Comparing Recursive Equilibrium in Economies with Dynamic Complementarities and Indeterminacy[1]

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Abstract

We prove the existence of recursive equilibrium (RE) for a broad class of infinite horizon dynamic general equilibrium models with positive externalities, dynamic complementarities, public policy, and equilibrium indeterminacy/sunspots. Our methods are based on new "multistep" monotone map methods, and apply to economies where no results on the existence of dynamic equilibria are known, and where existing methods for obtaining results on the presence of robust dynamic equilibrium comparative statics appear difficult to apply. Our methods are *global,* based on monotone operators defined on Euler equations, and we do not appeal to any methods in the theory of smooth dynamical systems that are commonly applied in the literature to characterize dynamic equilibria. Rather, using partial ordering methods, we are able to provide a qualitative theory of equilibrium comparative statics and/or comparative dynamics even in the presence of multiple equilibrium, and these comparison results are computable via successive approximations from upper and lower bounds on the set of recursive equilibria. We provide applications of our results to the extensive literature on the local indeterminacy of dynamic equilibria.

1 Introduction

Since the work of Lucas and Prescott ([1971]) and Prescott and Mehra ([1980]), recursive equilibrium (RE) have been a key focal point of both applied and theoretical work in characterizing sequential equilibrium for dynamic general equilibrium models in such fields as macroeconomics, international trade, growth theory, industrial organization, financial economies, and monetary theory.[1] When dynamic economies are Pareto optimal, in the case of homogeneous agent models and under standard concavity conditions, RE is unique, and can be computed using standard dynamic programming algorithms. In this case, equilibrium comparative statics analysis is reduced to either application of local or global implicit function theorem based smooth dynamical systems or applications of dynamic lattice programming methods to the social planner's problem. In nonoptimal economies, even the existence of dynamic equilibrium becomes complicated to prove, let alone obtain equilibrium comparative statics results. Although some recent extensions of dynamic lattice programming methods have been made for nonoptimal economies (including those with heterogeneous agents) in Mirman, Morand, and Reffett ([2008]) and Acemoglu and Jensen ([2015]), there are important nonoptimal homogeneous agent economies in which these tools are difficult to apply. Further, an extensive literature on monotone map methods has stemmed from the pioneering work of Coleman ([1991], [1997], [2000]) and Greenwood and Huffman ([1995]), but these methods are also known to fail in nonoptimal models (see, Santos ([2002], section 3.2).[2]

In this paper, we propose a new method for obtaining existence as well as equilibrium comparison in a well-studied class of nonoptimal homogeneous agent economies with dynamic complementarities. Our method is built on fixed point theory for parameterized monotone operators but, applicable to models with local indeterminacy, multiple equilibrium, and discontinuous minimal state space RE. We focus on dynamic general equilibrium models studied extensively in the literature with externalities and nonconvexities in production, public or monetary policy and monopolistic competition.[3] Very importantly, our results are based on *global* methods and make no appeal to local analysis, in contrast to the literature studying dynamic models with complementarities using the methods of smooth dynamical systems to characterize sequential equilibrium near steady states; see Benhabib and Farmer ([1994]), among numerous others.[4]

Our method involves "two-step" monotone maps on operator equations defined on partially ordered sets: each "step" of computation characterize the structure of a particular subclass of RE over individual and aggregate state variables. In essence, our two-step methods decompose the structural properties of RE

[1] See Stokey, Lucas, and Prescott ([1989]).

[2] See also Datta, Mirman, and Reffett ([2002]), Morand and Reffett ([2003]), and Datta, Mirman, Morand, and Reffett ([2005]) for extensions of monotone map methods.

[3] The "technology" we specify can be interpreted as a "reduced-form" for production in various nonoptimal economies, including models with monopolistic competition, taxes, learning, production externalities, and even some cash-in-advance models. See Benhabib and Farmer ([1994]), Greenwood and Huffman ([1995]) and Datta, Mirman, and Reffett ([2002]) for examples of other economies that fit this structure.

[4] For recent papers, see Beaudry and Poitier ([2007]), Jaimovich ([2007],[2008]), Wang and Wen ([2008]), Guo and Harrison ([2010]), Antoci, Galeotti, Russi ([2011]), d'Albis, Augeraud-Veron, and Venditti ([2012]), and Huang and Meng ([2012]), Braga, Modesto, and Seegmuller ([2014]).

over individual vs. aggregate states, allowing us to study RE over a large set of discontinuous functions. The key intuition that underlies these multi-step methods is rather simple: in the first step, we construct solutions to a parameterized fixed point problem that guarantee necessary structural restrictions implied by household optimization relative to *individual state variables*. Then, using a fixed point monotone comparative statics result on "first step" fixed points, we define a second step monotone operator, which verifies necessary *aggregate state consistency* conditions for an RE . The second stage fixed point structure allows us to compute *state asymmetric* RE, which is critical in the class of models with local indeterminacy e.g. The two-step procedure verifies the existence of RE via a *monotone* operator, and robust equilibrium comparative statics can be delivered in some deep parameters. An important implication is that the set of state asymmetric RE could be *huge* - consistent with results on local indeterminacy of sequential equilibrium in the literature. Note that, we do *not* need monotone RE for these methods to work, we need monotone operators defined on suitable chain complete partially ordered sets. Further, we do not need continuous RE. Indeed, our methods are designed specifically to allow for and construct discontinuous RE that are consistent with solutions to the household dynamic program.

It is, perhaps, important to point out that we obtain a rich set of robust RE comparative statics/dynamics without appealing directly to the lattice programming machinery of Topkis ([1978], [1998]) and Veinott ([1992]). Our methods can be interpreted as an iterative class of parameterized dynamic lattice programming problems built on the household program, not that of the social planner. Our work builds on Acemoglu and Jensen ([2015]), where a new approach to the existence of robust equilibrium comparative statics is proposed for in large dynamic models, and where they make significant progress in obtaining sufficient conditions for robust distributional equilibrium comparative statics. Although their methods are powerful for many important classes of dynamic economies (including situations where our methods do not apply), as we show in this paper, their sufficient conditions cannot be checked even in certain homogeneous agent economies. They apply dynamic lattice programming methods to the individual agent problem and obtain sufficient *partial* monotonicity of decision rules which are then exploited to deduce aggregate equilibrium comparative statics. In this sense, even though their results are more general, in some applications, the methods suffer from limitations similar to Mirman, Morand, and Reffett ([2008]).

The classical differential approach to equilibrium comparative statics goes far back to at least Samuelson ([1941]), and is perhaps best illustrated in the seminal work of Debreu ([1970], [1972]), who used powerful differential topology tools to bear on the question.[5] This method has been extended to dynamic economies by Kehoe, Levine and Romer ([1990]), and Santos ([1992]), among others. Most striking application of smooth equilibrium comparative statics is found in the extensive literature on "indeterminacy" of equilibrium in models of one-sector production with externalities e.g., see papers following the approach taken in Benhabib and Farmer ([1994]), Boldrin and Rustichini ([1994]), Benhabib and Perli ([1994]), and Farmer and Guo ([1994]).[6] These papers study determinacy of sequential equilibrium

[5] See also MasColell ([1986]) for a comprehensive discussion.

[6] E.g., for recent applications of these smooth dynamical systems methods, see Santos ([2002]), Jaimovich ([2007],[2008]), Wang and Wen ([2008]), Guo and Harrison ([2010]), Antoci, Galeotti, Russi ([2011]), Huang and Meng ([2012]), Nourry,

dynamics around a proposed hyperbolic point (e.g., the unique positive steady state of the model), and it is shown that if a smooth sequential equilibrium is present, the local dynamics would be consistent with a continuum of equilibrium paths leading to the steady state. An important new approach to the study of local (and global) indeterminacy is found in the Euler equation branching methods of Stockman ([2010]) and Raines and Stockman ([2010]). In methodology, our methods are very much in the spirit of these latter papers, but we ask a very different questions (i.e., we are concerned with existence of RE dynamics, and characterizing RE comparative statics; not a theory of the resulting RE dynamical system). These methods also cannot be applied to our RE (as we cannot prove the existence of continuous RE dynamics in the capital stock in aggregate states. But in principle, our methods seek to complement the results and methodological approach taken in these latter two papers.

The paper is laid out as follows. In the next section, we present a motivating example. In Section 3, we describe the class of homogeneous agent models analyzed in this paper. In section 4, we construct the RE operator and, in section 5, we prove existence and develop equilibrium comparative statics results. In the last section, we conclude with a discussion of our results with important applications.

2 A Motivating Example

We begin with a simple example of policy-induced indeterminacy[7] studied in Santos ([2002]), section 3.2. This class of economies is important as it is known that continuous RE do not exist within this class.[8] Further, from the numerical work of Peralta-Alva and Santos ([2010]), near an unstable steady-state of the model, there are a continuum of sequential equilibria. It bears mentioning, this economy is *identical* to those studied in Coleman ([1991], [2000]) and Mirman, Morand, and Reffett ([2008]), except the government imposes a *regressive* as opposed to a progressive income tax.

Time is indexed by $t \in \{0, 1, 2, ...\}$. Each period, agents are endowed with a unit of time which they supply inelastically to firms. There is no uncertainty. Discount factor is constant and given by $\beta \in (0, 1)$. The household's lifetime preferences are defined over consumption streams $\{c_t\}$, and given by:

$$\sum_{t=0}^{\infty} \beta^t u(c_t). \quad (1)$$

Usual assumptions are made on the period utility function u.

Assumption E1. *The utility function $u : \mathbf{R}_+ \to \mathbf{R}$ is strictly increasing, strictly concave, continu-*

Seegmuller, and Venditt ([2013]), Braga, Modesto, and Seegmuller ([2014]).

[7] The question of policy induced indeterminancies has been discussed extensively in the recent literature. The early literature considering this question includes papers by Schmidt-Grohe and Uribe ([1997]), Guo and Lansing ([1998]), Santos ([2002]), and Guo and Harrison ([2004]), while more recently, work by Nourry, Seegmuller, and Venditti ([2013]) and Nishimura, Seegmuller, and Venditti ([2015]), among others, has addressed this issue. In Datta, Reffett, and Woźny ([2015a]), we show how to extend the methods of this paper to more general problems involving policy-induced externalities.

[8] The "policy-induced" indeterminacy can identically arise in the model of Romer (1986) with "large" extermalities. That is, for $f(k, 1, K, 1) = k^{\alpha} K^{a}$ with $\alpha + a > 1$, $\alpha \geq 0$, $a \geq 0$, the arguments work exactly the same. We discuss this case in the last section of the paper.

ously differentiable, such that

$$\lim_{c\to 0} u'(c) = \infty \text{ and } \lim_{c\to\infty} u'(c) = 0,$$

or $u(c) = \ln c$.

There is a continuum of identical firms, operating in a competitive market, with technology $F(k, n)$, and we define $f(k) = F(k, 1)$, as the maximal possible output at state k. The following assumptions are imposed on technology:

Assumption E2. *The production function* $F : \mathbf{R}_+[0, 1] \to \mathbf{R}_+$ *is constant returns to scale, supermodular, increasing (but increasing strictly with each argument for the positive input of the other), weakly concave jointly (but strictly concave with each argument separately for the positive input of the other), and continuously differentiable in both arguments such that* $\lim_{k\to 0} F_1(k, 1) = \infty$. *Further, there exists a maximal sustainable capital stock* $k_{\max}$, *that is,* $F(k, 1) \leq k_{\max}$ *for all* $k \geq k_{\max}$ *and* $F(0, n) = F(k, 0) = 0$.

Assume that the government imposes a state contingent income tax, $\tau : \mathbf{K} \to [0, 1)$, where $\mathbf{K} = [0, k_{\max}]$. The household enters any given period with an individual level of capital $k \in \mathbf{R}_+$, facing an economy in aggregate state $K \in \mathbf{R}_+$where K is the per-capita capital stock. After-tax income for household in state (k, K) is,

$$(1 - \tau(K))y(k, K) = (1 - \tau(K))\{r(K)k + w(K)\},$$

where $r(K) = F_1(K, 1)$ is the rental rate for capital and $w(K) = F_2(K, 1)$ is the wage rate. Profits are zero by constant returns to scale. The income tax proceeds are redistributed as lump-sum transfer $J(K)$ back to households under a balance budget, $J(K) = \tau(K)y(K, K)$. Two broad classes of progressive and regressive state contingent tax policies will be considered next.

Assumption E3: *The tax function,* $\tau : \mathbf{K} \to [0, 1)$ *is either (i) progressive, i.e.* τ *is increasing and continuous; or (ii) regressive, i.e.* τ *is decreasing.*

Coleman ([1991]) analyzes case E3(i) of progressive taxation. E3(ii) includes the example in Santos ([2002], section 3.2), where it is claimed that continuous RE do not exist. In the Santo's example, the tax in Assumption E3(ii) is additionally Lipschitzian. We should also mention, Peralta-Alva and Santos ([2010]) show (numerically) the indeterminacy of sequential equilibria near the unstable steady state for the case that τ is decreasing and Lipschitz continuous.[9]

Denoting investment by x, the agent or household's budget constraint is,

$$c + x \leq \{(1 - \tau(K))(r(K)k + w(K)) + J(K)\} =: y_\tau(k, K). \tag{2}$$

The aggregate law of motion for capital in sequential equilibrium $\{K_t\}_{t=0}^\infty$ implies sequences of factor prices $\{r_t = r(K_t)\}_{t=0}^\infty$ and $\{w_t = w(K_t)\}_{t=0}^\infty$ from initial $K_0 > 0$. Further, the household assumes the

[9] Please note, that results for models under E3(i) here apply to Romer (1986) with "small externality." Suppose the production function is $\hat{f}(k, K) = f(k)e(K)$ where $e(K)$ is an externality and $\hat{f}_1(K, K)$ is decreasing in K. For models under assumptions E3(ii), we have the "large externality" case of Romer ([1986]), where $\hat{f}_1(K, K)$ could be increasing in K, for example, take $f(k) = k^\alpha$ and $e(K) = K^a$ for $a, \alpha \geq 0$ and $a + \alpha > 1$. This case has not been studied in the existing literature.

sequence of capital stock is recursively generated by a fixed function from initial capital $K_0 > 0$ as follows:

$$K_{t+1} = f(K_t) - C(K_t) =: g(K_t;C), \tag{3}$$

where $C \in B^f(\mathbf{K})$,

$$B^f(\mathbf{K}) = \{C : \mathbf{K} \to \mathbf{R}_+ | \ 0 \leq C(K) \leq f(K)\}$$

and $B^f(\mathbf{K})$ is the set of candidate recursive equilibrium consumption on $(\mathbf{K})$ equipped with pointwise partial order.[10]

In state $(k, K) \in \mathbf{K} \times \mathbf{K}$ the household faces aggregate capital dynamics generated by the candidate RE function $C \in B^f$ via (3). Under assumptions E1-E3, we construct a unique value function $V^* : \mathbf{K} \times \mathbf{K} \times B^f \to \mathbf{R}$ that satisfies the following Bellman equation for the household problem for each $g > 0$ and $K > 0$:

$$V^*(k, K; C) = \sup_{c \in [0, y_\tau(k,K)]} \{u(c) + \beta V^*(y_\tau(k, K) - c, g(K; C); C)\}, \tag{4}$$

Under assumptions E1 and E2, an optimal solution achieving the supremum on the right-hand side (4) is a unique continuous function to be denoted by $c^*(k, K; C)$.

A *minimal state space recursive equilibrium (RE)* is a consumption function $C^* \in B^f(\mathbf{D})$ and the corresponding law of motion g^* such that

$$\begin{aligned} C^*(K) &= c^*(K, K; C^*), \\ C^*(K) &= 0 \text{ else.} \end{aligned}$$

Note that, a RE is defined on the diagonal of the household's state space $\mathbf{K} \times \mathbf{K}$, i.e.,

$$\mathbf{D} = \{K | (K, K) \in \mathbf{K} \times \mathbf{K}\}$$

where $\mathbf{D}$ is the space $\mathbf{K}$ embedded into $\mathbf{K} \times \mathbf{K}$.

By a lattice programming argument applied to the household's dynamic program (4), we can further characterize a necessary condition for an RE. Under assumptions E1-E3, the unique optimal solution $c^*(k, K; C^*)$ in (4) has strong structural properties in the individual state k (although not in aggregate state K).

Proposition 1. Under assumptions E1, E2 and E3, in an RE with $k = K > 0$, (a) consumption function $c^*(k, K; C^*)$ is increasing and Lipschitz in k, and (b) investment function $x^*(k, K; C^*) = y_\tau(k, K) - c^*(k, K; C^*)$ is increasing and Lipschitz in k.

Proof. A standard argument shows that under E1, E2, and E3, the value function $V^*(k, K; C^*)$ is strictly concave and continuous in k, for each K, has a smooth envelope

$$V_1^*(k, K; C^*) = u'(c^*(k, K; C^*))r(K)(1 - \tau(K)),$$

[10] When referencing function spaces in the sequel, to minimize notation, after we define the function space, we will subsequently delete the domain for the set of functions when the context is clear. For example, $B^f(\mathbf{K})$ after definition will be referred to as B^f.

and $V_1^*(k,K;C)$ is decreasing in k; hence, consumption $c^*(k,K;g^*)$ is increasing in k which proves one part of (a). To prove (b), notice that the necessary and sufficient first order characterization of the unique optimal investment $x^* = x^*(k,K;C^*)$ is,

$$u'((y_\tau - x^*(k,K,C^*)) - \beta u'((y_\tau - x^*)(x^*, g^*(K); C^*)\ r(g^*(K))(1-\tau(g^*(K)) = 0. \tag{5}$$

where $g^*(K) = f(K) - C^*(K)$. From (a), a consumption $c^*(k,K;C^*) = (y_\tau - x^*)(k,K;C^*)$ is monotone increasing in k, when k rises, the left-hand side of (5) falls in k. As the continuation consumption $c^*(k',K';C^*) = (y_\tau - x^*)(k',K';C^*)$ is also monotone increasing in k, this implies $x^*(k,K;C)$ is increasing in k, which proves part of (b).

Finally, since $y_\tau(k,K)$ is Lipschitz for $K > 0$, $c^*(k,K;C)$ and $x^*(k,K;C)$ are both Lipschitz with a Lipschitz constant bounded by the Lipschitz constant of $y_\tau(k,K)$, namely $F_1(K,1)$ when $K > 0$. ■

We make a few remarks.

First, according to Proposition 1., *any* RE in the Santo's economy must be *continuous* in individual state k, so any discontinuities in a RE must occur only in aggregate states K.[11]

Second, if we approach the question of existence and characterization of RE via dynamic lattice programming methods (e.g., as discussed in Mirman, Morand, and Reffett ([2008]) and Acemoglu and Jensen ([2015]), and it is easy to see issues that arise under regressive taxation. Under Assumptions E1, E2, and E3(ii), the return on capital is not monotone in g. That is, a lattice programming argument cannot determine whether $c^*(k,K;C^*)$ is increasing or decreasing in C^*, and without further characterization of equilibrium single crossing properties; hence, existence of RE would need to verified, for example, by a topological argument. This would, for example, complicate the question of constructing monotone equilibrium comparative statics in the deep parameters of the economy (β, τ). Further, obtaining sharp characterizations of RE that have joint monotonicity properties of RE investment in both individual and aggregate states (i.e., monotonicity of $K \to c^*(K,K;C^*)$ in any RE C^*) directly via an application dynamic lattice programming, as in Mirman, Morand, and Reffett ([2008]), is not possible as requisite single-crossing properties are not evident. Actually, single-crossing properties are only shown to be held in *particular subclasses* of RE. This is discussed in section 4 of the paper.

Third, if we try to apply Coleman's monotone map method to verify existence of any RE with assumption E3(ii), we also run into serious problems. Following Coleman ([1991]), let us take a "guess" at future consumption function $C \in H$ where

$$H(\mathbf{D}) = \{C : D \to D | 0 \le C(K) \le f(K),\ C \text{ is continuous and increasing with } f - C \text{ increasing in } K\}.$$

Next, we define a mapping $Z_c(\hat{c}, k, K, C)$ based on the Euler equation as follows - for $C \in H$, $0 < C(K) < f(K)$, $K > 0$,

$$Z_c(\hat{c}, k, K, C) = u'(\hat{c}) - \beta u'(C(y_\tau(k,K) - \hat{c}))r(f(K) - \hat{c})[1 - \tau(f(K) - \hat{c})]. \tag{6}$$

[11] If you examine the sequential equilibrium from initial states (k_0, K_0), it can also be shown to require equilibrium investment choices to be locally Lipschitz from initial individual state k_0 and be consistent with the existence of an envelope theorem for the value function $V^*(k_0, K_0, \{K_t^*\}_{t=0}^{\infty})$ where $\{K_t^*\}_{t=0}^{\infty}$ is a sequential equilibrium path for capital.

The Coleman monotone map operator is,

$$
\begin{aligned}
A_c(C)(K) &= \hat{c}^*(K,K,C) \text{ such that } Z_c(\hat{c}^*(k,K,C),k,K,C)=0, \\
&= 0 \text{ if } C=0.
\end{aligned} \tag{7}
$$

Under assumption E3(i), everything works: that is, A_c is single-valued, isotone and has a nontrivial fixed point, which is a recursive equilibrium[12]. Also, the fixed point can be computed by successive approximations as the limits of non-stationary recursive equilibria for finite horizon economies. Further, one can show that the fixed point is increasing in β, and decreasing in τ.[13] However, under Assumption E3(ii), the Coleman monotone map operator A_c is *not single-valued*; rather, it is a nonempty upper hemicontinuous correspondence and does not necessarily admit a continuous selection in its first argument k, for each K, let alone a Lipschitz section as in Proposition 1.). Therefore, as Santos ([2002]) points out, the Coleman monotone map method cannot verify existence of a RE or characterize equilibrium comparative statics.

But the new methods in this paper do work for the simple case of economies under E3(ii). As we are interested in comparing RE, we will now specify explicitly how our new Euler equation operator depends on (β,τ), the deep parameters of the economy. This operator is a "two-step" extension of Coleman's monotone map method and the basis for a new construction applicable to more general economies. Our idea is to break up the construction of function $C^*(K)$ in the definition of a RE into two steps: i.e., let $h_1 \in H$, $h_2 \in B^f$, for $h_1 < f$, $k > 0$, define

$$
Z(\hat{c},k,K,h_1;h_2(K),\beta,\tau) = u'(\hat{c}) - \beta u'(h_1(f(k)-\hat{c}))r(f(k)-\hat{c})(1-\tau(f(K)-h_2(K))). \tag{8}
$$

Next, define the operator A as follows:

$$
\begin{aligned}
A(h_1,h_2,K;\beta,\tau)(k) &= \hat{c}^*(k,K,h_1,h_2;\beta,\tau) \text{ s.t. } Z(\hat{c}^*(k,K,h_1,h_2;\beta,\tau),k,K,h_1;h_2,\beta,\tau)=0,\ h_1>0, k>0 \\
&= 0 \text{ else.}
\end{aligned} \tag{9}
$$

Notice, the operator defined in (9) based in (8) differs from Coleman's operator defined in (7) based on (6) only by how it treats the "tax" in the second term of (6) versus (8). In this method, we add an additional step to the computation of the fixed point (compose τ with $h_2 \in B^f$) which allows us to study the complementarity structure of the household equilibrium Euler equation in "two-steps". In section 4, we show this "two-step" operator is monotone jointly in (h_1,h_2), and hence can be used to build monotone operator whose fixed points are RE. Further, as the operator is monotone, we can build a theory of robust equilibrium comparative statics in some deep parameters of the economy. In effect, the decomposition of the equilibrium fixed point problem deconstructs the single-crossing property for the household's problem into a single crossing property (in equilibrium) in two parts, one part isolating "individual" state dynamic complementarities, and a second part isolating "aggregate" state dynamic complementarities.

[12] Keep in mind, $C=0$ is a trivial fixed point.

[13] Here, "decreasing in τ" is in the pointwise partial order: $\tau_1 \geq \tau_2$ if $\tau_1(K) \geq \tau_2(K)$ for all K.

In terms of the example we are considering in this section: for any $h_2 \in B^f$, $h_1 \to A(h_1, h_2, K; \beta, \tau)$ is isotone. Also, the partial map $h_1 \to A(h_1, h_2, K; \beta, \tau)$ is *precisely* Coleman's "monotone map" operator embedded as a "first-step" operator in a "two-step" procedure. As H is a complete lattice under pointwise partial order, by Tarski's theorem, our first-step operator has a complete lattice of fixed points $\Psi_A \subset H$ for each (h_2, K, β, τ) with a trivial greatest fixed point at 0, and a *unique* (least) strictly "interior" fixed point $h^*(h_2, K, \beta, \tau) \in H$.[14]

Next, we can use the least fixed point to define a second step operator

$$\begin{aligned} A^*(h_2; \beta, \tau)(K) &= h^*(h_2, K; \beta, \tau)(K), \; h_2 \in B^f, K > 0 \\ &= 0 \text{ else} \end{aligned}$$

By Veinott's fixed point comparative statics result (e.g., Topkis ([1998], Theorem 2.5.2), $h^*(h_2, K; \beta, \tau)$ is increasing in h_2 on B^f for each (β, τ), with $h^*(h_2, K; \beta, \tau)$ also increasing in β, and decreasing in τ. By construction, $A^*(h_2; \beta, \tau) \in B^f$ but Notice B^f is a nonempty complete lattice. Therefore, by Tarski's theorem, the fixed point set of operator A^* denoted by $\Psi_{A^*}(\beta, \tau)$ is a nonempty complete lattice, where *each fixed point is an RE.*

Importantly, by construction for any $C^*(\beta, \tau) \in \Psi_{A^*}(\beta, \tau) \subset B^f$ we have that the optimal solution to the household dynamic program $(k, K) \to c^*(k, K, C^*; \beta, \tau) \in \mathbf{C}^*(\mathbf{D}, B^f(\mathbf{K}))$, where $\mathbf{C}^*(\mathbf{D}, B^f(\mathbf{K}))$ is the set of candidate RE consumption policies functions defined as:[15]

$$\mathbf{C}^*(\mathbf{D}, B^f(\mathbf{K})) = \{h : \mathbf{K} \times \mathbf{K} \to \mathbf{R}_+ | \text{s.t. for each } K \in \mathbf{K}, \; k \to h(k, K) \in H(\mathbf{D}), \text{ and } K \to h(K, K) \in B^f\}.$$

Further, as A^* is increasing in β and decreasing in τ, by a standard fixed point comparative statics argument, under *either* assumption E3(i) or E3(ii), the least RE, $\wedge \Psi_{A^*}(\beta, \tau)$, and the greatest RE, $\vee \Psi_{A^*}(\beta, \tau)$, are increasing in β and decreasing in τ.

Finally, notice we can provide even sharper characterization of *some* RE in this example (namely, some RE will have stronger monotonicity properties for investment). That is, assume additionally at function $K \to f(K) - h_2(K)$ is increasing and define a new space of candidate recursive equilibrium by

$$B^f_m = \{h_2 \in B^f | f(K) - h_2(K) \text{ is increasing}\}.$$

then, denoting by $A^*_m(h_2; \beta, \tau)$ the restriction of the mapping $A(h_2; \beta, \tau)$ to B^f_m. Now $A^*_m : B^f_m \to B^f_m$. As B^f_m is a subcomplete lattice of B^f, if we denote by $\Psi_{A^*_m}(\beta, \tau)$ the fixed point set of A^*_m, then, again by Tarski's theorem, $\Psi_{A^*_m}(\beta, \tau)$ is nonempty complete lattice of RE consumption functions having an associated RE investment monotone increasing (but, in general, discontinuous).

Finally, as $A^*_m(h_2; \beta, \tau)$ restricted B^f_m is also increasing in β, and decreasing in τ, we have the same robust equilibrium comparative statics relative to (β, τ) obtained in Coleman ([1991]) and Mirman, Morand, and Reffett ([2008]) not only for progressive taxes, but now for the case of regressive taxes

[14] Existence follows from Coleman ([1991]) and uniqueness of the strictly positive fixed point for the "first step" operator follows from Coleman ([2000]). Note, we show these results, as corollaries of our main theorems, in section.

[15] Subsequent to this, we shall again omit the domain of $\mathbf{C}^*$ from the notation. So, for example, $\mathbf{C}^*(\mathbf{D}, B^f(\mathbf{K}))$ will subsequently denoted by $\mathbf{C}^*(B^f)$.

as well. The difference between the cases is that with policy induced indeterminacy, we have multiple equilibria in general. We show these exist in multiple *subclasses* of RE, along with robust equilibrium comparative statics in all cases.

We make one final remark on this example: notice equilibrium consumption policies $\hat{c}^*$ in the "Santos economy" are *state asymmetric* relative to their structural properties on k versus K. That is, investment is continuous (Lipschitzian) in k with both investment and consumption being increasing in k; but, they are only bounded in K.

3 The General Framework

We study a general class of homogeneous agent dynamic general equilibrium models with externalities and complementarities with reduced-form production function that embeds numerous other non-optimal dynamic economies with complementarities including Benhabab and Farmer ([1994]) and Liu and Wang ([2014]), among others. Time is discrete and indexed by $t \in \{0, 1, 2, ...\}$. The economy has a continuum of identical infinitely-lived households with separable preferences over lifetime streams of consumption and leisure, with each household endowed with a unit of time each period which they supply elastically. Consumption and leisure streams are given by $\{c_t\}_{t=0}^{\infty}$and $\{l_t\}_{t=0}^{\infty}$, respectively, and

$$\sum_{t=0}^{\infty} \beta^t \{u(c_t) + v(l_t)\}, \tag{10}$$

is the lifetime utility for the representative household and $\beta \in (0, 1)$ is the discount factor.

Following regular assumptions are imposed on period utility functions (omitting time subscript):

Assumption A1: *The returns from consumption and leisure, $u : \mathbf{R}_+ \to \mathbf{R}_+$ and $v : [0, 1] \to \mathbf{R}_+$ are continuous, strictly increasing, strictly concave and C^1 with $u(0) = 0$, $v(0) = 0$ and Inada-type conditions are satysfied,*

$$\lim_{c \to 0} u'(c) \to \infty; \;\; \lim_{c \to \infty} u'(c) \to 0; \;\; \lim_{l \to 0} v'(l) \to \infty.$$

In addition to productive time, households are also endowed with an initial holding of capital denoted by $k_0 > 0$. Factor and goods markets are perfectly competitive. Households own factors of production, and rent them to firms who face production technologies with constant returns to scale (CRS) in private factors (k, n), where k is the individual firms decision on capital, n its decision on labor inputs, but we also allow for the economy to have externalities in the social production. As is standard in the literature, these social externalities depend on the per capital aggregate levels of capital K and labor N, respectively. We assume the following conditions on the production technology $F(k, n, K, N)$:

Assumption A2: *The production function $F : \mathbf{R}_+ \times [0, 1] \times \mathbf{R}_+ \times [0, 1] \to \mathbf{R}_+$ is multiplicatively separable in private returns and the social externalities: $F(k, n, K, N) = f(k, n)e(K, N)$; in addition, (a) f is constant returns to scale, supermodular, increasing (but increasing strictly with each argument for the positive input of the other), weakly concave jointly (but strictly concave with each argument separately* *for the positive input of the other), and continuously differentiable in both arguments with* $f(0,n) = 0 = f(k,0)$ *for* $n, k > 0$*; (b) the marginal products of* f *in capital and labor satisfy Inada-type conditions:*

$$\begin{aligned}
\lim_{k\to 0} f_1(k,n) &\to \infty \text{ for all } n \in (0,1], \\
\lim_{k\to \infty} f_1(k,n) &\to 0 \text{ for all } n \in [0,1], \\
\lim_{n\to 0} f_2(k,n) &\to \infty \text{ for all } k > 0;
\end{aligned}$$

and (c) the social externality e *is increasing and locally Lipschitz continuous jointly with* $e(0,N) = 0 = e(K,0) = 0$ *for* N*,* $K > 0$*. Also, (d) there exists a maximal sustainable capital stock* $k_{\max}$*, such that* $F(k,1,K,1) \leq k_{\max}$ *for all* $k, K \geq k_{\max}$*.*

We should make some remarks on assumptions A1 and A2.

First and foremost, there are no known results on the existence of *either* sequential or recursive equilibrium under these conditions.[16] In particular, no results are known on the existence of *smooth* sequential equilibrium. It bears mentioning that, in discrete time models without proving *smooth* sequential or recursive equilibria, at least locally near the steady-state, one *cannot* apply smooth dynamical systems methods (as is typically done in the literature to characterize local indeterminacy of equilibria).[17] Further, as the "Santos economy" ([2002], section 3.2) is embedded in our class of economies, we already know in our case that continuous (let alone *smooth*) RE do not exist.

Second, Coleman ([1997]) or Datta, Mirman, and Reffett ([2002]) do not handle assumption A2. In particular, these papers effectively do not consider the case of labor externalities, rather the case of elastic labor supply with income taxes and/or capital externalities under very strong restrictions. Further, Mirman, Morand, and Reffett ([2008]) only consider inelastic labor supply and no labor externality.

Third, equilibrium responses for labor supply is in general a *correspondence* under assumption A2. This plays a key role in our analysis. In particular, our methods for verifying RE involve *Euler equation branching methods* (see Raines and Stockman ([2010]) and Stockman ([2010]). That is, we construct a least and greatest selections of equilibrium labor supply in each period (contingent on consumption and the capital stock), and then parameterize "upper" and "lower" Euler equation operators. In general, though, these Euler equation branches are not ordered.

Fourth, if we allow for $e(K,0) > 0$ when $K > 0$, and $e(0,N) > 0$ when $N > 0$, our arguments still apply. The simplest case is $e(K,N) = (1-\tau(K))$, for $\tau(K) = [0,1)$, τ satisfying assumption E3(i) or E3(ii) in section 2 with $e(0,N) > 0$ when $K = 0$.

[16] Note that Feng et al ([2014]) do not apply to this economy.

[17] The Grobman and Hartmann theorem is the key result that is usually applied in this literature in justifying local methods to study determinacy of equilibrium via topological conjugacy arguments. In discrete time, this theorem requires sequential and/or recursive equilibrium dynamics near the steady state be *smooth* (e.g., to verify the steady state is hyperbolic and topological conjugacy). No such results on existence of smooth equilibrium dynamics in models with labor externalities (or large capital externalities) are known.

Further, given the results in Santos ([1991]) for Pareto optimal economies, one would assume such conditions would be *very strong*, requiring global strong concavity conditions for household dynamic programs along equilibrium paths. Hence, a global argument is needed to study indeterminancy.

3.1 Household Dynamic Programs

If aggregate RE labor supply is $N \in (0,1)$, then, by the profit maximization, the equilibrium factor prices along RE paths are given by:

$$\begin{aligned} r(K;N) &= f_1(K,N)\cdot e(K,N) \\ w(K;N) &= f_2(K,N)\cdot e(K,N) \end{aligned} \tag{11}$$

where r and w are the price of capital and labor, respectively. Let the maximal level of output in any state state K be given by $f^M(K) = f(K,1)e(K,1)$. Define the space of candidate, socially feasible consumption functions be given by

$$B^f(\mathbf{K}) := \{(C : \mathbf{K} \to \mathbf{R}_+ | 0 \le C(K) \le f^M(K)\} \tag{12a}$$

For the moment, endow B^f with the topology of pointwise convergence, as well as its pointwise partial order.

We can now develop a representation of the aggregate economy parameterized by $C \in B^f$ only. To generate the path for aggregate capital $\{K_t\}_{t=0}^{\infty}$ a RE, we need to further restrict the set of RE consumption and investment functions to reflect that fact that labor supply is endogenous in this economy. This fact will further restrict the space of possible RE investment/consumption functions we can consider. Anticipating the equilibrium conditions that govern the labor-leisure choice for a household in any RE, we posit the existence of a "contingent" representation of aggregate labor supply, given by $N(C,K) \in [0,1]$, where $N(C,K)$ represents the "static" equilibrium condition on RE labor supply, and is parameterized by both the current aggregate state K, and a level of consumption C. [18] This contingent aggregate labor supply mapping will imply further restriction on the attainable level of output in a RE, hence, shall restrict the possible laws of motion for the aggregate state variable in any RE. In particular, when developing the household's dynamic program in a candidate RE consumption function $C \in B^f$, for aggregate labor supply $N(C,K)$, we let households assume the law of motion on the aggregate capital stock is given by:

$$K' = g(K;C) = f(K, N(C,K))e(K; N(C,K)) - C(K). \tag{13}$$

Having that, and noting that the factor prices in (11) depend on K only, we can now generate the sequence of factor prices using the capital dynamics for $\{K_t\}_{t=0}^{\infty}$ using $g(K;C)$ from $K_0 > 0$.

Next, we develop the household's dynamic programming problem. First, given $N(C,K)$ for $C \in B^f$, appealing to zero profits under constant returns to scale, the household income process $y(k,n,K;N(C,K))$ is given by

$$y(k,n,K;N(C,K)) = r(K,N(C,K))k + w(K,N(C,K))n \tag{14}$$

where for $K > 0$,the income process is real-valued. The household's budget correspondence is then given by:

$$\Phi(k,K;N(C,K)) = \{c,n,x | c + x \le y(k,n,K;N(C,K)), c \ge 0, x \ge 0,\ n \in [0,1]\}$$

[18] As we will show in Lemma (4.), N is decreasing in C, and increasing in K.

where x denotes the household level of investment. Under Assumptions A1 and A2, as r, w are each continuous, the household's feasible correspondence Φ is a continuous correspondence, when $K > 0$.

Households can then use (C, N) to calculate the sequence of factor prices in a sequential equilibrium from initial states $K_0 > 0$. Then, the household's dynamic program can be stated as follows: given $C \in B^f$ such that $g(K;C) > 0$, with aggregate labor supply given by N, with $K \in \mathbf{K}_* = \mathbf{K}\backslash 0$, the household's value function $V^* : \mathbf{K} \times \mathbf{K}_* \times B^f \to \mathbf{R}_+$ satisfies the following parameterized Bellman equation:[19]

$$V^*(k, K; C) = \sup_{c,n \in \Phi(k,K;N(C,K))} \{u(c) + v(1-n) + \beta V^*(y(k, n, K; N(C, K)) - c, g(K; C); C)\}. \quad (15)$$

Let the optimal solutions for consumption and labor supply be given as $(c^*(k, K; C), n^*(k, K; C))$.

Definition 2. A *minimal state space RE* is pair of functions C^*, N^* for consumption and aggregate labor supply, as well as the associated value function $V^*(\cdot; C^*)$, law of motion g^* the optimal solutions $c^*(k, K; C^*)$ and $n^*(k, K; C^*)$ such that for any $k \in \mathbf{K}$:

$$\begin{aligned} c^*(K, K; C^*) &= C^*(K), \\ n^*(K, K; C^*) &= N^*(C^*(K), K), \\ g^*(K; C^*) &= f(K, N^*(C^*(K), K))e(K; N^*(C^*(K), K))) - C^*(K) \end{aligned}$$

with $C^*(0) = N^*(0, 0) = 0$.

Observe that for C^* and N^* the equilibrium law of motion on capital is given by:

$$K' = g^*(K; C^*) \quad (16)$$

when $K > 0$, $g^*(K; C^*(K)) > 0$, and $C^*(K) > 0$.

3.2 Necessary Properties of any RE

By a standard argument, $V^*(\cdot; C)$ is strictly concave and at least once-continuously differentiable in its first argument k (e.g., Coleman ([1991]) and the Mirman-Zilcha Lemma). This implies under Assumptions A1 and A2, objective function on the right-hand side of (15) is strictly concave in its control variable. Therefore, we can further characterize the optimal solutions in (15) by the first order conditions, which are necessary and sufficient. In particular, noting the Inada conditions, the optimal consumption $c^* = c^*(k, K; C)$ in any RE must satisfy the following functional equation: i.e, when $K > 0$, $g(K; C) > 0$ in (13):

$$Z(c^*, k, K; C) = u'(c^*) - \beta u'(c^*(y_{c^*}, y_{C^*}; C)) f_1(\frac{y_{C^*}}{N(C, y_{C^*})}) \cdot e(y_{C^*}, N(C, y_{C^*})) = 0 \quad (17)$$

[19] We could also allow for $u(c) = \ln(c)$. Denote $\mathbf{R}^*_- = \mathbf{R} \cup -\infty$. For the restrictions on the parameters stated on this problems, with log utility, the actual solution to the Bellman equation is an upper semicontious function $V^* : \mathbf{K} \times \mathbf{K} \times \mathbf{B}^f(D) \to \mathbf{R}^*_-$ that is continuous, when $K > 0$. Aside from that case, the range of the Bellman operator is actually $\mathbf{R}_+$.

where $y_{c^*} = y(k, n^*(k, K; C), K, N(C, K)) - c^*$, and where $r = f_1 \cdot e$ has been substituted in the Euler equation. Similarly, the first order condition associated with labor supply $n^*(k, K; C)$ in any RE is

$$Z_n(n^*, c^*, k, K; C) = \frac{v'(1 - n^*(k, K; C))}{u'(c^*(k, K; C))} - f_2(\frac{K}{N(C, K)}) \cdot e(K, N(C, K)) \tag{18}$$

where we have noted the assumption of homogeneity of production function in private returns in Assumption A2.

We can state some necessary continuity conditions along any RE.

LEMMA 3. *(Individual vs. Aggregate State Consistency in any RE).* Under A1 and A2, for any $C \in B^f$, and any $K \in \mathbf{K}_* := \mathbf{K}\backslash 0$, $g > 0$, the optimal solutions c^*, n^* are both continuous in k.

PROOF. The household's dynamic program in (15) can be written in terms on investment x as follows: given $C \in B^f$, with $K \in \mathbf{K}_* = \mathbf{K}\backslash 0$, $g(K; C) > 0$

$$V^*(k, K; C) = \max_{c, n \in \Phi(k, K; N(C, K))} \{u(c) + v(1 - n) + \beta V^*(y(k, n, K; N(C, K)) - c, g(K; C); C)\} \tag{19}$$

where by a standard argument, under A1 and A2, $V^*(k, K; C)$ is continuous and strictly concave in k. As the feasible correspondence $\Phi(k, K; N(C, L))$ is nonempty, compact and convex valued, and continuous in k, each (K, C), and the objective on the right-hand side of the Bellman equation at $V^*(k, K; C)$ in (19) is continuous and strictly concave in c each (k, K, C), by Berge's maximum theorem, $c^*(k, K; C)$ and $n^*(k, K; C)$ are both continuous in k, each (K, C). ■

Therefore, as in the motivating example, by Lemma 3., although RE must *necessarily* have structural properties in *individual* states, aside from resource feasibility, there are *no* required structural properties for a RE in aggregate states K. Our two step methods are going to allow us to exploit this fact, and decompose our fixed point arguments relative to individual vs. aggregate state variables, which will allow us to isolate the discontinuities of RE to only aggregate states. This means that although multiplicities of RE might be easy to construct, obtaining sufficient conditions for RE smoothness will be very difficult. This, therefore, casts concerns about applying smooth dynamical systems methods to our economies to characterize the multiplicity of equilibrium paths near any steady state associated with the model.

Also, Lemma 3. seems to pose serious challenges for developing rigorous applications of existing correspondence-based approaches to Generalized Markov equilibrium in the literature to compute RE in dynamic models (e.g., Kubler and Schmedders ([2003]) and Feng et. al. ([2014]). For example, existing methods do not work in the models in this paper. In fact any sequential equilibrium has to satisfy a version of Lemma 3. also. In particular, any sequential equilibrium that is being written recursively as in a Generalized Markov equilibrium on an enlarged state space involving "pseudo state variables" such as envelopes or shadow values for capital must deliver RE decision rules that are consistent with household value functions that are once-continuous differentiable in individual states along any sequential equilibrium path also.[20]

[20] Unfortunately, such a necessary condition is very difficult to check with correspondence-based approach in the spirit of Kubler and Schmedders ([2003]) and Feng et. al. ([2014]). In the former paper, the authors assume the correspondence of

4 Constructing the RE Operator Equation

We now construct our Euler equation operators. To do this, we first construct "contingent" equilibrium representations of labor supply $N(C,K)$, that we assumed when developing the household's dynamic program in (15). It bears mentioning, $N(C,K)$ represents the "static" necessary and sufficient condition for determining RE labor supply. In our economies, this static equilibrium relationship between consumption and labor supply is generally not unique. We will use the least and greatest selections from the correspondence N^*, each exhibiting the monotone comparative statics in (C,K) to parameterize "upper" and "lower" Euler equation operators.

4.1 Contingent RE Labor Supply

To construct contingent representation of labor supply in any RE, for any per capita equilibrium level of consumption C and any per capita aggregate labor supply N, define a new mapping :

$$Z_n(n,N,C,K) = \frac{v'(1-n)}{u'(C)} - f_2(K,n)\cdot e(K,N). \tag{20}$$

Define the mapping $\hat{n}^*(N;C,K)$ implicitly in equation (20):

$$Z_n(\hat{n}^*(N;C,K),N,C,K) = 0 \tag{21}$$

when $C>0$, all $K>0$. Noting, the Inada conditions on v and f_2 plus the strict concavity conditions, this root is well-defined and unique. Then extend it to include the boundaries by setting $\hat{n}^*(N;C,K)=1$, when $C=0, K>0$ and set it $=0$ otherwise. Lemma 4. characterizes solutions[21] to the equation $\hat{n}^*(N;C,K)=N$.

Lemma 4. Say Assumptions A1 and A2 are both satisfied. Then, (a) $\hat{n}^*$ is single-valued, and C^1 jointly when $(C,K)>>0$, $N\in(0,1)$. Further, (b) $\hat{n}^*(N,C,K)$ is increasing in N and K, and decreasing in C. Finally, (c) for each (C,K), $\hat{n}^*(N;C,K)$ has a nonempty compact set of fixed points $N^*(C,K)\subset[0,1]$, with the greatest selection $\vee N^*(C,K)$ and the least selection $\wedge N^*(C,K))$, both continuous, increasing in K, decreasing in C, and strictly positive, when $C>0, K>0$.

"pseudo state" variables is continuous; but a continuous correspondence need not admit an continuous selection for optimal decisions in a candidate sequential or recursive equilibrium. In Feng et. al. ([2014]), the Generalized Markov equilibrium on the expanded state space involves selecting from an *upper hemicontinuous correspondence*l that correspondence, of course, does *not* generally admit a locally Lipschitz selection in its first argument (let alone an increasing one).

[21] Under Assumption A2, when our technologies have the equilibrium wage rate w increasing in N (e.g., as in the case of Benhabib and Farmer ([1994]) and Liu and Wang ([2014]), among many others), the set of contingent RE labor supply decision will be a *correspondence.* We should note, in any *sequential equilibria* for Benhabib and Farmer models, for technologies evaluated at the so-called "indeterminacy parameters", this similar equilibrium labor supply decisions each period is also a *correspondence.* This means smoothness conditions near steady states required to apply the Grobman-Hartman Theorem and/or stable manifold theorem are going to be problematic. That is, its difficult to prove the existence of smooth equilibria, which is required to check the hypotheses of theorems needed to apply smooth dynamical systems methods to characterize the local determinacy of sequential equilibrium. This is true for both discrete and continuous time models.

Proof. (a) Note $Z_n(n, N, C, K)$ is strictly increasing in n, $\hat{n}^*(N; C, K)$ is unique for each (C, K)). By the Inada conditions on v and f in n, for all $N \in [0, 1]$, when $C > 0$, $K > 0$, $\hat{n}^*(N; C, K) \in (0, 1)$. Further, for all $N \in [0, 1]$, when $C > 0$, $K > 0$, $N \in (0, 1)$, as $|\partial_n Z_n(n^*(N; C, K), N, C, K)| \neq 0$ by strict concavity of v and f, by the global implicit function theorem, root $\hat{n}^*(N; C, K)$ is also globally C^1 (e.g., see Phillips ([2012], lemma 2)).

(b) The comparative statics result follows from the fact that under Assumptions A1 and A2, $Z_n(n, N, C, K)$ is strictly increasing in (N, C) and strictly decreasing in K.

(c) The set $[0, 1]$ is a complete lattice, $N \rightarrow \hat{n}^*(N, C, K)$ is a increasing function on $[0, 1]$ for each (C, K), hence, by Tarski's theorem (([1955]), Theorem 1), the fixed points of $\hat{n}^*(\cdot, C, K)$ denoted by $N^*(C, K)$ form a nonempty complete chain for all (C, K). As $\hat{n}^*$ is decreasing in K and increasing in C, by Veinott's fixed point comparative statics theorem ([1992]), Chapter 4, Theorem 14), the greatest fixed point $N^*(C, K)$ and least fixed point $\wedge N^*(C, K)$, each selection well-defined, are increasing in K, and decreasing in C.

Finally, the positivity of each selection, when $C > 0$, $K > 0$,. We have $n(0; C, K) > 0$ by the Inada conditions in Assumption A1. Finally, the continuity of the least and greatest selections $\vee N^*(C, K)$ and $\wedge N^*(C, K)$ follows from a modification of the tranversality argument in Raines and Stockman ([2010], Propositions 4 and 5) and C^1 assumption on u. ■

We make a few remarks on Lemma 4. . First, the existence of sunspot equilibria does not require capital externalities. That is, by a modification of the tranversality argument in Raines and Stockman ([2010], Proposition 4 and 5), for economies satisfying Assumptions A1 and A2, if we additionally assume the technologies are Cobb-Douglas (e.g., as in Benhabib and Farmer ([1994]) and Liu and Wang ([2014])), and the "indeterminacy parameters" in Benhabib and Farmer ([1994]), there are *two* continuous selections in N^*. This fact implies RE in our case will not be unique (even if each branch of our Euler equation operators we define in the next section of the paper have unique fixed points). Hence, sunspot equilibria will exist driven only by labor externalities.

For the case of more general assumptions on production technologies under Assumption A2, the Raines-Stockman results imply there are $0 < n < \infty$ solutions for contingent labor supply N^*, with n even. The problem with studying RE with each these selections is only the least and greatest selections are known to exhibit monotone comparative statics in (C, K); hence, for other selections, constructing any monotone map method (traditional or two-step methods) appear to be much more challenging.

Finally, its important to note that for a finite horizon version of our model, in the terminal period T, when equilibrium wage rates in our economies have $w(K, N) = f_2(K, N)e(K, N)$ are decreasing in N, the solution for the "lower bound" for RE labor supply, denoted by $N_f^*(K)$, is *unique,* where $N_f^*(K)$ is the unique n solving $Z_n(n, n, f(K, n)e(K, n), K) = 0$. This is the case, for example, in the models with elastic labor supply studied in Coleman ([1997]) and Datta, Mirman, and Reffett ([2002]). In these papers, the authors compute the (unique) RE as the "limit" of policy iteration type methods from (nonstationary) RE for finite horizon economies of length T. Then, the equilibrium for the terminal period economy implies a "lower bound" for RE labor supply (hence, RE output), namely, $f^*(K) = f(K, N_f^*(K))e(K, N_f^*(K))$, which is used as the "upper" bound for RE consumption for RE in the infinite horizon case, where the one period equilibrium labor supply $N_f^*(K)$ is the unique lower bound for labor supply in any RE for the infinite horizon economy.

This is *not* true in our models under Assumption A1 and A2. That is, we have *multiple equilibrium* in the terminal period for any finite horizon economy of length $T < \infty$. This fact requires us to be very careful when constructing the maximal level of output that given contingent labor supply is possible when defining the function spaces where RE can be shown to exist. That is, it turns out RE consumption cannot exceed the level of output $f^*(K) = f(K, N_f^*(K))e(K, N_f^*(K))$, but now we have many possible candidates for $N_f^*(K)$.

To see that, observe that under Assumptions A1 and A2, we can compute the set of terminal period equilibrium labor supplies $N_f^*(K)$ as the fixed point of the mapping $\hat{n}_f^*$ defined implicitly by $Z_n^f(\hat{n}_f^*(N, K), N, K) = 0$ for all $N \in [0, 1]$, $K > 0$, where

$$Z_n^f(n, N, K) = \frac{v'(1-n)}{u'(f(K, n, K, n))} - f_2(K, n) \cdot e(K, N) \tag{22}$$

and where we set $\hat{n}_f^*(N, 0) = 0$. In the general case of Assumption A2, where equilibrium wages $f_2(K, n)e(K, n)$ could be rising in n, we can have multiple (but finite) number of terminal period equilibria, each continuous in K.

4.2 RE via Two step Monotone Map Operators

We now define our Euler equation operators, one operator for each contingent aggregate RE labor supply $n_\vee^*(C, K) := \vee N^*(C, K)$ and $n_\wedge^*(C, K) := \wedge N^*(C, K)$.[22] We must also parameterize the space of feasible RE consumption function, in particular, in a manner that reflects the definition of these Euler equation operators. In particular, we shall always need to impose a restricted version of the upper bounded for output contingent on candidate RE consumption. In particular, we shall use the modified production function evaluated at $n_\vee^f(K) = \vee N_f^*(K)$ and $n_\wedge^f(K) = \wedge N_f^*(K)$:

$$f_\nu^*(K) = f_\nu^*(K, n_\nu^f(K))e(K, n_\nu^f(K)) \leq f^M(K)$$

where $\nu \in \{\vee, \wedge\}$ and $f^M(K) = f(K, 1, K, 1)$. Observe, we have a strict inequality here when $K > 0$ (as $n_\wedge^*(C, K) \leq n_\vee^*(C, K) \leq 1$ for all $K \in \mathbf{K}$, with equality when $K > 0$). Then, under Assumption A2, we also have

$$f_\vee^*(K, n_\vee^f(K))e(K, n_\vee^f(K)) \geq f_\wedge^*(K, n_\wedge^f(K)e(K, n_\wedge^f(K))) \tag{23}$$

for all $K \in \mathbf{K}$.

4.2.1 Some Useful Function Spaces

Our method proceeds in two steps In the first one, we will construct function spaces that guarantee RE have all the requisite continuity properties required of any RE per Lemma 3.; in the second step, we then

[22] That is, we will have Euler equation branching. See Raines and Stockman ([2010]), where the idea of Euler equation branching methods was first introduced.

seek all equilibria are consistent with have RE being simply feasible and bounded in aggregate states.

To provide a roadmap for how our construction in the remainder of this section will proceed, we briefly summarize the general structure of our two step fixed point methods. We shall define a mapping A on domain $H_1 \times H_2$, where H_2 is at least chain complete. Our "first step" operator will treat $h_2 \in H_2$ as fixed, and then study the fixed points of the partial map $h_1 \to A(h_1, h_2)$ in H_1. Then, in our "second step", we will define an operator that will use as their domain H_2, and map to a subset of H_2, where we shall prove RE exist. So we first define and discuss the domains for both the first and second steps; in the next section, we then define the operators we shall construct.

Recall the space B^f defined in equation (12a). Our "first step" operators will always use as their domain the following space: $h_1 \in H_\nu \subset B^f$, for $\nu = \{\vee, \wedge\}$:

$$\begin{aligned} H_\nu(\mathbf{D}) \quad := \quad & \{h_1 : \mathbf{D} \to \mathbf{R}_+ |, h_1 \text{ is increasing and continuous, such that} \\ & f^*_{h_1,\nu}(k) := f(k, n^*_\nu(h_1(k), k)) e(k, n^*_\nu(h_1(k), k)) - h_1(k) \text{ increasing in } k\}. \end{aligned}$$

Endow H_ν with its pointwise partial order, and the topology of uniform convergence. Notice also, as $n^*_\vee(C)(K) \geq n^*_\wedge(C)(K)$ pointwise all (C, K), by Assumption A2, we have for any $h_1 \in H_\nu$

$$f^*_{h_1,\vee}(k) \geq f^*_{h_1,\wedge}(k).$$

The space H_ν has desirable chain completeness and compactness properties, and noted in the following Proposition proved in Coleman ([1997]):

Proposition 5. Under A2, H_ν is compact in the space of continuous bounded functions endowed with the topology of uniform convergence (hence, chain complete under pointwise partial orders).

Proof. The compactness of follows H_ν from Coleman ([1997], Lemma 8), noting in Coleman's lemma, relative to our space H_ν, $u'(h_1(k)) = M(k)$ is falling in k for $h_1 \in H_\nu$. The chain completeness of H_ν follows as any compact partially ordered metric space is chain complete (e.g., Amann ([1977], Corollary 3.2)). ■

Although we will often use B^f as our second step domain, we will also use the following subset of B^f to prove the existence of RE where the aggregate consumption function h is decreasing in the aggregate state K, and the implied RE investment is increasing in K (i.e., we have $f^M(K) - h(K)$ is increasing in K)

$$B^f_m(\mathbf{D}) = \{h \in B^f | h \text{ is decreasing}\}.$$

Using our first and second step domains, can now define the ranges of our second step mappings when using the different domains B^f or B^f_m. In particular, we shall prove RE policies c^* exist for our economies in the following function spaces $\mathbf{C}^*_\nu$ defined as follows:

$$\mathbf{C}^*_\nu(\mathbf{D}, B^f(\mathbf{K})) = \{h : \mathbf{K} \times \mathbf{K} \to \mathbf{R}_+ | \text{s.t. for each } K \in \mathbf{K},\ k \to h(k, K) \in H_\nu(\mathbf{D}),\ \text{and } K \to h(K, K) \in B^f(\mathbf{K})\}. \tag{24}$$

for $\nu = \{\vee, \wedge\}$ indexes the set H_ν. Notice, RE policies in any of the spaces $\mathbf{C}_\nu^*(B)$ with $B = \{B^f, B_m^f\}$ is consistent with the necessary properties of any RE in Lemma 3.. We have following useful lemma.

Lemma 6. Under Assumptions A2 , (a) B^f and B_m^f are complete lattices. Moreover (b) $\mathbf{C}_\nu^*(B^f)$ is a complete lattice, $\mathbf{C}_\nu^*(B_m^f)$ is subcomplete.

Proof. (a) For $\nu = \{\vee, \wedge\}$, to see B^f is a complete lattice, consider any subset $B_1 \subset B^f$. As the pointwise inf and sup operations on the elements of B preserve pointwise bounds, we have $0 \leq \inf_x B_1 \leq f^M$, and $0 \leq \sup_x B_1 \leq f^M$; hence, $\wedge B_1 \in B^f$ and $\vee B_1 \in B^f$. Therefore, B_ν^f is a complete lattice. For $B_1 \subset B_m^f$, as the pointwise sup (resp, inf) operation preserves monotonicity, $\wedge B_1 \in B_m^f$ and $\vee B_1 \in B_m^f$. (b) For $B_1 \subset \mathbf{C}_\nu^*(B_m^f)$ as monotonicity in K (resp, equicontinuity at k), when $k = K$ are preserved also under arbitrary pointwise sup and inf operations on the compact set $\mathbf{D}$, $\wedge B_1 \in \mathbf{C}_\nu^*(B_m^f)$ and $\vee B_1 \in \mathbf{C}_\nu^*(B_m^f)$. Similarly, for $B_1 \subset \mathbf{C}_\nu^*(B^f)$. ■

4.2.2 Two Step Monotone Map Operators

We now construct our Euler equation operators. To do this, we first rewrite the equilibrium version of the household Euler equation in (15) on the larger collection of functions $(h_1, h_2) \in H_\nu \times B^f$. For the rest of this section, fix the index $\nu = \{\vee, \wedge\}$. Then, for $k = K > 0$, $h_1 > 0$, consider the following mapping:

$$Z_\nu(\hat{c}, k, h_1, h_2(K)) = u'(\hat{c}) - \beta u'(h_1(f_{\hat{c},\nu}^*(k))) r(\hat{c}, h_1(f_{\hat{c},v}^*(k)), h_2(K)) \tag{25}$$

where the distorted return on capital is given by:

$$r(\hat{c}, h_1(f_{\hat{c},v}^*(k)), h_2(K)) = f_1\left(\frac{f_{\hat{c},v}^*(k)}{n_\nu^*(h_1(f_{\hat{c},\nu}^*(k)), f_{h_2,\nu}^M(K))}\right) e(f_{h_2,\nu}^M(K), n_\nu^*(h_1(f_{\hat{c},\nu}^*(k)), f_{h_2,\nu}^M(K)))$$

Here, for $h_1 \in H_\nu$, $r(\hat{c}, h_1(f_{\hat{c},v}^*(k)), h_2(K))$ is increasing and continuous in $\hat{c}$, and decreasing in (K, h_1, h_2), noting we have defined

$$f_{\hat{c},\nu}^*(k) = f(k, n_\nu^*(\hat{c}, k)) e(k, n_\nu^*(\hat{c})(k)) - \hat{c},$$

and

$$f_{h_2}^M(K) = f^M(K) - h_2(K).$$

For $K > 0$, $h_1 \in H_\nu$, $h_1 > 0$, $h_2 \in B^f$, define the mapping $\hat{c}_\nu^*$ implicitly as follows:

$$Z_\nu(\hat{c}_\nu^*(k, h_1, h_2(K)), k, h_1, h_2(K)) = 0. \tag{26}$$

Then, the Euler equation operator defined as follows: when $(h_1, h_2) \in H_\nu \times B^f$:

$$\begin{aligned} A_\nu(h_1, h_2(K))(k) &= \hat{c}_\nu^*(k, h_1, h_2(K)),\ k > 0,\ h_1 > 0,\ h_2 < f^M \\ &= f^M(k) \text{ if } h_1 > 0,\ h_2 = f^M \text{ for any} \\ &= 0 \text{ otherwise.} \end{aligned} \tag{27}$$

We first study the monotonicity and order continuity properties of the operator A_ν. Before doing this, we define a few terms. Consider a mapping $f : X \to Y$, where X and Y are each countable chain complete partially ordered sets. We say F is *order continuous* if $A(\vee C) = \vee f(C)$ and $A(\wedge C){=}\wedge f(C)$ for all countable chains $C \subset X$. If X and Y are additionally Banach spaces, say $A(x)$ is a *compact operator* if it is (a) continuous (relative to the norm topologies on X and Y), and (b) for any bounded $C \subset X$, $A(C) \subset X$ is relative compact. We have the following useful result:

PROPOSITION 7. Say $X(S)$ a collection of functions on $S = [0,1]$, $X(S)$ compact in the topology of uniform convergence and endowed with the pointwise partial order, $f : X(S) \to X(S)$ is isotone and compact. Then, f is order continuous on $X(S)$.

PROOF. As $X(S)$ is compact in the topology of uniform convergence, $X(S)$ is compact in the topology of pointwise convergence (as pointwise and uniform convergence coincide in $X(S)$); hence, $X(S)$ is chain complete in the pointwise partial order on $X(S)$ (Amann, [1977], corollary 3.2). As $A : X(S) \to X(S)$ is isotone and continuous on $X(S)$ in the topology of uniform convergence, f is continuous in the pointwise topology. This implies f is continuous in the interval topology of $X(S)$ associated with pointwise partial orders (as interval topology in this case coincides with the uniform topology/pointwise topology in $X(S)$). Hence, f is order continuous in pointwise partial orders. ■

We now use Proposition 7. to prove the following Lemma concerning the monotonicity and order continuity properties of A_ν.

LEMMA 8. Consider the following collection of subsets of B^f index by $\bar{h}_2 \in B^f$: for any $0 \le \bar{h}_2 \le f^M$, with $\bar{h}_2(k) < f^M(k)$ when $k > 0$, $[0, \bar{h}_2] \subset B^f$. Then, under Assumptions A1 and A2, (a) for any $(h_1, h_2) \in H_\nu \times \{B^f \cap [0, \bar{h}_2]\}$, $A_\nu(h_1, h_2) \in H_\nu$. Further, (b) $A_\nu(h_1, h_2)$ order continuous on $H_\nu \times \{B^f \cap [0, \bar{h}_2]\}$.

PROOF. (a) We first prove for any $\bar{h}_2 \in B^f$, $0 \le \bar{h}_2 \le f^M$, with $\bar{h}_2(k) < f^M(k)$ when $k > 0$, for the order interval $[0, \bar{h}_2] \subset B^f$, $(h_1, h_2) \in H_\nu \times [0, \bar{h}_2]$, $A_\nu(h_1, h_2) \in H_\nu$.

Fix $h_2 \in [0, \bar{h}_2]$. By the continuity and monotonicity properties of $h_1 \in H_\nu$, under Assumption A1 and A2, when $k > 0$, $h_1 > 0$, $Z_\nu(\hat{c}, k, h_1, h_2(K))$ is decreasing and continuous in $\hat{c}$, increasing and continuous in k, for each (K, h_1, h_2). Further, noting the Inada conditions on u and f and the fact that for $h_2 \in [0, \bar{h}_2]$, $f^M(k) - h_2(k) > 0$, we therefore have (i) the existence of a unique root $\hat{c}^*_\nu(k, h_1, h_2(K))$ such that $Z_\nu(\hat{c}^*_\nu(k, h_1, h_2(K)), k, h_1, h_2(K)) = 0$, such that (ii) $\hat{c}^*_\nu$ is increasing and continuous in k for fixed (K, h_1, h_2). Therefore, noting the definition of $A(h_1; h_2)$ when $k = K = 0$, we have $k \to A_\nu(h_1; h_2)(k)$ continuous and increasing in k.

Further, if $k_1 \ge k_2$, $A_\nu(h_1; h_2)(k)$ is increasing in k, we have

$$u'(\hat{c}^*_\nu(k_1, h_1, h_2(K))) \le u'(\hat{c}^*_\nu(k_2, h_1, h_2(K)))$$

This implies by the definition of $\hat{c}^*_\nu(k, h_1, h_2(K))$, the second term of $Z_\nu(\hat{c}^*_\nu(k, h_1, h_2(K)), k, h_1, h_2(K))$ must decrease when $k_1 \ge k_2$. That is, $A_\nu(h_1; h_2)(k) = \hat{c}^*_\nu(k, h_1, h_2(K))(k)$ must be such that

$$f^*_{A_\nu(h_1;h_2),\nu}(k) = f(k, n^*_\nu(A_\nu(h_1; h_2)(k), k))e(k, n^*_\nu(A_\nu(h_1; h_2)(k), k)) - A_\nu(h_1; h_2)(k)$$

is increasing in k. Noting the definition of $A_\nu(h_1, h_2(K))(k)$ elsewhere, $A_\nu(h_1; h_2) \in H_\nu$ for each $h_2 \in [0, \bar{h}_2]$. As $\bar{h}_2 \in B^f$, $\bar{h}_2(k) < f^M(k)$ when $k > 0$ was arbitrary, that proves (a).

(b) Consider a function $A : X_1 \times X_2 \to Y$, where X_1, X_2 and Y nonempty and chain complete. We begin by mentioning two facts about order continuous operators. First, the operator A is order continuous jointly in $x = (x_1, x_2)$ iff it is order continuous in each argument (see Stoltenberg-Hansen et. al. ([1994], Proposition 2.4)). Therefore, for claim of the lemma, it suffices to check the order continuity of $A_\nu(h_1, h_2)$ in each argument separately. Second, an order continuous operator is necessarily isotone (e.g., see Dugundji and Granas ([1982], p15).

Therefore, for any $\bar{h}_2 \in B^f$, $[0, \bar{h}_2] \subset B^f$, consider $(h_1, h_2) \in H_\nu \times [0, \bar{h}_2]$. We first show A_ν is isotone in $(h_1, h_2) \in H_\nu \times [0, \bar{h}_2]$. To see this, observe for $k = K > 0$, $h_1 > 0$, $h_2 \in [0, \bar{h}_2]$, as Z_ν in (25) is decreasing and continuous in $\hat{c}$, and increasing in (h_1, h_2), the operator A_ν is isotone in (h_1, h_2). Noting the definition of $A_\nu(h_1, h_2)$ elsewhere in (27), A_ν is isotone on $H_\nu \times \{B^f \cap [0, \bar{h}_2]\}$.

By Lemma 6., $H_\nu \times$ B^f is a complete lattice; hence, countably chain complete. It follows that $H_\nu \times [0, \bar{h}_2]$ is countably chain subcomplete for any $\bar{h}_2 \in B^f$ that satisfies conditions of this lemma.

We first show A_ν preserves the supremum of countable chains in $h_1 \in H_\nu$, for each $h_2 \in [0, \bar{h}_2]$. When $h_2 \in [0, \bar{h}_2]$, $k = K > 0$, consider the countable chain $\{h_1^n\}$ each $h_1^n \in\subset H_\nu$. As for each $h_2 \in [0, \bar{h}_2]$, $h_1 \to A_\nu(h_1; h_2)$ is simply a special case of the nonlinear operator studied by Coleman ([1997]) (e.g., see Coleman ([1997], equation (9), lemma 5, 6, and 8). Therefore, by a result in Coleman ([1997], lemma 9), $h_1 \to A_\nu(h_1; h_2)$ is a compact operator (therefore, continuous in both the topology of uniform and the topology of pointwise convergence). Then, by Proposition 7., $h_1 \to A_\nu(h_1; h_2(K))$ is order continuous in its first argument.

Next, we next show of $A_\nu(h_1, h_2)$ preserves the supremum of countable chains in $\{h_2^n\}$ each $h_2^n \in [0, \bar{h}_2]$, for each $h_1 \in H_\nu$. For fixed $h_1 > 0$, $k = K > 0$, by the continuity assumptions on the derivatives of the primitives in A1 and A2, for all $k = K > 0$, we have

$$\begin{aligned} Z(\hat{c}_\nu^*(k_2, K, h_1, \vee h_2^n), k, h_1, \vee h_2^n) &= \vee Z(\hat{c}_\nu^*(k_2, K, h_1, \vee h_2^n), k, h_1, h_2^n) \\ &= Z(\vee \hat{c}_\nu^*(k_2, K, \vee h_1, h_2^n), k, h_1, h_2^n) \end{aligned}$$

where the first line follows from the fact that under assumption A2, Z is continuous pointwise in k', $n_\nu^*(C, K)$ is continuous C for $\nu = \{\vee, \wedge\}$, and $\vee(f - h_2^n)(K) = (f - \vee\, h_2^n)(K)$, for each K; the last line follows from the fact that under Assumptions A1 and A2, as for $h_1 \in H_\nu$, Z is continuous in $\hat{x}$. Therefore, we have

$$A(h_1; \vee h_2) = \vee A(h_1; h_2).$$

Noting the definition of $A_\nu(h_1, h_2)$ elsewhere, we have A is order continuous in $h_2 \in [0, \bar{h}_2]$ for all $\bar{h}_2 \in B^f$, $\bar{h}_2(k) < f^M(k)$ when $k > 0$.

The fact that $A_\nu(h_1, h_2)$ is meet preserving relative to countable chains in $\hat{H}_\nu \times [0, \bar{h}_2]$ for all $\bar{h}_2 \in B^f$, $\bar{h}_2 < f^M$ follows from a dual argument. ■

We use this lemma to show for each $h_2 \in B^f_\nu$, the operator $h_1 \to A_\nu(h_1; h_2)$ has a nontrivial strictly positive greatest fixed point, and this greatest fixed point is isotone in $h_2 \in [0, \bar{h}_2]$, where in the Lemma below, again, we always take $0 \leq \bar{h}_2 \leq f^M$, with $\bar{h}_2(k) < f^M(k)$ when $k > 0$, $\bar{h}_2 \in B^f$, $[0, \bar{h}_2] \subset B^f$:

LEMMA 9. Under Assumptions A1 and A2, for $h_2 \in [0, \bar{h}_2]$, and $\nu = \{\vee, \wedge\}$, (a) $h_1 \to A_\nu(h_1; h_2(K))$ has a greatest fixed point $h^*_\nu(h_2(K)) \in H_\nu$, with $h^*_\nu(h_2(K))(k) > 0$ when $k > 0$; (b) this fixed point can be computed by successive approximation from $h_0 = f^*_\nu$ as

$$\inf_n A^n_\nu(f^*_\nu; h_2(K)) = h^*_\nu(h_2(K)).$$

where $\inf_n A^n_\nu(f^*_\nu; h_2(K))(k) = \lim_n A^n_\nu(f^*_\nu; h_2(K))(k)$. Finally, (c) h^*_ν is isotone in $h_2 \in [0, \bar{h}_2]$.

PROOF. (a) Existence of greatest fixed point of $h_1 \to A_\nu(h_1; h_2)$: For each $h_2 \in [0, \bar{h}_2]$, as $A_\nu(h_1; h_2(K))$ is an isotone transformation of H_ν, and H_ν is a nonempty complete lattice, hence by Tarski's theorem ([1955], Theorem 1), the set of fixed points of $h_1 \to A_\nu(h_1; h_2(K))$ is a nonempty complete lattice.

(b) Computation of greatest fixed point $h_1 \to A_\nu(h_1; h_2)$: For each $h_2 \in [0, \bar{h}_2]$, as $h_1 \to A_\nu(h_1; h_2(K))$ is order continuous on H_ν, consider the iterations $A^n(f^*_\nu; h_2)$. Then, $\{A^n_\nu(f^*_\nu; h_2)\}^\infty_{n=0}$ is a decreasing chain. As pointwise and uniform convergence coincide in H_ν, and pointwise convergence⇒order convergence in H_ν by Proposition 7., we have

$$\begin{aligned}\lim_{n\to\infty} A^n_\nu(f^*_\nu; h_2(K)) &= \inf_n A^n_\nu(f^*_\nu; h_2(K)) \\ &= h^*_\nu(h_2(K)) \\ &= \vee\Psi_{A_\nu}(h_2(K))\end{aligned}$$

where $\Psi_{A_\nu}(h_2(K)) \in H_\nu$ for each $h_2 \in [0, \bar{h}_2]$ is the fixed point set of $h_1 \to A_\nu(h_1; h_2(K))$, with a trivial least fixed point $\wedge\Psi_{A_\nu} = 0$). That $h^*_{1,\nu}(h_2(K))(k)$ is strictly positive, when $k > 0$ follows from a modification of a standard argument involving the Inada conditions and iterations along RE paths (e.g., Coleman ([1997], lemma 11 and Theorem 12)).

(c) Isotonicity of $h^*_\nu(h_2(K))$: follows from Veinott's fixed point comparative statics result (i.e., Topkis ([1998], Theorem 2.5.2), noting $A_\nu(h_1; h_2(K))$ is increasing in $h_2 \in [0, \bar{h}_2]$.

Finally, noting that $\bar{h}_2$ satisfying the conditions of the theorem was arbitrary, this completes the proof. ■

Using Lemma 9., we can now define our RE operator A^*_ν based on the greatest fixed point of our first step operator $h^*_\nu(h_2(K)) \in H_\nu$. To do this, we first need to choose a suitable upper bound $\bar{h}_2 \in B^f$ (resp., $\bar{h}^m_2 \in B^f_m$) such that $A^*_\nu(\bar{h}_2) \leq \bar{h}_2$ (resp., $A^*_{m,v}(\bar{h}^m_2) \leq \bar{h}^m_2$) relate to the chain complete order interval $[0, \bar{h}_2] \subset B^f$ (resp, $[0, \bar{h}^m_2] \subset B^f_m$). Given that fact that $h^*_\nu(h_2(K)) \in H_\nu$, if this upper bound in B^f chosen such that $0 \leq \bar{h}_2 \leq f^M$, with $\bar{h}_2(k) < f^M(k)$ (resp., $0 \leq \bar{h}^m_2 \leq f^M$, with $\bar{h}^m_2(k) < f^M(k)$) when $k > 0$, $\bar{h}_2 \in B^f$, then by construction, we shall have $A^*_\nu(\bar{h}_2) \leq \bar{h}_2$ (resp., $A^*_{m,v}(\bar{h}^m_2) \leq \bar{h}^m_2$).

For the space B^f, take $\bar{h}_2 \in B^f$

$$f^M(k, 1, k, 1) \geq \bar{h}_2(k) = f^*_\nu(k) = f(k, n^f_\nu(k))e(k, n^f_\nu(k)) \tag{28}$$

with equality when $k > 0$. For the space B_m^f, as the greatest contingent labor supply $n_\vee^*(C, K)$ is decreasing in C, for any $h_2 \in B_m^f$, we have $n_\vee^*(h_2(K), K)$ is increasing in K. So, simply choose any $\bar{h}_2^m \in B_m^f$, such that $n_\nu^*(k) \leq n_\vee^*(\bar{h}_2^m(K), K) \leq 1$, with equality when $K > 0$.

Then, for example, we have

$$f^M(k, 1, k, 1) \geq \bar{h}_2^m(k) = f_\nu^u(k; \bar{h}_2) = f(k, n_\vee^*(\bar{h}_2^m(k), k))e(k, n_\vee^*(\bar{h}_2^m(k), k)) \tag{29}$$

with equality when $k > 0$. For $\bar{h}_2 \in B^f$ ($\bar{h}_2 \in B_m^f$), define $0 \leq \bar{h}_2 \leq f^M$, with $\bar{h}_2(k) < f^M(k)$ when $k > 0$, define a pair of "second step" operators as:

$$\begin{aligned} A_\nu^*(h_2)(k) &= h_\nu^*(h_2(k))(k) \text{ for } h_2 \in [0, \bar{h}_2] \\ &= 0 \text{ else.} \end{aligned} \tag{30}$$

where the restriction of $A_\nu^*(h_2)$ to the space $h_2 \in [0, \bar{h}_2] \subset B_m^f$ is denoted by $A_{m,\nu}^*(h_2)$. Then, a RE is any fixed point h^* of $A_\nu^*(h_\nu^*)$ such that when $k > 0$, $h^*(k) > 0$, and $g^* = (f_\nu^* - h^*) > 0$. Further, any such RE of that is a fixed point on B_m^f will additionally have RE investment g^* monotone.

5 Existence and Comparison of RE

We now consider the question of existence of RE. To simplify notation in this section, we shall always denote RE investment by

$$g_\nu^*(k) = f^*(k, n_\nu^*(C^*(k), k))e(k, n_\nu^*(C^*(k), k)) - C^*(k)$$

for any RE consumption function $C^* \in B^f$. Similarly, define $g_{m,\nu}^*$ for any RE consumption function in $C^* \in B_m^f$.

We now have our first main theorem of the paper which concerns the existence of RE using the operator A_ν^* in equation (30) in each of the spaces $[0, \bar{h}_2]$ (resp, $[0, \bar{h}_2^m]$) where the upper bounds $\bar{h}_2$ $\in B^f$ (resp., $\bar{h}_2^m \in B_m^f$) are given by equations (28) (resp., 29).

THEOREM 10. Under Assumptions A1, A2, for $h_2 \in [0, \bar{h}_2] \subset B^f$, (a) $A_\nu^* : [0, \bar{h}_2] \to [0, \bar{h}_2]$ has a nonempty complete lattice of RE $\Psi_{A_\nu^*} \subset [0, \bar{h}_2] \subset B^f$, with (b) least RE is $C_{\nu,L}^*$ and greatest RE is $C_{\nu,G}^*$ and associated RE labor supply $N_\nu^*(C_{\nu,L}^*(k), k)$ and $N_\nu^*(C_{\nu,G,}^*(k), k))$; (c) A_ν^* is order continuous on $[0, \bar{h}_2] \subset B^f$; (d) the least $C_{\nu,L}^*$ and the greatest $C_{\nu,G}^*$ RE can be computed as follows, for each k:

$$\vee(A_\nu^*)^n(0)(k) = C_{\nu,L}^*(k) \leq C_{\nu,G}^*(k) = \wedge(A_\nu^*)^n(\bar{h}_2)(k, k)$$
$$n_\nu^*(C_{\nu,L}^*(k), k) \geq n_\nu^*(C_{\nu,G}^*(k), k);$$

finally, (e) for $h_2 \in [0, \bar{h}_2^m] \subset B_m^f$, for the mapping $A_{m,\nu}^* : [0, \bar{h}_2^m] \rightarrow [0, \bar{h}_2^m]$ claims (a)-(d) hold for its nonempty complete lattice of RE $\Psi_{A_{m,\nu}^*} \subset [0, \bar{h}_2^m] \subset B_m^f$. Moreover, RE policies $c^*(\cdot, \cdot, C_\nu^*) \in \mathbf{C}^*(B^f)$ or $\mathbf{C}^*(B_m^f)$ respectively.

Proof. We prove parts (a)-(d) for RE in $[0,\bar{h}_2]$. The proof of (e) for RE in $[0,\bar{h}_2^m]$ relative to the claims in parts (a)-(d) for $A^*_{m,v}$ follows from a similar construction.

(a) First, observe by the Inada conditions on u and f, and the definition of the range of the first step fixed point, by construction, we have $0 \leq A^*_\nu(0) \leq A^*_\nu(\bar{h}_2) \leq \bar{h}_2$ with strict equality with $k = K > 0$, where $\bar{h}_2$ is defined in equation (29). Then, by Lemma 9.(c), as A_ν is isotone, and by Lemma 6. B^f is a nonempty complete lattice, the fixed point set $\Psi_{A^*_\nu} \subset [0,\bar{h}_2]$ is a nonempty complete lattice by Tarski's Theorem.

(c) From Lemma 8., A_ν is order continuous on $H_\nu \times [0,\bar{h}_2]$. We first show this implies greatest fixed point of the partial map $h_1 \to A_\nu(h_1;h_2)$, is order continuous in $h_2 \in [0,\bar{h}_2]$. To see this, consider the iterations from an initial point $h_1^0 := \vee H_\nu$, with the iterations given by $\{A^n_\nu(h_1^0;h_2)\}_{n=0}^\infty$. As order continuity is closed under composition, and evaluation maps and projections are order continuous in chain complete partially ordered sets, we conclude by the Tarski-Kantorovich theorem (Dugundji and Granas ([1982], Theorem 4.2)):

$$\begin{aligned} A^*_\nu(h_2) &= \wedge A^n_\nu(h_1^0;h_2) \\ &= A^n_\nu(\wedge h_1^n;h_2) \\ &= \lim_{n\to\infty} A^n_\nu(h_1^0;h_2) \\ &= h^*_\nu(h_2(K)) \\ &= \vee\Psi_{A_\nu}(h_2(K)) \end{aligned}$$

where the convergence to $\vee\Psi_{A_\nu}(h_2(K))$ is uniform, and $\vee\Psi_{A_\nu}(h_2(K))$ is order continuous in h_2, each K.

(b) and (d) By construction, we have $0 \leq A^*_\nu(0) \leq A^*_\nu(\bar{h}_2) \leq \bar{h}_2$ with strict equality with $k = K > 0$. Then, as by part (c), $A^*_\nu(h_2)$, $h_2 \in [0,\bar{h}_2]$ is order continuous, therefore, by the Tarski-Kantorovich theorem, we have

$$\begin{aligned} \vee(A^*_\nu)^n(0) &= C^*_{\nu,L} \\ &= \wedge\Psi_{A^*_\nu} \\ &\leq \vee\Psi_{A^*_\nu} \\ &= C^*_{v,G} \\ &= (A^*_\nu)^n(\bar{h}_2). \end{aligned}$$

■

We should mention, in Theorem 10., we characterize the comparative statics of any RE in individual and aggregate state variables for a *fixed* economy under Assumptions A1 and A2. That is, we prove all RE are have consumption and investment monotone in individual states (and continuous), as required by Lemma 3. Aside from these structural properties of RE, relative to aggregate states, there exist RE that have (a) monotone investment decision rules jointly in individual and aggregate states, with consumption decreasing in aggregate states, but both decision rules discontinuous in general in aggregate states; and (b) have both investment and consumption simply bounded. So RE in our economy are state asymmetric.

We now consider RE comparative statics on the space of deep parameters of the economy relative to the set of RE equilibrium in Theorem *10.* We first consider equilibrium comparative statics relative to discount rates (i.e., the question of "capital deepening"). Again, we shall do the comparative statics for RE using the operator A_ν^* in B^f and mention as a corollary the similar comparative statics in B_m^f.

THEOREM 11. *(Capital Deepening in discount rates). Under Assumptions A1 and A2, for* $\bar{h}_2 \in B^f$ given in expression (28), *we have (a) for the least fixed point* $C_{\nu,L}^*(\beta) \in [0, \bar{h}_2]$ (resp., greatest fixed point $C_{\nu,G}^*(\beta) \in [0, \bar{h}_2]$) *for* $\beta_1 \geq \beta_2$, $C_{\nu,L}^*(\beta_1) \leq C_{\nu,L}^*(\beta_2)$ (resp, $C_{\nu,G}^*(\beta_1) \leq C_{\nu,G}^*(\beta_2)$) with RE investment $g_{\nu,G}^*(\beta_1) \geq g_{\nu,G}^*(\beta_2)$ (resp., $g_{\nu,L}^*(\beta_1) \geq g_{\nu,L}^*(\beta_2)$), and the associated labor supply $N_{\nu,L}^*(\beta_2) := n_\nu^*(h_{\nu,L}^*(\beta_2), k) \geq n_\nu^*(h_{\nu,L}^*(\beta_1), k) =: N_{\nu,L}^*(\beta_1) =$ (resp, $N_{\nu,G}^*(\beta_2) \geq N_{\nu,G}^*(\beta_1)$. (b) these RE comparative statics can be computed by the successive approximations as follow:

$$\vee(A_\nu)^n(0;\beta_1) = C_{\nu,L}^*(\beta_1) \leq C_{\nu,L}^*(\beta_2) = \vee(A_\nu)^n(0;\beta_2)$$
$$\wedge(A_\nu)^n(\bar{h}_2;\beta_1) = C_{\nu,G}^*(\beta_1) \leq C_{\nu,G}^*(\beta_2) = \wedge(A_\nu)^n(\bar{h}_2;\beta_2)$$

Finally (c): the claims in (a) and (b) hold for the least and greatest fixed points of $A_{m,v}^*$ on B_m^f.

PROOF. Noting its dependence on the parameter β, we will do the case of RE in $C_\nu^*(\beta) \in [0, \bar{h}_2]$. The exact same argument works for $h_2 \in [0, \bar{h}_2^m]$.

Noting the definition of $A_\nu(h_1, h_2; \beta)$, it is decreasing in β. As by definition, $A_\nu^*(h_2;\beta)$ is the greatest fixed point of the partial map $h_1 \to A_\nu(h_1; h_2; \beta)$ by Veinott's fixed point comparative statics theorem, $A_\nu^*(h_2;\beta)$ is decreasing in β.

By Theorem 10.(c), the mapping $h_2 \to A_\nu^*(h_2;\beta)$ is order continuous in h_2. Then, by the Tarski-Kantorovich theorem, we have

$$\begin{aligned}\vee(A_\nu^*)^n(0;\beta_1) &= C_{\nu,L}^*(\beta_1) \\ &\leq C_{\nu,L}^*(\beta_2) \\ &= \vee(A_\nu^*)^n(0;\beta_2),\end{aligned}$$

Further, as $n_\nu^*(c,k)$ is decreasing and continuous in c, we have for RE labor supply:

$$n_\nu^*(C_{\nu,L}^*(\beta_1)(k), k) \geq n_\nu^*(C_{\nu,L}^*(\beta_2)(k), k)$$

But a dual argument, we could proceed for the greatest fixed point. Noting the definition of RE investment associated with least and greatest RE consumption $C_{\nu,L}^*(\beta)$ and $C_{\nu,G}^*(\beta)$, we have which RE investment $g_{\nu,G}^*(\beta_1) \geq g_{\nu,G}^*(\beta_2)$ (resp., $g_{\nu,L}^*(\beta_1) \geq g_{\nu,L}^*(\beta_2)$), which completes the proof of the claims in the theorem. ■

Notice, in Theorem 11., we prove comparisons of RE labor supply in the discount rate β, in addition to both investment and consumption. We should mention, Mirman, Morand, and Reffett ([2008]) and Acemoglu and Jensen ([2015]) deliver a similar monotone comparison result relative to dynamic economies with inelastic labor supply, small capital externalities, and no labor externalities, but is not clear how to extend their results to models with elastic labor supply, labor externalities, especially for the case of large capital externalities.

We next consider the robust equilibrium comparative statics for income tax example in Section 2, but now for our more general class of distorted economies with elastic labor supply. First note, it is well-known the RE problem for this economy with an income tax corresponding to a "reduced form" production specification in the spirit of assumption A2. In particular, we can rewrite the income process for our economy $y(k, n, K; N)$ in equation (14) as a "reduced form' production function $f(k, n)e(K, N)$ (e.g., see Greenwood and Huffman ([1995]), Coleman ([2000]) and Datta, Mirman, and Reffett ([2002])). In this case, you can define $e(K, N) = \hat{e}(K, N)(1 - \tau(K)) \in [0, 1)$, where $\hat{e}(K, N)$ satisfies assumption A2, and $\tau(K)$ satisfies either Assumption E3(i) or E3(ii) as in the motivating example. Then, the reduced-form production function is given by $f(k, n)e(k, n)$ and satisfies Assumption A2. Further, noting that under constant returns to scale, in equilibrium after imposing the balanced budget rule, $y(k, n, k; n) = f(k, n)\hat{e}(k, n)$, we can define our operator as in Section 4, and verify the existence of RE in Theorem 10..

For our tax economy, in all cases, assume the proceeds of the income tax are returned as lump-sum transfers $J(K)$ to the household, where these transfers satisfy a balanced-budget taxation rule

$$J(K) = \tau(K)y(K, N^*(C^*(K), K), K; N^*(C^*(K), K))$$

where N^* is any RE labor supply in Theorem 10. [23] Then, when $k = K$, noting constant returns to scale and zero profits in private returns for the firms, the income process in a RE for a household is

$$\begin{aligned} y_\tau(k, N^*(k), k, N^*(k)) &= (1 - \tau(k))\{r(k, N^*(k))k + w(k, N^*(k))N^*(k) + \Pi\} + J(k) \\ &= f(k, N^*(k))e(k, N^*(k)) \end{aligned}$$

where Π denotes profits.

We first consider the most general case of technologies and taxes under Assumption A2, and $\tau(K)$ satisfies Assumption E3(i) or E3(ii), In this case, RE exist and can be computed by Theorem 10.. To obtain our RE comparison results for this case of adding an income tax, if we substitute the equilibrium relationship $y_\tau(k, N^*(k), k; N^*(k)) = f(k, N^*(k))$, into the definition of the operator $A(h_1; h_2(K); \tau)$ in equations (25), (26), and (27), noting the dependence of $A(h_1, h_2; \tau)$ on the tax, we have the following RE comparative statics result:

THEOREM 12. *(Comparing RE under ordered changes in an income tax). Under Assumptions A1 and A2, and E3(i) or E3(ii), for* $\bar{h}_2 \in B^f$ *(a) for the least RE* $C^*_{\nu,L}(\tau) \in [0, \bar{h}_2]$ (resp., greatest RE $C^*_{\nu,G}(\tau) \in [0, \bar{h}_2]$ *we have for* $\tau_1(K) \geq \tau_2(K)$ for all $K \in \mathbf{K}$, *then* $C^*_{\nu,L}(\tau_1) \geq C^*_{\nu,L}(\tau_2)$ (resp, $C^*_{\nu,G}(\tau_1) \geq C^*_{\nu,G}(\tau_2)$) with RE investment $g^*_{\nu,G}(\tau_1) \leq g^*_{\nu,G}(\tau_2)$ (resp., $g^*_{\nu,L}(\tau_1) \leq g^*_{\nu,L}(\tau_2)$), and the associated RE

[23] Noice, our existence result here does *not* apply to the many important indeterminancy models in the existing literature, including Schmidt-Grohe and Uribe ([1997]), Guo and Lansing ([1998]), as well as more recent, work by Nourry, Seegmuller, and Venditti ([2013]) and Nishimura, Seegmuller, and Venditti ([2015]). It turns out our two step methods can be adapted to handle each of these papers also (i.e., balanced-budget income taxes, capital taxes, and wage taxes). See Datta, Reffett, and Wozny ([2015a]) for all these extensions.

labor supply $n_\nu^*(C_{\nu,L}^*(\tau_2)(k),k) \leq n_\nu^*(C_{\nu,L}^*(\tau_1)(k),k)$ (resp, $n_\nu^*(C_{\nu,G}^*(\tau_2)(k),k) \leq n_\nu^*(C_{\nu,G}^*(\tau_1)(k),k)$). (b) these RE comparative statics can be computed as follow:

$$\begin{aligned} \vee(A_\nu^*)^n(0;\tau_1) &= C_{\nu,L}^*(\tau_1) \geq C_{\nu,L}^*(\tau_2) = \vee(A_\nu^*)^n(0;\tau_2) \\ \wedge(A_\nu^*)^n(\bar{h}_2;\tau_1) &= C_{\nu,G}^*(\tau_1) \geq C_{\nu,G}^*(\tau_2) = \wedge(A_\nu^*)^n(\bar{h}_2;\tau_2). \end{aligned}$$

Finally (c) the claims in (a) and (b) hold for all the least and greatest RE computed using $A_{m,v}^*$ for $h_2 \in [0,\bar{h}_2^m] \subset B_m^f$.

Proof. We prove (b) first; then (a) follows from the argument directly. By the order continuity of $h_2 \to A_{m,v}^*(h_2;\tau)$ on B_m^f by the Tarski-Kantorovich theorem, we have

$$\begin{aligned} \vee(A_{m,\nu}^*)^n(0;\tau_1) &= h_\nu^*(\tau_1) \geq h_\nu^*(\tau_2) = \vee(A_{m,\nu}^*)^n(0;\tau_2) \\ \text{with } n_\nu^*(h_{m,\nu}^*(0;\tau_1) &\leq n_\nu^*(h_{m,\nu}^*(\tau_2)) \end{aligned}$$

with $g_{m,\nu}^*$ increasing in (k) for each τ, and the last line follows from the fact that $n_\nu^*(c,k)$ is continuous and decreasing in c. (a) For each $h_1 \in H_\nu$, $h_2 \in B_m^f$, using the fact that under lump-sum transfers, $y_\tau(k,N^*(k),k;N^*(k)) = f(k,N^*(k))e(k,N^*(k))$ is independent of τ in any RE, when $k>0$, $h_1>0$, $h_2 < y_{f,v}^*$, by from the definition $A(h_1,h_2;\tau)$ is increasing in τ. Then, as in the proof of previous theorem, by Veinott's fixed point comparative statics theorem has $A_{m,\nu}^*(h_2;\tau)$ is increasing in τ. ■

6 Application and Discussion

In conclusion, we briefly comment on the relationship between our results in this paper with Romer ([1986]), Santos ([2002]), Benhabib and Farmer ([1994]); previous analysis of monotone map methods in Coleman ([1997]) and Datta, Mirman and Reffett ([2002]).

6.1 Romer (1986) and Santos (2002)

To relate our results to Romer ([1986]) and Santos ([2002]), assume inelastic labor supply and no leisure-labor choice, or, $l=0$ and $N=1$, and for Assumption A2, take $e(K,N)=e(K,1)$ rising in K.

In Romer ([1986]), if $f_1(K,1)e(K,1)$ is falling in K, then there exists unique RE (e.g., Coleman ([1991], [2000]) and Mirman, Morand, and Reffett ([2008]). If $f_1(K,1)e(K,1)$ is rising in K, Theorem 10. can be applied to obtain existence of the least and the greatest RE (noting that in this case, we need to take $f^M(k,N)e(k,N)$ in the definition of the second step space B^f to be slightly higher: i.e., let $N>1$, so $f^M(K,N)e(K,N) > f(K,1)e(K,1)$. Notice, in this case, we have inelastic labor supply, so we only have a single Euler equation operator (as in the motivating example). For the space $H_\nu = H$ (where H is defined in section 2), as the first step operator uses the upper bound for output to be $f(k,1,k,1)$, the first step greatest fixed point $h^*(h_2(K))(K) < f^M(K,N)e(K,N)$; hence if you take $\bar{h}_2 = f(K,1)e(K,1) < f(K,N)e(K,N)$ in the second step, $A^*(\bar{h}_2) < \bar{h}_2$. Then, Theorem 10. provides all the existence results, and Theorem 11. provides all the RE comparison results.

Then, to obtain the results discussed in the motivating example for the economy with a state-contingent income tax, one uses the existence of a complete lattice of RE in both spaces $[0, \bar{h}_2]$ and $[0, \bar{h}_2^m]$ follows directly from Theorem 10. (noting here again, we do not have Euler equation branching with inelastic labor supply), while both of RE comparison theorems in Theorem 11. and Theorem 12. apply as well.

6.2 Benhabib-Farmer (1994)

The economies studied in Benhabib and Farmer ([1994]) are important special cases of our results, as well as the balanced-budget tax models of Guo and Lansing (2004), or the case of symmetric RE in the dynamic models with heterogeneous firms and credit constraints found in an recent paper by Liu and Wang ([2014]).

To obtain the results in Benhabib-Farmer ([1994]), noting they show the equivalence between monopolistic competition models and so-called "*laissez faire*" versions of their models, we simply the following:

Assumption A2-BF: *Benhabib-Farmer ([1994]):* (i) preferences $u(c) + v(l)$ are each power utility in consumption and leisure (or $u(c) = \ln c$), (ii) F is Cobb-Douglas, with $F(k, n, K, N) = k^a n^b K^c N^d$, where $f(k, n) = k^a n^b$, $e(K, N) = K^c N^d$, with $a + b = 1$, $a, b, c, d > 0$, such that $a + c > 1$, $b + d > 1$.

Then by Theorem 10. a complete lattice of RE exist in both $[0, \bar{h}_2]$ and $[0, \bar{h}_2^m]$, while by Theorem 11., and we can compare least and greatest RE in each of these function spaces in the discount rate. As Benhabib and Farmer ([1994]) show the equivalence of their "*laissez faire*" versions of their models to monopolistic competition versions of the models, our results apply to these decentralizations also. Liu and Wang (2014) recently produced a very interesting dynamic economy where credit constraints on heterogeneous firms generate a set of sequential equilibrium conditions in a symmetric equilibrium that are observationally equivalent to the model of Benhabib and Farmer ([1994]) without appealing to increasing returns directly. Our results apply to this class of models also.

Finally, Guo and Harrison ([2004]) show how models with state contingent proportional income taxes can be mapping the Benhabib and Farmer economies under balanced-budget rules. Their models differ from those models we considered in Section 2 and 5 relative to the question of existence and RE equilibrium comparative statics. Notice, in our models in these two sections, we lump-sum the proceeds of income taxes back to the household; alternatively, one could use the proceeds to finance government investment in to production (e.g., infrastructure ala Barro ([1990]).

In this latter case, under a balanced-budget rule, the resulting model can reduce in many cases to the model of Benhabib and Farmer ([1994]). In this case, the government spending works as an "infrastructure" externality. For these models, our existence results in Theorem 10. apply, as do our comparison results in Theorem 11. ; but our comparison results in Theorem 12. no longer directly apply (as the tax policies generate a reduction of income a change of income in the household's budget constraint that is not simple to order). In most cases of government policy induced indeterminacy, and government spending, our monotone methods can be extended to these setting per existence. In those models, versions of Theorem 11. can be produced. Further, in some cases, equilibrium comparative statics relative to changes in government spending and/or tax structure can also be compared. But these results require extensions of the methodology in this paper. In Datta, Reffett, and Woźny ([2015a]), we study both the existence of RE, and the question of RE comparison results for models with government spending and balanced-budget rules. The results in this paper do not necessarily apply to such economies.

Finally, we can develop a global theory of equilibrium comparisons of recursive sunspot equilibria for the models with multiple equilibria such as those in Benhabab and Farmer ([1994]). For example, one can extend the approach of Spear ([1991]) easily to our model, but in our case, as our construction is *global,* the resulting theory of stationary sunspot equilibrium is *global.* These results are studied in the paper by Datta, Reffett, and Woźny ([2015b]). It is not clear how to obtain such results using *local* approaches in the literature. For example, one immediate problem one encounters is that standard method in the literature cannot be applied. For example, Spear's results (Spear ([1991]) for the existence of *continuous* stationary sunspots in models with positive externalities such as Romer ([1986]) are not easy to extend. The complicate is clear from Lemma 4.. As contingent labor supply is a correspondence, it does not generally admit smooth selections; rather, only *continuous* selection that depend on steady-state capital; hence, the implicit function theorem cannot be applied at the steady-steady. Our global approach avoids this complication. Further, as we generate RE with monotone dynamics in capital, this allows us to study the question existence of stationary sunspot equilibria using monotone Markov process methods.

More generally, the problem when attempting to apply smooth dynamical system methods to further characterize equilibrium dynamics, relative to either recursive or sequential equilibrium, is that equilibrium dynamics near a steady state k^* are not necessarily smooth even locally. Further, its difficult to obtain sufficient conditions under which smooth local RE exist. To see this, the local RE dynamics are given by an function is *implicitly* as a mapping in equation (16) in section 3.2, i.e.,

$$k' = g_\nu^*(k) \tag{31}$$

for $\nu = \{\vee, \wedge\}$. By a fixed point comparative statics argument, under Assumption A2, we have $k_\vee^* > k_\wedge^*$ (i.e., the steady-state is *not unique* when the Benhabib-Farmer indeterminacy parameters are satisfied). Further, as the least and greatest contingent labor supplies N_ν^* is not necessarily smooth, the implicit mapping of $k' = g(k)$ computed in Spear's approach via the implicit function theorem cannot necessarily be applied; so even continuous local RE equilibria cannot be verified using this approach for models with elastic labor without further argument and/or sufficient conditions.

In particular, this situation applies to applications of either the Grobman-Hartman Theorem, or various versions of the stable manifold theorem to study the local properties of RE in Theorem 10.. For our economies under Assumption A1 and A2, this appears not possible given the results in the Santos' counterexample even for models with inelastic labor supply (see Santos ([2002], section 3.2). But more to the point, unless one can develop conditions such that solutions for contingent labor supply such that there is a *smooth* selection, it is not clear how to proceed with applying the methods of smooth dynamical systems to recursive equilibria near any steady state. In particular, we will not be able to prove the existence of an implicit function c^*, where g^* is given by equation (31) that satisfies on some closed set containing k^* for all steady states the local fixed point RE operator equation: $A^*_\nu(c^*) = c^*$. Actually, even proving the existence of continuous RE dynamics is not possible given his counterexample, although at some stable steady states, perhaps some results might be possible. We leave this question, among many others, for further work.

7 Appendix: Fixed Point Theory in Ordered Spaces

7.1 Mathematical Terminology

Posets and lattices: A *partially ordered set* (or poset) is a set P ordered with a reflexive, transitive, and antisymmetric relation. If any two elements of $C \subset P$ are comparable, C is referred to as a linearly ordered set, or chain. A *lattice* is a set L ordered with a reflexive, transitive, and antisymmetric relation $\geq$ such that any two elements x and x' in L have a least upper bound in L , denoted $x \wedge x'$, and a greatest lower bound in L, denoted $x \vee x'$. $L_1 \subset L$ is a *sublattice* of L if it contains the sup and the inf (with respect to L) of any pair of points in L_1. A lattice is *complete* if any subset L_1 of L has a least upper bound and a greatest lower bound. L_1 is *subcomplete* if it is complete and a sublattice. In a poset P, if every subchain in $C \subset P$ is complete, then C is referred to as a *chain complete poset* (or CPO). If every countably subchain in C is complete, then C is referred to as a *countably chain complete poset* (or CCPO). Let $[a) = \{x|x \in P, x \geq a\}$ be the *upperset* of a, and $(b] = \{x|x \in P, x \leq b\}$ the *lowerset* of b. We say P is an *ordered topological space* if $[a)$ and $(b]$ are closed in the topology on P. An order interval is defined to be $[a, b] = [a) \cap (b]$, $a \leq b$.

Isotone (or order preserving) mappings on a poset: Let $(X, \geq_X)$ and $(Y, \geq_Y)$ be Posets. A mapping $f : X \to Y$ is *increasing* (or isotone) on X if $f(x') \geq_Y f(x)$, when $x' \geq_X x$, for $x, x' \in X$. If $f(x') >_Y f(x)$ when $x' >_X x$, we say f is *strictly increasing.* [24] The mapping $f : X \to Y$ is *sup preserving* (resp, *inf preserving)* if we have for any countable chain C, $f(\vee C) = \vee f(C)$ (resp, $f(\wedge C) = \wedge f(C)$). A mapping that is both sup and inf preserving will be *order continuous.*

A correspondence (or multifunction) $F : X \to 2^Y$ is *ascending* in a binary set relation $\rhd$ on 2^Y if $F(x') \rhd F(x)$, when $x' \geq_X x$. Let $\mathbf{X}$ be a poset, $\mathbf{Y}$ a lattice, and define the relation $\rhd = \geq_v$on the range $\mathbf{L(Y)} = \{$ nonempty sublattices L of $\mathbf{Y}\}$, where for $L_1, L_2 \in \mathbf{L(X)}$, we say $L_1 \geq_v L_2$ in *Veinott's Strong Set order* if for all $x_2 \in L_2$, $x_1 \in L_1$, $x_1 \vee x_2 \in L_1$, $x_1 \wedge x_2 \in L_2$.

Fixed points. Let $F : X \to 2^X$ be a non-empty valued correspondence. $x \in X$ is a fixed point of F if $x \in F(x)$. If F is a function, a fixed point is $x \in X$ such that $x = F(x)$. Denote by $\Psi_F(t) : T \to 2^X$ the fixed point correspondence of F at $t \in T$.

[24] To avoid using references to "isotone mapping", we will often use the more traditional terminology in economics of "increasing". In the literature on partially ordered sets, an "increasing map often denotes something slightly different (e.g., $f(x') \geq_Y f(x)$ when $x' >_X x$ for x, $x' \in X$.)

7.2 Some Useful Fixed Point Theorems

One critical result we use through the paper is Veinott's version of Tarski's theorem. His result is stated in the next proposition as result (i) and (iii). Result (ii) in the theorem is essentially implied by (iii).

Proposition 13. Veinott ([1992], chapter 4, Theorem 14). Let X be a nonempty complete lattice, T a poset, $F(x,t) \in \mathbf{L}(X)$ a nonempty, subcomplete-valued correspondence that is strong set order ascending. Then, (i) $\Psi_F(t)$ is a nonempty complete lattice, and (ii) $\vee\Psi_F(t)$ and $\wedge\Psi_F(t)$ are isotone selections from $T \to X$.

Tarski's original theorem (Tarski ([1955]), Theorem 1) occurs as a special case of Proposition 13., where $F(x,t) = f(x)$, and $f : X \to X$ is a function. An important extension of Tarski's theorem is given my Markowsky ([1976], theorem 9) and is stated in the next proposition. The fixed point comparative statics result in the proposition per least (resp, greatest) fixed points is a corollary of a theorem proven in Heikkila and Reffett ([2006], Theorem 2.1), which in turn implies $\Psi_F(t)$ is weak-induced ascending upward and downward.

Proposition 14. Let X be a CPO, T a poset, $f : X \times T \to X$ increasing in x, each $t \in T$. Then, (i) $\Psi_F(t)$ is nonempty CPO, (ii) $\wedge\Psi_F(t)$ (resp, $\vee\ \Psi_F(t)$) are increasing selections.

For our results, we will need constructive versions of Proposition 13. and Proposition 14.. For this, we will assume for each $t \in T$, the partial map $f_t : X \to X$ is order continuous. For this case, we have the following version of Tarski-Kantorovich-Markowsky theorem. The characterization of the fixed point set in (i) is from Balbus, Reffett, and Woźny ([2014]). The computability result is the classic Tarski-Kantorovich theorem (e.g., Dujundgi and Granas ([1982], Theorem 4.2)). There is a dual version for the greatest selections.

Proposition 15. Let X be a CCPO, T a poset, $f : X \times T \to X$ order continuous in x, each t , and $\exists$ a $x_L \in X$ such that $x_L \leq f(x_L, t)$. Denote by $\Psi_f(t) : T \to 2^X$ the fixed point correspondence of f at $t \in T$. Then, (i) $\Psi_F(t)$ is nonempty CCPO. Further, the iterations $\sup_n f^n(x_L) = \wedge\Psi_f(t)$. Finally, if in addition, X and T are each continuous domains, and $f(x,t)$ is additionally order continuous on T, then the mapping $t \to \wedge\Psi_f(t)$ is order continuous.

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