geq 0$ for all (ρ, z) in Z (since $(0,0) \in \mathbb{R}_- \times \mathrm{L}_\infty^-$).

Assuming that $c < 0$, letting ρ grow large with z fixed leads to a contradiction, hence $c \geq 0$. Letting $x = x^*$ in the definition of Z allows choosing ρ arbitrarily small, which implies that $\Lambda \geq 0$.

Next, Slater's CQ is used to prove that (c, Λ) is a proper Lagrange multiplier. Indeed suppose that $c = 0$ and $\Lambda > 0$; by Slater's CQ, $(\rho, .5\phi(x'))$ belongs to Z (letting $x = x'$ in the definition of Z, and considering ρ large enough) since $\phi(x') < .5\phi(x') \leq -\varepsilon/2$, hence $\langle \Lambda, .5\phi(x') \rangle \geq 0$ but this contradicts $\Lambda > 0$. Thus $c > 0$, and $\Lambda^* = \frac{\Lambda}{c}$ satisfies the theorem. Clearly $\forall \Lambda \geq 0$ $\langle \Lambda^*, \phi(x^*) \rangle (= 0) \geq \langle \Lambda, \phi(x^*) \rangle$ (since $\phi(x^*) \leq 0$), and thus:

$$F(x) + \langle \Lambda^*, \phi(x) \rangle \geq F(x^*) + \langle \Lambda^*, \phi(x^*) \rangle \geq F(x^*) + \langle \Lambda, \phi(x^*) \rangle$$

for all $x \in L_\infty$ and all $\Lambda \geq 0$. □

Remark. A valuation Λ^* satisfies:

$$(F(x) + \langle \Lambda^*, \phi(x) \rangle) - (F(x^*) + \langle \Lambda^*, \phi(x^*) \rangle) \geq 0$$

for all $x \in L_\infty$, and therefore requires that $0 \in \partial \{F(x^*) + \langle \Lambda^*, \phi(x^*) \rangle\}$. If $x^* \in int(C)$, then the convex function F is continuous at x^* and $\partial(F(x^*) + \langle \Lambda^*, \phi(x^*) \rangle) = \partial F(x^*) + \partial \langle \Lambda^*, \phi(x^*) \rangle$ (see Ekeland and Temam (1999)) hence there exists $(\rho, \mu) \in \partial F(x^*) \times \partial \langle \Lambda^*, \phi(x^*) \rangle$ such that $\rho + \mu = 0$.

2.2 Existence of a price system

The norm dual of $L_\infty(S, \mathcal{F}, P)$ can be identified with $ba(S, \mathcal{F}, P)$, the set of bounded additive set functions ("finitely additive measures" in Yosida and Hewitt (1956)) that are absolutely continuous with respect to P. Furthermore, any finitely additive measure can be uniquely decomposed as the sum of a countably additive part and a purely finitely additive part. Our approach is to impose additional restrictions (beyond convexity and the standard CQ) on objective and constraints sufficient for the the purely finitely additive part of a valuation to be zero thus making it a price system.

Technically, these restrictions are expressed in terms of asymptotic properties of the primitive data along specific sequences constructed from arbitrarily small sets of elements A_n of $\mathcal{F}$ satisfying $\{A_n\} \downarrow 0$ (i.e., $A_{n+1} \subset A_n$ and $\lim_{n\to\infty} P(A_n) = 0$). Intuitively, they require that objective and constraints be largely unaffected by changes in the decision variable on arbitrarily small sets. Under these restrictions no value will be associated with commodities defined on such sets, a property which basically eliminates the purely finitely additive part of any valuation.

In the following assumption, given a set $B \in \mathcal{F}$, $\chi_B(s) = 1$ if $s \in B$ and $\chi_B(s) = 0$ if $s \in S \setminus B$.

Assumption 1. *Given* $\{A_n\} \downarrow 0$ *in* $\mathcal{F}$ *and any* $x, y \in L_\infty$ *define* $x^n = \chi_{A_n} y + (1 - \chi_{A_n}) x$. *Assume that:*

1(i). *If* $x, y \in C \cap \Gamma$ *then* $\exists N$, $\forall n > N$ $x^n \in C \cap \Gamma$.

1(ii). $\lim_{n\to\infty} F(x^n) = F(x)$.

1(iii). $\forall N' > N$ $\lim_{n\to\infty} \phi(s, x^n) = \phi(s, x)$ *on* $S \setminus A_{N'}$;

1(iv). $\exists M > 0$ *and* $\exists N \in \mathbb{N}$ *such that* $\forall n > N$, $\|\phi(x^n)\| \leq M$;

1(v). $\forall n > N$, $\forall \varepsilon > 0$, $\exists M > N : |\phi(s, x^n) - \phi(s, y)| \leq \varepsilon$ *on* A_M *(thus on* A_p *for* $p > M$*).*

Assumption 1(ii) is satisfied if the objective is Mackey continuous at $x \in C \cap \Gamma$; 1(iii) requires the constraint system to be essentially Asymptotically Non-Anticipatory (ANA), and is clearly satisfied if, for each s, $\phi(s, .)$ is weak* continuous; 1(iv), in combination with 1(i), can be viewed as strong version of the Exclusion Assumption in Bewley (1972).

Assumption 1(v) restricts how constraint values can be affected by changes in the choice variable: It requires that whenever s belongs to the arbitrarily small set A_p, almost surely the constraint value $\phi(s, x)$ may depend on choices $x(s)$ for $s \in A_n$ but is largely independent of any $x(s)$ for $s \notin A_n$. It is a form of Asymptotic Insensitivity (AI) which is satisfied in a large class of models, and it implies that $\forall n > N$, $\forall \varepsilon > 0$, $\exists q > 0 : \forall p \geq q$ $\left\|\chi_{A_p}\left(\phi(x^n) - \phi(y)\right)\right\| \leq \varepsilon$, hence the sequence $\{\chi_{A_p}\left(\phi(x^n) - \phi(y)\right)\}_{p=1}^\infty$ norm converges to 0.

2.2.1 Main existence result

Theorem 2. *Under Assumption 1 and Condition 1, any solution x^* to the convex program (1) has an associated price system, and all valuations associated with x^* are price systems.*

Proof. By Theorem 1 $\exists \Lambda^* \geq 0$ satisfying, for all $x \in L_\infty$:

$$F(x) + \langle \Lambda^*, \phi(x) \rangle \geq F(x^*)$$

By Yosida-Hewitt "decomposition" Theorems (see Appendix Section 5.4) the finitely additive measure Λ^* may be uniquely written as $\Lambda_p^* + \Lambda_c^*$ (with $\Lambda_p^* \geq 0$ and $\Lambda_c^* \geq 0$), and there exists $\{A_n\} \downarrow 0$ such that $\lim \Lambda_c(A_n) = 0$ and $\Lambda_p(S \setminus A_n) = 0$. As a result, defining $x^n = \chi_{A_n} x' + (1 - \chi_{A_n}) x^*$ (where x' satisfies Slater's CQ) as in Assumption 1, for all n:

$$F(x^n) + \langle \Lambda_c^*, \phi(x^n) \rangle + \left\langle \Lambda_p^*, \phi(x^n) \right\rangle \geq F(x^*) \tag{2}$$

in which all terms are well defined by Assumption 1(i). Taking limits in this expression leads to:

$$F(x^*) + \lim_{n \to \infty} \langle \Lambda_c^*, \phi(x^n) \rangle + \lim_{n \to \infty} \left\langle \Lambda_p^*, \phi(x^n) \right\rangle \geq F(x^*) \tag{3}$$

since the sequence $\{x^n\}$ Mackey converges to x^* and $\lim F(x^n) = F(x^*)$ by Assumption 2(i).

By Assumption 1(iii) on $S \setminus A_{N'}$:

$$\lim_{n \to \infty} \phi(s, x^n) = \phi(s, x^*)$$

hence, since $\Lambda_c \in L_1$, for all s in $S \setminus A_{N'}$:

$$\lim_{n \to \infty} \Lambda_c(s) \phi(s, x^n) = \Lambda_c(s) \phi(s, x^*)$$

Next, by Assumption 1(iv), whenever $n > N$, for all s in $S \setminus A_{N'}$:

$$|\Lambda_c(s) \phi(s, x^n)| \leq \|\Lambda_c\| \, M$$

and the conditions are therefore in place for an application of the Lebesgue Dominated Convergence Theorem (see, for instance, Royden (2010)), hence:

$$\lim_{n \to \infty} \int_{S \setminus B_{N'}} \Lambda_c(s) \left[\phi(s, x^n) - \phi(s, x^*) \right] P(ds) = 0$$

Given that $\lim_{N' \to \infty} \Lambda_c(B_{N'}) = 0$ and that $|\phi(s, x^n) - \phi(s, x^*)| \leq \|\phi(x^*)\| + M$:

$$\lim_{n \to \infty} \int_{S} \Lambda_c(s) \left[\phi(s, x^n) - \phi(s, x^*) \right] P(ds) = 0$$

that is, $\lim_{n \to \infty} \langle \Lambda_c, \phi(x^n) - \phi(x^*) \rangle = 0$.

By Assumption 1(v) the sequence $\{\chi_{A_k} \left(\phi(x^n) - \phi(x') \right)\}_{k=1}^\infty$ norm converges to 0 while Λ_p^* is norm continuous and linear. As a result, the sequence $\{\left\langle \Lambda_p^*, \chi_{A_k} \left(\phi(x^n) - \phi(x') \right) \right\rangle\}_{k=1}^\infty$ converges to 0 in $\mathbb{R}$. Recall that Λ_p has all its mass in the arbitrarily small set A_k thus, for all k:

$$\left\langle \Lambda_p^*, \chi_{A_k} \left(\phi(x^n) - \phi(x') \right) \right\rangle = \left\langle \Lambda_p^*, \phi(x^n) - \phi(x') \right\rangle$$

hence $\left\langle \Lambda_p^*, \phi(x^n) \right\rangle = \left\langle \Lambda_p^*, \phi(x') \right\rangle$.

As a result, inequality (3) becomes:

$$\left\langle \Lambda_p^*, \phi(x') \right\rangle + \langle \Lambda_c^*, \phi(x^*) \rangle \geq 0$$

Recall that $\langle \Lambda^*, \phi(x^*) \rangle = \langle \Lambda_c^*, \phi(x^*) \rangle + \left\langle \Lambda_p^*, \phi(x^*) \right\rangle = 0$, which implies that $\langle \Lambda_c^*, \phi(x^*) \rangle = 0$, since $\Lambda_c^* \geq 0$, $\Lambda_p^* \geq 0$ and $\phi(x^*) \leq 0$. As a result, the above inequality together with $\phi(x') < 0$ imply that $\Lambda_p^* = 0$. All valuations therefore must be price systems. □

Remark. It is clear from the proof above that the asymptotic properties of F and ϕ in Assumption 1 need only be verified for sequences $\{x^n = \chi_{A_n} x' + (1 - \chi_{A_n}) x^*\}$.

Remark. When $x^* \in int(C)$ (which insures the non-emptiness of $\partial F(x^*)$) an alternative to Assumption 1(ii) is $\partial F(x^*) \subset L_1$. Indeed, as noted above, if $x^* \in int(C)$, the existence of a valuation implies that $\rho + \mu = 0$ for some $(\rho, \mu) \in \partial F(x^*) \times \partial \langle \Lambda^*, \phi(x^*) \rangle$, thus, by convexity of $x \longmapsto \langle \Lambda^*, \phi(x) \rangle$:

$$\langle \Lambda^*, \phi(x^n) \rangle - \langle \Lambda^*, \phi(x^*) \rangle \geq \mu(x^n - x^*) = -\rho(x^n - x^*)$$

But $\lim_{n \to \infty} \rho(x^n - x^*) = 0$ whenever $\rho \in L_1$, and we arrive again at inequality 3 in the proof of Theorem 2, and the rest of that proof follows hence $\Lambda_p^* = 0$.

We show next that, on $int(C)$, the assumption $\partial F(x) \subset L_1$ is more general (less restrictive) than Mackey continuity, a result therefore extending Lemma 1 in Araujo, Novinski and Pascoa (2011) to L_∞.[3]

Lemma 1. *If $x \in int(C)$ and F is decreasing, then the Mackey continuity of F at x implies $\partial F(x) \subset L_1$, which itself implies the Mackey lower semicontinuity of F at x.*

Proof. F is subdifferentiable at any $x \in int(C)$ and decreasing, therefore for any $\lambda \in \partial F(x)$ there exists in some small open ball centered on x such that for all $y \geq 0$ in this ball:

$$0 \geq F(x + y) - F(x) \geq \langle \lambda, y \rangle$$

hence $\lambda \leq 0$. By a now standard argument λ may be uniquely written as $\lambda_p + \lambda_c$ (with $\lambda_p^* \leq 0$ and $\lambda_c^* \leq 0$), and there exists $\{A_n\} \downarrow 0$ such that $\lim \lambda_c(A_n) = 0$ and $\lambda_p(S \setminus A_n) = 0$. Note that $\{A_n\} \downarrow 0$ implies that the sequence $\{x^n = x + \chi_{A_n} y\}$ Mackey (and thus weak*) converges to x.

At $x \in int(C)$, by definition of $\partial F(x)$:

$$F(x^n) - F(x) \geq \langle \lambda_p, x^n - x \rangle + \langle \lambda_c, x^n - x \rangle$$

The weak* convergence of $\{x^n\}$ to x implies that $\lim_{n \to \infty} \langle \lambda_c, x^n - x \rangle = 0$. By definition of x^n, $\langle \lambda_p, x^n - x \rangle = \langle \lambda_p, \chi_{A_n} y \rangle$ and by Mackey continuity of F at x, $\lim_{n \to \infty} F(x^n) = F(x)$, hence the above inequality implies that:

$$\forall y \in L_\infty, \lim_{n \to \infty} \langle \lambda_p, \chi_{B_n} y \rangle \leq 0$$

As a result, $\lambda_p = 0$ and $\partial F(x) \subset L_1$.

[3] As correctly noted by LeVan and Saglam (2004) (Remark 2, page 403),$\partial F(x^*) \subset l^1$ (an hypothesis made in Dechert (1982)) is weaker than Mackey continuity when $x^* \in int(C)$.

Next, if $\delta \in \partial F(x) \subset L_1$ and if $\{x^n\}$ Mackey converges (thus weak* converges) to x, then $\lim \langle \delta, x^n - x \rangle = 0$. By definition, for such a sequence:

$$F(x^n) \geq F(x) + \langle \delta, x^n - x \rangle$$

which implies that $\liminf_{n \to +\infty} F(x^n) \geq F(x)$, hence F is Mackey/weak* lower semicontinuous at x. □

3 Special Cases and Examples

The objective and constraint system of Program (1) in many cases take a simpler form for which the hypothesis of Theorem 2 are easily satisfied. This is the case, for instance, in models of decision making under uncertainty where the objective is an expected value (e.g. expected utility, or expected profits) or where the constraint system takes a much simpler form than $\varphi(s, x) \leq 0$. We examine these two cases below.

It is also quite common for the convex functions F and ϕ to have differentiable properties beyond their subdifferentiability; we prove in this section that a simple constraint qualification is sufficient to guarantee the uniqueness of price systems in Gateaux differentiable programs.

3.1 Expected values and Mackey semicontinuity of the objective

Programs in which objectives take the following form:

$$F(x) = \int_S f(s, x(s)) P(ds)$$

are standard in economics, and sufficient conditions for F to be Mackey continuous have been clearly spelled out by Bewley in his seminal work on prices in infinite dimensional commodity spaces (Bewley (1972)).

On the interior of the effective domain of F the hypothesis $\partial F(x^*) \subset L_1$ is more general than that of Mackey continuity, so it is natural to search for reasonable conditions under which that hypothesis is satisfied. In doing so, we first prove that under a weaker set of conditions than those of Bewley (1972) (they allow f to take the value $+\infty$ and do not require f to be monotone), F is Mackey lower semicontinuous.[4]

Proposition 1. *Suppose that (i)* $\forall s \in S$, $u \in \mathbb{R} \longmapsto f(s, u) \in \mathbb{R} \cup \{+\infty\}$ *is proper convex and lower semicontinuous and has interior points in its effective domain* $\{u \in \mathbb{R},\ f(s, u) < +\infty\}$*, and (ii)* $\forall u \in \mathbb{R}$, $s \in S \longmapsto f(s, u)$ *is Borel measurable and* $f(s, u)$ *is integrable on* S*. Then* $x \in L_\infty \mapsto F(x) \in \overline{\mathbb{R}}$ *is weak* lower semicontinuous (equivalently, Mackey lsc).*

Proof. f is a normal convex integrand on $S \times \mathbb{R}$ by Lemma 2 in Rockafellar and Wets (1968). Since $f(s, u)$ is integrable on S for every $u \in \mathbb{R}$, by Corollary 2A in Rockafellar and Wets (1971) F and F^* (its conjugate) are well-defined (on L_∞ and L_1, respectively) and conjugate to each other. But if F is the conjugate of F^*, then for any $\mu \in \mathbb{R}$ and any sequence $\{x_n\}$ in L_∞ weak*

[4] Equivalently, for a concave maximization problem under appropriate corresponding conditions, F would be Mackey upper semicontinuous.

converging to $x \in L_\infty$, by definition of the conjugate:

$$F(x_n) = \sup\{\langle x_n, u^* \rangle - F^*(u^*),\, u^* \in L_1\}$$

and, therefore:

$$F(x_n) \leq \mu \Rightarrow \forall u^* \in L_1,\, \langle x_n, u^* \rangle - F^*(u^*) \leq \mu$$

By weak* convergence of x_n to x, for all u^* in L_1, $\langle x_n, u^* \rangle \to \langle x, u^* \rangle$ hence $\langle x, u^* \rangle - F^*(u^*) \leq \mu$ and:

$$F(x) = \sup\{\langle x, u^* \rangle - F^*(u^*),\, u^* \in L_1\} \leq \mu$$

The set $\{z,\, F(z) \leq \mu\}$ is thus weak* closed, and the convex function F is weak* (thus Mackey) lower semicontinuous. It is then norm lower semicontinuous and therefore norm continuous on the interior of its domain (see Appendix Section 5.1). □

The Mackey/weak* lower semicontinuity of the objective in a minimization program is an important property; when combined with the Mackey or weak* compactness of the feasible domain it insures the existence of solutions.

Next, it is only a matter of adding a simple condition on the integrability of f to prove that $\partial F(x^*) \subset L_1$.

Proposition 2. *Under the assumptions of Proposition 1 suppose that there exist $r > 0$ such that $f(s, x^*(s) + u)$ is integrable whenever $|u| < r$, $u \in \mathbb{R}$. Then $\partial F(x^*)$ is a non empty weakly (i.e. $\sigma(L_1, L_\infty)$) compact subset of L_1.*

Proof. See Corollary 2C to Theorem 2 in Rockafellar and Wets (1971). □

Alternatively, we could seek conditions under which $\partial F(x^*) \cap L_1 = \emptyset$, so that no price system could exist since the countably additive part of any valuation would be zero.

Rather than pursuing this technical task, we present simple but non-trivial objectives of *maximization* programs who fail to be Mackey lower semicontinuous, while still being upper semicontinuous. Failure to be Mackey lsc has been tied by Araujo, Novinski and Pascoa (2011) to the concept of wariness, the willingness to neglect gains but not losses on arbitrarily small sets of events, as illustrated in the next two examples.

Example 1. Consider a consumer who worries about the worst possible outcomes but willing to disregard events of measure 0. In the measure space $(S, \mathcal{F}, P)$ with $S = [0.1]$, $\mathcal{F}$ the Borel algebra, and P the Lebesgue measure, the utility function U defined on $L^+_\infty(S, \mathcal{F}, P)$, is:

$$U(x) = \sup_{A \in \mathcal{F}}\{\inf_{s \in A} x(s);\, A \in \mathcal{F},\, P(A) = 1\}$$

captures this behavioral trait. For $n = 1, ...$ let $A_n = [0, \frac{1}{n}]$ so that for any $x \in int(L^+_\infty)$, the sequence $\{x_n = x(1 - \chi_{A_n})\}_{n=1}^\infty$ Mackey converges to x (as $\lim_{n\to\infty} P(A_n) = 0$). However $\forall n \geq 1$, $U(x_n) = 0 < U(x)$, hence U cannot be Mackey lsc at x.

Example 2. Consider the utility function representing the preferences of an "uncertainty averse" consumer, as in Gilboa and Schmeider (1989). Specifically, the decision maker's "vague" beliefs about the likelihood of states are assumed to be represented by a set P of finitely additive probability measures (with $card(P) > 1$), and its aversion to uncertainty by the utility function:

$$U(x) = \inf_{m \in P} \left\{ \int u(x(s))m(ds) \right\}$$

in which $u : \mathbb{R}^+ \to \mathbb{R}$ is continuous, increasing, and concave, and P is a convex and weakly closed set. Function U is Mackey usc, but, as shown by Epstein and Wang (1995) (Theorem 2.1) it is Mackey lsc if and only P is continuous at certainty, that is:

$$\inf\{m(A_n);\ m \in P\} \uparrow 1$$

whenever $A_n \uparrow S$, $A_n \in \mathcal{F}$.

3.2 Simple constraint systems

In its most general form, our result on existence of price systems requires the constraint system $(s, x) \in S \times L_\infty \to \phi(s, x) \in \mathbb{R}$ to be both essentially ANA and AI.

It is easy to see that constraints systems in programs:

$$\begin{aligned} &\min F(x) \\ &\phi(s, x(s)) \leq 0 \\ &x \in L_\infty \end{aligned} \tag{4}$$

satisfy these requirements, so that no additional assumption beyond the Mackey continuity of the objective is required for a valuation to be a price system. Indeed in these programs, in any given state s' the constraint $\phi(s', x(s'))$ imposes restrictions only on the decision $x(s')$ and not on $x(s)$ for $s \neq s'$; it can therefore be called "fully" Insensitive and "fully" Non-Anticipatory.

Proposition 3. *Under Assumptions 1(i), 1(ii), and 1(iv) and Condition 1, for any solution $x^*$to the convex program (4) there exists a price system on L_∞ associated with x^*, and all valuations must be price systems.*

Proof. The constraint system satisfies Assumption 1(iii) and 1(v). Indeed, if $x^n = \chi_{A_n} x' + (1 - \chi_{A_n})x^*$ in which $\{A_n\} \downarrow 0$, then $\forall M > n\ s \in A_M \Rightarrow s \in A_n$ thus $x^n(s) = x'(s)$ and $\phi(s, x^n) = \phi(s, x^n(s)) = \phi(s, x'(s)) = \phi(s, x')$ on A_M hence Assumption 1(v) is satisfied. On the other hand, for any $N' > N\ s \in S \setminus A_{N'} \Rightarrow \forall n > N',\ s \in S \setminus A_n$ thus $\phi(s, x^n) = \phi(s, x^n(s)) = \phi(s, x^*(s)) = \phi(s, x^*)$ and 1(iii) is satisfied.

The proof of Theorem 2 follows easily since $\forall n > N'$, $\langle \Lambda_c, \chi_{S \setminus A_n} \phi(x^n) \rangle = \langle \Lambda_c, \chi_{S \setminus A_n} \phi(x^*) \rangle$ while $P(A_n) \to 0$, so that $\langle \Lambda_c, \phi(x^n) \rangle = \langle \Lambda_c, \phi(x^*) \rangle$, and $\forall n$, $\langle \Lambda_p, \phi(x^n) \rangle = \langle \Lambda_p, \phi(x') \rangle$. Slater's condition requires that $\phi(x') < 0$ but $\langle \Lambda_c, \phi(x^*) \rangle = 0$ and:

$$\langle \Lambda_p, \phi(x') \rangle + \langle \Lambda_c, \phi(x^*) \rangle \geq 0$$

hence $\Lambda_p = 0$. □

3.3 Static Examples

We provide two illustrations of our results in static economies. The first is the exchange economy of Bewley (1972), and the second a simple model of optimal portfolio selection.

Example 3. Consider an exchange economy populated by I consumers with concave and strictly increasing utility functions, $U_i : L_\infty^+ \to \overline{\mathbb{R}}$, $i = 1, ..., I$. Consumer i is endowed with $w_i \in L_\infty^+$ and purchases the "contingent commodity" x_i. A valuation equilibrium is as an element $x^* = (x_1^*, ..., x_N^*)$, $x_i^* \in L_\infty$ $i = 1, ..., I$ together with an element $\pi \geq 0$ of $(L_\infty)'$ with $\pi \neq 0$, such that for each $i = 1, ..., N$, x_i solves:

$$\begin{gathered} \min -U_i(x_i) \\ \langle \pi, x_i - w_i \rangle \leq 0 \, and \, x_i \in L_\infty^+ \end{gathered} \tag{5}$$

with the market clearing condition $\sum_{i=1}^N x_i \leq w = \sum_{i=1}^N w_i$ also holding true. Note that the Mackey/weak* lower semicontininuity of the convex functions $-U_i$ is a sufficient sufficient for the existence of a solution to program (5), given any $\pi \geq 0$ and $w \ggg 0$[5].

Proposition. *Suppose that all U_i are Mackey continuous and strictly increasing, and that $w \ggg 0$. Then if (x^*, π) is an equilibrium with $x_i^* \in int(L_\infty^+)$ for all i, necessarilyπ must be in L_1^+.*

Proof. Although the proof follows easily from the application of Proposition 3 to the minimization problem of agent i, it nevertheless is interesting to have a close look at it. The condition $w_i \gg 0$ implies that Slater's CQ is satisfied, hence there exists a Lagrange multiplier vector $(\lambda_i, \mu_i) \in \mathbb{R}^+ \times (L_\infty)'$ and $\mu_i \geq 0$ satisfying:

$$0 \in \partial\{-U_i(x_i^*) + \lambda_i \langle \pi, x_i^* - w_i \rangle + \left\langle \mu_i, \sum_{j=1}^N x_j^* - w \right\rangle\} + N_{L_\infty^+}(x_i^*)$$

Assuming that $x_i^* \in int(L_\infty^+)$ implies that the set $N_{L_\infty^+}(x_i^*)$ is empty so that $\lambda_i \pi + \mu_i \in \partial U_i(x_i^*)$.

If all the λ_i are 0, then $\pi = 0$ is a price system associated with the equilibrium allocation $\{x_i^*\}_{i=1}^I$, in which case $\mu_i \in \partial U_i(x_i^*)$ and the strict monotonicity of U_i implies that $\mu_i \neq 0$ hence $\sum_{j=1}^N x_j^* - w = 0$. Alternatively, there exists some $\lambda_j \neq 0$ and the Mackey continuity of U_j implies by Lemma 1 that $\partial U_j(x_j^*) \subset L_1^+$. As a result $\pi \in L_1$, a result familiar to readers of Bewley (1972). □

Thus, for the price system π not to be in L_1 while $x_j^* \in int(L_\infty^+)$, necessarily $-U_j$ must fail to be usc at x_j^*. Additionally, for the price not to be in L_1 when U_j is Mackey continuous, x_j^* must be on the boundary of L_+^∞.

Example 4. Consider a representative investor with an initial portfolio holding $A \in \mathbb{R}^J$ of J different assets, with p_j denoting the price of asset j. Seeking to maximize utility derived exclusively from asset returns, the agent has the opportunity to purchase a new portfolio θ. Asset returns are completely described by the linear return function $R : L_\infty \to \mathbb{R}$; portfolio θ thus returns $(R.\theta)(s) = \sum_{i=1}^J \theta_i R_i(s)$ in state s.

The investor's problem is therefore:

$$\begin{gathered} \min -U(R.\theta) \\ \langle P, \theta - A \rangle \leq 0 \\ -(R.\theta)(s) \leq 0 \, \text{almost surely} \end{gathered} \tag{6}$$

[5] $w \ggg 0$ if there exists $r > 0$ such that $w(s) > r$ almost surely

An arbitrage is a portfolio θ such that $\langle P, \theta \rangle < 0$ and $(R.\theta)(s) \geq 0$ almost surely, or $\langle P, \theta \rangle \leq 0$ and $(R.\theta)(s) > 0$ almost surely.

Lemma. *There is no arbitrage iff there exists $\pi \in (L_\infty)'$ such that $\langle P, \theta \rangle = \langle \pi, R.\theta \rangle$ for all $\theta \in \mathbb{R}^J$.*

Proof. Consider the set $M = \{(-P\theta, R.\theta) : \theta \in \mathbb{R}^J\} \subset \mathbb{R} \times L_\infty$ and the cone $K = \mathbb{R}^+ \times L_\infty^+$. There is no arbitrage is and only if $K \cap M = \{0\}$. Suppose that $K \cap M = \{0\}$ then the Hahn Banach separation theorem implies the existence of a linear function $F : \mathbb{R} \times L_\infty \to \mathbb{R}$ such that $F(\theta) < F(\alpha)$ for all $\theta \in M$ and all non zero $\alpha \in K$. In particular this implies that $F(\alpha) > 0$ for all non zero $\alpha \in K$. The existence of a linear functional F implies that there exists some $c > 0$ in $\mathbb{R}$ and some $\pi' \in (L_\infty)'$, $\pi > 0$, such that $F(x, y) = cx + \langle \pi', y \rangle$ for any $(x, y) \in \mathbb{R} \times L_\infty$. As a result:

$$-cP\theta + \langle \pi', R.\theta \rangle = 0$$

for all $\theta \in \mathbb{R}^J$. By setting $\pi = \frac{\pi'}{c}$ we just established the existence of $\pi \in (L_\infty)'$, with $\pi > 0$, such that $\langle P, \theta \rangle = \langle \pi, R.\theta \rangle$. Conversely, the existence of such π easily implies no arbitrage.

When there is no arbitrage:

$$\langle P, \theta - A \rangle = \langle \pi, R.\theta - R.A \rangle$$

hence if $x^* = R.\theta^*$ solves program (6), then, necessarily x^* solves:

$$\min -U(x)$$

subject to:

$$\langle \pi, x - R.A \rangle \leq 0$$

When markets are complete, i.e, $R.L_\infty = L_\infty$, and $R.A > 0$ almost surely, Slater's condition is satisfied hence by Theorem 1 there exists $\Lambda^* \in \mathbb{R}^+$ with $\Lambda^* > 0$ such that:

$$0 \in \partial\{-U(x^*) + \Lambda^* \langle \pi, x^* - R.A \rangle\}$$

Hence $\Lambda^*\pi \in \partial U(x^*)$ therefore $\pi \in L_1$ whenever $\partial U(x^*) \subset L_1$. □

Remark. For objective of the form $U(x) = \int_S u(x(s))P(ds)$ where $u : \mathbb{R} \to \mathbb{R}$ is increasing, concave and norm usc (U is then weak* and Mackey usc) $\partial U(x^*)$ is indeed a non empty subset of L_1 by Proposition 2.

3.4 Differentiable programs and uniqueness of the multiplier

Convex programs in which both functions $x \in L_\infty \mapsto f(x) = f(., x(.)) \in L_\infty$ and $x \in L_\infty \longmapsto \phi(x) = \phi(., x) \in L_\infty$ are (at least) Gateaux differentiable have additional special properties. In particular, we show next that each interior solution of such programs is associated with a unique price system whenever a simple constraint qualification is met, a result which has important implications for establishing the uniqueness of competitive equilibrium in many economic models.

First, note that if the convex function ϕ is Gateaux differentiable, then it is subdifferentiable regular (see Appendix, Section 5.2), that is if $x^* \in int(\Gamma)$:

$$\partial\left(\langle \Lambda^*, \phi(x^*)\rangle\right) = \langle \Lambda^*, D\phi(x^*)\rangle$$

hence $card\{\partial\left(\langle \Lambda^*, \phi(x^*)\rangle\right)\} = 1$. Second, since F is finite and norm continuous on the interior of its effective domain, if $x^* \in int(C)$ then:

$$\partial F(x^*) = \partial \int f(s, x^*(s))dP(s) = \int \partial f(s, x^*(s))dP(s)$$

(See Rockafellar and Wets (1982)), hence the Gateaux differentiability of f implies that $Card(\partial F(x^*)) = 1$.[6]

Next, recall that a standard constraint qualification for differentiable programs is Robinson's Constraint Qualification (RCQ), satisfied at $x^* \in C \cap \Gamma$ if:

$$0 \in int\{\phi(x^*) + \langle D\phi(x^*), L_\infty\rangle + L_\infty^+\}$$

We will assume, however, that the following stronger condition holds:

$$0 \in int\{\phi(x^*) + \langle D\phi(x^*), L_\infty\rangle + A_{\Lambda^*}\} \tag{7}$$

where $A_{\Lambda^*} = \{k \in L_\infty^+,\ \langle \Lambda^*, k\rangle = 0\}$. This condition, which implies RCQ, has been tied to the uniqueness of the multiplier in both finite and infinite dimensional programs respectively by Kyparisis (1985) and by Bonnans and Shapiro (2000), and named in both papers the Strict (Mangasarian) Constraint Qualification (SCQ).

Theorem 3. *If $x^* \in int(C \cap \Gamma)$ is a solution of Program (1), if ϕ and F are Gateaux differentiable and if condition (7) holds, then there is at most one price system.*

Proof. Suppose there are two multipliers $\Lambda_i^*\ i = 1, 2$ associated with the solution $x^* \in int(C)$, in which case, for $i = 1, 2$:

$$0 \in \partial F(x^*) + \partial \langle \Lambda_i^*, \phi(x^*)\rangle)$$

As shown above both subdifferential sets are singletons therefore:

$$\langle \Lambda_1^* - \Lambda_2^*, D\phi(x^*)\rangle = 0$$

By the constraint qualification (7) above, $\forall y \in X$, there exists $\varepsilon > 0$, $x \in X$ and $k \in A_{\Lambda_1^*}$ such that $\varepsilon y = \phi(x^*) + \langle D\phi(x^*), x\rangle + k$. Thus:

$$\begin{aligned}\langle \Lambda_1^* - \Lambda_2^*, \varepsilon y\rangle &= \langle \Lambda_1^* - \Lambda_2^*, \langle D\phi(x^*), x\rangle\rangle + \langle \Lambda_1^* - \Lambda_2^*, \phi(x^*) + k\rangle \\ &= \langle \Lambda_1^* - \Lambda_2^*, \phi(x^*) + k\rangle\end{aligned}$$

But $\langle \Lambda_2^* - \Lambda_1^*, \phi(x^*)\rangle = 0$ (both Λ_1^*, Λ_2^* are multipliers) and $\langle \Lambda_1^* - \Lambda_2^*, k\rangle = \langle -\Lambda_2^*, k\rangle$ (by definition of $A_{\Lambda_1^*}$, $\langle \Lambda_1^*, k\rangle = 0$), hence:

$$\langle \Lambda_1^* - \Lambda_2^*, \varepsilon y\rangle = \langle -\Lambda_2^*, k\rangle \leq 0$$

[6] Alternative conditions for the interchangeability of the integration and subdifferentiation operations when $x^* \notin int(C)$ are discussed in Rockafellar and Wets (1982).

because $\Lambda_2^* \geq 0$ and $k \geq 0$. But y was arbitrary, thus $\Lambda_1^* - \Lambda_2^* = 0$ and the Lagrange multiplier must therefore be unique. □

4 Applications and Extensions

We presents two direct applications of our results to one sector dynamic growth models. First, we extend the results of LeVan and Saglam (2004) and Dechert (1982) to distorted versions of the standard deterministic neoclassical one sector growth model, where non-convexities arise because aggregate level variables affect the decision making process of individual agents. Our success rests upon the ability of re-writing the representative agent's problem as a "modified" convex program to which our result on existence of price systems readily applies.

Next, because of the flexibility in the choice of the probability space $(S, \mathcal{F}, P)$, the choice of $L_\infty(S, \mathcal{F}, P)$ as commodity spaces allows for the application of our results to economies with countably many periods and an infinite number of states in each period, a "double infinity" featured in stochastic one sector growth models, the focus of our second example.

Finally, we also note that our results directly apply to the analysis of so called convex "non-linear programs" (NLP), that is, convex programs defined on $\mathbb{R}^n$. Letting $S = \{1, ..., n\}$, $\mathcal{F}$ be the σ-algebra of all subsets of S, and P the probability measure on $(S, \mathcal{F})$ defined as $P(i) = 1/n$, clearly:

$$L_\infty(S, \mathcal{F}, P) = \{x : S \to \mathbb{R}, \sup |x(i)| < +\infty\} = \mathbb{R}^n$$

Program 1 is then simply a standard convex NLP of the form:

$$\begin{gathered} \min F(x) \\ \phi_i(x) \leq 0, \; i = 1, ..., n \\ x \in \mathbb{R}^n \end{gathered}$$

where F and ϕ_i, $i = 1, ..., n$, are proper convex functions. Assuming that F is continuous at some feasible point, and that Slater's CQ is satisfied, (i.e., $\exists x' \in \mathbb{R}^n$, $\forall i = 1, ..., n \, \phi_i(x') < 0$), Theorem 1 implies that for any solution x^* there exists a valuation, and thus a price system. This is precisely the traditional result on optimality conditions in convex NLP (See, for instance, Theorem 3.34 in Ruszczynski (2006)).

4.1 Distorted deterministic one-sector growth models

4.1.1 Countable number of goods and infinite horizon deterministic economies

Letting $S = \mathbb{N}$, $\mathcal{F} = 2^{\mathbb{N}}$, and P the counting measure on $(S, \mathcal{F})$, the decision space becomes $L_\infty(\mathbb{N}, 2^{\mathbb{N}}, P) = l^\infty$, a case thoroughly analyzed by LeVan and Saglam (2004) and Dechert (1982); we relate our assumptions to these two papers.

It is important to recall that purely finitely additive measures in $ba(\mathbb{N})$ assign zero mass to finite subsets $D_n = \{0, 1, ..., n-1\}$ of $\mathbb{N}$.[7] Price systems are therefore valuations which have no mass "at infinity", so asymptotic properties of the objective and constraints sufficient for a valuation to be a price system may be stated in terms of limiting behavior along sequences of

[7] See Subsection 5.4 in the Appendix.

the form $\{x^n = \chi_{E_n}.y + (1-\chi_{E_n}).x\}$, where $E_n = \{n, n+1...\}$. Note that the sequence $\{x^n\}$ Mackey converge to x since $E_{n+1} \subset E_n$ and $\cap_{n\in\mathbb{N}} E_n = \emptyset$ (see Bewley (1972) Appendix I [24]), and the sets E_n play the role of the sets A_n in Assumption 1.[8]

In the commodity space l^∞ Assumption 1(ii)-(v) respectively become:

$$\begin{array}{ll} 1'(ii) & \lim_{n\to\infty} F(x^n) = F(x) \\ 1'(iii) & \lim_{n\to\infty} \phi(t, x^n) = \phi(t, x) \\ 1'(iv) & \exists M > 0,\, \exists N \in \mathbb{N},\, \forall n > N\ \|\phi(x^n)\| \leq M \\ 1'(v) & \forall n > N,\, \forall \varepsilon > 0,\, \exists q > 0 : \forall t > q,\, |\phi(t, x^n) - \phi(t, y)| < \varepsilon \end{array} \tag{8}$$

Assumptions 1'(ii) and 1'(iii) correspond to Assumptions 1 and 2(a) in LeVan and Saglam (2004) (and ANA in Dechert (1982)), and Assumption 1'(v) is precisely the statement that $\lim_{t\to\infty} |\phi(t, x^n) - \phi(t, y)| = 0$ of Assumption 2(c) in LeVan and Saglam (2004) (and AI in Dechert (1982)) with the appropriate change in notations (x^n is precisely $x^{n+1}(x, y)$).

The application of our results on the existence of price systems (Theorem 2) to the analysis of standard one-sector optimal growth models simply duplicate the analysis LeVan and Saglam (2004). While we refer the reader to Example 1 in LeVan and Saglam (2004), we also note that a more precise result can be obtained whenever utility and production functions are differentiable and if the optimal solution $x^* = (c^*, k^*)$ is interior to $l^\infty_+ \times l^\infty_+$. In that case, it can easily be shown that both F and $\phi = (\phi^1, \phi^2, \phi^3)$ are Gateaux differentiable at x^*, where $\phi^1_t(x^*) = c^*_t + k^*_{t+1} - f(k^*_t)$, $\phi^2_t(x^*) = -c^*_t$, and $\phi^3_t(x^*) = -k^*_{t+1}$. We show next that the Strict Constraint Qualification is satisfied at x^*.

Proposition: Standard one sector growth model with bounded state space, Inada conditions (including and $f'(0) > 1$), increasing, concave and differentiable utility and production functions. Then there exists a unique price system associated with the optimal solution.

4.1.2 Definition of competitive equilibrium in distorted economies

We extend the results of LeVan and Saglam (2004) to distorted models common in the literature on existence of equilibrium in one sector non-optimal economies with inelastic labor supply (such as Coleman (1991), Coleman (2000) and Greenwood and Huffman (1995)).

In these economies, there is a continuum of infinitely lived identical households. Each household has the same production technology for producing the unique good and enters period t with capital stock k_t and an endowment of one unit of time. Production takes the form $F(k_t, n_t, K_t, N_t)$, with F exhibiting constant returns to scale in private in inputs (k_t, n_t), and thus also depends on per capita capital stock K_t and per capita labor N_t.

Labor is supplied inelastically to firms (hence $n_t = N_t = 1$) and we denote by $f(k_t, K_t) = F(k_t, 1, K_t, 1)$ the reduced form production function. As discussed in Greenwood and Huffman (1995) this reduced form production technology encompasses many environments such as income taxes, externalities in production, and monopolistic competition.

We make the following assumptions:

Assumption 2. *Function* $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ *is strictly increasing, continuously differentiable and strictly concave in its first argument;* $f(k, 0) = 0$ *for all* $k \geq 0$*,* $1/\beta < \lim_{k\downarrow 0} f_1(k, k) \leq +\infty$*,*

[8] Alternatively, and equivalently, one can exploit the property that $L_\infty(\mathbb{N}, 2^{\mathbb{N}}, \mu) = l^\infty$ where $\mu\{n\} = 2^{-(n+1)}$, in which case sets E_n satisfy $E_{n+1} \subset E_n$ and $\lim_{n\to\infty} \mu(E_n) = 0$.

$\lim_{k\uparrow\infty} f_1(k,K) = 0$, *$f_1(k,K)$ is increasing in K, and $\exists \bar{k} > 0$ such that $f(\bar{k},\bar{k}) = \bar{k}$ and $f(k,k) < k$ for all $k > \bar{k}$.*

The utility representing a household's preferences over sequences of consumptions $\{c_t\}$ in l_+^∞ takes the form:

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

in which u satisfies the following standard assumption.

Assumption 3. *Function $u : \mathbb{K} = [0,\hat{k}] \to \mathbb{R}_+$ is strictly increasing, strictly concave and continuously differentiable; $0 < \beta < 1$, $u(0) = 0$ and $\lim_{c\downarrow 0} u'(c) = +\infty$.*

A competitive equilibrium is defined as a set of allocations and prices satisfying the usual market clearing and optimality requirements together with the restriction that the price sequences must be in $l_+^1 \setminus \{0\}$. More formally:

Definition 1. Given $k_0^* = k_0 = K_0$, the allocations $\{k_t^*\}_{t=0}^\infty, \{K_t^*\}_{t=0}^\infty, \{c_t^*\}_{t=0}^\infty$ in l_+^∞ and the prices $\{p_t^*\}_{t=0}^\infty, \{r_t^*\}_{t=0}^\infty, \{w_t^*\}_{t=0}^\infty$ in $l_+^1 \setminus \{0\}$ form a competitive equilibrium if:

(i). Given $\{p_t^*\}_{t=0}^\infty, \{r_t^*\}_{t=0}^\infty, \{w_t^*\}_{t=0}^\infty$ and $\{K_t^*\}_{t=0}^\infty$, the sequences $\{k_t^*\}_{t=0}^\infty$ and $\{c_t^*\}_{t=0}^\infty$ solve the representative firm's problem:

$$\max \left\{ \prod = \sum_{t=0}^{\infty} p_t^* f(k_k, K_t^*) - r_t^* k_t - w_t^* \right\}$$

(ii) Given $\{p_t^*\}_{t=0}^\infty, \{r_t^*\}_{t=0}^\infty, \{w_t^*\}_{t=0}^\infty$ the sequences $\{k_t^*\}_{t=0}^\infty$ and $\{c_t^*\}_{t=0}^\infty$ solve the representative consumer's problem:

$$\min \{-\textstyle\sum_{t=0}^\infty \beta^t u(c_t)\}$$
$$\text{s.t.} \textstyle\sum_{t=0}^\infty p_t^*(c_t + k_{t+1}) \le \sum_{t=0}^\infty (r_t^* k_t + w_t^*) + \prod$$

(iii) The goods markets clear, $f(k_t^*, K_t^*) = c_t^* + k_{t+1}^*$ for all t, and the equilibrium condition $k_t^* = K_t^*$ holds for all t.

We prove that a competitive equilibrium exists in two steps. In the first step we follow the monotone approach developed by Coleman (1991) and applied in Morand and Reffett (2003) to prove existence of a stationary recursive optimal allocation policy (using the terminology of Prescott and Mehra (1980)) , a law of motion for the per capita capital stock such that the representative agent's decisions are consistent with this law of motion.

In the second step we consider the "modified" program associated with a representative consumer who is constrained to use such a stationary recursive optimal allocation policy to compute all future aggregate per capita per capita capital stocks, and therefore all equilibrium relative input prices (i.e., period t wage and interest in terms of units of period t consumption good, for all t). The modified program is convex, and we know from step 1 that it has a solution. Given that objective and constraints satisfy Assumption 1, by Theorem 2 there exists a price system. It is easy to verify that the stationary recursive optimal allocation policy together with this price system define a competitive equilibrium.

4.1.3 Step 1: Existence of stationary recursive optimal allocation through monotone methods

In a competitive equilibrium factor prices necessarily satisfy:

$$r_t^* = p_t^* f_1(K_t^*, K_t^*)$$
$$w_t^* = p_t^* \left[f(K_t^*, K_t^*) - K_t^* f_1(K_t^*, K_t^*) \right]$$

These prices, together with the assumption of constant returns to scale imply 0 profits so the representative consumer's problem may be written as:

$$\begin{gathered} \min \{ -\textstyle\sum_{t=0}^{\infty} \beta^t u(c_t) \} \\ \text{s.t. } c_t + k_{t+1} \leq f(K_t, K_t) + (k_t - K_t) f_1(K_t, K_t) \text{ for all} t \in \mathbb{N} \end{gathered} \tag{9}$$

given $k_0 = K_0$ and the sequence $\{K_t = K_t^*\}$. We search for a sequence $\{K_t^*\}$ such that the solution precisely satisfies $k_t^* = K_t = K_t^*$, and latter establish that it is the allocation part of a competitive equilibrium.

We prove next that such a specific sequence exists, and that it can be expressed in the form of a simple (Markovian) decision rule $K_{t+1}^* = h(K_t^*)$. First, a standard application of the classic machinery of dynamic programming to Program (9) implies that, given any bounded continuous $h : \mathbb{K} \to \mathbb{K}$ generating the sequence of per capita capital stocks $\{K_{t+1} = h(K_t)\}$, there exists a unique value function $V : \mathbb{K} \times \mathbb{K} \to \mathbb{R}$ satisfying Bellman's equation:

$$V(k, K; h) = \max_{0 \leq y \leq m(k,K)} \{ u(m(k, K) - y) + \beta V(y, h(K); h) \}$$

in which $m(k, K) = f(K, K) + (k - K) f_1(k, K)$.

Definition 2. A law of motion $h : \mathbb{K} \to \mathbb{K}$ for the per capita capital stock is a stationary recursive optimal allocation policy if the corresponding optimal investment policy in equilibrium precisely coincides with h, that is:

$$\begin{gathered} \forall (k, K) \in \mathbb{K} \times \mathbb{K},\ y(k, K) \in \arg\max_{0 \leq y \leq m(k,K)} \{ u(m(k, K) - y) + \beta V(y, h(K); h) \} \\ \forall k \in \mathbb{K},\ y(k, k) = h(k) \end{gathered}$$

Our proof of existence of such h is based on the property that stationary recursive optimal allocation policies are precisely the fixed points of a monotone order continuous operator A defined implicitly in the Euler equation. This operator A is shown to map the complete lattice $(H, \leq)$ where:

$$H = \{ h : \mathbb{K} \to \mathbb{K},\ \forall k \in \mathbb{K},\ 0 \leq h(k) \leq f(k, k),\ h \,\text{increasing},\ f - h \text{ increasing} \}$$

and $\leq$ is the pointwise partial order, and existence follows from Tarski's fixed point theorem. We refer the readers to Morand and Reffett (2003) for the detailed proof of the following result:

Theorem 4. *The set of stationary recursive optimal allocation policies is a non-empty complete sublattice in H, denoted Ψ; the sequence $\{A^n f\}$ converges uniformly to the greatest one.*

We note the following additional property shared by all members of Ψ.

Proposition 4. *Given any $h \in \Psi$, there exists $\hat{k} \in K$ such that $h(\hat{k}) = \hat{k}$ and $\forall k_0$ $0 < k_0 < \hat{k}$, $k^*_{t+1} = h(k^*_t) > 0$ for all $t \in \mathbb{N}$ and $\lim_{t\to\infty} k^*_t = \hat{k}$.*

Proof. Consider $k_0 \in \mathbb{K}$ and $k_0 > 0$, and suppose that $k^*_1 = h(k_0) \leq k_0$. Since h is increasing, the sequence $\{k^*_t\}$ is decreasing. At the same time, because the optimal consumption policy associated with h (the function $f - h$) is also an increasing function, $\{c^*_t\}$ is also a decreasing sequence. Consider then the feasible sequences $\{k^{'*}_t\}$ and $\{c^{'*}_t\}$ constructed by altering the sequences $\{k^*_t\}$ and $\{c_t{}^*\}$ as follows.

In period 0, allocate a little less consumption (knowing that $c^*_0 > 0$ since $\lim_{c\downarrow 0} u'(c) = +\infty$) and a little more to savings, consume all the additional product in period 1, and keep everything else the same hence:

$$c^{'*}_0 = c^*_0 - \Delta$$
$$k^{'*}_1 = k^*_1 + \Delta$$
$$c^{'*}_1 = c^*_1 + (f(k^*_1 + \Delta, k^*_1 + \Delta) - f(k^*_1, k^*_1)$$
$$k^{'*}_t = k^*_t \text{ and } c^{'*}_t = c^*_t \text{ for all } t \geq 2$$

in which Δ is chosen sufficiently small so that $c^*_0 - \Delta > 0$. Thus, for all Δ sufficiently small:

$$\frac{U(c^{'*}) - U(c^*)}{\Delta} = \frac{u(c^{'*}_0) - u(c^*_0)}{\Delta} + \beta\left(\frac{u(c^{'*}_1) - u(c^*_1)}{\Delta}\right)$$

By definition:

$$\lim_{\Delta\downarrow 0}\left(\frac{u(c^{'*}_0) - u(c^*_0)}{\Delta}\right) = -u'(c^*_0)$$

and:

$$\lim_{\triangle\downarrow 0}\beta\left(\frac{u(c^{'*}_1) - u(c^*_1)}{\triangle}\right) = \beta u'(c^*_1) f_1(k^*_1, k^*_1)$$

The concavity of u and the assumption that $c^*_1 \leq c^*_0$ implies that $u'(c^*_1) \geq u'(c^*_0)$ hence by Assumption 2 $\beta f_1(k^*_1, k^*_1) > 1$ whenever k_0 is sufficiently small (and given the initial assumption $k^*_1 \leq k^*_0$). Thus there exists $\varepsilon > 0$ such that, $\forall k_0 \in]0, \varepsilon[$ and for all $\triangle > 0$ sufficiently small:

$$\beta\left(\frac{u(c^{'*}_1) - u(c^*_1)}{\triangle}\right) > u'(c^*_0)$$

As a result:

$$\frac{U(c^{'*}_t) - U(c^*)}{\Delta} > 0$$

but this contradicts the hypothesis that c^* maximizes utility.

This established that there must exists some there must exists some $\varepsilon > 0$ such that $\forall k_0 \in]0, \varepsilon[$ necessarily $h(k_0) > k_0$. By monotonicity of h the sequence $\{k^*_t\}$ is increasing; since it is bounded (it is in $\mathbb{K}$) it must be convergent, and its limit satisfies $h(\hat{k}) = \hat{k}$ by continuity of h. It is then easy to show that $h(k_0) \geq k_0$ for all $0 < k_0 \leq \hat{k}$. A corresponding argument applies to the sequence $\{c^*_t\}$ using the monotonicity of $f - h$. □

4.1.4 Step 2: Existence of competitive equilibrium

That a competitive equilibrium exists is a direct application of Theorem 2 to the consumer's program:

$$\min\{-U(c) = -\sum_{t=0}^{\infty} \beta^t u(c_t)\}$$
$$\forall t \in \mathbb{N},\, c_t + k_{t+1} - f(K_t^*, K_t^*) - (k_t - K_t^*) f_1(K_t^*, K_t^*) \leq 0$$
$$-c_t \leq 0$$
$$-k_{t+1} \leq 0,\, k_0 \text{ given}$$

in which $\forall t \in \mathbb{N}$, $K_{t+1}^* = h(K_t^*)$ where h is a stationary recursive optimal allocation policy, and $K_0^* = k_0$. We state this result and then prove it.

Theorem 5. *For each $h \in \Psi$ there exists a unique price system $\{p_t\}_{t=0}^{\infty}, \{w_t\}_{t=0}^{\infty}, \{r_t\}_{t=0}^{\infty}$ such that the sequences $\{k_t^* = h^{(t)}(k_0)\}_{t=0}^{\infty}, \{K_t^* = h^{(t)}(k_0)\}_{t=0}^{\infty}$, and $\{c_t^* = f(k_t^*, k_t^*) - k_{t+1}^*\}_{t=0}^{\infty}$ together with this price system form a competitive equilibrium.*

Proof. Recalling that, by construction, the unique solution to the consumer's program above is precisely the sequence $\{k_{t+1}^* = h^{(t)}(k_0)\}_{t=0}^{\infty}$ generated by h itself, we verify first that objective and constraints satisfy the conditions of Theorem 2 evaluated along sequences of the form $\{X^n = \chi_{E_n}.Y + (1 - \chi_{E_n}).X\}$ with $E_n = \{n, n+1...\}$ (i.e., Assumption 1'(ii)-1'(v) in (8)).

Indeed, setting $X^* = (c^*, k^*)$ where $c* = (c_0^*, c_1^*, ...)$ and $k^* = (k_1^*, k_2^*, ...)$, the objective U is Mackey continuous at X^*. The second and third sets of constraints $t \to \phi^2(t, X) = -c_t$ and $t \to \phi^3(t, X) = -k_{t+1}$ are clearly Insensitive and Non-Anticipatory. The first constraint system satisfies $\forall t > n$ $\phi^1(t, X^n) = \phi^1(q, Y)$ (hence Assumption 1'(v)) and for all $t > 0$, $\forall p > 0$, $\phi^1(t, X^{t+p}) = \phi^1(t, X)$ (hence Assumption 1'(iii)). Assumption 1'(iv) clearly holds because of the compactness of the state space.

Consequently, by Theorem 2 there exist sequences $\{\lambda_{1,t}\}_{t=0}^{\infty}, \{\lambda_{2,t}\}_{t=0}^{\infty}, \{\lambda_{3,t}\}_{t=0}^{\infty}$ in l_+^1 such that, for all $c = \{c_t\}_{t=0}^{\infty}$ and $k = \{k_{t+1}\}_{t=0}^{\infty}$ in l_+^{∞}:

$$\begin{gathered} U(c^*) - \sum_{t=0}^{\infty} \lambda_{1,t}(c_t^* + k_{t+1}^* - f(k_t^*, k_t^*)) + \sum_{t=0}^{\infty} \lambda_{2,t} c_t^* + \sum_{t=0}^{\infty} \lambda_{3,t} k_{t+1}^* \\ \geq \\ U(c) - \sum_{t=0}^{\infty} \lambda_{1,t}(c_t + k_{t+1} - f(k_t^*, k_t^*) - (k_t - k_t^*) f_1(k_t^*, k_t^*)) + \sum_{t=0}^{\infty} \lambda_{2,t} c_t + v\lambda_{3,t} k_{t+1} \end{gathered} \tag{10}$$

and:

$$\forall t,\, \lambda_{1,t}(c_t^* + k_{t+1}^* - f(k_t^*, k_t^*)) = 0,\, \lambda_{2,t} c_t^* = 0,\, \lambda_{3,t} k_{t+1}^* = 0$$

Since we have previously established that $\{k_{t+1}^*\}$ converges to $k^* > 0$ and $\forall t$, $k_t^* > 0$, necessarily $\lambda_{3,t} = 0$ for all t and, by a similar argument, $\lambda_{2,t} = 0$. Thus $\{\lambda_{1,t}\}_{t=0}^{\infty} \in l_1^+ \setminus \{0\}$, or else the multiplier Λ^* in Theorem 2 would be identical to zero.

Setting $p_t^* = \lambda_{1,t}$, $r_t^* = p_t^* f_1(k_t^*, k_t^*)$ and $w_t^* = p_t^*(f(k_t^*, k_t^*) - k_t^* f_1(k_t^*, k_t^*))$, the sequence $\{p_t^*\}_{t=0}^{\infty}$ is in $l_1^+ \setminus \{0\}$ (as is $\{\lambda_{1,t}^*\}_{t=0}^{\infty}$) and so are the sequences of factor prices, given the convergence of k_t^* to k^* established previously. Thus, as $k_t^* r_t^* + w_t^* = p_t^* f(k_t^*, k_t^*) = p_t^*(c_t^* + k_{t+1}^*)$, inequality (10) implies that for $\{c_t\}_{t=0}^{\infty}$ and $\{k_{t+1}\}_{t=0}^{\infty}$ in l_+^{∞} satisfying $\sum_{t=0}^{\infty} p_t^*(c_t + k_{t+1}) \geq \sum_{t=0}^{\infty}(k_t r_t^* + w_t^*)$, necessarily:

$$U(c^*)$$
$$\geq$$
$$U(c) - [\sum_{t=0}^{\infty} p_t^*(c_t + k_{t+1}) - \sum_{t=0}^{\infty}(k_t r_t^* + w_t^*)]$$
$$\geq$$
$$U(c)$$

This means that given prices $\{p_t^*\}_{t=0}^{\infty}, \{w_t^*\}_{t=0}^{\infty}, \{r_t^*\}_{t=0}^{\infty}$, the sequences $\{k_t^*\}_{t=0}^{\infty}$ and $\{c_t^*\}_{t=0}^{\infty}$ yield a higher utility than any other set of sequences satisfying the budget constraints (at these prices). It implies that $(\{k_{t+1}^* = h(k_t^*) = K_{t+1}\}_{t=0}^{\infty}, \{c_t^*\}_{t=0}^{\infty}, \{p_t\}_{t=0}^{\infty}, \{w_t\}_{t=0}^{\infty}, \{r_t\}_{t=0}^{\infty})$ form a competitive equilibrium, noting that at prices $\{p_t^*\}_{t=0}^{\infty}, \{w_t^*\}_{t=0}^{\infty}, \{r_t^*\}_{t=0}^{\infty}$ profits are maximized at $\{k_{t+1}^* = h(k_t^*) = K_{t+1}\}_{t=0}^{\infty}$ and that profits are zero since:

$$\Pi = \sum_{t=0}^{\infty} [p_t^* F(k_t^*, 1, K_t, 1) - k_t^* r_t^* - w_t^*] = 0$$

Finally, recall also that u is assumed to be strictly increasing, hence for all t, $c_t^* + k_{t+1}^* = f(k_t^*, k_t^*)$ i.e., the goods market clears in all periods.

To prove uniqueness, note that the solution $X^* = (c^*, k^*) \in l_+^\infty \times l_+^\infty$ to the consumer's program is interior, a consequence of Proposition 4. The Mackey continuity and monotonicity of U then imply that $\partial_c U(X^*) \subset l^1$ (see Lemma 1) and if period utility u is differentiable it is then easy to see that $\partial_c U(X^*)$ is a singleton (i.e, U is Gateaux differentiable at X^*).

Any price system $(\Lambda_1^*, \Lambda_2^*, \Lambda_3^*)$ associated with the solution $X^* = (c^*, k^*) \in int(L_\infty^+ \times L_\infty^+)$ satisfies $\Lambda_2^* = \Lambda_3^* = 0$ and:

$$0 \in \{-DU(X^*) + \partial \left\langle \Lambda_1^*, \phi^1(X^*) \right\rangle\}$$

or, equivalently:

$$DU(X^*) \in \partial \left\langle \Lambda_1^*, \phi^1(X^*) \right\rangle$$

Clearly $\partial \left\langle \Lambda_1^*, \phi^1(X^*) \right\rangle \subset \partial_c \left\langle \Lambda_1^*, \phi^1(X^*) \right\rangle \times \partial_k \left\langle \Lambda_1^*, \phi^1(X^*) \right\rangle$; but since $\phi_i^1(X) = c_i + k_{i+1} - f(k_i)$, ϕ^1 is obviously regular subdifferentiable in c at X^* with $\partial_c \left\langle \Lambda_1^*, \phi^1(X^*) \right\rangle = \langle \Lambda_1^*, 1 \rangle$ hence:

$$D_c U(X^*) = \langle \Lambda_1^*, 1 \rangle$$

and Λ_1 must be unique. □

Remark. The above uniqueness also results from Theorem 3, given that the surjectivity of $D_c \phi^1(X^*)$ implies that the strict constraint qualification (i.e., condition (7)) is satisfied.

4.2 Stochastic optimal one sector bounded growth models

4.2.1 The commodity space

Commodity spaces in economic models with a countable number of periods and with an infinite number of states in each period, such as the one sector stochastic optimal growth model, can also be conveniently rewritten as L_∞ Banach spaces for which our sufficient conditions for the existence of a price system can easily be checked.

Consider a stochastic process $\{s_t\}_{t=0}^{+\infty}$, each random variable s_t taking its value in some compact subset Z of $\mathbb{R}$, and denote $S = \{s = (s_0, s_1...),\ s_t \in Z\,\forall t\}$ the set of all possible values of s. The σ-algebra generated by S, denoted $\mathcal{F}$, is by definition the smallest σ-algebra on S such that all random variables s_t are measurable. A measure P is defined on $(S, \mathcal{F})$ so that $(S, \mathcal{F}, P)$ becomes the underlying measure space of the problem.

The stochastic process $\{s_t\}_{t=0}^{+\infty}$ generates the natural filtration $\{\mathcal{F}_t\}_{t\in\mathbb{N}}$ in the following manner: $\mathcal{F}_t$ is the smallest σ-algebra on S such that all the random variables s_i, $i \leq t$ are measurable. Clearly $\mathcal{F}_t \subset \mathcal{F}_{t+1} \subset \mathcal{F}$. Denote by $\sigma : S \to S$ the "shift operator" function, defined as $\sigma(s)_t = s_{t+1}$.

A commodity is a stochastic process $x = \{x_t\}_{t=0}^{\infty}$ that is adapted to the filtration of σ-algebra $\{\mathcal{F}_t\}_{t=0}^{\infty}$, in the sense that for all $i \in \mathbb{N}$, $x_i : S \to \mathbb{R}$ is $\mathcal{F}_i$-measurable. Intuitively, this means that the value $x_t(s)$ is essentially unaffected by changes in s_i for $i > t$.[9] The decision space for is thus $\prod_{t=0}^{\infty} L_\infty(S, \mathcal{F}_t, P)$, but can be conveniently re-written as $L_\infty(M, \mathcal{M}, P')$ by defining the measure space $(M, \mathcal{M}$,P') in the following way:

- ⋆ $M = S \times \mathbb{N}$;
- ⋆ $\mathcal{M}$ is the smallest σ-algebra generated by the family of sets $A_i \times \{i\} :\ A_i \in \mathcal{F}_i$;
- ⋆ P' is the probability measure defined as: $P'(A_i \times \{i\}) = P(A_i)$ if $A_i \in \mathcal{F}_i$.

By construction, $x : M \to \mathbb{R}$ is $\mathcal{M}$-measurable iff $x_i = x(.,i) : S \to \mathbb{R}$ is $\mathcal{F}_i$-measurable (i.e. x is adapted to $\{\mathcal{F}_t\}_{t=0}^{\infty}$). The norm is defined as $\|x\| = inf\{N > 0;\ P'\{m \in M :\ |x(m)| > N\} = 0\}$. Since:

$$\sum_{i=0}^{\infty} P\{s \in S :\ |x_i(s)| > N\} = 0 \Rightarrow \forall i \in \mathbb{N},\ P\{s \in S :\ |x_i(s)| > N\} = 0$$

the set $L_\infty(M, \mathcal{M}, P')$ only includes essentially bounded commodities.

4.2.2 Stochastic one sector optimal bounded growth

Given $k_0 > 0$, consider the program:

$$\begin{gathered} \min -U(c) \\ c_t + k_{t+1} - f(k_t, s_t) \leq 0 \\ -c_t \leq 0 \\ -k_{t+1} \leq 0 \end{gathered}$$

where the objective $U : L_\infty \to \mathbb{R}$, is such that $U(c) = E\left\{\sum_{t=0}^{\infty} \beta^t u(c_t(s), \sigma^t(s))\right\}$ in which $\{c_t\}_{t=0}^{\infty}$ is an adapted consumption program. We assume that the standard assumptions are met, (in particular that the the capital stock is essentially bounded) and go on to prove that the valuations must be price systems.

Mackey continuity of the objective. Assuming that function $u : \mathbb{R}_+ \times S \to \mathbb{R}$ is continuous, strictly increasing, and concave in its first argument, as well as $\Sigma(\mathbb{R}_+) \otimes \mathcal{F}_0$-measurable, the Mackey continuity of the objective U (when considering the product-Mackey topology defined by the Mackey topology on each $L_\infty(S, \mathcal{F}_t, P)$ space) results from the application of a theorem in Bewley (1972) (Appendix II, page 535). Indeed, recall that a net c^α converges to c in the product-Mackey topology if and only if for all $t \in \mathbb{N}$, c_t^α converges to c_t in the Mackey topology

[9] In the language of Rockafellar and Wets x is said to be (essentially) Non-Anticipative.

topology on $L_\infty(S, \mathcal{F}_t, P)$. But by Bowler's Theorem each function $\int_S \beta^t u(c_t(s), \sigma^t(s))P(ds)$ is Mackey continuous therefore $U(c^\alpha)$ converges to $U(c) = \sum_{t=0}^\infty \int_S \beta^t u(c_t(s), \sigma^t(s))P(ds) < +\infty$.

The decision variable is $x = (c, k)$ where $c = (c_0, c_1, ...)$ and $k = (k_1, k_2, ...)$ are both stochastic process adapted to the filtration $\{\mathcal{F}_i\}_{i=0}^\infty$, i.e. elements of $L_\infty(M, \mathcal{M}, P')$. For $m = (s, i)$, we write the first constraint as $\phi^1(m, x) \leq 0$ with $\phi^1((s, i), (c, k)) = \phi_i^1(s, (c, k)) = c_i(s) + k_{i+1}(s) - f(k_i(s), \sigma^i s)$ and the last two as $\phi^2(m, x) = -c_i(s)$ and $\phi^3(m, x) = -k_{i+1}(s)$.

The standard Slater condition is easily verified, hence, given the Mackey continuity of the objective, for a valuation to be a price system, only the conditions concerning the constraint system need to be checked. We propose to prove this in a manner analogous to Lucas and Prescott (1972) (Section 4) by demonstrating first that the valuation is a countable sum of (period) valuations and showing next that each period valuation is itself a price system. A valuation Λ^* associated with the solution $x^* = (c^*, k^*)$ satisfies, for all x:

$$-U(x) + \langle \Lambda^*, \phi(x) \rangle \geq -U(x^*)$$

Step 1: Valuations are countable sums of period valuations. It is a consequence of the (now familiar) Yosida-Hewitt decomposition Theorem that for a valuation Λ^* to be a price system, it must have 0 mass in any arbitrarily μ-small subsets of the measurable space $(M, \mathcal{M})$ *whatever the measure* μ defined on the measurable space $(M, \mathcal{M})$, or in decreasing sequences $\{E_n\}$ of sets satisfying $\cap_{n\in\mathbb{N}} E_n = \emptyset$. In particular, if μ is defined as:

$$\mu(A_i \times \{i\}) = \begin{cases} 0 \text{ if } P(A_i) = 0 \\ 1 \text{ otherwise} \end{cases}$$

then the purely finitely additive part of Λ^* must have no mass in the sets $E_n = S \times \{n, n+1, ...\}$. Again, sufficient conditions for Theorem 2 to apply are trivially satisfied for constraints ϕ^2 and ϕ^3, and for ϕ^1 as well since $\forall n > i$ and $\forall s \in S$:

$$\phi^1((s, i), x^n) = \phi^1((s, i), x)$$

while $\forall i > n + 1$ and $\forall s \in S$:

$$\phi^1((s, i), x^n) = \phi^1((s, i), y)$$

Thus by Theorem 2 Λ^* belongs to $L_1(M, \mathcal{M}, \mu)$ and $\langle \Lambda^*, \phi(x) \rangle = \sum_{i=0}^\infty \langle \Lambda_i^*, \phi_i(x) \rangle$ where Λ_i^* is a positive element of $ba(S, \mathcal{F}_i, P)$ for each $i \in \mathbb{N}$, $\phi_i = (\phi_i^1, \phi_i^2, \phi_i^3)$, and:

$$\langle \Lambda^*, \phi(x^*) \rangle = \sum_{i=0}^\infty \langle \Lambda_i^*, \phi_i(x^*) \rangle = 0$$

implying in turn that $\langle \Lambda_i^*, \phi_i(x^*) \rangle = 0$ for each i.

Remark. Equivalently, one could work with the probability measure $\mu'(A_i \times \{i\}) = 2^{-(i+1)} P(A_i)$ for which the decreasing sequence of sets $\{E_n = S \times \{n, n+1, ...\}\}$ satisfies $\lim_{n\to\infty} \mu'(E_n) = 0$ and just as well establish the existence of a valuation Λ^* satisfying:

$$\langle \Lambda^*, \phi(x^*) \rangle = \sum_{i=0}^\infty 2^{-(i+1)} \langle \Lambda_i^*, \phi_i(x^*) \rangle = 0$$

In the second step, we call on the Yosida-Hewitt Theorems once again to assert that for each $i \in \mathbb{N}$, $\Lambda_i^* = \Lambda_{i,p}^* + \Lambda_{i,c}^*$ and that there exists a sequence $\{A_n^i\}$ of elements of $\mathcal{F}_i$ such that $\lim_{n\to\infty} P(A_n^i) = 0$, $\lim_{n\to\infty} \Lambda_{i,c}^*(A_n^i) = 0$ and $\Lambda_{i,p}^*(S \setminus A_n^i) = 0$.

Step 2. The purely finitely additive of each period valuation is zero. We prove by a recursive argument that $\Lambda_{i,p} = 0$ for all $i \in \mathbb{N}$. That is we prove that $\Lambda_{0,p} = 0$ and $\Lambda_{i,p} = 0$ for $i = 0, ..., k-1$ implies $\Lambda_{k,p} = 0$. Associated with the sequence $\{A_n^0\}$ in $\mathcal{F}_0$ we first construct the sequence $\{x^{0,n} = \chi_{A_n^0} x' + (1 - \chi_{A_n^0}) x^*\}$. By definition of the valuation, for all n sufficiently large:

$$-U(c^{0,n}) + \sum_{i=0}^{\infty} \left\langle \Lambda_i^*, \phi_i(x^{0,n}) \right\rangle \geq -U(c^*)$$

In addition, for any $j \in \mathbb{N}$:

$$\phi((s,j), x^{0,n}) = \begin{cases} \phi((s,j), x'(s)) = \phi((s,j), x') \text{ if } s \in A_n^0 \\ \phi((s,j), x^*(s)) = \phi((s,j), x^*) \text{ if } s \in A_n^0 \end{cases}$$

As a result $\left\langle \Lambda_0^*, \phi_0(x^{0,n}) \right\rangle = \left\langle \Lambda_{0,c}^*, \phi_0(x^*) \right\rangle + \left\langle \Lambda_{0,p}^*, \phi_0(x') \right\rangle$ while $\forall i > 0$:

$$\left\langle \Lambda_i^*, \phi_i(x^{0,n}) \right\rangle = \left\langle \Lambda_i^*, \chi_{A_n^0} \phi_i(x^n) + (1 - \chi_{A_n^0}) \phi(x') \right\rangle \leq 0$$

since $\phi_i(x') < 0$ and $\phi_i(x^*) \leq 0$. Consequently, the convergence of $U(c^{0,n})$ to $U(c^*)$ (which results from the Mackey convergence of $\{x^{0,n}\}$ to x^*) implies that:

$$\left\langle \Lambda_{0,c}^*, \phi_0(x^*) \right\rangle + \left\langle \Lambda_{0,p}^*, \phi_0(x') \right\rangle \geq 0$$

which implies that $\Lambda_{0,p}^* = 0$, just like in the proof of Theorem 2.

Assume then that $\Lambda_{i,p}^* = 0$ for $i = 0, ..., k-1$ and consider the sequence $\{A_n^k\}$ of elements of $\mathcal{F}_i$ such that $\lim_{n\to\infty} P(A_n^k) = 0$, $\lim_{n\to\infty} \Lambda_{k,c}^*(A_n^k) = 0$ and $\Lambda_{k,p}^*(S \setminus A_n^k) = 0$. Constructing the sequence $\{x^{k,n} = \chi_{A_n^k} x' + (1 - \chi_{A_n^k}) x^*\}$ which product-Mackey converges to x^*, it is easy to see that for any $j \in \mathbb{N}$:

$$\phi((s,j), x^{k,n}) = \begin{cases} \phi((s,j), x'(s)) = \phi((s,j), x') \text{ if } s \in A_n^k \\ \phi((s,j), x^*(s)) = \phi((s,j), x^*) \text{ if } s \in A_n^k \end{cases}$$

Thus $\left\langle \Lambda_k^*, \phi_k(x^{k,n}) \right\rangle = \left\langle \Lambda_{k,c}^*, \phi_k(x^*) \right\rangle + \left\langle \Lambda_{k,p}^*, \phi_k(x') \right\rangle$ and $\forall i > k$:

$$\left\langle \Lambda_i^*, \phi_i(x^{k,n}) \right\rangle = \left\langle \Lambda_i^*, \chi_{A_n^k} \phi_i(x^n) + (1 - \chi_{A_n^k}) \phi(x') \right\rangle \leq 0$$

since $\phi_i(x') < 0$ and $\phi_i(x^*) \leq 0$. For $i < k$, $\left\langle \Lambda_i^*, \phi_i(x^{k,n}) \right\rangle = \left\langle \Lambda_{i,c}^*, \phi_i(x^{k,n}) \right\rangle$ so that $\lim_{n\to\infty} \left\langle \Lambda_i^*, \phi_i(x^{k,n}) \right\rangle = \left\langle \Lambda_i^*, \phi_i(x^*) \right\rangle = 0$ since $\phi_i(x^{k,n})$ and $\phi_i(x^*)$ differ only on the set A_n^k and $\lim_{n\to\infty} P(A_n^k) = 0$. Again we have:

$$\left\langle \Lambda_{k,c}^*, \phi_k(x^*) \right\rangle + \left\langle \Lambda_{k,p}^*, \phi_k(x') \right\rangle \geq 0$$

thus $\Lambda_{k,p}^* = 0$, which concludes the proof.

Example 5. In the commodity space $L_\infty(M, \mathcal{M}, \mu)$ discussed above, the Mackey lower semi-continuity of the objective (in the context of a maximization problem) implies some restrictions concerning, in particular, the asymptotic behavior of the utility function along the sequence $\{E_n = S \times \{n, n+1...\}\}$ in $\mathcal{M}$. For instance, the objective:

$$F(x) = \lim_{i\to\infty} \int_S \left(\frac{x_1(s) + ... + x_i(s)}{i} \right) P(ds)$$

in which utility is average expected consumption, fails to be Mackey (lower) continuous since $F(y) > F(x)$ but $F((1-\chi_{E_n})y) \leq F(x)$ for $x = 0$ and $y = 1$ since $F((1-\chi_{E_n})y) = 0$.

5 APPENDIX: Mathematical tools and results

5.1 Convex Functions

Definition 3. Given a Banach space X (with norm $\|.\|$) and its norm dual X', a functional $f : X \longmapsto \mathbb{R}$ is:

⋆ convex if $\forall (x,y) \in X^2$, $\forall \lambda \in [0,1]$:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

⋆ proper if $\forall x \in X,\ f(x) > -\infty$ and if f is not the constant function $+\infty$.

⋆ lower semicontinuous at x in the norm topology if:

$$f(x) = \liminf_{y \to x} f(y)$$

where the convergence of $y \to x$ is in the norm topology. Equivalently, $f : (X, \|.\|) \to (\overline{\mathbb{R}}, \text{lower topology})$ is continuous at x, where the lower topology is generated by the basis of open sets of the form $]a, +\infty[$, $a \in [-\infty, +\infty[$. The function f is lower semicontinuous lsc iff $epi(f) = \{(x,a) \in X \times \mathbb{R}, f(x) \leq a\}$ is a norm-closed set. Equivalently, for all $\mu \in \mathbb{R}$ the sets $\{x \in X,\ f(x) \leq \mu\}$ are norm closed. Equivalently, $\forall x \in X$, $\forall c \in \mathbb{R}$, $\lim x_i = x$ and $\lim f(x_i) \leq c$ implies $f(x) \leq c$.

The effective domain of f is the convex set:

$$Dom(f) = \{x \in X,\ f(x) < +\infty\}$$

Remark. The definition of convexity can be generalized to functions $F : X \to Y$ where X and Y are both Banach spaces, and where the order on Y is defined as: $y \geq y'$ iff $y - y' \in A_Y$, where A_Y is any closed convex cone of Y

Unlike in finite dimensional spaces, when X is infinite dimensional, proper convex functions are not necessarily norm-continuous on the interior of its domain. However:

Proposition 5. *A proper convex function f is continuous on the interior of its domain (Lipschitz near a point $x \in int(dom(f))$) if and only if it is bounded from above on a neighborhood of an interior point of its domain (resp. bounded from above in a neighborhood of x).*

Lower semicontinuous convex functionals are “better” behaved:

Proposition 6. *A lower semicontinuous proper convex function is continuous at every algebraic interior point (and thus at every interior point) of its domain.*

A lower semicontinuous proper convex functions is therefore bounded from above on a neighborhood of any interior point of its domain, since it is continuous at such points. Note also that if the proper convex function $f : X \to \mathbb{R}$ is finite at some $x \in X$, then it is directionally differentiable at such x in the sense that:

$$f'(x, v) = lim_{t \downarrow 0} \left(\frac{f(x + tv) - f(x)}{t} \right)$$

always exists for any $v \in X$.

5.2 Differentiability and Subdifferentiability

Definition 4. The subdifferential of a convex functional $g : X \to \overline{\mathbb{R}}$ at $x \in X$ is the set $\partial g(x)$ of bounded linear maps $\Lambda : X \to \mathbb{R}$ satisfying $\forall y \in X,\ g(y) - g(x) \geq \langle \Lambda, (y - x) \rangle$.

The set $\partial g(x)$ may be empty. However, if g is finite and continuous at some $y \in X$, then $\partial g(x) \neq \emptyset$ for all x in the interior of the domain of g.

This definition can be generalized to convex functions $G : X \to Y$ as follows:

$$\partial G(x) = \{\Lambda \in L(X, Y),\ \forall u \in X,\ G(u) - G(x) \geq \langle \Lambda, (u - x) \rangle\}$$

where $L(X, Y)$ denotes the vector space of all continuous linear mappings from the Banach space X into the Banach space Y.

Definition 5. $G : X \to Y$ is Gateaux differentiable at x_0 if there exists $DG(x_0) \in L(X, Y)$ such that, for all $y \in X$:

$$lim_{t \downarrow 0} \left(\frac{G(x_0 + ty) - G(x_0)}{t} - \langle DG(x_0), y \rangle \right) = 0$$

where the convergence of $\frac{G(x_0+ty)-G(x_0)}{t}$ to $\langle DG(x_0), y \rangle$ is in the norm topology on Y.

Proposition 7. *If $G : X \to Y$ is convex and Gateaux differentiable at x^* then (i) $\partial G(x^*) = DG(x^*)$, and (ii) $\forall y \in L(Y, \mathbb{R})$, $\partial \langle y, G(x^*) \rangle = \langle y, DG(x^*) \rangle$ (i.e, G is said to be regular subdifferentiable).*

Proof. (i) By definition, for all $\varepsilon > 0$ and all $y \in X$, there exists T such that $\forall 0 < t < T$:

$$\langle DG(x^*), y \rangle \leq \frac{G(x^* + ty) - G(x^*)}{t} + \varepsilon$$

By convexity of G, setting $y + x^* = y'$, for any $0 < t < 1$:

$$G(x^* + ty) = G(ty' + (1 - t)x^*) \leq tG(x^* + y) + (1 - t)G(x^*)$$

and, therefore:

$$\frac{G(x^* + ty) - G(x^*)}{t} \leq G(x^* + y) - G(x^*)$$

As a result, for all $\varepsilon > 0$ and all y:

$$\langle DG(x^*), y \rangle \leq G(x^* + y) - G(x^*) + \varepsilon$$

which proves that $DG(x^*) \in \partial G(x^*)$.

Reciprocally, suppose that $u^* \in \partial G(x^*)$. By definition, for all t>0 and all $y \in X$:

$$\frac{G(x^* + ty) - G(x^*)}{t} \geq \langle u^*, y \rangle$$

Taking limits as $t \downarrow 0$ implies that, for all $y \in X$ $\langle DG(x^*), y \rangle \geq \langle u^*, y \rangle$ and therefore that $DG(x^*) = u^*$. Thus, by uniqueness of the Gateaux derivative, $\partial G(x^*)$ is a singleton which coincides with $DG(x^*)$. In fact, on $int(dom(G))$, Gateaux differentiability and $Card(\partial G(x)) = 1$ are equivalent.

(ii). Gateaux differentiability implies that:

$$G(x^* + ty) - G(x^*) = \langle DG(x^*), ty \rangle + w(x^*, ty)$$

in which $lim_{t \to 0} \left\| \frac{w(x^*, ty)}{t} \right\| = 0$. Consequently, $\forall y' \in Y'$ (the norm dual of Y):

$$\langle y', G(x^* + ty) \rangle - \langle y', G(x^*) \rangle = \langle y', \langle DG(x^*), ty \rangle \rangle + \langle y', w(x^*, ty) \rangle$$

and by linearity of y', $\frac{\langle y', w(x^*, ty) \rangle}{t} = \left\langle y', \frac{w(x^*, ty)}{t} \right\rangle$ and since $lim_{t \to 0} \left\| \frac{w(x^*, ty)}{t} \right\| = 0$, necessarily $lim_{t \to 0} \left\langle y', \frac{w(x^*, ty)}{t} \right\rangle = 0$. This establishes that:

$$lim_{t \to 0} \left(\frac{\langle y', G(x^* + ty) \rangle - \langle y', G(x^*) \rangle}{t} - \langle y', \langle DG(x^*), y \rangle \rangle \right) = 0$$

and therefore that $x \to \langle y', G(x) \rangle \in \mathbb{R}$ is Gateaux differentiable and $\langle D(\langle y', G \rangle)_{x^*}, y \rangle = \langle y', \langle DG(x^*), y \rangle \rangle$. To simplify the notation we write this equality as $D(\langle y', G \rangle)_{x^*} = \langle y', DG(x^*) \rangle)$. Note also that for all $y' \in Y'$ the function $x \in X \to \langle y', G(x) \rangle \in \mathbb{R}$ is convex (in addition to be Gateaux differentiable), so by (i) its sub-differential is a singleton equal to its Gateaux derivative hence:

$$\partial \langle y', G(x^*) \rangle = \langle y', DG(x^*) \rangle$$

□

5.3 Essentially bounded functions

Consider a set S, a σ-algebra $\mathcal{F}$ of subsets of S, and a probability measure P defined on $(S, \mathcal{F})$. Recall the following definitions and results (see, for instance, Aliprantis and Border (1999) for more details):

- ⋆ $L_\infty(S, \mathcal{F}, P) = \mathrm{L}_\infty = \{x : S \to \mathbb{R},\ x \text{ measurable and } \|x\|_\infty < \infty\}$, where $\|x\|_\infty = Inf\{M,\ P\{s \in S\, |x(s)| > M\} = 0\}$, is the space of essentially bounded real valued measurable functions defined on S.
- ⋆ $\mathrm{L}_1(S, \mathcal{F}, P) = L_1 = \{x : S \to \mathbb{R},\ x \text{ measurable and } \|x\|_1 < \infty\}$, where $\|x\|_1 = \int_S |x(s)|\, P(ds) < \infty$, is the subspace of integrable functions ($\mathrm{L}_1 \subset \mathrm{L}_\infty$).

- ⋆ $(L_1)' = L_\infty$, hence the weak* topology $\sigma(L_\infty, L_1)$ on L_∞, (weaker than the $\|.\|_\infty$ topology).
- ⋆ $(L_\infty)' = ba(S, \mathcal{F}, P)$ (an isomorphism), where$ba(S, \mathcal{F}, P)$ is the set of bounded additive set functions on $(S, \mathcal{F})$ absolutely continuous with respect to P. This permits defining hence the weak topology $\sigma(L_\infty, ba)$ on L_∞. A set function λ is absolutely continuous w.r.t. to P if $P(A) = 0$ implies $\lambda(A) = 0$.
- ⋆ The Mackey topology $\tau(L_\infty, L_1)$, the strongest (i.e. largest) topology for which the dual of L_∞ is L_1. Overall:

$$\sigma(L_\infty, L_1) \subset \tau(L_\infty, L_1) \subset \sigma(L_\infty, ba) \subset \|.\|_\infty\text{-topology}$$

5.4 Yosida-Hewitt Theorems

Proofs of existence of price systems commonly rely on two important results from Yosida and Hewitt (1956), the first of which characterizes purely finitely additive measures as being those concentrated on arbitrarily small sets of events. Given a measure space $(S, \mathcal{F})$, recall that a measure λ is countably additive if for every decreasing sequence $\{A_n\}$ in $\mathcal{F}$ such that $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$, $lim_{n\to\infty}\lambda(A_n) = 0$.[10]

Theorem 6. *(Yosida and Hewitt (1956) Theorem 1.22). If $\pi \geq 0$ is purely finitely additive and $\psi \geq 0$ is countably additive, there exists a decreasing sequence $\{E_n\}$ of elements of $\mathcal{F}$ such that $\lim \psi(E_n) = 0$ and $\pi(S \setminus E_n) = 0$.*

As a direct application of this result it can be shown that a purely finitely additive measure π in $ba(\mathbb{N})$ assigns zero mass to any finite subset of $\mathbb{N}$. Indeed, a set $W \subset \mathbb{N}$ is finite iff $\exists n \in \mathbb{N}$ such that $\Omega \subset E_n = \{1, ..., n\}$. Denoting θ the probability measure assigning equal mass $1/n$ to each point in E_n, Theorem 6 implies the existence of a decreasing sequence $\{A_p\}_{p=0}^{\infty}$ in $2^{\mathbb{N}}$ satisfying $\lim_{p\to\infty} \theta(A_p) = 0$, and $\pi(\mathbb{N} \setminus A_p) = 0$. But $\lim_{p\to\infty} \theta(A_p) = 0$ implies that $E_n \cap A_p = \emptyset$ for p large enough, and therefore that $E_n \subset \mathbb{N} \setminus A_p$. In combination with $\pi(\mathbb{N} \setminus A_p) = 0$, this implies that π has no mass in E_n and therefore no mass on Ω.

The second result shows that any positive measure can be written as the sum of positive countably additive measure and a positive purely finitely additive measure, and that such "decomposition" is unique.

Theorem 7. *(Yosida and Hewitt (1956) Theorem 1.23). Let $\phi \geq 0$ be any measure. Then ϕ can be uniquely written as the sum of a countably additive measure $\phi_c \geq 0$ and a purely finitely additive measure $\phi_p \geq 0$.*

These two theorems have the following important consequences.

Corollary. *A finitely additive measure π in $ba(\mathbb{N})$ assigns 0 mass to any finite subset of $\mathbb{N}$.*

Proof. Given any finite subset A of $\mathbb{N}$, there exists $i \in \mathbb{N}$ such that $\forall x \in A$, $x < i$. Consider then the countably additive measure θ_i assigning equal weights (1/i) to $n = 1, ..., i$. By Theorem 6 there exists a decreasing sequence $\{A_n\}$ $A_n \in 2^{\mathbb{N}}$ such that $lim_n\theta_i(A_n) = 0$, and $\pi(\mathbb{N} \setminus A_n) = 0$.

[10] $\{A_n\}$ is decreasing if $A_n \supset A_{n+1}$ for all n.

Thus any $\{p\}$ for $p = 1, ..., i$ cannot belong to A_n for n large enough, since $\theta_i(\{p\}) = 1/i$, hence $\{1, 2, ..., i\} \cap A_n = \emptyset$. As a result, $\pi(\{1, 2, ..., i\}) = 0$, and π has no mass in A and thus no mass in any finite subsets of $\mathbb{N}$. □

Given a probability measure P defined on the measure space $(S, \mathcal{F})$, and using the standard notation $\{A_n\} \downarrow 0$ to indicate that the decreasing sequence $\{A_n\}$ satisfies $\lim P(A_n) = 0$, the following result combines the two theorems of Yosida and Hewitt (1956).

Corollary. *Letting $\phi \geq 0$ be any measure, and $\phi = \phi_c + \phi_p$ its unique decomposition according to Theorem 7, there exists $\{A_n\} \downarrow 0$ in $\mathcal{F}$ such that* $\lim \phi_c(A_n) = 0$ *and* $\phi_p(S \setminus A_n) = 0$.

Proof. By Theorem 6, $\exists\{E_n\}$ such that $\lim P(E_n) = 0$ and $\phi_p(S \setminus E_n) = 0$ and $\exists\{U_n\}$ such that $\lim \phi_c(U_n) = 0$ and $\phi_p(S \setminus U_n) = 0$. Consider the sets $A_n = E_n \cap U_n$. Clearly $\lim P(A_n) = \lim \phi_c(A_n) = 0$, and $0 \leq \phi_p((S \setminus E_n) \cup (S \setminus U_n)) \leq \phi_p(S \setminus E_n) + \phi_p(S \setminus U_n) = 0$ hence $\phi_p((S \setminus A_n) = \phi_p((S \setminus E_n) \cup (S \setminus U_n)) = 0$. □

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