Robust Estimation of the Birnbaum-Saunders Distribution

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**Key Words** — Fatigue data, Influence function, Optimal bias-robust estimator.

**Summary & Conclusions** — The Birnbaum-Saunders distribution is prevalent in the engineering sciences as an effective means of modeling fatigue life. In practice however, there is no guarantee that the collected data follow such a model. Consequently, this paper considers the robust estimation of the parameters & quantiles of this distribution. This robust procedure is a powerful alternative to commonly used procedures, such as MLE (maximum-likelihood estimate), which are sensitive to model deviations that often occur in practice hence yielding severely distorted parameter estimates. Our robust estimation technique is based on OBRE (optimal bias-robust estimator) and assigns a weight to each observation and gives estimation technique is based on OBRE (optimal bias-robust estimator) and assigns a weight to each observation and gives estimates of the parameters and quantiles based on data which are well modeled by the distribution. Thus, observations which are not consistent with the proposed distribution can be identified and the validity of the model assessed.

An 'application to aluminum fatigue data' and 'simulation results' provide strong evidence in support of OBRE. OBRE is extremely competitive with the MLE at the model. As well, in the presence of bad data, OBRE provides good estimates with modest standard deviations based on 'the bulk of the data' and 'insightful weights identifying observations lying outside the model'. The OBRE efficiency deteriorates with increasing \( \alpha \) (shape parameter) but for the usual range of \( 0 < \alpha < 1 \), OBRE performs more than adequately for practical purposes. Furthermore, efficiency in many ways becomes a non-issue as we move away from the model. We must give up some degree of efficiency to gain robustness, and OBRE provides a powerful method of doing so. The simulation study shows that compromise can be made which are effective in both regards.

Since statistical-confidence intervals can be calculated for OBRE, robust statistical-confidence interval estimates for the critical time of the hazard rate can also be obtained. These techniques are fundamental in describing, analyzing, and comparing fatigue data so that engineers can achieve the desired reliability on a rational basis and at the same time avoid serious consequences stemming from incorrect inference.

**INTRODUCTION**

**Acronyms & Abbreviations**

- **IF** influence function
- **LMS** least median squares
- **LS** least squares
- **ML** maximum likelihood
- **MLE** ML estimator
- **2CI** 2-sided symmetrical \( s \)-confidence interval
- **OBRE** optimal bias-robust estimator
- **rms** root mean square

BSD is widely applied in the engineering sciences as an effective means of modeling stochastic wear-out. Derived in the context of fatigue [I, 2], BSD can be used to model the life-lengths of structural components in critical applications such as aerospace & bridge design [1, 2, 5, 6, 16]. Typically, the parameters of such distributions are estimated using classical procedures such as ML. Unfortunately, in many practical situations we are faced with deviations from the assumed model especially when considering fatigue data. Even small deviations can greatly affect ML parameter estimates and have far reaching repercussions on both engineering design and reliability. Design engineers, for instance, use lower quantiles which are functions of the parameter estimates. Poorly estimated quantiles can lead to serious consequences such as structural failure in buildings and bridges or premature failure of mechanical components.

This paper develops robust techniques, based on the optimal bias-robust estimator, for estimating BSD. These techniques perform approximately as well as MLE at the model but are far superior in the presence of contamination. Efficient parameter-estimates are obtained based on observations agreeing with the assumed model as well as an insightful set of weights. These weights are used to assess the validity of BSD since observations with low final weights do not fit BSD. To ignore this information can jeopardize the validity of any inferences.

Section 2 shows that both the MLE and the Mean Mean-Estimator [2] have unbounded influence functions and hence poor robustness; then it introduces the powerful alternative, OBRE, and robustifies the graphical approach of Chang & Tang [3] to obtain effective starting values for its iterative computation. Section 3 analyzes aluminum fatigue data. Section 4 presents simulation results. Appendix A.1 gives details of OBRE; appendix A.2 summarizes the fitting algorithm.
Notation
- \(t\) time \(t \geq 0\) unless otherwise specified
- \(\alpha, \beta\) parameters of the BSD; \(\alpha > 0, \beta > 0\)
- \(F(t; \alpha, \beta)\) Cdf of BSD; see (1)
- \(p\) number of parameters in the model
- \(\Theta\) p-dimensional parameter
- \(s(t; \Theta)\) score function
- \(c\) robustness constant
- \(S\) arithmetic mean
- \(H\) reciprocal of the harmonic mean
- \(\bar{\beta}\) mean mean-estimator of \(\beta\)
- \(\Psi\) Fisher consistent score function
- \(F_0\) Cdf\{\} with parameter \(\Theta\)
- \(V(\Psi, F_0)\) asymptotic covariance matrix
- \(W_c(t; \Theta)\) weight function
- \(F^{(\alpha)}(\cdot)\) empirical Cdf\{\}
- \(z_q\) gaufl\((z_q) = q\)
- \(t_q\) quantile of BSD \(q\)
- \(\tau\) implies: data-based estimator
- \(\eta\) precision threshold

Standard Notation
- \(\text{gaufl}(\cdot), \text{gauf}(\cdot), \text{gaufc}(\cdot)\): pdf, Cdf, Sf of the standard Gaussian distribution.

2. OPTIMAL BIAS-ROBUST ESTIMATORS

The BSD has Cdf:

\[
F(t; \alpha, \beta) = \text{gaufl} \left( \frac{t}{\alpha} \left( \sqrt{\frac{\beta}{t}} - \sqrt{t} \right) \right). 
\]

The BSD involves a 2-dimensional parameter \(\Theta = [\alpha, \beta]^T\); \(\beta\) is both a location parameter (median of the distribution) and a scale parameter, and \(\alpha\) is a shape parameter (and consequently the most critical of the two). As \(\alpha \to 0\), the distribution tends to s-normality.

The robustness of an estimator can be assessed by means of its IF [12, 13] which describes the effect of a small amount of contamination on the estimate. The IF of an MLE is proportional to its score function. Specifically, for the BSD, the MLE has score function:

\[
s(t; \Theta) = s(t; \alpha, \beta) = \left[ \frac{\partial \ln \left( f(t; \alpha, \beta) \right)}{\partial \alpha} \frac{\partial \ln \left( f(t; \alpha, \beta) \right)}{\partial \beta} \right]. 
\]

which is unbounded in \(t\). Hence the corresponding IF must also be unbounded. This explains why the MLE become s-biased and inefficient when the model does not hold exactly, and motivates the need for an alternative estimation procedure.

A desirable robustness property for an estimator is that it have a bounded IF. Such an estimator is called bias-robust. Bounding the IF can ensure that 'small deviations from the model distribution' do not cause 'large changes in the estimates'. OBRE [14] is constructed to have bounded influence and provides the weights, to indicate which observations, if any, are not well fit by the model. OBRE belongs to the class of M-estimators which are a generalization of the MLE. The estimator for \(\Theta\) is given in Appendix A.1.

The 'weights assigned by OBRE' \(\leq 1\), and are less than 1 when the norm of the scaled score function exceeds the cutoff, \(c\). The weights are determined by the value of a suitably normalized score function. If the score function is large in the appropriate metric, it indicates that the observation is somewhat distant from the main body of the observations, based on the current parameter estimates. As a result, observations with low final weights are those that do not fit the model, as determined by the robust parameter estimates.

To assess the weights properly, remember that OBRE might down-weight points generated from a BSD. To understand this behavior, we ran a small simulation: generate samples of size 100 from a BSD and run 500 simulations at each of 5 values of \(\alpha; \beta = 1\). Table 1 gives the mean and standard deviation of the weights of the ordered observations; it provides calibration for determining when the down-weighting indicates a serious lack of fit. (These results show no change in the first 3 decimal places when the random number generator seed is changed).

The OBRE are computed from an algorithm originally used in personal-income models [20]. OBRE has been successfully applied in the context of extreme-value theory [7, 8]. We have altered the algorithm to apply it to BSD. The basic idea is to begin with the score function and modify
Table 1: Mean (Standard Deviation) of Weights for OBRE \((c = 4)\) for BSD Data

<table>
<thead>
<tr>
<th>Rank of Observation</th>
<th>(\alpha)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>...</th>
<th>96</th>
<th>97</th>
<th>98</th>
<th>99</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.2</td>
<td>.80</td>
<td>.96</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>.96</td>
<td>.81</td>
<td></td>
</tr>
<tr>
<td></td>
<td>((.18))</td>
<td>(.09)</td>
<td>(.02)</td>
<td>(.00)</td>
<td>(.00)</td>
<td>(.00)</td>
<td>(.02)</td>
<td>(.09)</td>
<td>(.18)</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>0.6</td>
<td>.77</td>
<td>.94</td>
<td>.99</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>.99</td>
<td>.94</td>
<td>.78</td>
<td></td>
</tr>
<tr>
<td></td>
<td>((.19))</td>
<td>(.10)</td>
<td>(.03)</td>
<td>(.01)</td>
<td>(.00)</td>
<td>(.00)</td>
<td>(.03)</td>
<td>(.10)</td>
<td>(.18)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>.73</td>
<td>.92</td>
<td>.98</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>.98</td>
<td>.92</td>
<td>.74</td>
<td></td>
</tr>
<tr>
<td></td>
<td>((.19))</td>
<td>(.12)</td>
<td>(.04)</td>
<td>(.01)</td>
<td>(.00)</td>
<td>(.01)</td>
<td>(.04)</td>
<td>(.12)</td>
<td>(.19)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.4</td>
<td>.69</td>
<td>.88</td>
<td>.97</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>.98</td>
<td>.89</td>
<td>.70</td>
<td></td>
</tr>
<tr>
<td></td>
<td>((.20))</td>
<td>(.14)</td>
<td>(.06)</td>
<td>(.02)</td>
<td>(.00)</td>
<td>(.01)</td>
<td>(.06)</td>
<td>(.14)</td>
<td>(.19)</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>1.8</td>
<td>.65</td>
<td>.85</td>
<td>.96</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>.96</td>
<td>.86</td>
<td>.66</td>
<td></td>
</tr>
<tr>
<td></td>
<td>((.20))</td>
<td>(.15)</td>
<td>(.08)</td>
<td>(.03)</td>
<td>(.00)</td>
<td>(.02)</td>
<td>(.08)</td>
<td>(.15)</td>
<td>(.19)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

it so that: 1) the influence function is bounded, and 2) it is Fisher consistent. The steps are detailed in appendix A.2.

Our OBRE algorithm is convergent, provided the starting value of \(\Theta\) is near the solution. Having already established that the MLE performs poorly in the presence of contamination, one may argue that the use of the MLE as starting values is not robust enough. Ref [2] asserts the great utility of \((\Psi)\) However, \((\Psi)\) is sensitive to s-outliers\(^3\). The IF of \(S\) is unbounded while that for \(H\) is bounded. Thus, \(\widetilde{\beta}\) possesses an unbounded influence function and is therefore also unsuitable as a starting value for the OBRE algorithm.

Another estimation procedure for the BSD parameters was proposed in [3]. That procedure involved transforming & parameterizing the data to obtain a pair of variables with a linear relationship and then using LS regression to fit a straight line. From the estimated parameters of this fitted line, the BS parameters can be estimated. Since LS estimates also have poor robustness, we suggest using LMS regression [19] instead. The LMS estimates remain consistent with the majority of the data, even in the presence of contamination, helping to ensure convergence of the OBRE algorithm. When no s-outliers are present, this procedure yields results which differ little from the LS solution. This modified graphical method is called the LMS method.

One may question the necessity of going further to compute OBRE, given the strong robustness properties of simply using the LMS method. To justify going further, we reemphasize some of the more important aspects of OBRE:

- unlike the LMS method, OBRE enables calculation of the asymptotic covariance matrix [14] of the estimates useful for obtaining 2CI and comparing s-efficiencies. Thus, we advocate use of the LMS method only for obtaining starting values.

OBRE is asymptotically s-normally distributed when \(\Psi\) is bounded & continuous as it is in the case at hand. Exact regularity conditions are in [15].

\[
\begin{align*}
V(\Psi, F_\Theta) &= M(\Psi, F_\Theta)^{-1} \cdot Q(\Psi, F_\Theta) \cdot M(\Psi, F_\Theta)^{-T}, \\
M(\Psi, F_\Theta) &= -\int \frac{\partial}{\partial \Theta} \Psi(t, \Theta) \ dF_\Theta(t) \\
Q(\Psi, F_\Theta) &= \int \Psi(t, \Theta) \cdot \Psi(t, \Theta)^T \ dF_\Theta(t)
\end{align*}
\]

This exact asymptotic covariance matrix is very difficult to compute analytically due to the implicit definition of many terms. We apply the plug-in principle and obtain estimates by replacing \(F_\Theta\) with \(F^{(n)}\). The plug-in principle works quite well in general [10]. There is empirical evidence that the approach is good for (4) when \(F_\Theta\) is the generalized extreme-value distribution [9] and we anticipate similar behavior when \(F_\Theta\) is (1).

3. APPLICATION TO ALUMINUM FATIGUE DATA\(^4\)

We use OBRE to analyze aluminum fatigue data which has been used as a bench mark for tests regarding BSD [1, 2]. We address:

- model adequacy, and
- detection of s-outliers,
both of which are very relevant in many statistical analyses of this kind. The data, collected by members of the Instrument Development Unit of the Physical Research Staff, Boeing Airplane Company, are the number of cycles-to-failure of 101 strips of 6061-T6 aluminum sheeting, cut parallel to the direction of rolling. Each had been subjected to periodic loading with a frequency of 18 cycles/sec.

\(^3\)The term s-outlier is not related at all to the physical validity of a datum (whether or not it was an experimental mistake); it is related only to how well the data fit a particular model, regardless of the adequacy of that model.

\(^4\)The number of significant figures is not intended to imply any accuracy in the estimates, but to illustrate the arithmetic.
Table 2: Aluminum Fatigue Data, lifetimes (kilo-cycles)

<table>
<thead>
<tr>
<th>Maximum stress/cycle 31 kpsi</th>
</tr>
</thead>
<tbody>
<tr>
<td>70</td>
</tr>
<tr>
<td>104</td>
</tr>
<tr>
<td>112</td>
</tr>
<tr>
<td>120</td>
</tr>
<tr>
<td>124</td>
</tr>
<tr>
<td>130</td>
</tr>
<tr>
<td>132</td>
</tr>
<tr>
<td>136</td>
</tr>
<tr>
<td>141</td>
</tr>
<tr>
<td>144</td>
</tr>
<tr>
<td>151</td>
</tr>
<tr>
<td>158</td>
</tr>
<tr>
<td>168</td>
</tr>
</tbody>
</table>

Table 3: Parameter Estimates (Standard Deviations) for the Aluminum Fatigue Data

<table>
<thead>
<tr>
<th>MLE</th>
<th>OBRE</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>0.17(0.01)</td>
</tr>
<tr>
<td>( \beta )</td>
<td>131.82(2.23)</td>
</tr>
</tbody>
</table>

Assumption

1. The observations in table 2 are i.i.d. following (1) with unknown parameters \( \alpha, \beta \).

Table 3 lists the MLE and OBRE \((c = 4.0)\). Estimates of standard deviation for the OBRE are computed from (4). Estimates of standard deviation for the MLE are computed from their asymptotic covariance matrix \([11]\). Parameter estimates have changed and the standard deviations for the robust estimates are reduced since they are less influenced by deviations from the model.

A complete analysis requires inspection of the weights. All weights are 1 with the exception of the two smallest & largest observations which are assigned weights 0.32, 0.92, and 0.75, 0.53, respectively. When we compare the assigned weights with those in table 1, only the smallest and the two largest are a concern. The OBRE, in down-weighting these observations, suggests that they are perhaps not well fit by a BSD model. In many engineering applications such s- outliers are often the observations of greatest interest.

OBRE, which fits the bulk of the data, estimates \( \beta \) of the aluminum sheeting at approximately 133 kilocycles. Using the MLE instead yields an estimate of \( \beta \) of 131 kilocycles. However, MLE does not fully account for all observed behavior so we must question the validity of inferences drawn from it and perhaps seek alternatives.

The \( t_q \) is often the quantity of interest for reliability evaluation; \( t_q \) for BSD is the solution of:

\[
F(t_q; \alpha, \beta) - q = 0
\]

(5)

Table 4: Point and 95% CI Estimates of an Example Lower Quantile for Aluminum Fatigue Data

<table>
<thead>
<tr>
<th>MLE</th>
<th>OBRE</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_{0.01} )</td>
<td>78.32(83.63)</td>
</tr>
<tr>
<td>2CI</td>
<td>(77.71, 78.93)</td>
</tr>
</tbody>
</table>

\( F(\cdot) \) is from (1). From [4]:

\[
t_q = \frac{z_q}{\sqrt{\frac{\alpha^2}{\beta} + 4}}
\]

(6)

\[
t_{1-q} = \frac{z_q}{\sqrt{\frac{\alpha^2}{\beta} + 4}}
\]

(7)

Table 4 gives point and ‘95% CI’ estimates of \( t_{0.01} \). The CI were computed using the delta method. These CI were compared with those from [4]. The latter were found to be more conservative as [4] had suggested they would be. A thorough investigation of which CI are more appropriate will be the subject of a separate paper. Regardless, the results show an important difference between the quantile obtained using the OBRE vs that obtained using the MLE.

To reduce the probability of making a poor choice of estimate, it is best to determine whether the two smallest and largest observations should remain part of the data-set. If this is the case, then alternative model fitting procedures, such as mixtures should be used. Unfortunately, determining these models is not easy. If instead, they are special cases to be handled separately we would be secure reporting the OBRE of the quantile.

The aluminum fatigue data contained no gross errors. Gross errors (by definition) are caused by physical or recording errors; they do not follow the proposed model and, unlike good observations that simply do not follow the model, one should not modify the model to accommodate them. It is important to identify these errors and avoid any impact that they might have on the analysis. To illustrate further the usefulness of the OBRE, we insert a gross error in the data and carry out the computations as usual.

Let the observation 51 be misrecorded as 633 (instead of 133). We can easily imagine such an event occurring. The MLE are heavily distorted and yield:

\[
\hat{\alpha} = 0.24(0.02), \quad \hat{\beta} = 134.77(3.22), \quad t_{0.01} = 64.94(3.55).
\]

The reader is reminded of the 2 kinds of errors. 1) making it stronger when not needed, and 2) not making it stronger when needed.
OBRE, on the other hand, assigns a weight of 0.05 to the gross error and yields the more reasonable values:

\[ \hat{\alpha} = 0.15(0.01), \]
\[ \hat{\beta} = 133.40(1.68), \]
\[ t_{0.01} = 84.36(1.43). \]

Similar results are found with other gross errors.

### 4. A SIMULATION STUDY

To investigate OBRE further, a simulation study was performed; it was designed to gage the effectiveness of OBRE when the BSD model holds exactly as well as when there is contamination. Samples of size \( n = 25, 50, 100 \) were generated. Sample sizes larger than this were not considered since, in practice, data-sets larger than this seldom exist. Fatigue data, for example, are not only expensive, but very difficult to obtain.

The \( \alpha \) in the BSD was permitted to have values in (0, 2]; this range is consistent with that in the literature. For example, fatigue studies indicate that (0, 0.5] is the usual range [2], aircraft-engine data showed that \( \alpha > 2 \) is atypical in practice [18], and general practical engineering applications suggest that (0, 1] is the range of interest [18].

Both the MLE & OBRE of \( \beta \) are invariant under linear transformations of the data; so (without loss of generality) set \( \beta = 1 \). Three values of \( c \) were considered: 2, 3, 4.

Table 5 reports the s-bias and the rms deviation of the estimators for various values of \( c \). Overall, the estimators are quite similar in behavior with the MLE showing slightly less bias and slightly smaller rms deviation. As anticipated, the results improve with increasing \( n \) — validating the consistency of the OBRE of \( \alpha \). These results suggest that OBRE is in close competition with the MLE at the model.

We are aware that the sensitivity of OBRE to contamination depends on the choice of \( c \). However, lowering \( c \) leads to an efficiency loss at the model. To quantify this loss, we can measure the overall relative efficiency at the model using the ratio of the traces of the asymptotic covariance matrices of OBRE and the MLE. Table 6 shows these results. With \( \alpha = 0.2 \), the OBRE \((c = 4)\) is 95% as efficient as the MLE. These overall relative efficiencies deteriorate both with decreasing \( c \) and increasing \( \alpha \). One
Table 7: Effects of Contamination on MLE and OBRE, n=100, \( \alpha=0.5, \beta=1.0 \)

<table>
<thead>
<tr>
<th>Model</th>
<th>Contamination</th>
<th>Bias (( \alpha, \beta )) MLE</th>
<th>OBRE</th>
<th>Bound c</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1%[upper]</td>
<td>(.110,.062)</td>
<td>(.012,.003)</td>
<td>4.0</td>
<td>(.117,.081)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(.009,.002)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1%[lower]</td>
<td>(.110,.058)</td>
<td>(.011,.002)</td>
<td>4.0</td>
<td>(.118,.055)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(.008,.001)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>5%[mixed]</td>
<td>(.297,.198)</td>
<td>(.031,.039)</td>
<td>4.0</td>
<td>(.305,.204)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(.029,.036)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 8: Mean (standard deviation) of weights for the OBRE (\( c = 4.0 \)) for contaminated BSD Data

<table>
<thead>
<tr>
<th>Model</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>...</th>
<th>96</th>
<th>97</th>
<th>98</th>
<th>99</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.80</td>
<td>.96</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>.99</td>
<td>.95</td>
<td>.90</td>
<td>.81</td>
<td>.72</td>
</tr>
<tr>
<td></td>
<td>(.18)</td>
<td>(.09)</td>
<td>(.02)</td>
<td>(.00)</td>
<td>(.00)</td>
<td>(.00)</td>
<td>(.03)</td>
<td>(.10)</td>
<td>(.15)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>.08</td>
<td>.95</td>
<td>.99</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>.96</td>
<td>.81</td>
<td>.72</td>
<td>.63</td>
</tr>
<tr>
<td></td>
<td>(.01)</td>
<td>(.10)</td>
<td>(.03)</td>
<td>(.01)</td>
<td>(.00)</td>
<td>(.00)</td>
<td>(.02)</td>
<td>(.09)</td>
<td>(.18)</td>
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</tr>
<tr>
<td>3</td>
<td>.23</td>
<td>.34</td>
<td>.45</td>
<td>.62</td>
<td>.87</td>
<td>1.00</td>
<td>1.00</td>
<td>.99</td>
<td>.90</td>
<td>.81</td>
<td>.72</td>
</tr>
<tr>
<td></td>
<td>(.09)</td>
<td>(.12)</td>
<td>(.15)</td>
<td>(.19)</td>
<td>(.16)</td>
<td>(.00)</td>
<td>(.00)</td>
<td>(.02)</td>
<td>(.04)</td>
<td>(.14)</td>
<td></td>
</tr>
</tbody>
</table>

cannot anticipate keeping the efficiency when increasing the robustness. However, table 6 illustrates that the efficiency is still within a very acceptable range in the region of greatest practical application: \( 0 < \alpha < 1 \).

4.2 OBRE with Contamination

We designed contaminated models in the context of fatigue, because this is the application of primary concern. Data were generated according to the 3 models:

1. Model with 1% of very bad contamination; the upper 1% of the observations are multiplied by 5 \( [\alpha = 0.5, \beta = 1.0] \).

2. Model with 1% of very bad contamination; the lower 1% of the observations are multiplied by 1/5 \( [\alpha = 0.5, \beta = 1.0] \).

3. Mixture model with 5% contamination given by

\[
\{ 0.95 \cdot F(t; \alpha, \beta) + 0.05 \cdot F(t; \alpha, 0.1) \beta \}
\]

\( [\alpha = 0.5, \beta = 1.0] \).

Model 1 introduces observations into the upper tail of the distribution; they pose as those structures that just won't fail, but instead survive much longer than one would usually anticipate. These longer life-lengths might be the result of faulty machinery, ineffective testing, incorrect mixing, etc.

Model 2 introduces observations into the lower tail of the distribution. Such unlikely events could represent severely shortened life-lengths, perhaps as a result of machine malfunction or power outage. In either case if these observations are accepted as part of the model they increase the skewness of the original distribution. Hence we anticipate the MLE to accept these extreme observations and consequently over-estimate \( \alpha \). Alternatively, the OBRE, in identifying these extreme observations as s-outliers, should remain relatively unaffected. Our results substantiate this as shown in table 7.

The \( \beta \) (median) should show little change since a few extreme observations have little impact on such a robust estimate. Our hypotheses are validated by the results.

With 1% contamination in the upper tail of the distribution (model 1), there is a small but positive s-bias associated with the MLE of \( \beta \). When the contamination is in the left tail of the distribution (model 2) there is a small negative s-bias associated with the MLE of \( \beta \). This phenomenon is not shown by the OBRE of \( \beta \) which consistently slightly underestimates \( \beta \) as a result of the residual down-weighting that has occurred.

It is also quite feasible that parts of a structure exhibit a different life-length distribution than that of the bulk of the structure. This could be a consequence of incomplete mixing of components, or perhaps even faulty machinery. Such a model is one in which the majority of the data follows one distribution, but a small fraction follows some other distribution; it is a mixture model. In this case we intend for the OBRE to help identify the part of the structure following a different life-length distribution. Once identified, the corresponding part of the data-set can be removed, and modeled separately; thus leading to a more accurate model of both parts of the structure. Model 3 is one such situation; again OBRE shows little change while
the MLE are heavily distorted.

It is instructive to examine the weights that have been assigned to the contaminated models; see table 8. For model 1, the largest observation (1% upper contamination) has been substantially down-weighted. For model 2, the smallest observation (1% lower contamination) has been substantially down-weighted. The 5% contamination of model 3 is in the lower tail of the distribution since \( \frac{1}{3} \) was reduced to 1/10 of its original size, to generate this contamination. This contamination (the smallest 5 observations) has been effectively down-weighted. Evidently, OBRE has successfully picked up the contamination and placed little importance on it when calculating the parameter estimates.

It is important to ascertain that OBRE is still competitive with regard to its variance in the presence of contamination. It is no longer appropriate to report relative efficiencies since this term is usually reserved for comparing the variance of one estimate with another known, a priori, to be optimal, eg, the MLE at the model. However, here we are no longer at the model so the MLE is not necessarily optimal. Consequently, we report the ratio of the variances, and carefully interpret them. Table 9 gives the variance ratios for some of the contaminated models; it is clear that the OBRE are far less variable than the MLE in the presence of contamination. As \( c \) decreases, the variances of OBRE increase. This is as we anticipate since we are sacrificing efficiency for robustness by decreasing \( c \).

One could also look at a ratio of mean squared deviations (doing so would add the squared s-bias terms to the variance). However, since we have already shown the OBRE to have appreciably reduced s-bias in comparison to the MLE, we can effectively compare their variability by using the variance only.

### Table 9: Variance Ratios of the MLE to OBRE for Contaminated BSD data

<table>
<thead>
<tr>
<th>Model</th>
<th>( c )</th>
<th>1</th>
<th>2</th>
<th>2</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>5.42</td>
<td>2.87</td>
<td>5.44</td>
<td>2.88</td>
<td></td>
</tr>
<tr>
<td>( \beta )</td>
<td>1.80</td>
<td>1.35</td>
<td>1.41</td>
<td>1.04</td>
<td></td>
</tr>
</tbody>
</table>

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**Appendix**

### A.1 Optimal Bias — Robust Estimators

The OBRE are the solution \( T_n \) of the equation-set:

\[
\sum_{i=1}^{n} \Psi(t_i; T_n) = 0,
\]

for some function \( \Psi: T \times \mathbb{R}^p \rightarrow \mathbb{R}^p \). The IF of an M-estimator \( \Psi \) at \( \theta_0 \) is:

\[
\text{IF}(t; \Psi, \theta_0) = M(\Psi, \theta_0)^{-1} \cdot \Psi(t, \theta_0),
\]

\[
M(\Psi, \theta_0) = - \int \frac{\partial}{\partial \theta} \Psi(t, \theta) \, dF_\theta(t).
\]  

The OBRE is optimal in the sense that it is the M-estimator which minimizes the trace of the asymptotic covariance matrix under the constraint that it has a bounded IF. There are several versions of the OBRE: each different in the way they choose to bound the influence function. Here, we use the standardized OBRE. For a given bound \( c \) on the IF \( (c \geq \sqrt{p}) \), these estimators are defined implicitly by:

\[
\sum_{i=1}^{n} \Psi(t_i, \theta) = \sum_{i=1}^{n} \{ s(t_i, \theta) - a(\theta) \} \cdot W_c(t_i, \theta) \quad (A-2)
\]

\[
W_c(t, \theta) = \min \left[ 1; \| A(\theta) \cdot [s(t, \theta) - a(\theta)] \| \right] \quad (A-3)
\]

\[
s(t, \theta) \text{ is as in (2),}\]

\[
\| \ldots \| \text{ denotes the Euclidean norm.}
\]

The \( p \times p \) matrix, \( A(\theta) \), and the \( p \times 1 \) vector, \( a(\theta) \) are defined implicitly in:

\[
E \{ \Psi(t, \theta) \cdot \Psi(t, \theta)^T \} = \{ A(\theta)^T \cdot A(\theta) \}^{-1},
\]

\[
E \{ \Psi(t, \theta) \} = 0.
\]  

**A.2 Summary of OBRE Fitting Algorithm**

There are 5 steps for an observed sample \( t_1, \ldots, t_n \).

**Step 1**

Fix an initial value for \( \Theta = [\alpha, \beta]^T \) by computing initial estimates (starting values) of \( \alpha, \beta \).

**Step 2**

Fix \( \eta, c, a = 0, A = J^{1/2}(\theta)^{-T} \).

\[
J(\theta) = \int s(t, \theta) \cdot s(t, \theta)^T \, dF_\theta(t)
\]

is the Fisher information matrix.

**Step 3**

Solve the following 2 equations for \( a \) and \( A \):

\[
A^T \cdot A = M_2^{-1},
\]

\[
a = \frac{\int s(t, \theta) \cdot W_c(t, \theta) \, dF_\theta(t)}{\int W_c(t, \theta) \, dF_\theta(t)},
\]

\[
M_k = \int \Phi(t) \cdot \Phi(t)^T \cdot W_c(t, \theta)^k \, dF_\theta(t), \quad k = 1, 2;
\]

\[
\Phi(t) = s(t, \theta) - a
\]

The current values of \( \Theta, a, A \) are used as starting values to solve the equations in this step.

**Step 4**

Compute \( M_1 \) using \( a \) & \( A \) from step 3, and:

\[
\Delta \Theta = M_1^{-1} \left[ \frac{1}{n} \sum_{i=1}^{n} \Phi(t_i) \cdot W_c(t_i, \theta) \right].
\]
Step 5

If \( \max_j \left( \frac{\Delta \theta_j}{\delta \theta_j} \right) > \eta \), for \( j = 1, 2 \);

Then \( \Theta \rightarrow \Theta + \Delta \Theta \);

and Return to step 3;

Else Stop;

End If.

The integration can be avoided in step 2 and in computing \( M_4 \) (steps 3 - 4) by replacing \( F_0 \) with its empirical Cdf. However, numerical integration must be used to compute \( a \), otherwise (A.2) will be satisfied by all estimates.

REFERENCES


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