Dynamic Games in Macroeconomics

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Abstract

In this chapter, we survey how the methods of dynamic and stochastic games have been applied in macroeconomic research. In our discussion of methods for constructing dynamic equilibria in such models, we focus on strategic dynamic programming, which has found extensive application for solving macroeconomic models. We first start by presenting some prototypes of dynamic and stochastic games that have arisen in macroeconomics and their main challenges related to both their theoretical and numerical analysis. Then, we discuss the strategic dynamic programming method with states, which is useful for proving existence of sequential or subgame perfect equilibrium of a dynamic game. We then discuss how these methods have been applied to some canonical examples in macroeconomics, varying from sequential equilibria of dynamic nonoptimal economies to time-consistent policies or policy games. We conclude with a brief discussion and survey of alternative methods that are useful for some macroeconomic problems.

Keywords

Strategic dynamic programming • Sequential equilibria • Markov equilibria • Perfect public equilibria • Non-optimal economies • Time-consistency problems • Policy games • Numerical methods • Approximating sets • Computing correspondences

1 Introduction

The seminal work of Kydland and Prescott (1977) on time-consistent policy design initiated a new and vast literature applying the methods of dynamic and stochastic games in macroeconomics and has become an important landmark in modern macroeconomics.[1] In their paper, the authors describe a very simple optimal policy design problem in the context of a dynamic general equilibrium model, where government policymakers are tasked with choosing an optimal mixture of policy instruments to maximize a common social objective function. In this simple model, they show that the consistent policy of the policymaker is not optimal because it does not take account of the effect of his future policy instrument on economic agents' present decision. In fact, Kydland and Prescott (1977) make the point that a policy problem cannot be dealt with just optimal control theory since there a policymaker

[1] Of course, there was prior work in economics using the language of dynamic games that was related to macroeconomic models (e.g., Phelps and Pollak 1968; Pollak 1968; Strotz 1955) but the paper of Kydland and Prescott changed the entire direction of the conversation on macroeconomic policy design.

is interacting with economic agents having rational expectations. In other words, as the successive generations of policymakers cannot commit to the future announced plans of the current generation, they argue that one cannot assume that optimal plans that are designed by any current generation of government policymakers will ever be followed if they are not required to be additionally dynamically consistent. This observation gave rise to a new and very important question of how to construct *credible* government policies, as well as raising the question of whether discretion vs. rules were more important to the design of optimal policy, and the study of dynamic macroeconomic models with strategically interacting agents and limited commitment begun (and has continued for the last four decades).

In subsequent work, Kydland and Prescott (1980) proposed a new set of recursive methods for constructing time-consistent optimal policies in decentralized dynamic equilibrium models with capital and labor. Their methods actually were an integration of new dynamic optimization techniques under additional constraints (i.e., constraints that were added to guarantee decision-makers would look forward or backward in a manner that the resulting optimal decisions for future policy were time consistent). Their methods in this paper introduced the idea of using set-valued operators to construct time-consistent sequential equilibrium solutions defined recursively on an *expanded* set of endogenous state variables that could be used to provide the needed dynamic incentives for them to choose time-consistent solutions.

Their methods, although not explicitly game theoretic, provided an important preamble to the introduction of more general, powerful, and systematic game theoretic approaches that are now central to much work in macroeconomics. These new methods are referred in the literature as "strategic dynamic programming methods" and are built upon the seminal work of Abreu et al. (1986, 1990) (APS) for solving for the equilibrium value set of very general classes of repeated games. As in the original Kydland-Prescott approach (e.g., Kydland and Prescott 1980), they introduce new state variables (in this case, either value functions or envelope theorems) and in essence are set-valued generalizations of standard dynamic programming methods. This approach (especially since the pioneering paper of Atkeson 1991) has found many important implementations to solve macroeconomic models with limited commitment or dynamically inconsistent preferences and is (in their structure) basically APS method extended to models with state variables.[2] These methods both verify the existence of subgame perfect equilibrium in a large class of dynamic/stochastic games, and they provide a systematic method for constructing all the sequential or subgame perfect equilibria in many dynamic macroeconomic models that can be formulated as a dynamic game.

[2]Strategic dynamic programming methods were first described in the seminal papers of Abreu (1988) and Abreu et al. (1986, 1990), and they were used to construct the entire set of sequential equilibrium values for repeated games with discounting. These methods have been subsequently extended in the work of Atkeson (1991), Judd et al. (2003), and Sleet and Yeltekin (2016), among others.

In this chapter, we survey some of the important literature on macroeconomic models that use the methods of dynamic and stochastic games. We first discuss the literature and how dynamic and stochastic games naturally arise in dynamic general equilibrium models that are the workhorse of macroeconomic modeling. We then discuss strategic dynamic programming methods extending to setting with state variables that are very important for solving these models. We focus on strategic dynamic programming with states, as when these methods apply, they provide a systematic method for constructing *all* dynamic equilibria in the models. At the end of the chapter, we also discuss alternative optimization and Euler equation-based methods for solving these models, which have also been studied in the literature. These latter methods, although in some cases not explicitly game theoretic, provide powerful alternatives to the set-theoretic approaches that APS methods with state variables provide.

There are many prototype problems in macroeconomics that require the tools of dynamic game theory, and there are a number of alternative methods for studying these models. Take, for example, the paper of Phelan and Stacchetti (2001), where they consider optimal taxation in a model first described in Kydland and Prescott (1980), where the structure of optimal taxation in their model was studied as a sequential equilibrium of a dynamic game played between overlapping generations of government policymakers who are collectively tasked with choosing an optimal sequence of capital and/or labor taxes to finance a stream of government spending over an infinite horizon, where government policymakers maximize the representative agent's lifetime utility function in a sequential equilibrium. Further, as Kydland and Prescott (1980) showed, as labor and capital decisions in the private economy are made endogenously by households and firms, the resulting dynastic social objective function for the collective government is not dynamically consistent. This raised the interesting question of studying sustainable (or credible) optimal taxation policies, where constraints forcing the government to make time-consistent choices further restricted the set of optimal government policies (i.e., forced optimal government policies to satisfy a further restriction that all current plans about decisions by future generations of government policymakers are actually optimal for those successor generations of policymakers when their decisions have to be made). This situation was distinct from previous work in dynamic general equilibrium theory (as well as much of the subsequent work on optimal policy design over the decade after their paper) which assumed perfect commitment on the part of government policymakers.[3] In showing this (far from innocuous) assumption of perfect commitment in dynamic economies, Kydland and Prescott (1980) asked the question of how to resolve this fundamental credibility issue for optimal policy design. Their construction of dynamic equilibria incorporated explicitly the strategic considerations between current and future policy agents into the design of sequential equilibrium optimal plans.

[3]The model they studied turned out to be closely related to the important work on optimal dynamic taxation in models with perfect commitment in the papers of Judd (1985) and Chamley (1986). For a recent discussion, see Straub and Werning (2014).

The papers of Kydland and Prescott (1980) and Phelan and Stacchetti (2001) also provide a nice comparison and contrast of methods for studying macroeconomic models with dynamic strategic interaction, dynamically inconsistent preferences, or limited commitment. Basically, the authors study very related dynamic economies (i.e., so-called „Ramsey optimal taxation models"), but their approaches to constructing time-consistent solutions are very different. Kydland and Prescott (1980) viewed the problem of constructing time-consistent optimal plans from the vantage point of optimization theory (with a side condition that is a fixed point problem that is used to guarantee time consistency). That is, they forced the decision-maker to respect the additional implicit constraint of time consistency by adding new endogenous state variables to further restrict the set of optimal plans from which government policymakers could choose, and the structure of that new endogenous state variable is determined by a (set-valued) fixed point problem. This "recursive optimization" approach has a long legacy in the theory of consistent plans and time-consistent optimization.[4]

Phelan and Stacchetti (2001) view the problem somewhat differently, as a dynamic game between successive generations of government policymakers. When viewing the problem this way, in the macroeconomics literature, the role for strategic dynamic programming provided the author a systematic methodology for both proving existence of, and potentially computing, sequential equilibria in macroeconomic models formulated as a dynamic/stochastic game.[5] As we shall discuss in the chapter, this difference in viewpoint has its roots in an old literature in economics on models with dynamically inconsistent preference beginning with Strotz (1955) and subsequent papers by Pollak (1968), Phelps and Pollak (1968), and Peleg and Yaari (1973).

One interesting feature of this particular application is that the methods differ in a sense from the standard strategic dynamic programming approach of APS for dynamic games with states. In particular, they differ by choice of expanded state variables, and this difference in choice is intimately related to the structure of dynamic macroeconomic models with strategically interacting agents. Phelan and Stacchetti (2001) note, as do Dominguez and Feng (2016a,b) and Feng (2015) subsequently, that an important technical feature of the optimal taxation problem is the presence of Euler equations for the private economy. This allows them to develop for optimal taxation problems a hybrid of the strategic dynamic programming methods of APS. That is, like APS, the recursive methods these authors develop

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[4]Indeed, in the original work of Strotz (1955), this was the approach taken. This approach was somehow criticized in the work of Pollak (1968), Phelps and Pollak (1968), and Peleg and Yaari (1973). See also Caplin and Leahy (2006) for a very nice discussion of this tradition.

[5]In some cases, researchers also seek further restrictions of the set of dynamic equilibria studied in these models, and they focus on Markov perfect equilibria. Hence, the question of memory in strategic dynamic programming methods has also been brought up. To answer this question, researchers have sought to generate the value correspondence in APS type methods using nonstationary Markov perfect equilibria. See Doraszelski and Escobar (2012) and Balbus and Woźny (2016) for discussion of these methods.

employ enlarged states spaces, but unlike APS, in this particular case, these additional state variables are Karush-Kuhn-Tucker (KKT) multipliers or envelope theorems (e.g., as is also done by Feng et al. 2014).

These enlarged state space methods have also given rise to a new class of recursive optimization methods that incorporate strategic considerations and dynamic incentive constraints explicitly into dynamic optimization problems faced by social planners. For early work using this recursive optimization approach, see Rustichini (1998a) for a description of so-called primal optimization methods and also Rustichini (1998b) and Marcet and Marimon (1998) for related "dual" recursive optimization methods using recursive Lagrangian approaches.

Since Kydland and Prescott's work was published, over the last four decades, it had become clear that issues related to time inconsistency and limited commitment can play a key role in understanding many interesting issues in macroeconomics. For example, although the original papers of Kydland and Prescott focused on optimal fiscal policy primarily, the early papers by Fischer (1980a) and Barro and Gordon (1983) showed that similar problems arise in very simple monetary economies, when the question of optimal monetary policy design is studied. In such models, again, the sequential equilibrium of the private economy can create similar issues with dynamic consistency of objective functions used to study the optimal monetary policy rule question, and therefore the sequential optimization problem facing successive generations of central bankers generates optimal solutions that are not time consistent. In Barro and Gordon (1983), and subsequent important work by Chang (1998), Sleet (2001), Athey et al. (2005), and Sleet and Yeltekin (2007), one can then view the problem of designing optimal monetary policy as a dynamic game, with sequential equilibrium in the game implementing time-consistent optimal monetary policy.

But such strategic considerations have also appeared outside the realm of policy design and have become increasingly important in explaining many important phenomena observed in macroeconomic data. The recent work studying consumption-savings puzzles in the empirical data (e.g., why do people save so little?) has focused on hyperbolic discounting and dynamically inconsistent choice as a basis for an explanation. Following the pioneering paper by Strotz (1955), where he studied the question of time-consistent plans for decision-makers whose preferences are changing overtime, many researchers have attempted to study dynamic models where agents are endowed with preferences that are dynamically inconsistent (e.g., Harris and Laibson 2001, 2013; Krusell and Smith 2003, 2008; Laibson 1997). In such models, at any point in time, agents make decisions on current and future consumption-savings decisions, but their preferences exhibit the so-called present-bias. These models have also been used to explain sources of poverty (e.g., see Banerjee and Mullainathan 2010; Bernheim et al. 2015). More generally, the question of delay, procrastination, and the optimal timing of dynamic choices have been studied in O'Donoghue and Rabin (1999, 2001), which has started an important discussion of how to use models with dynamically inconsistent payoffs to explain observed behavior in a wide array of applications, including dynamic asset choice.

Additionally, when trying to explain the plethora of defaults that we observe in actual macroeconomies, and further address the question of how to sustain sovereign debt arrangements and debt repudiation, a new theory of asset markets with strategic default has emerged, where the role of limited commitment has generated a wide array of new models of dynamic insurance under incomplete markets with strategic default. These models have been applied to many important problems in international lending, where limited commitment plays a key role in understanding financial arrangements. Strategic default also plays a key role in the construction of dynamic models with endogenous borrowing constraints. These models have played a critical role in explaining various asset pricing puzzles in the macroeconomics literature. This literature began with the important early paper by Atkeson (1991) which studies international lending and debt repudiation; but the problem of sustainable debt under limited commitment has been studied in the early work of Kehoe and Levine (1993, 2001), as well as in Alvarez and Jermann (2000), and Hellwig and Lorenzoni (2009). Further, the issue of sovereign debt repudiation has been studied in a number of papers including Arellano (2008), Benjamin and Wright (2009), Yue (2010), and Broner et al. (2014, 2010).

One final prototype of a dynamic game in macroeconomics arises in models of economic growth with limited commitment. One common version of this sort of model arises in models of strategic altruism, where a dynastic household faces a collective choice problem between successive generations of families. Models in this spirit were first introduced in Phelps and Pollak (1968) and subsequently studied in Bernheim and Ray (1983), Leininger (1986), Amir (1996b), Nowak (2006c), Balbus et al. (2012, 2014, 2015a,b,c) and Woźny and Growiec (2012), among others. Another classic example of strategic growth models arises in the seminal work of Levhari and Mirman (1980), where the "great fishwar" was originally studied. In this model, a collection of agents face the problem of managing a common resource pool, where each period agents can consume from the existing stock of resources, with the remainder of that stock being used as input to a regeneration process (i.e., as investment into a social production function) that produces next period stock of resources. This problem has been extensively studied (e.g., Mirman (1979), Sundaram (1989a), Amir (1996b), Balbus and Nowak (2004), Nowak (2006a,b), Jaśkiewicz and Nowak (2015), and Fesselmeyer et al. (2016) among others).

As dynamic games have been introduced more extensively into macroeconomics, researchers have developed some very powerful methods for studying sequential or Markovian equilibrium in such models. For example, in the macroeconomic models where sequential optimization problems for agents have preferences that are *changing* over time, when searching for time-consistent optimal solutions, since the work of Strotz (1955) it has been known that additional constraints on the recursive optimization problem must be imposed. These constraints can be formulated as either backward- or forward-looking constraints. In Kydland and Prescott (1980), they proposed a very interesting resolution to the problem. In particular, they reformulate the optimal policy design problem recursively in the presence of additional *endogenous* state variables that are used to force optimal plans of the government decision-makers to be time consistent. That is, one can formulate an agent's incentive to deviate from a candidate dynamic equilibrium future choices by imposing a sequence of incentive constraints on current choices. Given these additional incentive constraints, one is led to a natural choice for a set of new endogenous state variables (e.g., value functions, Kuhn-Tucker multipliers, envelope theorems, etc.). Such added state variables also allow one to represent sequential equilibrium problems recursively. That is, they force optimal policies to condition on lagged values of Kuhn-Tucker multipliers. In these papers, one constructs the new state variable as the fixed point of a set-valued operator (similar, in spirit, to the methods discussed in Abreu et al. (1986, 1990) adapted to dynamic games. See Atkeson (1991) and Sleet and Yeltekin (2016)).[6]

The methods of Kydland and Prescott (1980) have been extended substantially using dynamic optimization techniques, where the presence of strategic interaction creates the need to further constrain these optimization problems with period-by-period dynamic incentive constraints. These problems have led to the development of "incentive-constrained" dynamic programming techniques (e.g., see Rustichini (1998a) for an early version of "primal" incentive-constrained dynamic programming methods and Rustichini (1998b) and Marcet and Marimon (1998) for early discussions of "dual" methods). Indeed, Kydland and Prescott's methodological approach was essentially the first "recursive dual" approach to a dynamic consistency problem. Unfortunately, in either formulation of the incentive-constrained dynamic programming approach, these optimization methods have some serious methodological issues associated with their implementation. For example, in some problems, these additional incentive constraints are often difficult to formulate (e.g., for models with quasi-hyperbolic discounting. See Pollak 1968). Further, when these constraints can be formulated, they often involve punishment schemes that are ad hoc (e.g., see Marcet and Marimon 1998).

Now, additionally, in "dual formulations" of these dynamic optimization approaches, problems with dual solutions not being primal feasible can arise even in convex formulations of these problems (e.g., see Messner and Pavoni 2016), dual variables can be very poorly behaved (e.g., see Rustichini 1998b), and the programs are not necessarily convex (hence, the existence of recursive saddle points is not known, and the existing duality theory is poorly developed. See Rustichini (1998a) for an early discussion and Messner et al. (2012, 2014) for a discussion of problems with recursive dual approaches). It bears mentioning, all these duality issues also arise in the methods proposed by Kydland and Prescott (1980). This dual approach has been extended in a number of recent papers to related problems, including Marcet and Marimon (2011), Cole and Kubler (2012), and Messner et al. (2012, 2014).

[6]The key difference between the standard APS methods and those using dual variables such as in Kydland and Prescott (1980), and Feng et al. (2014) is that in the former literature, value functions are used as the new state variables; hence, APS methods are closely related to "primal" methods, not dual methods.

In addition to recursive dual approaches, incentive-constrained dynamic programming methods using "primal" formulations have been also proposed, and these methods do not exploit the dynamic structure of the set of Karush-Kuhn-Tucker multipliers associated with the recursive dual approach. As with dual dynamic optimization approaches, these primal methods also suffer from the problem that they are not concave programs. Further, characterizing optimal solutions can be very problematic.

Because of issues related to these "dynamic optimization" approaches, strategic dynamic programming has emerged as a systematic approach to this problem of constructing sequential equilibrium in dynamic macroeconomic models that can be formulated as a dynamic (or stochastic) game. For example, for the optimal taxation economy in Kydland and Prescott, where time-consistent optimal policies are viewed as subgame perfect equilibrium in a dynamic game played by successive generations of government policymakers, one can first construct a sequential equilibrium for the private economy for each sequential path for policy and then considers in the second stage a dynamic game played between successive generations of short-lived policymakers assuming no commitment (e.g., Dominguez and Feng 2016b; Phelan and Stacchetti 2001). This method, in some broad sense, can be thought of as a generalization of a "primal" incentive-constrained dynamic programming method, and this method has played a key role in the study of sustainable optimal government policy (e.g., see Sleet (2001) for an early discussion of using strategic dynamic programming methods for studying optimal monetary policy). In either case, one can additionally consider the role of reputation in the sustainability of optimal government plans (e.g., see Rogoff (1987) for an early discussion of this approach). In this latter approach, strategic dynamic programming methods that extend the seminal work of Abreu (1988) and Abreu et al. (1986, 1990) have been typically employed. We shall focus primarily on these strategic dynamic programming methods for studying strategic interaction in macroeconomic models that are formulated as a dynamic game in this chapter.

The rest of this chapter is laid out as follows: in the next section, we survey the application of dynamic and stochastic games in macroeconomics. In Sect. 3, we discuss the strategic dynamic programming approach to studying sequential equilibrium (and subgame perfect equilibrium) in these models more formally. We discuss both the extension of APS methods to models with states, as well as in Sect. 4 discuss some computational issues associated with strategic dynamic programming methods. In Sect. 5, we return to particular versions of the models discussed in Sect. 2 and discuss how to formulate sequential equilibrium in these models using strategic dynamic programming. In Sect. 6, we briefly discuss some alternative approaches to dynamic games in macroeconomics, and in the last section, we conclude.

2 Dynamic and Stochastic Games in Macroeconomics

The literature on dynamic and stochastic games in macroeconomics is extensive. These models often share a common structure and are dynamic general equilibrium models where some (or all) of the economic agents have dynamically inconsistent preferences or limited commitment generating a source of strategic interaction. In some models, the dynamic inconsistency problems stem from the primitive data of the model (e.g., models where agents have lifetime preferences that exhibit hyperbolic discounting). In other models, strategic interactions emerge because of the lack of commitment (i.e., as in dynastic models of economic growth where current generations care about future generations, but cannot control what future generations actually decide, or asset accumulations models with strategic default where one cannot assume borrowers will reply unless it is in their incentive to do so). Still in other models, the source of dynamic inconsistency comes from the structure of sequential equilibrium (e.g., preferences for government decision-makers designing optimal fiscal or monetary policy which are time inconsistent because of how the private economy responses in a sequential equilibrium to government policy). We now describe few prototypes of these models that we shall discuss in Sect. 4 of the chapter.

2.1 Hyperbolic Discounting

One prototype for dynamic games in macroeconomics is infinite horizon model of optimal economic growth or asset allocation where households have dynamically inconsistent preferences. The most studied version of this problem is economy where agents have preferences that exhibit hyperbolic discounting. This problem was first studied in Strotz (1955), subsequently by Pollak (1968) and Phelps and Pollak (1968), and has become the focus of an extensive literature in macroeconomics (e.g., see Barro 1999; Bernheim et al. 2015; Harris and Laibson 2001, 2013; Krusell et al. 2010; Krusell and Smith 2003; Laibson 1997).

The classical approach to studying the existence of time-consistent optimal plans for these problems has emphasized the language of recursive decision theory, as was discussed in the original paper by Strotz (1955). Unfortunately, as is well known, optimal dynamically consistent (including Markov) plans for such models need not exist, so the question of sufficient conditions for the existence of time-consistent optimal plans is a question of a great deal of study (e.g., see Pollak (1968), Peleg and Yaari (1973), and Caplin and Leahy (2006) for discussions of the nonexistence question).

One reason time-consistent plans may be nonexistent lies in the seemingly inherent presence of discontinuities in intertemporal preferences that arise very naturally in these problems when the recursive decision theory approach is applied. The reason for this lack of continuity is found in the inherent lack of commitment between the current "versions" of the dynamic decision-maker and all her continuation "selves." For example, from a decision theoretic perspective, when a "current" decision-maker is indifferent between some alternatives in the future, the earlier decision-maker ("planner") can still strictly prefer one of such alternatives in advance. As a result, he is willing to commit, yet lack access to a reasonable "commitment device" that would impose discipline on the choices of her future "selves" when tomorrow actually arrives. Due to this discontinuity, the optimal level of "commitment" may be nonexistent, and the dynamic maximization problem can turn out to be poorly defined (see, for example, Caplin and Leahy (2006) for an excellent discussion of this point).

An alternative way of obtaining a set of consistent plans for a dynamic choice problem with hyperbolic discounting is to view the dynamic choice problem as a dynamic game among different generations of "selves." In this formulation of the decision problem, at any current period, the current "self" takes as given a set of continuation strategies of all her "future selves" and best responds to this continuation structure in the game. For example, in the context of optimal growth, one could search for Markov perfect equilibrium in this dynamic game played between successive "selves." This is the approach advocated in the early work of Peleg and Yaari (1973) and in subsequent work by Laibson (1997), Barro (1999), Harris and Laibson (2001, 2013), Krusell and Smith (2003), Krusell et al. (2010), Balbus et al. (2015d), Balbus and Woźny (2016), and Bernheim et al. (2015). In this setting, one could take a candidate pure strategy continuation policy for savings/investment of one's future "self" as given, generate a value from the program from tomorrow onward, and given this value function could determine an optimal savings/investment decision problem for the current self. A fixed point in this mapping between continuation savings/investment and current savings/investment would be a Markov perfect equilibrium.

The problem is finding a space with sufficient continuity to study this fixed point problem. For example, if you take the continuation decision on savings/investment as continuous, the value function it generates need not be concave in the income state; this then means the current decision problem is not concave (hence, the best reply correspondence does not generally admit a continuous selection). If the continuation policy is only semicontinuous, then the current generations best reply correspondence need not contain a semicontinuous selection. So finding sufficient continuity for the existence of even pure strategy Markov perfect plans is problematic. Similar issues arise when considering subgame perfect equilibrium.[7] Finally, when Markovian time-consistent plans do exist, they are difficult to characterize and compute, as these models often suffer from an indeterminacy of equilibria (e.g., Krusell and Smith 2003).

[7]It bears mentioning that this continuity problem is related to difficulties that one finds in looking for continuity in best reply maps of the stage game given a continuation value function. It was explained nicely in the survey by Mirman (1979) for a related dynamic game in the context of equilibrium economic growth without commitment. See also the non-paternalistic altruism model first discussed in Ray (1987).

Perhaps the most well-studied version of hyperbolic discounting involves models where preferences exhibit quasi-hyperbolic discounting. In the quasi-hyperbolic model, agents have "$\beta - \delta$" preferences, where they have a "long-run" discount rate of $\delta \in (0, 1)$, and a "short-run" discount rate of $\beta \in (0, 1)$. In such models, agents have changing preferences, where at each period the preferences exhibit a bias toward current consumption. Such preferences often lead to an important role for public policy (e.g., Krusell et al. 2002, 2010). One class of models where the introduction of quasi-hyperbolic discounting has been shown to be important are models of asset accumulation (e.g., see the series of papers by Laibson (1994, 1997), Harris and Laibson (2001, 2013), as well as the recent paper by Bernheim et al. 2015). In these papers, the authors have shown using various methods that Markovian equilibrium savings behavior of models where agents have dynamically inconsistent preferences differ a great deal from models with standard time separable, dynamically consistent preferences. It is well known that in models with present-bias, savers consume more as a fraction of income than in models with dynamically consistent, time-separable preferences (also, see Diamond and Koszegi (2003) for examples of this in overlapping generations/life cycle models). In a very important paper, Laibson (1997) showed that in a standard asset accumulation model where agents possess preferences with quasi-hyperbolic preferences, and models enhanced with illiquid assets, the impact of present-bias preference can be mitigated by the presence of the illiquid asset. Indeed, illiquidity of assets can help constrain time-inconsistent behavior by working as a commitment device. His work suggests that financial innovation, therefore, can have a profound influence on equilibrium savings rates.

These models have also been used in the study of equilibrium economic growth. For example, Barro (1999) shows that in a version of the optimal growth model, under full commitment, and isoelastic period utility, agents save more and consume less; under imperfect commitment, saving rates and capital accumulation are lower. Krusell and Smith (2003) study a version of the optimal growth model and find additionally there exists a continuum of Markovian equilibria in their model without commitment. Krusell et al. (2002) produce a very interesting result for a particular parametric class of models. In particular, they show that for this particular parametric case, social planning solutions are strictly worse in welfare terms than a recursive equilibrium solution.

Extensions of this work of dynamic inconsistency in dynamic models have been numerous. The paper by O'Donoghue and Rabin (1999) extends class of Strotzian models to encompass models of procrastination. In their model, decision-makers are sophisticated or naive about their future structure of preferences (i.e., the nature of their future self-control problem), must undertake a single activity, and face intermediate costs and rewards associated with this activity. In the baseline model, "naive" decision-makers suffer procrastination ("acting too late") about undertaking a future activities with intermediate cost, while they act too soon relative to activities with intermediate future rewards. Sophistication about future self-control problems mitigates procrastination problems associated with dynamic inconsistency, while it makes the problem of preproperation ("acting too early") worse. In O'Donoghue and Rabin (2001), they extend this model to more general choice problems (with "menus" of choices).

In another line of related work, Fudenberg and Levine (2006) develop a "dual-selves" model of dynamically inconsistent choice and show that this model can explain both the choice in models with dynamically inconsistent preferences (e.g., Strotz/Laibson "$\beta - \delta$" models) and the O'Donoghue/Rabin models of procrastination. In their paper, they model the decision-maker as a "dual self," one being a long-run decision-maker, and a sequence of short-run myopic decision-makers, the dual self sharing preferences and playing a stage game.

There have been many different approaches in the literature to solve this problem. One approach is a recursive decision theory (Caplin and Leahy 2006; Kydland and Prescott 1980). In this approach, one attempts to introduce additional (implicit) constraints on dynamic decisions in a way that enforces time consistency. It is known that such decision theoretic resolutions in general can fail in some cases (e.g., time-consistent solutions do not necessarily exist).[8] Alternatively, one can view time-consistent plans as sequential (or subgame perfect) equilibrium in a dynamic game between successive generations of "selves." This was the approach first proposed in Pollak (1968) and Peleg and Yaari (1973). The set of subgame perfect equilibria in the resulting game using strategic dynamic programming methods is studied in the papers of Bernheim et al. (2015) and Balbus and Woźny (2016). The existence and characterization of Markov perfect stationary equilibria is studied in Harris and Laibson (2001), Balbus and Nowak (2008), and Balbus et al. (2015d, 2016). In the setting of risk-sensitive control, Jaśkiewicz and Nowak (2014) have studied the existence of Markov perfect stationary equilibria.

2.2 Economic Growth Without Commitment

Models of economic growth without commitment provide another important example of dynamic and stochastic games in macroeconomics. These models have arisen in many forms since the pioneering papers of Phelps and Pollak (1968), Peleg and Yaari (1973), Ray (1987), and Levhari and Mirman (1980).[9] For example, consider the model of altruistic growth without commitment as first described in Phelps and Pollak (1968) and Peleg and Yaari (1973) and extended in the work of Bernheim and Ray (1983), Leininger (1986), Amir (1996b), and Nowak (2006c). The model consists of a sequence of identical generations, each living one period, deriving utility from its own consumption, as well as the consumption of its successor generations. In any period of the economy, the current generation begins the period with a stock of output goods which it must either consume or invest in a

[8] For example, see Peleg and Yaari (1973), Bernheim and Ray (1983), and Caplin and Leahy (2006).

[9] For example, models of economic growth with strategic altruism under perfect commitment have also been studied extensively in the literature. For example, see Laitner (1979a,b, 1980, 2002), Loury (1981), and including more recent work of Alvarez (1999). Models of infinite-horizon growth with strategic interaction (e.g., "fishwars") are essentially versions of the seminal models of Cass (1965) and Brock and Mirman (1972), but without commitment.

technology that reproduces the output good tomorrow. The reproduction problem can be either deterministic or stochastic. Finally, because of the demographic structure of the model, there is no commitment assumed between generations. In this model, each generation of the dynastic household cares about the consumption of the continuation generation of the household, but it cannot control what the future generations choose. Further, as the current generation only lives a single period, it has an incentive to deviate from a given sequence of bequests to the next generation by consuming relatively more of the current wealth of the household (relative to, say, past generations) and leaving little (or nothing) of the dynastic wealth for subsequent generations. So the dynastic household faces a time-consistent planning problem.

Within this class of economies, conditions are known for the existence of semicontinuous Markov perfect stationary equilibria, and these conditions have been established under very general conditions via nonconstructive topological arguments (e.g., for deterministic versions of the game, in Bernheim and Ray 1987; Leininger 1986), and for stochastic versions of the game, by Amir (1996b), Nowak (2006c), and Balbus et al. (2015b,c). It bears mentioning that for stochastic games, existence results in spaces of continuous functions have been obtained in these latter papers. In recent work by Balbus et al. (2013), the authors give further conditions under which sharp characterizations of the set of pure strategy Markov stationary Nash equilibria (MSNE, henceforth) can be obtained. In particular, they show that the set of pure strategy MSNE forms an antichain, as well as develop sufficient conditions for the uniqueness of Markov perfect stationary equilibrium. This latter paper also provides sufficient conditions for globally stable approximate solutions relative to a unique nontrivial Markov equilibrium within a class of Lipschitz continuous functions. Finally, in Balbus et al. (2012), these models are extended to settings with elastic labor supply.

It turns out that relative to the set of subgame perfect equilibria, strategic dynamic programming methods can also be developed for these types of models (e.g., see Balbus and Woźny 2016).[10] This is interesting as APS type methods are typically only used in situations where players live an infinite number of periods. Although the promised utility approach has proven very useful in even this context, for models with altruistic growth without commitment, they suffer from some well-known limitations and complications. First, they need to impose discounting typically in this context. When studying the class of Markov perfect equilibria using more direct (fixed point) methods, one does not require this. Second, and more significantly, the presence of "continuous" noise in our class of dynamic games proves problematic for existing promised utility methods. In particular, this noise introduces significant complications associated with the measurability of value correspondences that represent continuation structures (as well as the possibility of constructing and characterizing measurable selections which are either equilibrium value function or pure strategies). We will discuss how this can be handled in versions of this model with discounting. Finally, characterizations of pure strategy equilibrium values

[10] Also see Balbus et al. (2012) section 5 for a discussion of these methods for this class of models.

(as well as implied pure strategies) is also difficult to obtain. So in this context, more direct methods studying the set of Markov perfect stationary equilibria can provide sharper characterizations of equilibria. Finally, it can be difficult to use promised utility continuation methods to obtain any characterization of the long-run stochastic properties of stochastic games (i.e., equilibrium invariant distributions or ergodic distributions).[11]

There are many other related models of economic growth without commitment that have also appeared in the literature. For example, in the paper of Levhari and Mirman (1980), the authors study a standard model of economic growth with many consumers but without commitment (the so-called great fishwar). In this model, in each period, there is a collective stock of output that a finite number of players can consume, with the remaining stock of output being used as an input into a productive process that regenerates output for the next period. This regeneration process can be either deterministic (e.g., as in Levhari and Mirman 1980 or Sundaram 1989a) or stochastic (as in Amir 1996a; Nowak 2006c).

As for results on these games in the literature, in Levhari and Mirman (1980), the authors study a parametric version of this dynamic game and prove existence of unique Cournot-Nash equilibrium. In this case, they obtain unique smooth Markov perfect stationary equilibria. In Sundaram (1989a), these results are extended to symmetric semicontinuous Markov perfect stationary equilibria in the game, but with more standard preferences and technologies.[12] Many of these results have been extended to more general versions of this game, including those in Fischer and Mirman (1992) and Fesselmeyer et al. (2016). In the papers of Dutta and Sundaram (1992) or Amir (1996a), the authors study stochastic versions of these games. In this setting, they are able to obtain the existence of continuous Markov perfect stationary Nash equilibrium under some additional conditions on the stochastic transitions of the game.

2.3 Optimal Policy Design Without Commitment

Another macroeconomic model where the tools of dynamic game theory play a critical role are models of optimal policy design where the government has limited commitment. In these models, again the issue of dynamic inconsistency appears. For example, there is a large literature studying optimal taxation problem in models under perfect commitment (e.g., Chamley 1986; Judd 1985. See also Straub and Werning 2014). In this problem, the government is faced with the problem of financing dynamic fiscal expenditures by choosing history-contingent paths for future taxation policies over capital and labor income under balanced budget constraints. When viewing the government as a dynastic family of policymakers, they collectively face a common agreed upon social objective (e.g., maximizing

[11] For competitive economies, progress has been made. See Peralta-Alva and Santos (2010).

[12] See also the correction in Sundaram (1989b).

the representative agent's objective function along sequential equilibrium paths for the private economy). As mentioned in the introduction, this problem is often studied under limited commitment (e.g., in Kydland and Prescott 1980; Pearce and Stacchetti 1997; Phelan and Stacchetti 2001, and more recently Dominguez and Feng 2016a,b, and Feng 2015). As the objective function in this class of models is generally not time consistent, the question of credible optimal government policy immediately arises.

The existence of time-consistent optimal plans for capital and labor income taxes was first studied in Kydland and Prescott (1980). In their formulation of the problem, the game was essentially a dynamic Stackelberg game, that is, in the "first stage," the agents in the private economy take sequences of tax instruments and government spending as given, and a sequential equilibrium for the private economy is determined. Then, in the second stage, this sequential equilibrium induces dynastic social preferences of government policymakers over these sequences of tax instruments and government spending (under a balanced budget rule). These preferences are essentially the discounted lifetime utility of a representative agent, and are maximized over the government's fiscal choice, which is not time consistent (therefore, as successive generations of government policymakers possess limited commitment across time, announced future plans will not necessarily be implemented by future government policymakers). To illustrate the basic problems of their model, we consider the following example:

Example 1. Consider a two-period economy with preferences given by

$$u(c_1) + \delta(u(c_2) + \gamma u(g)),$$

with linear production, full depreciation, and initial capital $k_0 > 0$. Then the economy's resource constraint is $c_1 + k = k_0$ and $c_2 + g = k$, where g is a public good level. Suppose that $\delta = 1$, $\gamma = 1$, and $u(c) = \log(\alpha + c)$ for some $\alpha > 0$.

The optimal, dictatorial solution (benevolent government choosing nonnegative k and g) to the welfare maximization problem is given by FOC:

$$u'(k_0 - k) = u'(k - g) = u'(g),$$

which gives $2g = k = \frac{2}{3}k_0$ with $c_1 = c_2 = g = \frac{1}{3}k_0$.

Now consider a competitive equilibrium economy, where the government finances public good g by levying a linear tax $\theta \in [0, 1]$ on capital income. The household budget is $c_1 + k = k_0$ and $c_2 = (1 - \theta)k$. Suppose that consumers have rational expectations and we look for a credible tax level θ under the balanced budget condition $g = \theta k$. For this reason suppose that θ is given and solve for competitive equilibrium investment k. The FOC gives:

$$u'(k_0 - k) = (1 - \theta)u'((1 - \theta)k),$$

which gives

$$k(\theta) = \frac{(1-\theta)(\alpha + k_0) - \alpha}{2(1-\theta)},$$

with $k'(\theta) < 0$. Now, knowing this reaction curve, the government chooses the optimal tax level solving the competitive equilibrium welfare maximization problem:

$$\max_{\theta \in [0,1]} u(k_0 - k(\theta)) + u((1-\theta)k(\theta)) + u(\theta k(\theta)).$$

Here the first-order condition requires

$$[-u'(k_0 - k(\theta)) + (1-\theta)u'((1-\theta)k(\theta))]k'(\theta) + [-u'((1-\theta)k(\theta)) \\ + u'(\theta k(\theta))]k(\theta) + \theta u'(\theta k(\theta))k'(\theta) = 0.$$

The last term (strictly negative) is the credibility adjustment which distorts the dynamically consistent solution from the optimal one. It indicates that in the dynamically consistent solution, when setting the tax level in the second period, the government must look backward for its impact on the first-period investment decision.

Comment: to achieve the optimal, dictatorial solution the government would need to promise $\theta = 0$ in the first period (so as not to distort investment) but then impose $\theta = \frac{1}{2}$ to finance the public good. Clearly it is a dynamically inconsistent solution.

This problem has led to a number of different approaches to solving it. One idea, found in the original paper of Kydland and Prescott (1980), was to construct optimal policy rules that respect a "backward"-looking endogenous constraint on future policy. This, in turn, implies optimal taxation policies must be defined on an enlarged set of (endogenous) state variables. That is, without access to a "commitment device" for the government policymakers, for future announcements about optimal policy to be credible, fiscal agents must constrain their policies to depend on additional endogenous state variables. This is the approach that is also related to the recursive optimization approaches of Rustichini (1998a), Marcet and Marimon (1998), and Messner et al. (2012), as well as the generalization of the original Kydland and Prescott method found in Feng et al. (2014) that is used to solve this problem in the recent work of Feng (2015).

Time-consistent polices can also be studied as a sequential equilibrium of a dynamic game between successive generations to determine the optimal mixture of policy instruments, where commitment to planned future policies is guaranteed in a sequential equilibrium in this dynamic game between generations of policymakers. These policies are credible optimal policies because these policies are subgame perfect equilibrium in this dynamic game. See also Chari et al. (1991, 1994) for a related discussion of this problem. This is the approach taken in Pearce and Stacchetti (1997), Phelan and Stacchetti (2001), Dominguez and Feng (2016a,b), among others.

Simple optimal policy problems also arise in the literature that studies optimal monetary policy rules; similar papers have been written in related macroeconomic models. This literature began with the important papers of Fischer (1980b) and Barro and Gordon (1983) (e.g., see also Rogoff (1987) for a nice survey of this work). More recent work studying the optimal design of monetary policy under limited commitment includes the papers of Chang (1998), Sleet (2001), Athey et al. (2005), and Sleet and Yeltekin (2007).

3 Strategic Dynamic Programming Methods

In this section, we lay out in detail the theoretical foundations of strategic dynamic programming methods for repeated and dynamic/stochastic games.

3.1 Repeated Models

An original strategic dynamic programming method was proposed by Abreu et al. (1986) and further developed in Abreu et al. (1990) for a class of repeated games with imperfect public information and perfect public equilibria. As the game is repeated, the original APS methods did not have "states" (e.g., in addition to promised utility). These methods have been used in macroeconomics (especially in dynamic contract theory, but also in policy games when considering the question of sustainable optimal monetary policy (e.g., see Chang 1998).[13]

Consider an infinitely repeated game between n-players with imperfect public information. Let $N = \{1, 2, \ldots, n\}$ be a set of players. In each period, each of n-players chooses simultaneously a strategy so that the strategy profile is $a = (a_1, a_2, \ldots, a_n) \in A$, where $A = \times_{i=1}^{n} A_i$, i.e., a Cartesian product of individual action sets. Each $a \in A$ induces a distribution over the realization of publicly observable signals $y \in Y$, where $Y \subset \mathbb{R}^k$ $(k \in \mathbb{N})$ given by $Q(dy|a)$. Each player $i \in N$ has a one-stage payoff given by $u_i(y, a_i)$, and its expectation is $g_i(a) := \int_Y u_i(y, a_i) Q(dy|a)$.

Remark 1. A repeated game with observable actions is a special case of this model, if $Y = A$ and $Q(\{a\}|a) = 1$ and zero otherwise.

[13] Also, for repeated games with quasi-hyperbolic discounting, see Chade et al. (2008) and Obara and Park (2013).

For each $t > 1$, let H_t be a public history at the beginning of period t. Mathematically, it is a sequence of the signals before t, i.e., $H_t := Y^t$ with generic element $h^t := (y_0, y_1, \ldots, y_{t-1})$. A public strategy of player i is a sequence of functions $\sigma_i := (\sigma_{i,t})_{t=0}^{\infty}$, where each $\sigma_{i,t}$ maps histories H_t to probability distributions on A_i. A strategy $\sigma_{i,t}$ is pure if it maps histories H_t into A_i. A strategy profile is a product of strategies, i.e., $\sigma := (\sigma_t)_{t=0}^{\infty}$, where $\sigma_t := (\sigma_{1,t}, \ldots, \sigma_{n,t})$.

Let $H := Y^{\infty}$ be a set of all public histories with generic element $h := (y_0, y_1, \ldots)$. By Ionescu-Tulcea theorem, a transition probability Q and strategy σ induce the unique Borel probability measure on H. Let E^{σ} be an expectation associated with this measure.

Assume a common discount factor $\delta \in (0, 1)$; then the player i's expected payoff from the repeated game is given by:

$$U_i(\sigma) := (1 - \delta)E^{\sigma}\left(\sum_{t=0}^{\infty} \delta^t g_i(\sigma_t(h^t))\right),$$

where $(1 - \delta)$ normalization is used to make payoffs of the stage game and infinitely repeated game comparable.

We impose the following set of assumptions:

Assumption 1. *(i) A_i is finite for each $i \in N$,*

(ii) for each $a \in A$, $Q(\cdot|a)$ is absolutely continuous probability measure with density $q(\cdot, a)$,

(iii) the support of $Q(\cdot|a)$ is independent of a, and without loss of generality assume that it is Y. That is

$$Y := \{y \in Y : q(y|a) > 0, \textit{ for all } a \in A\},$$

(iv) for each $i \in N$ and $a_i \in A_i$, $u_i(\cdot, a_i)$ is a continuous function,

(v) the one-shot strategic form game $(N, (A_i, g_i)_{i \in N})$ has a pure strategy Nash equilibrium.

Let $\mathcal{V} := L^{\infty}(Y, \mathbb{R}^n)$ be a set of all equivalence classes of essentially bounded Lebesgue measurable functions from Y into $\mathbb{R}^n$. Endow $\mathcal{V}$ with its weak star topology. Similarly, denote the measurable functions from Y to any subsets of $\mathbb{R}^n$. Moreover, with a slight abuse of notation, we will denote the i-th component of $v \in \mathcal{V}$ by $v_i : Y \to \mathbb{R}$, and hence $v = (v_1, v_2, \ldots, v_n) \in \mathcal{V}$.

A standard tool to deal with discounted n-player repeated games is the class of one-shot auxiliary (strategic form) games $\Gamma(v) = (N, (A_i, \Pi_i(v_i))_{i \in N})$, where

$$\Pi_i(a)(v_i) = (1 - \delta)g_i(a) + \delta \int_Y v_i(y)Q(dy|a)$$

is player i's payoff. Let $W \subset \mathbb{R}^n$. By $B(W)$ denote the set of all Nash equilibrium payoffs of the auxiliary game for some vector function $v \in \mathcal{V}$ having image in W. Formally:

$$B(W) := \{w \in \mathbb{R}^n : \text{ there is } a^* \in A, v \in \mathcal{V} \text{ such that}$$

$$\Pi_i(a^*)(v_i) \geq \Pi_i(a^*_{-i}, a_i)(v_i),$$

$$\text{for all } a_i \in A_i, i \in N, \text{ and } v(y) \in W, \text{ for almost all } y \in Y\}.$$

By the axiom of the choice, there is an operator $\eta : B(W) \to L^\infty(Y, W)$ and $\xi : W \to A$ such that for each $i \in N$ it holds

$$w_i = \Pi_i(\xi(w))(\eta(w, \cdot)) \geq \Pi_i(\xi_{-i}(w), a_i)(\eta(w, \cdot)).$$

Modifying on null sets if necessary, we may assume that $\eta(w, y) \in W$ for all y. We say that $W \subset \mathbb{R}^n$ is self-generating if $W \subset B(W)$. Denoting by $V^* \subset \mathbb{R}^n$ the set of all public perfect equilibrium vector payoffs and using self-generation argument one can show that $B(V^*) = V^*$. To see that we proceed in steps.

Lemma 1. *If W is self-generating, then $W \subset V^*$.*

Self-generation is an extension of the basic principle of optimality from dynamic programming. Let W be some self-generating set. Then, if $w \in W$, by self-generation, $w \in B(W)$. Consequently, we may find a sequence of functions $(v^t)_{t=1}^\infty$ such that $v^t : Y^t \to W$, for each $t > 1$ such that $v^1(y) := \eta(w, y)$ and for $t > 1$ $v^t(y_1, y_2, \ldots, y_t) := \eta(v^{t-1}(y_1, \ldots, y_{t-1}), y_t)$ and a sequence of functions $(\sigma_t)_{t=0}^\infty$ such that $\sigma_1 := \xi(w)$ and for $t > 0$ $\sigma_{t+1}(y_1, \ldots, y_t) := \xi(\eta(v^t(y_1, \ldots, y_t)))$. We claim that $\sigma := (\sigma_t)_{t=0}^\infty$ is a perfect Nash equilibrium in public strategies and $w_i = U_i(\sigma)$. Indeed, if player i deviates from σ until time T, choosing $\tilde{a}_i^t$ instead of a_i^t, then by definition of ξ and η:

$$w_i = U_i(\sigma) \geq (1 - \delta) \sum_{t=1}^{T} \delta^t g_i(\tilde{a}_i^t) + \delta^{T+1} U_i\left(J^T(\sigma)\right).$$

Here $J^T(\sigma) := (\sigma_{T+1}, \sigma_{T+2}, \ldots)$. Taking a limit with $T \to \infty$, we may conclude σ is a perfect Nash equilibrium in public strategies and $w \in V^*$. To formalize this thinking, we state the next theorem. Let $V \subset \mathbb{R}^n$ be some large set of possible payoffs, such that $V^* \subset V$. Then:

Theorem 1. *Suppose that Assumption 1 holds. Then,*

(i) $\bigcap_{t=1}^{\infty} B^t(V) \neq \emptyset$,

(ii) $\bigcap_{t=1}^{\infty} B^t(V)$ *is the greatest fixed point of* B,

(iii) $\bigcap_{t=1}^{\infty} B^t(V) = V^*$.

To see (i) of the aforementioned theorem, observe that V is a nonempty compact set. By Assumption 1 (ii), (iv) and (v), we may conclude that $B(V)$ is nonempty compact set and consequently that each $B^t(V)$ is nonempty and compact. Obviously, B is an increasing operator, mapping V into itself. $B^t(V)$ is a decreasing sequence; hence, its intersection is not empty. To see (ii) and (iii) of this theorem, observe that all sets $B^t(V)$ include any fixed point of B and, consequently, its intersection also. On the other hand, $\bigcap_{t=1}^{\infty} B^t(V)$ is self-generating, hence by Lemma 1

$$\bigcap_{t=1}^{\infty} B^t(V) \subset V^*. \tag{1}$$

By Assumption 1 (iii), we may conclude that V^* is self-generating; hence, $B(V^*) \subset V^*$. Consequently, $V^* = B(V^*)$; hence, $V^* \subset \bigcap_{t=1}^{\infty} B^t(V)$. Together with (1), we have points (ii) and (iii). Moreover, observe that V^* is compact. To see that, observe that V^* is bounded and its closure $cl(V^*)$ is compact. On the other hand, $V^* = B(V^*) \subset B(cl(V^*))$. By Assumption 1 (ii) and (iv), we have compactness of $B(cl(V^*))$ and consequently $cl(V^*) \subset B(cl(V^*))$; hence, by Lemma 1, $cl(V^*) \subset V^*$. As a result, V^* is closed and hence compact.

An interesting property of the method is that the equilibrium value set can be characterized using some extremal elements of the equilibrium value set only. Abreu et al. (1990) call it a bang-bang property. Cronshaw and Luenberger (1994) (and Abreu (1988) for some early examples) push this fact to the extreme and show that the equilibrium value set of a strongly symmetric subgame perfect equilibrium can be characterized using the worst punishment only. This observation has important implications on computation algorithms and applications.[14]

For each $W \subset \mathbb{R}^n$, let $co(W)$ be a convex hull of W. By $ext(W)$, we denote the set of extreme points of $co(W)$.

[14]See Dominguez (2005) for an application to models with public dept and time-consistency issues, for example.

Definition 1. We say that the function $v \in L^{\infty}(Y, W)$ has bang-bang property if $v(y) \in ext(W)$ for almost all $y \in Y$.

Using Proposition 6.2 in Aumann (1965), we have:

Theorem 2. *Let $W \subset \mathbb{R}^n$ be a compact set. Let $a^* \in A$ and $v \in L^{\infty}(Y, co(W))$ be chosen such that a^* is Nash equilibrium in the game $\Gamma(v)$. Then, there exists a function $\tilde{v} \in L^{\infty}(Y, ext(W))$ such that a^* is Nash equilibrium of $\Gamma(\tilde{v})$.*

Corollary 1. *If $W \subset \mathbb{R}^n$ is compact, then $B(W) = B(ext(W))$.*

Theorem 2 and its corollary show that we may choose $\eta(w, \cdot)$ to have bang-bang property. Moreover, if that continuation function has bang-bang properties, then we may easily calculate continuation function in any step. Especially, if Y is a subset of the real line, the set of extreme points is at most countable.

Finally, Abreu et al. (1990) present a monotone comparative statics result in the discount factor. The equilibrium value set V^* is increasing in the set inclusion order in δ. That is, the higher the discount factor, the larger is the set of attainable equilibrium values (as cooperation becomes easier).

3.2 Dynamic and Stochastic Models with States

We now consider an n-player, discounted, infinite horizon, stochastic game in discrete time. This is the basic APS tool used in numerous applications in macroeconomics (e.g., all the examples discussed in Sect. 2, but others too). Along these lines, consider the primitives of a class of stochastic games given by the tuple:

$$\left\{S, (A_i, \tilde{A}_i, \delta_i, u_i)_{i=1}^{N}, Q, s_0\right\},$$

where S is the state space, $A_i \subset \mathbb{R}^{k_i}$ is player i's action space with $A = \times_i A_i$, $\tilde{A}_i(s)$ the set of actions feasible for player i in state s, δ_i is the discount factor for player i, $u_i : S \times A \to \mathbb{R}$ is the one-period payoff function, Q denotes a transition function that specifies for any current state $s \in S$ and current action $a \in A$, a probability distribution over the realizations of the next period state $s' \in S$, and finally $s_0 \in S$ is the initial state of the game. We assume that $S = [0, \bar{S}] \subset \mathbb{R}$ and that $\tilde{A}_i(s)$ is a compact Euclidean subset of $\mathbb{R}^{k_i}$ for each s, i.

Remark 2. A dynamic game is a special case of this model, if Q is a deterministic transition.

Using this notation, a formal definition of a (Markov, stationary) strategy, payoff, and a Nash equilibrium can be stated as follows. A set of all possible histories of player i till period t is denoted by H_i^t. An element $h_i^t \in H_i^t$ is of the form $h_i^t = (s_0, a_0, s_1, a_1, \ldots, a_{t-1}, s_t)$. A *pure strategy* for a player i is denoted by $\sigma_i = (\sigma_{i,t})_{t=0}^{\infty}$ where $\sigma_{i,t} : H_i^t \to A_i$ is a measurable mapping specifying an action to be taken at stage t as a function of history, such that $\sigma_{i,t}(h_i^t) \in \tilde{A}_i(s_t)$. If, for some t and history $h_i^t \in H_i^t$, $\sigma_{i,t}(h_i^t)$ is a probability distribution on $\tilde{A}(s_t)$, then we say σ_i is a *behavior strategy*. If a strategy depends on a partition of histories limited to the current state s_t, then the resulting strategy is referred to as *Markov*. If for all stages t, we have a Markov strategy given as $\sigma_{i,t} = \gamma_i$, then a strategy identified with γ_i for player i is called a *Markov stationary strategy* and denoted simply by γ_i. For a strategy profile $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$, and initial state $s \in S$, the expected payoff for player i can be denoted by:

$$U_i(\sigma, s_0) = (1 - \delta_i) \sum_{t=0}^{\infty} \delta_i^t \int u_i(s_t, \sigma_t(h^t)) d m_i^t(\sigma, s_0),$$

where m_i^t is the stage t marginal on A_i of the unique probability distribution (given by Ionescu-Tulcea theorem) induced on the space of all histories for σ. A strategy profile $\sigma^* = (\sigma_{-i}^*, \sigma_i^*)$ is a *Nash equilibrium* if and only if σ^* is feasible, and for any i, and all feasible σ_i, we have

$$U_i(\sigma_{-i}^*, \sigma_i^*, s_0) \geq U_i(\sigma_{-i}^*, \sigma_i, s_0).$$

Assumption 2. *(i) S is a standard Borel space,*

(ii) A_i is a separable metric space and $\tilde{A}$ is a compact-valued measurable correspondence,

(iii) Each u_i is a uniformly bounded and jointly measurable function such that for each $s \in S$, $u_i(s, \cdot)$ is continuous on $\tilde{A}(s)$,

(iv) For each Borel measurable subset D of S, $(s, a) \mapsto Q(D|s, a)$ is jointly measurable and for each $s \in S$

$$\lim_{n \to \infty} \sup_D |Q(D|s, a_n) - Q(D|s, a)| = 0$$

whenever $a_n \to a$.

When dealing with discounted n-player dynamic or stochastic games, the main tool is again the class of one-shot auxiliary (strategic form) games $\Gamma_s(v) = (N, (A_i, \Pi_i(s, \cdot)(v_i))_{i \in N})$, where $s \in S \subset \mathbb{R}^n$ is the current state, while $v = (v_1, v_2, \ldots, v_n)$, where each $v_i : S \to \mathbb{R}$ is the integrable continuation value and the payoffs are given by:

$$\Pi_i(s, a)(v_i) = (1 - \delta_i) u_i(s, a) + \delta_i \int_S v_i(s') Q(ds'|s, a).$$

Then, by $K \subset \mathbb{R}^n$ denote some initial compact set of attainable payoff vectors and consider the large compact valued correspondence $V : S \rightrightarrows K$. Let $W : S \rightrightarrows K$ be any correspondence. By $B(W)(s)$ denote the set of all payoff vectors of $\Gamma_s(v)$ in K, letting v varying through all integrable selections from W. Then showing that $B(W)$ is a measurable correspondence, and denoting by $a^*(s)(v)$ a Nash equilibrium of $\Gamma_s(v)$, one can define an operator B such that:

$$B(W)(s) := \{w \in K : \text{ there is integrable selection } v \text{ of } W$$

$$\text{and a measurable Nash equilibrium } a^*(s)(v) \text{ of } \Gamma_s(v) \text{ such that}$$

$$\text{for each } i \in N \text{ it holds } w_i = \Pi_i(s, a^*(s)(v))(v)\}.$$

It can be shown that B is an increasing operator; hence, starting from some large initial correspondence V_0, one can generate a decreasing sequence of sets $(V_t)_t$ (whose graphs are ordered under set inclusion) with $V_{t+1} = B(V_t)$. Then, one can show using self-generation arguments that there exists the greatest fixed point of B, say V^*. Obviously, as V is a measurable correspondence, $B(V)$ is a measurable correspondence. By induction, one can then show that all correspondences V_t are measurable (as well as nonempty and compact valued). Hence, by the Kuratowski and Ryll-Nardzewski selection theorem (Theorem 18.13 in Aliprantis and Border 2005), all of these sets admit measurable selections. By definition, $B(V^*) = V^*$; hence, for each state $s \in S$ and $w \in B(V^*)(s) \subset K$, there exists an integrable selection v'^* such that $w = \Pi(s, a^*(s)(v'))(v')$. Repeating this procedure in the obvious (measurable) way, one can construct an equilibrium strategy of the initial stochastic game. To summarize, we state the next theorem:

Theorem 3. *Suppose that Assumption 2 holds. Then,*

(i) $\bigcap_{t=1}^{\infty} B^t(V) \neq \emptyset$,

(ii) $\bigcap_{t=1}^{\infty} B^t(V)$ *is the greatest fixed point of* B,

(iii) $\bigcap_{t=1}^{\infty} B^t(V) = V^*$,

where V^* is the set of all values of subgame perfect behavior strategies.

The details of the argument are developed in Mertens and Parthasarathy (1987), restated in Mertens and Parthasarathy (2003), and nicely summarized by Mertens et al. (2015) (pages 397–398). See also Fudenberg and Yamamoto (2011) for similar concepts used in the study of irreducible stochastic games with imperfect monitoring, or Hörner et al. (2011) with a specific and intuitive characterization of equilibria payoffs of irreducible stochastic games, when discount factor tends to 1. See also Baldauf et al. (2015) for the case of a finite number of states.

3.3 Extensions and Discussion

The constructions presented in Sects 3.1 and 3.2 offer the tools needed to analyze appropriate equilibria of repeated, dynamic, or stochastic games. The intuition, assumptions, and possible extensions require some comments, however.

The method is useful to prove existence of a sequential or subgame perfect equilibrium in a dynamic or stochastic economy. Further, when applied to macroeconomic models, where Euler equations for the agents in the private economy are available, in other fields like in economics (e.g., industrial organization, political economy, etc.), the structure of their application can be modified (e.g., see Feng et al. (2014) for an extensive discussion of alternative choices of state variables). In all cases, when the method is available, it allows one to characterize the entire set of equilibrium values, as well as giving a constructive method to compute them.

Specifically, the existence of some fixed point of B is clear from Tarski fixed point theorem. That is, an increasing self-map on a nonempty complete lattice has a nonempty complete lattice of fixed points. In the case of strategic dynamic programming, B is monotone by construction under set inclusion, while the appropriate nonempty complete lattice is a set of all bounded correspondences ordered by set inclusion on their graphs (or simply value sets for a repeated game).[15] Further, under self-generation, it is only the largest fixed point of this operator that is of interest. So the real value added of the theorems, when it comes to applications, is characterization and computation of the greatest fixed point of B. Again, it exists by Tarski fixed point theorem.

However, to obtain convergence of iterations on B, one needs to have stronger continuity type conditions. This is easily obtained, if the number of states S (or Y for a repeated game) is countable, but typically requires some convexification by sunspots of the equilibrium values, when dealing with uncountably many states. This is not because of the fixed point argument (which does not rely on convexity); rather, it is because the weak star limit belongs pointwise only to the convex hull of the pointwise limits. Next, if the number of states S is uncountable, then one needs to work with correspondences having *measurable* selections. Moreover, one needs to show that B maps into the space of correspondences having some measurable selection. This can complicate matters a good bit for the case with uncountable states (e.g, see Balbus and Woźny (2016) for a discussion of this point). Finally, some Assumptions in 1 for a repeated game or Assumption 2 for a stochastic game are superfluous if one analyzes particular examples or equilibrium concepts.

As already mentioned, the convexification step is critical in many examples and applications of strategic dynamic programming. In particular, convexification is important not only to prove existence in models with uncountably many states but also to compute the equilibrium set (among other important issues). We refer the reader to Yamamoto (2010) for an extensive discussion of a role of public randomization in the strategic dynamic programming method. The paper is

[15] See Baldauf et al. (2015) for a discussion of this fact.

important not only for discussing the role of convexification in these methods but also provides an example with a *non-convex* set of equilibrium values (where the result on monotone comparative statics under set inclusion relative to increases in the discount rate found in the original APS papers does not hold as well).

Next, the characterization of the entire set of (particular) equilibrium values is important as it allows one to rule out behaviors that are not supported by any equilibrium strategy. However, in particular games and applications, one has to construct carefully the greatest fixed point of B that characterizes the set of all equilibrium values obtained in the public perfect, subgame perfect, or sequential equilibrium. This requires assumptions on the information structure (for example, assumptions on the structure of private signals, the observability of chosen actions, etc.). We shall return to this discussion shortly.

In particular, for the argument to work, and for the definition of operator B to make sense, one needs to guarantee that for every continuation value v, and every state s, there exists a Nash equilibrium of one-shot game $\Gamma_s(v)$. This can be done, for example, by using mixed strategies in each period (and hence mapping to behavior strategies of the extensive form game). Extensive form mixed strategies in general, when players do possess some private information in sequential or subgame perfect equilibria, cannot always be characterized in this way (as they do not possess recursive characterization). To see that, observe that in such equilibria, the strategic possibilities at every stage of the game is not necessarily common knowledge (as they can depend arbitrarily on private histories of particular players). This, for example, is not the case of public perfect equilibria (or sequential equilibrium with *full support* assumption as required by Assumption 1 (iii)) or for subgame perfect equilibria in stochastic games with no observability of players' past moves.

Another important extension of the methods applied to repeated games with public monitoring and public perfect equilibria was proposed by Ely et al. (2005). They analyze the class of repeated games with private information but study only the so called "belief-free" equilibria. Specifically, they consider a strong notion of sequential equilibrium, such that the strategy is constant with respect to the beliefs on others players' private information. Similarly, as Abreu et al. (1990), they provide a recursive formulation of all the belief-free equilibrium values of the repeated game under study and provide its characterizations. Important to mention, general payoff sets of repeated games with private information lack such recursive characterization (see Kandori 2002).

It is important to emphasize that the presented method is also very useful when dealing with nonstationary equilibrium in macroeconomic models, where an easy extension of the abovementioned procedure allows to obtain comparable existence results (see Bernheim and Ray (1983) for an early example of this fact for an economic growth model with altruism and limited commitment). But even in stationary economies, the equilibria obtained using APS method are only stationary as a function of the current state and future continuation value. Put differently, the equilibrium condition is satisfied for a set or correspondence of values, but not necessarily its particular selection,[16] say $\{v^*\} = B(\{v^*\})$. To map it on the histories of the game and obtain stronger stationarity results, one needs to either consider (extensive form) correlated equilibria or sunspot equilibria or semi-Markov equilibria (where the equilibrium strategy depends on both current and the previous period states). To obtain Markov stationary equilibrium, one needs to either assume that the number of states and actions is essentially finite or transitions are nonatomic or concentrate on specific classes of games.

One way of restricting to a class of nonstationary Markov (or conditional Markov) strategies is possible by a careful redefinition of an operator B to work in function spaces. Such extensions were applied in the context of various macroeconomic models in the papers of Cole and Kocherlakota (2001), Doraszelski and Escobar (2012), or Kitti (2013) for countable number of states and Balbus and Woźny (2016) for uncountably many states. To see that, let us first redefine operator B to map the set of bounded measurable functions $\mathcal{V}$ (mapping $S \to \mathbb{R}^N$) the following way. If $W \subset \mathcal{V}$, then

$$
\begin{aligned}
B^f(W) := \{&(w_1, w_2, \ldots, w_n) \in \mathcal{V} \text{ and} \\
&\text{for all } s, i \text{ we have } w_i(s) = \Pi_i(s, a^*(s)(v))(v_i), \text{ where} \\
&v = (v_1, v_2, \ldots, v_n) \in W \text{ and each } v_i \text{ is an integrable function}\}.
\end{aligned}
$$

Again one can easily prove the existence of and approximate the greatest fixed point of B^f, say V_f^*. The difference between B and B^f is that B^f maps between spaces of functions not spaces of correspondences. The operator B^f is, hence, not defined pointwise as operator B. This difference implies that the constructed equilibrium strategy depends on the current state and future continuation value, but the future continuation value selection is constant among current states. This can be potentially very useful when concentrating on strategies that have more stationarity structure, i.e., in this case, they are Markov but not necessarily Markov stationary, so the construction of the APS value correspondence is generated by sequential or subgame perfect equilibria with *short memory*.

To see that formally, observe that from the definition of B and characterization of V^*, we have the following:

$$
\begin{aligned}
&(\forall s \in S)(\forall \text{ number } w \in V^*(s))(\exists \text{ measurable function} \\
&v'^* \text{ s.t. } w = \Pi(s, a^*(s)(v'))(v')).
\end{aligned}
$$

[16]However, Berg and Kitti (2014) show that this characterization is satisfied for (elementary) paths of action profiles.

Specifically, observe that continuation function v' can depend on w and s, and hence we shall denote it by $v'_{w,s}$. Now, consider operator B^f and its fixed point V_f^*. We have the following property:

$$(\forall \text{ function } w \in V^*)(\exists \text{ measurable function } v_f'^* \text{ s.t. } (\forall s \in S)\, w(s)$$
$$= \Pi(s, a^*(s)(v'))(v')).$$

Hence, the continuation v' depends on w only, and we can denote it by v'_w.

Observe that in both methods, the profile of equilibrium decision rules: $a^*(s)(v')$ is *generalized Markov*, as it is enough to know state s and continuation function v' to make an optimal choice. In some cases in macroeconomic applications, this generalized Markov equilibrium can be defined using envelope theorems of continuation values (not the value function itself).[17] In construction of B^f, however, the dependence on the current state is direct: $s \to a^*(s)(v'_w)$. So we can easily verify properties of the generalized Markov policy, such as whether it is continuous or monotone in s. In the definition of operator B, however, one has the following: $s \to a^*(s)(v'_{w,s})$. So even if the Nash equilibrium is continuous in both variables, (generally) there is no way to control continuity of $s \to v'_{w,s}$. The best example of such discontinuous continuation selection in macroeconomics application of strategic dynamic programming is, perhaps, the time-consistency model (see Caplin and Leahy 2006) discussed later in the application section. These technical issues are also important when developing a computational technique that uses specific properties of (the profile) the equilibrium decision rules with respect to s (important especially when the state space is uncountable).

4 Numerical Implementations

4.1 Set Approximation Techniques

Judd et al. (2003) propose a set approximation techniques to compute the greatest fixed point of operator B of the APS paper. In order to accomplish this task, they introduce public randomization that technically convexifies each iteration on the operator B, which allows them to select and coordinate on one of the future values that should be played. This enhances the computational procedure substantially.

More specifically, they propose to compute the inner V^I and outer V^O approximation of V^*, where $V^I \subset V^* \subset V^O$. Both approximations use a particular approximation of values of operator B, i.e., an inner approximation B^I and an outer approximation B^O that are both monotone. Further, for any set W, the approximate operators preserve the order under set inclusion, i.e., $B^I(W) \subset B(W) \subset B^O(W)$.

[17] See Kydland and Prescott (1980), Phelan and Stacchetti (2001), Feng et al. (2014), and Feng (2015).

Having such lower and upper approximating sets, Judd et al. (2003) are able to compute the error bounds (and a stopping criterion) using the Hausdorff distance on bounded sets in $\mathbb{R}^n$, i.e.:

$$d(W^O, W^I) = \max_{w_O \in W^O} \min_{w_I \in W^I} ||w_I - w_O||.$$

Their method is particularly useful as they work with convex sets at every iteration and map them on $\mathbb{R}^m$ by using its m extremal points. That is, if one takes m points, say $Z \subset W \subset \mathbb{R}^n$, define $W^I = \text{co}Z$. Next, for the outer approximation, take m points for Z on the boundary of a convex set W, and let $W^O = \bigcap_{l=1}^m \{z \in \mathbb{R}^n : g_l \cdot z \leq g_l \cdot z_l\}$ for a vector of m subgradients oriented such that $(z_l - w) \cdot g_l > 0$. To start iterating toward the inner approximation, one needs to find some equilibrium values from V^*, while to start iterating toward the outer, one needs to start from the largest possible set of values, say given by minimal and maximal bounds of the payoff vector.

Importantly, recent work by Abreu and Sannikov (2014) provides an interesting technique of limiting the number of extreme points of V^* for the finite number of action repeated games with perfect observability. In principle, this procedure could easily be incorporated in the methods of Judd et al. (2003). Further, an alternative procedure to approximate V^* was proposed by Chang (1998), who uses discretization instead of extremal points of the convex set. One final alternative is given by Cronshaw (1997), who proposes a Newton method for equilibrium value set approximation, where the mapping of sets W and $B(W)$ on $\mathbb{R}^m$ is done by computing the maximal weighted values of the players' payoffs (for given weights).[18]

Finally, and more recently, Berg and Kitti (2014) developed a method for computing the subgame perfect equilibrium value of a game with perfect monitoring using fractal geometry. Specifically, their method is interesting as it allows computation of the equilibrium value set with no public randomization, sunspot, or convexification. To obtain their result, they characterize the set V^* using (elementary) subpaths, i.e., (finite or infinite) paths of repeated action profiles, and compute them using the Hausdorff distance.[19]

4.2 Correspondence Approximation Techniques

The method proposed by Judd et al. (2003) was generalized to dynamic games (with endogenous and exogenous states) by Judd et al. (2015). As already mentioned, an appropriate version of the strategic dynamic programming method uses correspon-

[18] Both of these early proposals suffer from some well-known issues, including curse of dimensionality or lack of convergence.

[19] See Rockafellar and Wets (2009), chapters 4 and 5, for theory of approximating sets and correspondences.

dences V^* defined on the state space to handle equilibrium values. Then authors propose methods to compute inner and outer (pointwise) approximations of V^*, where for given state s, $V^*(s)$ is approximated using original Judd et al. (2003) method. In order to convexify the values of V^*, the authors introduce sunspots.

Further, Sleet and Yeltekin (2016) consider a class of games with a finite number of exogenous states S and a compact set of endogenous states K. In their case, correspondence V^* maps on $S \times K$. Again the authors introduce sunspots to convexify the values of V^*; this, however, does not guarantee that $V^*(s, \cdot)$ is convex. Still the authors approximate correspondences by using step (convex-valued) correspondences applying constructions of Beer (1980).

Similar methods are used by Feng et al. (2014) to study sequential equilibria of dynamic economies. Here, the focus is often also on equilibrium policy functions (as opposed to value functions). In either case (of approximating values or policies), the authors concentrate on outer approximation only and discretize both the arguments and the spaces of values. Interestingly, Feng et al. (2014), based on Santos and Miao (2005), propose a numerical technique to simulate the moments of invariant distributions resulting from the set of sequential equilibria. In particular, after approximating the greatest fixed point of B, they convexify the image of $B(V)$ and approximate some invariant measure on $A \times S$ by selecting some policy functions from the approximated equilibrium value set V^*.

Finally, Balbus and Woźny (2016) propose a step correspondence approximation method to approximate function sets without the use of convexification for a class of short-memory equilibria. See also Kitti (2016) for a fractal geometry argument for computing (pointwise) equilibria in stochastic games without convexification.

5 Macroeconomic Applications of Strategic Dynamic Programming

In this section, we apply strategic dynamic programming methods to the canonical examples discussed in Sect. 2.

5.1 Hyperbolic Discounting

As already mentioned in Sect. 2.1, one important application of strategic dynamic programming methods that is particularly useful in macroeconomics is finding the time-consistent solutions to the quasi-hyperbolic discounting optimization problem.[20] We now present this application in more detail and provide sufficient conditions to construct all the consistent plans for this class of models.

Our environment is a version of a $\beta - \delta$ quasi-hyperbolic discounting model that has been studied extensively in the literature. We envision an agent to be a sequence

[20] See, e.g., Harris and Laibson (2001) or Balbus et al. (2015d).

of selves indexed in discrete time $t \in \mathbb{N} \cup \{0\}$. A "current self" or "self t" enters the period in given state $s_t \in S$, where for some $\bar{S} \in \mathbb{R}_+$, $S := [0, \bar{S}]$, and chooses an action denoted by $c_t \in [0, s_t]$. This choice determines a transition to the next period state s_{t+1} given by $s_{t+1} = f(s_t - c_t)$. The period utility function for the consumer is given by (bounded) utility function u that satisfies standard conditions. The discount factor from today (t) to tomorrow $(t+1)$ is $\beta\delta$; thereafter, it equals δ between any two future dates $t + \tau$ and $t + \tau + 1$ for $\tau > 0$. Thus, preferences (discount factor) depend on τ.

Let $h^t = (s_0, s_1, \ldots, s_{t-1}, s_t) \in H_t$ be the history of states realized up to period t, with $h^0 = \emptyset$. We can now define preferences and a subgame perfect equilibrium for the quasi-hyperbolic consumer.

Definition 2. The sequence of functions $\sigma := (\sigma_t)_{t \in \mathbb{N}}$ is subgame perfect, if there is a sequence $(v_t)_{t \in \mathbb{N}}$, such that for each $t \in \mathbb{N}$ and $s \in S$

$$\sigma_t(h^t) \in \arg\max_{c \in [0, s_t]} \left\{ (1-\delta)u(c) + \beta\delta v_{t+1}((h^t, f(s_t - c))) \right\},$$

and

$$v_t(h^t) = (1-\delta)u(\sigma_t(h^t)) + \delta v_{t+1}((h^t, f(s_t - \sigma_t(h^t)))).$$

Here, for uniformly bounded v_t, we have the following payoffs:

$$v_t(h^t) = \sum_{\tau=1}^{\infty} \delta^{\tau-1} u(\sigma_{t+\tau}(h^{t+\tau})). \tag{2}$$

Intuitively, current self best responds to the value v_{t+1} discounted by $\beta\delta$ and that continuation value v_{t+1} summarizes payoffs from future "selfs" strategies $(\sigma_\tau)_{\tau=t+1}^{\infty}$. Such a best response is then used to update v_{t+1} discounted by δ to v_t.

In order to construct a subset of SPNE, we proceed with the following construction. Put:

$$\Pi^\kappa(s, c)(v) := (1-\delta)u(c) + \kappa v(f(s-c))$$

for $\kappa \in [0, 1]$. The operator B defined for a correspondence $W : S \rightrightarrows \mathbb{R}$ is given by:

$$B(W)(s) := \Big\{ v \in \mathbb{R} : \ v = \Pi^\delta(s, a(s))(w), \ \text{for some } a, w$$

$$\text{s.t. } a \in \arg\max_{c \in [0,s]} \Pi^{\beta\delta}(s, c)(w), \ \text{and } w \in W(s) \Big\}.$$

Based on the operator B, one can prove the existence of a subgame perfect equilibrium in this intrapersonal game and also compute the equilibrium correspondence. We should note that this basic approach can be generalized to include nonstationary transitions $\{f_t\}$ or credit constraints (see, e.g., Bernheim et al. (2015), who compute the greatest fixed point of operator B for a specific example of CIES utility function). Also, Chade et al. (2008) pursue a similar approach for a version of this particular game, where at each period n, consumers play a strategic form game. In this case, the operator B must be adopted to require that a is not the optimal choice, but rather a Nash equilibrium of the stage game.

Finally, Balbus and Woźny (2016) show how to generalize this method to include a stochastic transition and concentrate on short-memory (Markov) equilibria (or Markov perfect Nash equilibria, MPNE henceforth). To illustrate the approach to short memory discussed in Sect. 3.3, we present an application of the strategic dynamic programming method for a class of strategies where each σ_t depends on s_t only, but in the context of a version of the game with stochastic transitions. Let $s_{t+1} \sim Q(\cdot|s_t - c_t)$, and CM be a set of nondecreasing, Lipschitz continuous (with modulus 1) functions $h : S \to S$, such that $\forall s \in S\ h(s) \in [0, s]$. Clearly, as CM is equicontinuous and closed, it is a nonempty, convex, and compact set when endowed with the topology of uniform convergence. Then the discounted sum in (2) is evaluated under E_s^{σ} that is an expectation relative to the unique probability measure (existence and uniqueness of such a measure follows from standard Ionescu-Tulcea theorem) on histories h^t determined by initial state $s_0 \in S$ and a strategy profile σ.

For given $\bar{S} \in \mathbb{R}_+$, $S = [0, \bar{S}]$, define a function space:

$$V := \{v : S \to \mathbb{R}_+ : v \text{ is nondecreasing and u.s.c. bounded by } u(0) \text{ and } u(\bar{S})\}.$$

And let:

$$V^* = \{v \in V : \exists \text{ MPNE } (\sigma_t)_{t\in\mathbb{N}}, \\ \text{where each } \sigma_t \in CM,\ \text{s.t } v(s) = U((\sigma_t)_{t\in\mathbb{N}})(s)\ \forall s \in S\}.$$

In such a case, the stage payoff is

$$\Pi^{\kappa}(s, c)(v) := (1 - \delta)u(c) + \kappa \int_S v(s')Q(ds'|s - c),$$

and operator B^f defined on 2^V is given by:

$$B^f(W) := \bigcup_{w\in W} \Big\{ v \in V : (\forall s \in S)\ v(s) = \Pi^{\delta}(s, a(s))(w),\ \text{for some } a : S \to S, \\ \text{s.t. } a(s) \in \arg\max_{c\in[0,s]} \Pi^{\beta\delta}(s, c)(w) \text{ for all } s \in S \Big\}.$$

Balbus and Woźny (2016) prove that the greatest fixed point of B^f characterizes the set of all MPNE values in V generated with short memory, and they also discuss how to compute the set of all such equilibrium values.

5.2 Optimal Growth Without Commitment

Similar methods for (nonstationary) Markov perfect equilibrium can be developed for a class of optimal growth models without commitment between consecutive generations. As discussed in Sect. 2.2, this is often formalized using a class of paternalistic bequest games. We now present a detailed application of this class of models.

For the sake of illustration, consider a simple model of stochastic growth without commitment where there is an infinite sequence of generations labeled by $t \in \mathbb{N}$. In the economy, there is one commodity which may be either consumed or invested. Every generation lives one period and derives utility u from its own consumption and utility v from consumption of its immediate descendant. Generation t receives the endowment $s_t \in S$ and chooses consumption level $c_t \in A(s_t) := [0, s_t]$. The investment of $y_t := s_t - c_t$ determines the endowment of its successor according to some stochastic transition probability Q_t from S to S which depends on y_t.

Let P be the set of (bounded by a common bound) Borel measurable functions $p : S \mapsto \mathbb{R}_+$. A strategy for generation t is a function $\sigma_t \in \Sigma$, where Σ is a set of Borel measurable functions such that $\sigma(s) \in A(s)$ for each $s \in S$. The expected utility of generation t is defined as follows:

$$u(c) + \int_S v(\sigma_{t+1}(s'))Q(ds'|s - c, s), \tag{3}$$

where $u : S \mapsto \mathbb{R}_+$ is a bounded function, whereas $v : S \mapsto \mathbb{R}_+$ is bounded and Borel measurable. We endow P with its weak star topology and order 2^P (the set of all subsets of P) by set inclusion order.

Then, in section 5 of their paper, Balbus et al. (2012) define an operator B^f on 2^P:

$$B^f(W) = \bigcup_{p \in W} \Big\{p' \in \mathcal{P} : p'(s) = v(a_p^*(s)), \text{ where}$$

$$a_p^*(s) \in \arg\max_{c \in A(s)} \{u(c) + \int_S p(s')Q(ds'|s - c)\}\Big\}.$$

Clearly, each selection of values $\{v_t^*\}$ from the greatest fixed point $V^* = B^f(V^*)$ generates a MPNE strategy $\{\sigma_t^*\}$, where $\sigma_t^*(s) \in \arg\max\{u(c) + \int_S v_t^*(s')Q(ds'|s - c)\}$. Hence, using operator B^f, not only is the existence of MPNE established, but also a direct computational procedure can be used to compute the entire set of sustainable MPNE values. Here we note that a similar technique was used by Bernheim and Ray (1983) to study MPNE of a nonstationary bequest game.

Another direct application of the strategic dynamic programming presented above was proposed by Atkeson (1991) to study the problem of international lending with moral hazard and risk of repudiation. Specifically, using the currently available income (net of repayment) as a state variable and correspondences of possible continuation utilities, he characterizes the set of Pareto optimal allocations constrained to satisfy individual rationality and incentive compatibility (including no repudiation constraint), using the techniques advocated in Sect. 3.1.

5.3 Optimal Fiscal Policy Without Commitment

As already introduced in Sect. 2.3, in their seminal paper, Kydland and Prescott (1980) have proposed a recursive method to solve for the optimal tax policy of the dynamic economy. Their approach resembles APS method for dynamic games, but is different as it incorporates dual variables as states of the dynamic program. Such a state variable can be constructed because of the dynamic Stackelberg structure of the game. That is, equilibrium in the private economy is constructed first, and these agents are "small" players in the game, and take as given sequences of government tax policies, and simply optimize. As these problems are convex, standard Euler equations govern the dynamic equilibrium in this economy. Then, in the approach of Kydland-Prescott, in the second stage of the game, successive generations of governments design time-consistent policies by forcing successive generations of governments to condition optimal choices on the lagged values of Lagrange/KKT multipliers.

To illustrate how this approach works, consider an infinite horizon economy with a representative consumer solving:

$$\max_{\{a_t\}} \sum_{t=0}^{\infty} \delta^t u(c_t, n_t, g_t),$$

where $a_t = (c_t, n_t, k_{t+1})$ is choice of consumption, labor, and next period capital, subject to the budget constraints $k_{t+1} + c_t \leq k_t + (1 - \theta_t) r_t k_t + (1 - \tau_t) w_t n_t$ and feasibility constraint $a_t \geq 0$, $n_t \leq 1$. Here, θ_t and τ_t are the government tax rates and g_t their spendings. Formulating Lagrangian and writing the first-order conditions, together with standard firm's profit maximization conditions, one obtains $u_c(c_t, n_t, g_t) = \lambda_t$, $u_n(c_t, n_t, g_t) = -\lambda_t(1 - \tau_t) w_t$ and $\delta[1 + (1 - \theta_{t+1}) f_k(k_{t+1}, n_{t+1})]\lambda_{t+1} = \lambda_t$.[21]

Next the government solves:

$$\max_{\{\pi_t\}} \sum_{t=0}^{\infty} \delta^t u(c_t, n_t, g_t),$$

[21] Phelan and Stacchetti (2001) prove that a sequential equilibrium exists in this economy for each feasible sequence of tax rates and expenditures.

where $\pi_t = (g_t, \tau_t, \theta_t)$ under the above-given first-order conditions of the consumer and budget balance constraint: $g_t \leq \theta_t f_k(k_t, n_t)k_t + \tau_t f_n(k_t, n_t)n_t$. It is well known that the solution to this problem (on the natural state space k_t) is time inconsistent. That is, the solution of the problem, e.g., π_{t+s} chosen at time t, is different from the solution of the same problem at time $t + s$. Hence, standard dynamic programming techniques cannot be applied.

Kydland and Prescott (1980) then propose, however, a new method to make the problem recursive by adding a pseudo-state variable λ_{t-1}. Relative to this new state space, one can then develop a recursive optimization approach to the time-consistency problem that resembles the strategic dynamic programming methods of APS. To see this, omitting the time subscripts, Kydland and Prescott (1980) rewrite the problem of the government recursively by

$$v(k, \lambda_{-1}) = \max_{a,\pi,\lambda}\{u(c, n, g) + \delta v(k', \lambda)\}$$

under the budget balance, the first-order conditions of the consumer, and requiring that $(k'^*, \lambda^*) \in V^*$. Here V^* is the set of such $\{k_t, \lambda_t\}$, for which there exists an equilibrium policy $\{a_s, \pi_s, \lambda_s\}_{s=t}^{\infty}$ consistent with or supportable by these choices. This formalizes the constraint needed to impose time-consistent solutions on government choices. To characterize set V^*, Kydland and Prescott (1980) use the following APS type operator:

$$\begin{aligned} B(W) = \{(k, \lambda_{-1}) \in [0, k_{\max}] \times [\lambda_{\min}, \lambda_{\max}] :\ &\text{there exists} \\ &(a, \pi, \lambda)\text{satisfying budget balance constraints and consumer FOCs, with} \\ &(k', \lambda) \in W\}. \end{aligned}$$

They show that V^* is the largest fixed point of B and this way characterize the set of all optimal equilibrium policies. Such are time consistent on the expanded state space (k, λ_{-1}), but not on the natural state space k.

This approach was later extended and formalized by Phelan and Stacchetti (2001). They study the Ramsey optimal taxation problem in the symmetric sequential equilibrium of the underlying economy. They consider a dynamic game and also use Lagrange multipliers to characterize the continuation values. Moreover, instead of focusing on optimal Ramsey policies, they study symmetric sequential equilibrium of the economy and hence incorporate some private state variables. Specifically, in the direct extension of the strategic dynamic programming technique with private states, one should consider a distribution of private states (say capital) and (for each state) a possible continuation value function. But as all the households are ex ante identical, and sharing the same belief about the future continuations, they have the same functions characterizing the first-order conditions, although evaluated at different points, in fact only at the values of the Lagrange multipliers that keep track of the sequential equilibrium dynamics. In order to characterize the equilibrium conditions by FOCs, Phelan and Stacchetti (2001) add a public sunspot

$s \in [0, 1]$ that allows to convexify the equilibrium set under study. The operator B in their paper is then defined as follows:

$$B(W)(k) = \text{co}\{(\lambda, v) : \text{ there exists } (a, \lambda', v') \text{ satisfying consumer equilibrium FOCs, with } (\lambda', v') \in W(k) \text{ and the government deviation is punished by the minimax (worst) equilibrium value}\}.$$

Here, λ is the after tax marginal utility of consumption and v is household equilibrium value. Notice that, as opposed to the methods in Kydland and Prescott (1980), these authors integrate the household and government problem into one operator equation. Phelan and Stacchetti (2001) finish with the characterization of the best steady states of the symmetric sequential equilibrium.

Finally, this approach was more recently extended by Feng et al. (2014) (and Santos and Miao (2005) earlier), as a generalization of the strategic dynamic programming method to characterize all sequential equilibria of the more general dynamic stochastic general equilibrium economy. They follow the Phelan and Stacchetti (2001) approach and map the sequential equilibrium values to the space of continuation Lagrange multipliers values. Specifically, they consider a general economy with many agents in discrete time, with endogenous choices $a \in A$ and countably many exogenous shocks $s \in S$, drawn each period from distribution $Q(\cdot|s)$. Denoting the vector of endogenous variables by y, they assume the model dynamics is given by a condition $\phi(a', a, y, s) = 0$ specifying the budget and technological constraints. Next, denoting by $\lambda \in \Lambda$ the marginal values of all the investments of all the agents, they consider a function $\lambda = h(a, y, s)$. Finally, the necessary and sufficient first-order conditions for the household problems are given by

$$\Phi(a, y, s, \sum_{s' \in S} \lambda'(s') Q(ds'|s)) = 0,$$

where λ' is the next period continuation marginal value as a function on S. Next, they characterize the correspondence V^* mapping $A \times S$ to Λ, as the greatest fixed point of the correspondence-based operator:

$$B(W)(a, s) := \{\lambda : \lambda = h(a, y, s) \text{ for some } y, a', \lambda' \text{ with } \Phi(a, y, s, \sum_{s' \in S} \lambda'(s') Q(ds'|s)) = 0,\ \phi(a', a, y, s) = 0 \text{ and } \lambda'(s') \in W(a', s')\}.$$

To characterize V^*, they operate on the set of all upper hemi-continuous correspondences and under standard continuity conditions show that B maps W with compact graph into correspondence $B(W)$ with compact graph. Using the intersection theorem, along with a standard measurable selection theorem, they select a policy function a' as function of (a, s, λ) (hence, Markovian on the expanded state space including Lagrange multipliers λ).[22]

In order to compute the equilibrium correspondence, they use Beer (1980) algorithm of approximating correspondences by step correspondences on the discretized domain and co-domain grids as already discussed. Feng et al. (2014) conclude their paper with applications of the above framework to nonoptimal growth models with taxes, monetary economies, or asset prices with incomplete markets. See also Dominguez and Feng (2016b) and Feng (2015) for a recent application of the Feng et al. (2014) strategic dynamic programing method to a large class of optimal Ramsey taxation problems with and without constitutional constraints. In these papers, the authors are able to quantify the value of commitment technologies in optimal taxation problems (e.g., constitutional constraints) as opposed to imposing simply time-consistent solutions.

Finally, it is worth mentioning that the Feng et al. (2014) method is also useful to a class of OLG economies, hence with short-lived agents. See also Sleet (1998) (chapter 3) for such a model.

5.4 Optimal Monetary Policy Without Commitment

We should briefly mention that an extension of the Kydland and Prescott (1980) approach in the study of policy games was proposed by Sleet (2001). He analyzes a game between the private economy and the government or central bank possessing some private information. Instead of analyzing optimal tax policies like Kydland and Prescott (1980), he concentrates on optimal, credible, and incentive-compatible monetary policies. Technically, similar to Kydland and Prescott (1980), he introduces Lagrange multipliers that, apart from payoffs as state variables, allow to characterize the equilibrium set. He then applies the computational techniques of Judd et al. (2003) to compute dynamic equilibrium and then recovers the equilibrium allocation and prices. This extension of the methods makes it possible to incorporate private signals, as was later developed by Sleet and Yeltekin (2007).

We should also mention applications of strategic dynamic programming without states to optimal sustainable monetary policy due to Chang (1998) or Athey et al. (2005) in their study of optimal discretion of the monetary policy in a more specific model of monetary policy.

[22]It bears mentioning that in Phelan and Stacchetti (2001) and Feng et al. (2014) the authors actually used envelope theorems essentially as the new state variables. But, of course, assuming a dual representation of the sequential primal problem, this will then be summarized essentially by the KKT/Lagrange multipliers.

6 Alternative Techniques

We conclude with a few remarks concerning alternative methods to strategic dynamic programming for constructing dynamic equilibria in the macroeconomic models with strategically interacting agents. In particular, we focus on two widely used approaches that have been proposed in the literature, each providing significant advantages relative to strategic dynamic programming per characterizing *some* dynamic equilibrium (when applicable).

6.1 Incentive-Constrained Dynamic Programming

The first alternative approach to strategic dynamic programming methods are incentive-constrained dynamic programming methods (and their associated dual methods, often referred to in the literature as "recursive saddle point" or "recursive dual" methods). These methods develop recursive representations of sequential incentive-constrained optimization problems that are used to represent dynamic equilibria in macroeconomic models that are also dynamic/stochastic games. The methods are in the spirit of the recursive optimization approaches we discussed in Sect. 2 (e.g., the recursive optimization approaches to models with dynamically inconsistent payoffs or limited commitment such as models that are studied as interpersonal games between successive generations as in models with quasi-hyperbolic agents or growth models with limited commitment). The seminal early work on these methods is found in Rustichini (1998a,b) and Marcet and Marimon (1998), but a wealth of recent work has extended many of their ideas. These methods are used in models where agents face sequential optimization problems, but have incentives to change future optimal continuation plans when future states actually arise. Therefore, incentive-constrained programming methods add further constraints on sequential optimal decisions that agents face in the form of period-by-period dynamic incentive and participation constraints. These constraints are imposed to guarantee optimal decisions are time consistent (or, in some cases, subgame perfect) along equilibrium paths and therefore further restrict sequential optimal choices of economic agents. Then, incentive-constrained dynamic programming methods seek to find recursive primal or dual representations of these sequential incentive-constrained optimization problems. Such recursive representations help sharpen the characterization of dynamic equilibria strategies/policies.

Many applications of these incentive-constrained programming methods have arisen in the macroeconomic literature. For example, in dynamic asset pricing models with limited commitment and strategic default, where incentive constraints are used to model endogenous borrowing constraints that restrict current asset-consumption choices to be consistent with households not defaulting on outstanding debt obligations in any state the continuation periods (e.g., see Alvarez and Jermann 2000; Hellwig and Lorenzoni 2009; Kehoe and Levine 1993, 2001). Such solvency constraints force households to make current decisions that are consistent with them being able to credibly commit to future repayment schemes, and not to default on their debt obligations, making repayment schemes self-enforcing and sustainable. Similar recursive optimization approaches to imposing dynamic incentives for credible commitment to future actions arise in models of sustainable plans for the government in models in dynamic optimal taxation. Such problems have been studied extensively in the literature, including models of complete information and incomplete information (e.g., see the work of Chari et al. 1991; Chari and Kehoe 1990; Farhi et al. 2012; Sleet and Yeltekin 2006a,b).

Unfortunately, the technical limitations of these methods are substantial. In particular, the presence of dynamic incentive constraints greatly complicates the analysis of the resulting incentive-constrained sequential optimization problem (and hence recursive representation of this sequential incentive-constrained problem). For example, even in models where the primitive data under perfect commitment imply the sequential optimization problem generates value functions that are concave over initial states in models with limited commitment and state variables (e.g., capital stocks, asset holdings, etc.), the constraint set in such problems is no longer convex-valued in states. Therefore, as value function in the sequential problem ends up generally not being concave, it is not in general differentiable, and so developing useful recursive primal or recursive dual representations of the optimal incentive-constrained solutions (e.g., Euler inequalities) is challenging (e.g., see Rustichini (1998a) and Messner et al. (2014) for a discussion). That is, given this fact, an immediate complication for characterizing incentive-constrained solutions is that value functions associated with recursive reformulations of these problems are generally not differentiable (e.g., see Rincón-Zapatero and Santos (2009) and Morand et al. (2015) for discussion). This implies that standard (smooth) Euler inequalities, which are always useful for characterizing optimal incentive-constrained solutions, fail to exist. Further, as the recursive primal/dual is not concave, even if necessary first-order conditions can be constructed, they are not sufficient. These facts, together, greatly complicate the development of rigorous recursive primal methods for construction and characterization of optimal incentive-constrained sequential solutions (even if conditions for the existence of a value function in the recursive primal/dual exist). This also implies that even when such sequential problems can be recursively formulated, they cannot be conjugated with *saddle points* using any known recursive dual approach. See Messner et al. (2012, 2014) for a discussion.[23]

Now, when trying to construct recursive representations of the sequential incentive-constrained primal problem, new problems emerge. For example, these problems cannot be solved generally by standard dynamic programming type

[23]It is worth mentioning that Messner et al. (2012, 2014) often do not have sufficient conditions on primitives to guarantee that dynamic games studied using their recursive dual approaches have recursive saddle point solutions for models with state variables. Most interesting applications of game theory in macroeconomics involve states variable (i.e., they are dynamic or stochastic games).

arguments (e.g., standard methods for solving Bellman equations in dynamic programming). In particular, the resulting operator equation that must be solved is not, in general, a contraction (e.g., see Rustichini 1998a). Rustichini (1998a) shows that although the standard dynamic programming tools do not apply to sequential optimization problems with dynamic incentive constraints, one can develop a monotone iterative method based on a nonlinear operator (that has the spirit of a Bellman operator) that computes recursive solutions to the sequential incentive-constrained optimization problems. When sufficient conditions for a fixed point for the resulting functional equation in the recursive primal problem can be given, the recursive primal approach provides an alternative to APS/strategic dynamic programming methods as some dynamic equilibrium value in the model can be computed *if* the Markovian policies can be computed and characterized. Unfortunately, without a dual method for its implementation, computing the set of incentive-constrained optimal solutions that achieve this value (i.e., in these models, dynamic equilibria) is in general very difficult.

One other technical limitation of this method relative to strategic dynamic programming is often the punishment scheme used to sustain dynamic equilibria in general is ad hoc. That is, in strategic dynamic programming/APS methods, the punishment schemes used to construct sequential/subgame values are *endogenous*; in the standard version of an incentive-constrained dynamic programming problem, the punishment schemes are exogenous.

The primal formulation of incentive-constrained dynamic programming has been applied to many important macroeconomic models. In his original paper, Rustichini (1998a) shows how by adding period-by-period incentive constraints to the relevant decision-makers' problems in some important macroeconomic models with limited commitment incentive-constrained dynamic programming can be used to prove the existence of time-consistent or sustainable optimal policies. If the question is the existence of dynamic equilibria in such models, the primal versions of these recursive primal methods are very powerful. The problem with these methods is that it is challenging to compute the incentive-constrained optimal solutions themselves. Two interesting applications he makes in his paper are to optimal Ramsey taxation problems under limited commitment and models of economic growth without commitment. Since the publication of his paper, other applications of these methods have arisen. For example, they have been applied to studying optimal solutions to household's problem in dynamic asset accumulation models with limited commitment, a government optimal taxation problem with time inconsistent preferences, sustaining sovereign debt in models of international finance, and contract enforcement problems in models with human capital. See, for example, Koeppl (2007), Durdu et al. (2013), and Krebs et al. (2015), among many others, for a discussion.

To address the question of the computation of incentive-constrained optimal solutions, an extensive new literature has arisen. This recent work was motivated by the original paper of Kydland and Prescott (1980), as well as Marcet and Marimon (1998), where "dual variables" were used as "pseudo-state" variables to construct time-consistent optimal solutions. Indeed, in these two papers, the authors show

how to apply recursive dual approaches to a plethora of dynamic macroeconomic and dynamic contracting problems. In the original paper by Kydland and Prescott (1980), a recursive method for constructing generalized Markov equilibria was proposed where by adding the lagged values of Karush-Kuhn-Tucker multipliers to the set of state variables to optimal taxation rules, which forced the resulting government policymaker's taxation policies to respect a "backward-looking" constraint, would in turn force the resulting optimal solution to the time-inconsistent Euler equations (under the additional implied constraints) to be time consistent. See also Feng et al. (2014) for a significant extension of this method.

In the important paper by Marcet and Marimon (1998), the authors extend the ideas of Kydland and Prescott (1980) to the setting of a recursive dual optimization method, where the restrictions implied in the Kydland-Prescott method were explored more systematically. In the approach of Marcet and Marimon (1998), this restriction was embedded more formally into an extended dual recursive optimization approach, where KKT multipliers are added as state variables, and where in principle sufficient conditions can be developed such that this dual method will deliver incentive-constrained solutions to the primal recursive optimization methods ala Rustichini (1998a). The success of this dual approach critically relies on the existence of a recursive representation of saddle points, and in dynamic models where their dual recursive saddle point methods remain strictly concave, it can be proven that the methods of Marcet and Marimon (1998) compute primal incentive-constrained optimal solutions. The problem with this method is that in very simple concave problems, serious issues with duality can arise (e.g., see Cole and Kubler (2012) and Messner and Pavoni (2016) for discussion). The first problem is that in simple dynamic contracting problems, dual solutions can fail to be primal feasible (e.g., see the example in Messner and Pavoni 2016). In some cases, this issue can be resolved by extending the recursive saddle point method to weakly concave settings by introducing lotteries into the framework. In particular, see Cole and Kubler (2012). So even when recursive saddle points exist, some technical issues with the method can arise. Very importantly, Rustichini (1998b) shows even in concave settings, the dual variables/KKT multipliers can be poorly behaved from a duality perspective (see also Le Van and Saglam (2004) and Rincón-Zapatero and Santos (2009) for details).

In a series of recent papers by Messner et al. (2012, 2014), the authors further develop this recursive dual method. In these papers, they develop sufficient conditions for the equivalence of sequential primal and recursive dual formulations. For example, similar technical issues arise for recursive dual methods per existence of value functions that satisfy the functional equation that must be solved to represent the dual sequential incentive-constrained programming problem with a recursive dual (e.g., see Messner et al. 2014). Relative to the question of the existence of a recursive dual version of the recursive primal problem, Messner et al. (2014) provide the most general conditions under which a recursive dual formulation exists for a large class of dynamic models with incentive constraints.[24] In this paper, also many important questions concerning the equivalence of recursive primal and recursive dual solutions are addressed, as well as the question of sequential and recursive dual equivalence. In Messner et al. (2012), for example, the authors provide equivalence in models without backward-looking constraints (e.g., constraints generated by state variables such as capital in time-consistent optimal taxation problems á la Kydland and Prescott 1980) and many models with linear forward-looking incentive constraints (e.g., models with incentive constraints, models with limited commitment, etc.). In Messner et al. (2014), they give new sufficient conditions to extend these results to settings with backward-looking states/constraints, as well as models with general forward-looking constraints (including models with nonseparabilities across states). In this second paper, they also give new conditions for the existence of recursive dual value functions using contraction mapping arguments (in the Thompson metric). This series of papers represents a significant advancement of the recursive dual approach; yet, many of the results in these papers still critically hinge upon the existence of recursive saddle point solutions, and conditions on primitives of the model are not provided for these critical hypotheses. But critically, in this recursive dual reformulations, the properties of Lagrangians can be problematic (e.g., see Rustichini 1998b).

6.2 Generalized Euler Equation Methods

A second class of methods that have found use to construct Markov equilibrium in macroeconomic models that are dynamic games are generalized Euler equation methods. These methods were pioneered in the important papers by Harris and Laibson (2001) and Krusell et al. (2002), but have subsequently been used in a number of other recent papers. In these methods, one develops a so-called generalized Euler equation that is derived from the local first- and second-order properties relative to the theory of derivatives of local functions of bounded variation of an equilibrium value function (or value functions) that govern a recursive representation of agents' sequential optimization problem. Then, from these local representations of the value function, one can construct a generalized first-order representation of any Markovian equilibrium (i.e., a generalized Euler equation, which is a natural extension of a standard Euler) using this more general language of nonsmooth analysis. From this recursive representation of the agents' sequential optimization problem, plus this related generalized Euler equation, one can then construct an approximate solution to the actual pair of functional equations that are used to characterize a Markov perfect equilibrium, and Markov perfect equilibrium values and pure strategies can then be computed. The original method based on the

[24] For example, it is not a contraction in a the "sup" or "weighted sup" metric. It is a contraction (or a local contraction) under some reasonable conditions in the Thompson metric. See Messner et al. (2014) for details.

theory of local functions of bounded variation was proposed in Harris and Laibson (2001), and this method remains the most general, but some authors have assumed the Markovian equilibrium being computed is continuously differentiable, which greatly sharpens the generalized Euler equation method.[25]

These methods have applied in a large number of papers in the literature. Relative to the models discussed in Sect. 2, Harris and Laibson (2001), Krusell et al. (2002), and Maliar and Maliar (2005) (among many others) have used generalized Euler equation methods to solve dynamic general equilibrium models with a representative quasi-hyperbolic consumers. In Maliar and Maliar (2006b), a version of the method is applied to dynamic economies with heterogeneous agents, each of which has quasi-hyperbolic preferences. In Maliar and Maliar (2006a, 2016), some important issues with the implementation of generalized Euler equations are discussed (in particular, they show that there is a continuum of smooth solutions that arise using these methods for models with quasi-hyperbolic consumers. In Maliar and Maliar (2016), the authors propose an interesting resolution to this problem by using the turnpike properties of the dynamic models to pin down the set of dynamic equilibria being computed.

They have also been applied in the optimal taxation literature.[26] For example, in Klein et al. (2008), the authors study a similar problem to the optimal time-consistent taxation problem of Kydland and Prescott (1980) and Phelan and Stacchetti (2001). In their paper, they assume that a *differentiable* Markov perfect equilibrium exists, and then proceed to characterize and compute Markov perfect stationary equilibria using a generalized Euler equation method in the spirit of Harris and Laibson (2001). Assuming that such smooth Markov perfect equilibria exist, their characterization of dynamic equilibria is much sharper than those obtained using the calculus of functions of bounded variation. The methods also provide a much sharper characterization of dynamic equilibrium than obtained using strategic dynamic programming. In particular, Markov equilibrium strategies can be computed and characterized directly. They find that only taxation method available to the Markovian government is capital income taxation. This appears in contrast to the findings about optimal time-consistent policies using strategic dynamic programming methods in Phelan and Stacchetti (2001), as well as the findings in Klein and Ríos-Rull (2003). In Klein et al. (2005), the results are extended to two country models with endogenous labor supply and capital mobility.

There are numerous problems with this approach as it has been applied in the current literature. First and foremost, relative to the work assuming that the Markov perfect equilibrium is smooth, this assumption seems exceedingly strong as in very

[25]By "most general", we mean has the weakest assumptions on the *assumed* structure of Markov perfect stationary equilibria. That is, in other implementations of the generalized Euler equation method, authors often assume *smooth* Markov perfect stationary equilibria exist. In none of these cases do the authors actually appear to prove the existence of Markov perfect stationary equilibria within the class postulated.

[26]See for example Klein and Ríos-Rull (2003) and Klein et al. (2008).

few models of dynamic games in the literature, when Markov equilibria are known to exist, they are smooth. That is, conditions on the primitives of these games that guarantee such smooth Markov perfect equilibria exist are never verified. Further, even relative to applications of these methods using local results for functions of bounded variation, the problem is, although the method can solve the resulting generalized Euler equation, that solution cannot be tied to any particular value function in the actual game that generates this solution as satisfying the *sufficient* condition for a best reply map in the actual game. So it is not clear how to relate the solutions using these methods to the actual solutions in the dynamic or stochastic game.

In Balbus et al. (2015d), the authors develop sufficient conditions on the underlying stochastic game that a generalized Bellman approach can be applied to construct Markov perfect stationary equilibria. In their approach, the Markov perfect equilibria computing in the stochastic game can be directly related to the recursive optimization approach first advocated in, for example, Strotz (1955) (and later, Caplin and Leahy 2006). In Balbus et al. (2016), sufficient conditions for the uniqueness of Markov perfect equilibria are given. In principle, one could study if these equilibria are smooth (and hence, rigorously apply the generalized Euler equation method). Further, of course, in some versions of the quasi-hyperbolic discounting problem, closed-form solutions are available. But even in such cases, as Maliar and Maliar (2016) note, numerical solutions using some type of generalized Euler equation method need not converge to the actual closed-form solution.

7 Conclusion

Strategic interactions play a critical role in many dynamic models in macroeconomics. The introduction of such strategic elements into dynamic general equilibrium models has expanded greatly since the seminal work of Kydland and Prescott (1977), as well as early papers by Phelps and Pollak (1968), Peleg and Yaari (1973), Bernheim and Ray (1983), and Levhari and Mirman (1980). It is now a common feature of many models in macro, including models of economic fluctuations, public policy, asset pricing, models of the behavioral aspects of consumption-savings problems, models of economic growth with limited commitment or strategic altruism, among others. In this chapter, we have presented a number of canonical situations, where strategic considerations arise in the study of dynamic equilibria in macroeconomics. Then, we have discussed how the tools of dynamic and stochastic game theory can be used to study equilibria in such problems.

The introduction of such strategic dimensions into macroeconomics greatly complicates the analysis of equilibria. Still, rigorous and general methods are available for constructing, characterizing, and computing them. We have argued that strategic dynamic programming methods, first pioneered in Abreu et al. (1986, 1990) for repeated games, when extended to settings with state variables, provide a powerful systematic set of tools to construct and compute equilibria in such macroeconomic models. Also, we have mentioned that in some cases, for particular subclasses of sequential or subgame perfect equilibria (e.g., Markov perfect equilibria), these methods can be improved upon using recursive primal/dual methods or generalized Euler equation methods. Unfortunately, relative to strategic dynamic programming methods, these methods are known to suffer from serious technical limitations in some dynamic models with state variables. As the majority of the models studied in macroeconomics are dynamic, and include states, strategic dynamic programming offers the most systematic approach to such models; hence, in this chapter, we have discussed what these methods are and how they can be applied to a number of interesting models in dynamic macroeconomics.

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