# A Nonsmooth Approach to Envelope Theorems: A Correction

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**Abstract**

In the published version of this paper, we applied our results to minimax lattice programming to study normality of demand in the consumer's problem. In all the versions of the paper circulated before publication, the sufficient conditions were stated correctly (i.e., we assumed the the cardinal complementarity condtions in Quah ([13]). In the published version, inadvertantly, the conditions are not correct. In particular, we need the utility function for the consumer $u(c)$ supermodular on $\mathbf{E}^n_+$ (not the weaker "increasing difference" conditions mentioned in the published paper).

In this note, we reproduce the original proof of our claims under the Quah cardinal complementary conditions from the earlier drafts. The original draft of the paper from before the final corrections for publication with the correct conditions/proof is also now available on the second author's webpage.

## 1 Lipschitz Optimization Problems

In the paper, we considered the following family of *Parameterized Lipschitz Optimization Problems.* Let $A$ be space of actions (or controls), $S$ the

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parameter space, and $f : A \times S \to \mathbf{R}$ the objective function. We seek to characterize envelope theorems in the following parameterized program:

$$V(s) = \max_{a \in D(s)} f(a, s) \tag{1.0.1}$$

where the feasible correspondence is given by

$$D(s) = \{a | g^i(a, s) \leq 0, \quad i = 1, ........, p, \ h^j(a, s) = 0, \quad j = 1, ........, q\}$$

and the optimal solution correspondence is:

$$A^*(s) = \arg \max_{a \in D(s)} f(a, s)$$

We shall often refer to program (1.0.1) as the "primal" problem.

We will maintain some baseline assumptions throughout the paper:

*Assumption 1:*

*(a) The metric spaces $(A, \rho_A)$ and $(S, \rho_S)$ are each convex in* $\mathbf{R}^n$ and $\mathbf{R}^m$*, respectively,*

*(b) the constraints $g^i$, $i = 1, .., p$ and $h^j = 0$, $j = 1, .., q$ are jointly $C^1$, and $n \geq q$,*

*(c) the objective function $f : A \times S \to \mathbf{R}$ is continuous in $(a, s)$,*

*(d) the feasible correspondence $D : S \rightrightarrows A$ is nonempty, continuous and compact-valued.*

*Assumption 2: The objective $f(a, s)$ and the constraints* $g(a, s)$ and $h(a, s)$ *both admit differential extensions over the boundary of $A \times S$.*

To obtain unique KKT multipliers for each optimal solution, we impose as slightly stronger constraint qualification than MFCQ (along with some standard regularity conditions on $D(s)$, e.g., uniform compactness), namely the Strict Mangasarian-Fromowitz Constraint Qualification (henceforth, SM-FCQ), which is defined as follows:

**Definition 1** *A feasible point $a \in D(s)$ satisfies the Strict Mangasarian-Fromovitz Constraint Qualifier (SMFCQ) if:*

*(i) the following vectors are linearly independent,*

$$g_1^i(a, s),\ i \in I_b; \ \ h_1^j(a, s),\ j = 1, ..., q$$

*(ii) there exist $y \in \mathbf{R}^n$ such that,*

$$g_1^i(a,s)y < 0,\ i \in I_s;\ \ g_1^i(a,s)y = 0,\ i \in I_b$$
$$h_1^j(a,s)y = 0, \quad j = 1, ..., q$$

*where $I_b = \{i \in I : \lambda_i > 0\}$, $I_s = \{i \in I : \lambda_i = 0\}$ and $I = \{i : g^i(a,s) = 0\}$.*

**Lemma 2** *Under Assumptions 1 and 2, if additionally $f$ is locally Lipschitz, $D(s)$ is nonempty and uniformly compact near $s$, and MFCQ holds at the optimal solution $a^*(s) \in A^*(s)$, then, in any direction $x \in \mathbf{R}^m$, we have:*

$$\liminf_{t \to 0^+} \frac{V(s+tx) - V(s)}{t} \geq \inf_{(\lambda,\mu) \in K(a^*(s),s)} L_s^{-o}(a^*(s), s, \lambda, \mu; x)$$

$$\limsup_{t \to 0^+} \frac{V(s+tx) - V(s)}{t} \leq \sup_{(\lambda,\mu) \in K(a^*(s),s)} \{L_s^o(a^*(s), s, \lambda, \mu; x)\}$$

*therefore,*

$$\begin{aligned} &\sup_{a^*(s) \in A^*(s)} \min_{(\lambda,\mu) \in K(a^*(s),s)} \left\{L_s^{-o}(a^*(s), s, \lambda, \mu; x)\right\} \\ &\leq D_+V(s;x) \leq D^+V(s;x) \\ &\leq \sup_{a^*(s) \in A^*(s)} \max_{(\lambda,\mu) \in K(a^*(s),s)} \{L_s^o(a^*(s), s, \lambda, \mu; x)\} \end{aligned}$$

*where*

$$\begin{aligned} L_s^{-o}(a^*(s), s, \lambda, \mu; x) &= \min_{\varsigma_s \in \partial_s f(a^*(s),s)} \left[\left(\varsigma_s - (\lambda^T g_2 + \mu^T h_2)(a^*(s), s)\right) \cdot x\right] \\ L_s^o(a^*(s), s, \lambda, \mu; x) &= \max_{\varsigma_s \in \partial_s f(a^*(s),s)} \left[\left(\varsigma_s - (\lambda^T g_2 + \mu^T h_2)(a^*(s), s)\right) \cdot x\right] \end{aligned}$$

Notice that although Lemma 2 only provides differential bounds, these bounds are easily calculated directly from the Lagrangian at the optimum. Under MFCQ, tighter (or exact) bounds are not possible, as one loses uniqueness of the KKT multipliers for each optimal solution and therefore cannot pin down the values of the directional derivatives in any direction.

We proved the following useful result we use in the minimax programming example (see the published paper for proof):

**Corollary 3** *Let Assumption 1 and 2 hold, and assume additionally the following: $f$ is $C^1$, concave in $a$, $g$ is convex, $h$ is affine in $a$, $D(s)$ is nonempty and uniformly compact near $s$, and MFCQ holds for every optimal solution $a^*(s) \in A^*(s)$. Then, for any direction $x \in \mathbf{R}^m$, the directional envelope is given by:*

$$V^{'}(s;x) = \max_{a^*(s)\in A^*(s)} \min_{(\lambda,\mu)\in K(a^*(s),s)} L_2(a^*(s), s, \lambda, \mu) \cdot x$$

**Proof.** Lemma 2 part (i) provides the lower bounds of The Dini derivative. With additional structure the upper limit is tighter. Let us first choose a sequence $\{t_n\}$ converging to 0 such that:

$$\limsup_{t\to 0^+} \frac{V(s+tx) - V(s)}{t} = \lim_{n\to\infty} \frac{V(s+t_nx) - V(s)}{t_n}$$

Since $D(s)$ is uniformly compact near $s$, for $n$ large, there exists $a_n^*(s) \in D(s+t_nx)$ such that $V(s+t_nx) = f(a_n^*(s), s+t_nx)$. Since the sequence $\{a_n^*(s)\}$ is in a compact domain, there exists a convergent subsequence. Therefore, without loss of generality, assume that $a_n^*(s) \to a^*(s) \in D(s)$. Then, by the continuity of $V$, $V(s) = f(a^*(s), s)$. As any $a^*(s) \in A^*(s)$ is a global maxima, appealing to strong duality, the Lagrangian also has a global saddle point at $(a^*(s), \lambda, \mu)$ where $(\lambda, \mu) \in K(a^*(s), s)$. Then, for any $((\lambda, \mu) \in K(a^*(s), s)$ , we have the following,

$$\begin{aligned}
&\limsup_{t\to 0^+} \frac{V(s+tx) - V(s)}{t} \\
&= \lim_{n\to\infty} \frac{V(s+t_nx) - V(s)}{t_n} \\
&= \lim_{n\to\infty} \frac{L(a_n^*(s) + t_n\overline{y}, s+t_nx, \lambda, \mu) - L(a^*(s), s, \lambda, \mu)}{t_n}
\end{aligned}$$

As $n \to \infty$, $t_n \to 0^+$, and thus,

$$\begin{aligned}
&\limsup_{t\to 0^+} \frac{V(s+tx) - V(s)}{t} \\
&\leq \lim_{t_n\to 0^+} \frac{L(a_n^*(s) + t_n\overline{y}, s+t_nx, \lambda, \mu) - L(a^*(s), s, \lambda, \mu)}{t_n} \\
&= L_1(a^*(s), s, \lambda^*(s), \mu^*(s)) \cdot \overline{y} + L_2(a^*(s), s, \lambda^*(s), \mu^*(s)) \cdot x \\
&= L_2(a^*(s), s, \lambda^*(s), \mu^*(s))
\end{aligned}$$

where last equality follows from the first order condition. Note, the above hold, for all $(\lambda, \mu) \in K(a^*(s), s)$, Therefore

$$\begin{aligned} &\limsup_{t \to 0^+} \frac{V(s+tx) - V(s)}{t} \\ &\leq \inf_{\lambda, \mu} [L_2(a_I^*(s), s, \lambda, \mu) \cdot x] \end{aligned}$$

Thus, the upper bound of Lemma 2 Part (ii) is more tight for all optimal solutions $a^*(s) \in A^*(s)$. Noting that for any $a^*(s) \in A^*(s)$, the lower and upper bound of Lemma 2 parts (i) and (ii) coincide, thus as in Lemma 2 (iii), we have

$$\begin{aligned} &\sup_{a^*(s) \in A^*(s)} \min_{(\lambda,\mu) \in K(a^*(s),s)} L_2(a^*(s), s, \lambda, \mu) \cdot x \\ &= D_+ V(s; x) = D^+ V(s; x) \\ &= \sup_{a^*(s) \in A^*(s)} \min_{(\lambda,\mu) \in K(a^*(s),s)} L_2(a^*(s), s, \lambda, \mu) \cdot x \end{aligned}$$

Hence the result ■

# 2 Minimax Lattice Programming Application

We next apply our results to a class of constrained lattice programming problems. For example, it is well-known that standard lattice programming techniques (e.g., Topkis [15]) in the consumer's problem to obtain conditions for normal demand for a single good (as ordered changes in income do not induce strong set ordered changes in the budget correspondence in the standard Euclidean order $\mathbf{E}^n$,[1] so the standard lattice programming approach does not apply). In recent work, many authors have each proposed interesting new approaches that do obtain conditions for some version of normal demand.[2] In many of these cases, the comparative statics obtained are "weak" comparative statics (e.g., one identifies conditions under which there exists a monotone selection for demand, as opposed to "strong" comparative statics where one has sharp characterizations of monotone selections that are typically obtained in standard lattice programming formulations). In this

[1] The space $\mathbf{E}^n$ is the pair $(\mathbf{R}^n, \geq_e)$ where $\geq_e$ is the standard componentwise (product) Euclidean order.

[2] See, for example, [1][2][13][12], among others.

section, we use our nonsmooth envelope theorems and duality to propose a new method to provide answers to this question to obtain strong comparative statics.

To understand our dual approach to constrained lattice programming, consider a standard monotone comparative statics in the consumer's problem (in particular, income effects). Let $A = C \subset \mathbf{E}_{+}^{n}$ be the commodity space, $C$ a nonempty sublattice, $s \in S = [0, \infty)$ be the level of income. For this entire section, fix the vector of prices to be $p \in \mathbf{S}_{n-1}$, with $p >> 0$, and $\mathbf{S}_{n-1}$ is the $n-1$ dimensional simplex. When formalizing our aggregation procedure, we will need to work on sections of the commodity space $C$. Therefore, define following sections of $C$ : (i) $C_i = \{c \in C | c_{-i} = (c_1, c_2, ..., c_{i-1}, c_{i+1}, ..c_n)$ fixed$\} \subset \mathbf{E}_{+}$, and (ii) $C_{-i} = \{c \in C | c_i \in \mathbf{E}_{+}$ fixed$\} \subset \mathbf{E}^{n-1}$. Further, as we shall seek strong monotone comparative statics, we need to define the following Veinott sublattice powerdomains (iii) $\mathcal{L}(E_i) = \{L_i \subset E_{+} | L_i$ a nonempty sublattice of $E_{+})$, and (iv) $\mathcal{L}(E_{-i}) = \{L_{-i} | L_{-i}$ a nonempty sublattice of $E_{+}^{n-1}\}$. Endow all these sets with Veinott's strong set order.[3]

We shall assume the cardinal complementarity conditions in Quah [13]. That is, we assume the consumer's preferences can be represented by a real-valued utility function $u : C \to \mathbf{R}$, where $u(c)$ satisfies the following assumption:

*Assumption 3: ($C_i-$supermodularity). The function $u(c)$ is (a) $C^1$ in* $c_{-i}$ for each $c_i$, *(b) supermodular on $C \subset \mathbf{E}^n$, (c) concave in $c_{-1}$ for each $c_1$ (where $c_{-1}$ denotes the vector $c$ with the ith coordinate deleted), (d) $u(c)$ increasing, (e) $u(c)$ is bounded below.*

The conditions in Assumption 3 are the cardinal complementarity conditions for constrained optimization problems first proposed in Quah [13], and we shall refer to them as the *Quah cardinal complementarity conditions.*[4]

We ask the following question: for a fixed price vector $p >> 0$, what are sufficient conditions for the consumer's demand correspondence for good $c_i$

[3] That is, we seek standard monotone comparative statics. Also, note we can define similar Veinott sublattice powerdomains can be defined for the dual variable $\lambda \subset \mathbf{E}_{+}$ in the Lagrangian problem we write down in a moment.

[4] Quah requires only locally Lipschitz structure of payoffs in $c_{-i}$ in his cardinal complementarity conditions (as he imposes partial concavity in the cardinal case). We do not need $C^1$ either (and can just work with directionals and subdifferentials for the value functions, if needed). The point of this section is to show the structure of our dual argument. Hence, for the sake of exposition, we assume smooth primitive data.

to be isotone in income $s \in S$ in *Veinott's strong set order* to $\mathcal{L}(C_i)$. For the cardinal case, one answer for these *strong* comparative statics will be Quah's cardinal complementarity conditions.

The consumer's problem can be stated as follows: for fixed price vector $p = (p_1, p_2, .., p_{i-1}, 1, p_{i+1} ..., p_n) >> 0$, $p \in \mathbf{S}_{n-1}$, at income level $s \in S$, the consumer solves[5]

$$V(s) = \max_{c \in D(s)} u(c) \tag{2.0.2}$$

where $V(s)$ is the value function at $s$, the budget correspondence is

$$D(s) = \{c | p \cdot c \leq s, c \in C\}$$

and the optimal solutions for the problem are

$$C^*(s) = \arg\max_{c \in D(s)} u(c) \tag{2.0.3}$$

Under Assumption 3 clearly this program is well-defined (e.g., via a standard application of Berge's theorem). We shall refer to program (2.0.2) as the *primal problem.*

Define the $C_i-$*aggregated primal* (or, simply the "*aggregated primal*") as follows: for any $\infty > s > 0$, and $c_i \in [0, s]$, solve:

$$V^{ap}(c_i, s) = \max_{c_{-1} \in D(c_i, s)} u(c_{-i}; c_i) \tag{2.0.4}$$

The optimal solutions for the aggregated primal are

$$C^{ap*}_{-i}(c_i, s) = \arg\max_{c_{-1} \in D(c_i, s)} u(c_{-i}, c_i)$$

We can now recast this monotone comparative statics problem (under aggregation) as the following problem:

$$V(s) = \max_{c_i \in [0, s]} V^{ap}(c_i, s) \tag{2.0.5}$$

where $V(s)$ is the value function of the primal problem (2.0.2). Notice, program (2.0.5) is a *standard* lattice programming problem if for each $p \in$

[5] As our arguments must hold for each fixed $p$, to economize on notation, we suppress the $p$ in the notation for parameters except where emphasis is needed, and/or the context is not clear.

$\mathbf{S}_{n-1}$ with $p >> 0$, the value function $V^{Ap}(c_i, s)$ is supermodular in $(c_i, s)$ (as the feasible correspondence in program (2.0.5) is Veinott strong set order isotone in $s$).

To study the complementarity structure of the value function $V^{ap}$ in (2.0.4), first notice as the feasible correspondence is generated by a single constraint $g(c;p) = p \cdot c$ that is smooth, convex (and increasing and submodular also), the feasible correspondence $\Psi(s) = \{c_i | c_i \in [0, s]\}$ has a convex-graph. Therefore, as the objective function in the aggregated primal is concave in the choice variables $c_{-i}$ for each $c_i \in [0, s]$, the value function $V^{ap}(c_i, s)$ is partially concave function in $s$ for each $c_i \in [0, s]$.[6] Further, as the Slater condition is trivially satisfied for this case, for each $c_i \in [0, s]$, the value function $V^{ap}(c_i, s)$ has a non-empty partial subdifferential in $s$ (and, actually, it is directionally differentiable by Corollary 3). So our monotone comparative statics question reduces to characterizing a (nonsmooth) envelope theorem from Corollary 3 in the aggregated problem (2.0.4).

To characterize this nonsmooth envelope, we first conjugate the aggregated primal with a classical Lagrangian dual. Under Assumption 3, as the aggregated primal in (2.0.4) is a standard convex programming problem in $c_{-i}$, we use the following standard duality scheme: for each $c_i \in [0, s]$, and $s > 0$, define

$$\begin{aligned} L(c_{-i}, \lambda; c_i, s) &= u(c_{-i}; c_i) - \lambda(p \cdot c - s) \text{ if } c_{-i} \in C_{-i}, \phi \in \Omega \\ &= +\infty \text{ if } c_{-i} \in C_{-i}, \phi \notin \Omega \\ &= -\infty \text{ if } c_{-i} \notin C_{-i} \end{aligned} \tag{2.0.6}$$

Using this Lagrangian, define the following "$C_i-$aggregated dual" problem (or, simply "aggregated dual") as follows: for $c_i \in [0, s]$, $s > 0$,

$$V^{ad}(c_i, s) = \inf_{\lambda} \sup_{c_{-i}} L(c_{-i}, \lambda; c_i, s) \tag{2.0.7}$$

where $\lambda \in \Lambda \subset \mathbf{R}_+$, $c_{-i} \in C_{-i}$, and $V^{ad}(c_i, s)$ is the value function for the aggregated dual. By strong duality, we have the following in place: (i) zero duality gap (i.e., $V^{ap}(c_i, s) = V^{ad}(c_i, s)$), (ii) standard necessary and sufficient Lagrange multiplier rules, and (iii) saddlepoint stability for optimal solutions, among other properties [7]. The objective function in the Lagrangian dual is

$$V^L(\lambda, c_i, s) = \sup_{c_{-i}} L(c_{-i}; \lambda, c_i, s) \tag{2.0.8}$$

[6] e.g., see Fiacco and Kyparisis ([8], Proposition 2.5).

[7] See Rockafellar [14] for discussion.

and its set of optimal solutions for the Lagrangian dual is given by

$$C_{-i}^{L*}(\lambda, c_i, s) = \arg \sup_{c_{-i} \in C_{-i}} L(c_{-i}; \lambda, c_i, s)$$

We first characterize the correspondence $C_{-i}^{L*}(\lambda, c_i, s)$ using standard lattice programming. We have the following result for $p >> 0$:

**Lemma 4** $C_{-i}^{L*} : \Omega \times C_i \times S \times \mathbf{S}_{n-1} \to \mathcal{L}(\mathcal{C}_i)$ *is (i) descending in* $\lambda \in \Omega$ *(with all selections antitone), for each* $(c_i, s) \in C_i \times S$*, and (ii) ascending in* $c_i$ *(with least and greatest selection increasing), each* $(\lambda, s) \in \Omega \times S$*.*

**Proof.** As $C_{-i} \subset \mathbf{E}^{n-1}$ is a section of $C$, $C_{-i}$ is a sublattice (e.g., Topkis ([15], p. 16). Further, for each $(\lambda, c_i, s) \in \Omega \times C_i \times S$, noting the presence of an indicator function for the sublattice $C_{-i}$ in the definition of the Lagrangian in (2.0.8), we first consider the parameterized Lagrangian $L_{(\lambda,c_i,s)}$: $\mathbf{E}^{n-1} \mapsto \mathbf{R}_{-}^{*} = \mathbf{R} \cup -\infty$. As (i) $(\mathbf{R}_{-}^{*}, *)$ is a posemigroup (with identity element) for $* = +$, and (ii) the indicator function for any sublattice $C_i^{'} \subset C_i$ super$*$ (e.g., Veinott [17], Chapter 6, p. 18) with $*$=+ properly increasing in this posemigroup, under assumption 4.2.1, the Lagrangian is super$*$ in $c_i$ for $* = +$ (and, hence, $L_{(\lambda,c_i,s)}(c_{-i})$ supermodular in $c_{-i}$, for each $(\lambda, c_i, s)$). Further, $L_{(\lambda,c_i,s)}(c_{-i})$ has (i) strict decreasing differences between $(c_{-i}; \lambda)$ for each $(c_i, s)$, and increasing differences between $(c_{-i}; c_i)$ each $(\lambda, s)$ (and is a valuation in $(c_i; s)$ for each $(\lambda, c_i)$). Therefore, by Topkis' theorem (Topkis [15], Theorem 2.8.3), the correspondence $C_{-i}^{L*}(\lambda, c_i, s) = C_{-i}^{L*}(\lambda, c_i)$ is a Veinott strong set order descending in $\lambda$, each $c_i$, and ascending in $c_i$, each $\lambda$. Noting as in each case, we have strict decreasing (respectively, increasing) differences, $C_{-i}^{L*}(\lambda, c_i)$ has every selection (a) antitone in $\lambda$, for each $c_i$, and (b) isotone in $c_i$, for each $\lambda$ (and independent of $s$). This proves the result.

■

By a standard argument, the value function in $V^L(\lambda, c_i, s)$ convex in $\lambda$, each $(c_i, s)$. It is also submodular (noting that for the posemigroup $(\mathbf{R}_{+}^{*}, *)$ with $\mathbf{R}_{+}^{*} = \mathbf{R} \cup \infty$ and $* = +$, indicator functions of sublattices in the space are sub$*$ with $*$ properly increasing). Further, by the convexity and submodularity of $V^L$ in $\lambda$, the effective domain for $V^L$ (i.e., the subset $\Lambda^e \subset \Lambda$, $\Lambda^e = \{\lambda | V^L(\lambda, c_i, s) < \infty\}$ is a convex sublattice. Finally, by a version of Danskin's theorem (e.g., Grinold [9], Lemma, p. 186), we have the right directional envelope $V^L(\lambda, c_i, s)$ in $\lambda$ in direction $e^+ > 0$ given by

$$V_{\lambda}^{L}(\lambda, c_i; e^+) = \min_{c_{-i} \in C_{-i}^{L*}} (s - p_{-i} c_{-i}(\lambda, c_i, s) - c_i) \cdot e^+$$

Further, for $e^- < 0$, the left directional is

$$V_\lambda^L(\lambda, c_i; e^-) = \max_{c_{-i} \in C_{-i}^{L*}} (s - p_{-i} c_{-i}(\lambda, c_i, s) - c_i) \cdot e^-$$

Let $\Lambda^*(c_i, s)$ be the set of KKT multipliers in the aggregated dual problem. We now state the following lemma:

**Lemma 5** *Under Assumption 3, in the dual problem (2.0.8), the KKT multipliers $\Lambda^*(c_i, s)$ are (1.a) ascending in Veinott's strong set order in $c_i$, each $s \in S$ with $\wedge\Lambda^*(c_i, s)$ and $\vee\Lambda^*(c_i, s)$ isotone selections in $c_i$, each $s \in [0, s]$; (1.b) descending in Veinott's strong set order in $s$, each $c_i \in [0, s]$ with all selections antitone selections in $s$, each $c_i \in [0, s]$. Finally, (1.c) the aggregated primal $V^{ad}(c_i, s)$ has increasing differences (and, hence, is supermodular) in $(c_i, s)$.*

**Proof.** Noting the definition of the (partial) directional derivative $V_\lambda^L(\lambda, c_i, s)$ in any direction $e$, as $C_{-i}^L$ is ascending in $c_i$, each $s$, and independent of $s$, each $c_i$, we conclude $V^L(\lambda, c_i, s)$ has (i) decreasing differences between $(\lambda, c_i)$ each $s$,and (ii) strict increasing differences between $(\lambda, s)$, each $c_i$. Therefore,

(1.a) and (1.b) follows directly from Topkis's theorem (Topkis, [15], Theorem 2.8.3).

To see (1.c), appealing to the partial concavity of $V^{ad}(c_i, s)$ in $s$, for each $c_i \in [0, s]$, we can simplify the calculation of the Clarke gradient of $V^{ap}(c_i, s)$ in program (2.0.2) using strong duality and Corollary 3 as the following right directional for $e^+ > 0$

$$\begin{aligned} V_s^{ap}(c_i, s; e^+) &= \min_{\lambda \in \Lambda^*} \max_{c_{-i} \in C_{-i}^{d*}} \partial_s L(c_{-i}^d(c_i, s), \lambda^*(c_i, s), c_i, s) \cdot e^+ \\ &= \min_{\lambda \in \Lambda^*} \Lambda^*(c_i, s) \cdot e^+ \\ &= \wedge\Lambda^*(c_i, s) \cdot e^+ \end{aligned}$$

which is well-defined as $\Lambda^*(c_i, s)$ is subchained. Further, as the selection $\wedge\Lambda^*(c_i, s)$ is also isotone in $c_i$, each $s$, by duality we have increasing differences in $V^{ap}(c_i, s)$ for any right perturbation. Dually, from the left directional is

$$\begin{aligned} V_s^{ap}(c_i, s; e^-) &= \max_{\lambda \in \Lambda^*} \min_{c_{-i} \in C_{-i}^{d*}} \partial_s L(c_{-i}^d(c_i, s), \lambda^*(c_i, s), c_i, s) \cdot e^- \\ &= \max_{\lambda \in \Lambda^*} \Lambda^*(c_i, s^-) \cdot e^- \\ &= \vee\Lambda^*(c_i, s^-) \cdot e^- \end{aligned}$$

which is also well-defined as $\Lambda^*(c_i, s)$ is subchained. Again, as $\wedge\Lambda^*(c_i, s)$ is also isotone in $c_i$, each $s$, we have increasing differences for $V^{ap}(c_i, s)$ for perturbations from the left. Therefore, $V^{ap}(c_i, s)$ has increasing differences in $(c_i, s)$, and hence, is supermodular. ■

We now have the main comparative statics result of this section:

**Theorem 6** *Under Assumption 3, (1.a) $C_i^* : S \to \mathcal{L}(C_i)$ is ascending in Veinott strong set order, (1.b) $\vee C_i^*(s)$ and $\wedge C_i^*(s)$ isotone selections. Further, if Assumption 1 holds for all $i \in \{1, 2, .., n\}$, then statements (1.a) and (1.b) are true for all $i$, and $\vee C_i^*(s)$ and $\Lambda C_i^*(s)$ each Lipschitz selections of modulus 1.*

**Proof.** Claims (1.a) and (1.b) follow from Topkis' theorem (e.g., Topkis ([15], Theorem 2.8.2) noting that the feasible set $[\wedge \sum_{j \neq i} c_j^*(\lambda^*(s, c_i), c_i), s]$ is strong set order increasing in $s$ for all selection $c_j^* \in C_j^*(\lambda^*, c_i)$, and $\lambda^* \in \Lambda^*(s, c_i)$. The last claim in the theorem follows from Curtat ([5], Theorem 2.3) noting that, $\vee C_i^*(s) = s - \sum_{-i} p_{-i} \cdot \vee C_{-i}^*(s)$ and $\sum_{-i} p_{-i} \cdot \vee C_{-i}^*(s)$ (resp, $\wedge C_i^*(s) = s - \sum_{-i} p_{-i} \cdot \wedge C_{-i}^*(s)$ and $\sum_{-i} p_{-i} \cdot \wedge C_{-i}^*(s)$) are both increasing in $s$, each $i$, at $\vee C_i^*(s)$ (resp, $\wedge C_i^*(s)$) and $s$ is of Lipschitz of modulus 1. ■

A few remarks about Theorem 6. First, Theorem 6 delivers a nonsmooth version of Chipman's result (e.g., Chipman ([3]). Second, a similar approach could be taken to examining monotone comparative statics per price effects. That is, in our setting, issues of (gross) substitutability or complementarity in demand reduces to sufficient condition in the dual program (2.0.8), $\lambda^*(c_i, p)$ is falling or rising in $p_i$ (for own substitution effects or cross substitution effects). This comparative static is rather easily incorporated into our dual formulation, but appears somewhat difficult to accommodate in Quah [13]. Third, under Assumption 3, the optimal solutions $C^*(s) \notin \mathcal{L}(\mathbf{E}^n)$ (i.e, sublattices in $\mathbf{E}^n$); rather, the optimal solutions are only chain subcomplete in $\mathbf{E}_+^n$. Actually, under strict mononicity assumptions for $u(c)$, $C^*(s)$ is *antichained-valued* (as the optimal solutions each are elements of the budget line, and, hence, an antichain for each $s$). Fourth, if we only need good $i$ strong set order isotone in $s$, we do not need the choice space for $C_i$ to be convex. That is, the section $C_i \in \mathbf{E}_+$ could be a discrete lattice. It is not clear how Quah's flexible set ordering approach can handle this case.

Further, in principle, our methods can be extended to integer programming problems (at least in static problems) using conjugation schemes developed in the literature on Lagrangian relaxation methods. Finally, to study the ordinal case using dual methods involves conjugating quasiconcave programs. For this, we need to pick a different duality scheme with an "ordinal" Lagrangian (e.g., for discussion, see Crouzeix [4] or Trach [16], and references therewithin).

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