

# Competitive Advertising and Pricing\*

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## Abstract

We consider an oligopoly model in which each firm chooses not only its price but also its advertising strategy regarding how much product information to provide. To highlight firms' strategic incentives, we impose no structural restriction on feasible advertising content, so that each firm can disclose or conceal any information. We provide a general characterization of the equilibrium advertising content and show that intense competition induces firms to provide precise product information. We also demonstrate that strategic advertising has ambiguous implications for market prices and consumer surplus.

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## 1 Introduction

One of the fundamental questions in the economics of advertising is how much, and what, product information a firm would provide for consumers.<sup>1</sup> Offering more product information can allow the firm to be more aggressive in pricing or try some form of price discrimination. However, it comes at the cost of losing some consumers who do not find the revealed product characteristics

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<sup>1</sup>We restrict attention to this unbiased information transmission role of advertising. However, the literature has identified several other roles. See [Bagwell \(2007\)](#) and [Renault \(2015\)](#) for comprehensive overviews of the literature.

appealing. This essential trade-off has been extensively studied in the monopoly context.<sup>2</sup> A common insight in the literature is that in the absence of varying advertising costs, a monopolist wishes to provide either no information or full information: the former enables her to serve all consumers without charging too low a price, while the latter allows her to extract the most from high-value consumers.

In this paper, we study firms' incentives to provide product information in an oligopoly environment. We employ the canonical random-utility discrete-choice framework of [Perloff and Salop \(1985\)](#): there are  $n(\geq 2)$  firms that engage in Bertrand competition for a consumer whose value for each product is independently and identically drawn from a certain distribution  $F$ . We extend the basic framework to allow for strategic advertising by the firms: each firm chooses how much information to provide about its own product.<sup>3</sup> Therefore, the firms compete not only on price but also through advertising (information provision).

Unlike most classical studies on advertising, we impose no structural restriction on the set of feasible advertising strategies and allow each firm to disclose or conceal any information.<sup>4</sup> As is well known in the literature on information design (e.g. [Ostrovsky and Schwarz, 2010](#); [Gentzkow and Kamenica, 2016](#); [Kolotilin et al., 2017](#); [Roesler and Szentes, 2017](#)), with a risk-neutral consumer, this fully flexible advertising can be modeled as each firm being able to choose any mean-preserving contraction (MPC hereafter)  $G_i$  of the true value distribution  $F$ .<sup>5</sup> Since it is as if each firm chooses the consumer's value distribution for its product, our model can be interpreted as one in which the consumer's value distribution—exogenously given in [Perloff and Salop \(1985\)](#)—is endogenously determined by firms' advertising choices.

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<sup>2</sup>[Lewis and Sappington \(1994\)](#) is a seminal contribution to this literature. Subsequent important works include [Che \(1996\)](#) (return policies as a means of facilitating consumer experimentation), [Ottaviani and Prat \(2001\)](#) (the value of revealing a signal associated with the buyer's private signal), [Anderson and Renault \(2006\)](#) (optimal advertising for search goods), [Johnson and Myatt \(2006\)](#) (U-shaped profit function based on the rotation order), and [Roesler and Szentes \(2017\)](#) (buyer-optimal signal for experience goods). See also [Boleslavsky et al. \(2017\)](#), who study the interaction between advertising (“demonstration”) and pricing in the entry game context (where consumers' values for the incumbent's product are known, while their idiosyncratic values for the entrant's product are unknown).

<sup>3</sup>We do not allow for “comparative advertising,” in which a firm not only controls its own product information but can also provide information about rivals' products. See [Anderson and Renault \(2009\)](#) for an important contribution on comparative advertising. We also assume that advertising is costless. This is clearly at odds with a sizable advertising expenditure in reality, estimated to be around \$300 billion in North America, according to [statista.com](#). However, it is a common assumption in the theoretical literature and enables us to focus on strategic incentives. It would be an interesting but technically challenging exercise to introduce explicit advertising (information) costs into our model.

<sup>4</sup>See [Anderson and Renault \(2006\)](#) for an earlier application of this general formulation to advertising. The assumption can be interpreted as each firm having access to numerous advertising channels and fine-grained information on various product attributes, which enables them to adjust advertising contents without restriction. However, note that the main advantage of our information design approach is not that its full flexibility is particularly realistic, but that it enables us to focus on firms' strategic incentives, not subject to any structural assumptions.

<sup>5</sup>Intuitively, each firm can induce a degenerate distribution by revealing no product information and  $F$  itself by revealing all product information. It obtains any distribution between these two extremes by selectively disclosing information but cannot induce a more dispersed distribution than  $F$ .

We first demonstrate that intense competition induces—or, forces—firms to provide precise product information. Specifically, we consider a restricted game, termed *the advertising game*, in which the firms compete by choosing their advertising strategies given symmetric prices and show that the full information equilibrium—in which each firm chooses the underlying distribution  $F$ —exists if and only if  $F^{n-1}$  is convex over its support. Notice that  $F^{n-1}$  represents the distribution of the consumer’s best alternative to a given product. So from an individual firm’s perspective,  $F^{n-1}$  being convex means that the consumer likely has an attractive outside option. In this case, a firm should provide the highest possible value to the consumer, which can be done by disclosing sufficient information in our informative-advertising environment.

One important corollary of the above result—in fact, this paper’s main economic insight—is that the full information equilibrium exists whenever there are sufficiently many firms.<sup>6</sup> Mathematically, this is simply because  $F^{n-1}$  is convex for  $n$  sufficiently large, irrespective of the shape of  $F$ . Economically, when there are many competitors, a firm should aggressively take a risk and maximize the probability that the consumer highly values its product. Importantly, this result is robust along multiple dimensions; in [Section 6](#), we show that even if the result may not hold for any finite  $n$ , the equilibrium distribution  $G^*$  often converges to  $F$  as  $n$  tends to infinity.

In general, the unique symmetric equilibrium advertising strategy, denoted by  $G^*$ , features the following two properties: (i)  $(G^*)^{n-1}$  is convex, and (ii) locally, either  $G^*$  coincides with  $F$  or  $(G^*)^{n-1}$  is linear. The first property ensures that a firm has no incentive to contract its distribution by providing less information: if  $(G^*)^{n-1}$  is convex, then a firm is “risk-loving” and so unwilling to reduce risk. The second property prevents the opposite incentive to disperse its distribution by providing more information: a firm would not disclose more information if there is no further information to reveal (i.e.,  $G^* = F$ ), or it is “risk neutral” and so has no incentive to do so (i.e.,  $(G^*)^{n-1}$  is linear). These two necessary properties are effectively sufficient for equilibrium advertising and provide a tractable method to construct or verify the equilibrium distribution.<sup>7</sup>

We then evaluate the economic effects of strategic advertising by comparing the equilibrium outcome to that of the full information benchmark; this comparison is not only of theoretical interest but also has direct implications for policies regarding firms’ disclosure requirements. Clearly, strategic advertising reduces social surplus through information loss. However, it has ambiguous implications for both producer and consumer surplus; each can increase or decrease depending on the shape of  $F$ . We analyze this ambiguity by establishing a new result ([Proposition 3](#)) that reveals a crucial distributional property for the equilibrium price in the Perloff-Salop model.

One technical aspect worth contemplating regards the existence of pure-price equilibria.<sup>8</sup> In

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<sup>6</sup>As elaborated shortly, this result complements the seminal result by [Ivanov \(2013\)](#).

<sup>7</sup>For example, the aforementioned full information result is immediate from this general characterization. If  $F^{n-1}$  is concave then  $(G^*)^{n-1}$  should be linear over its support. See [Section 4](#) for more examples.

<sup>8</sup>The existence of pure-price equilibrium is a classical problem in the Perloff-Salop model. The main difficulty is

our model, a firm can engage in a compound deviation, changing both its price and advertising strategy from its equilibrium pair  $(p^*, G^*)$ . This suggests that given the same  $F$ , a pure-price equilibrium may exist in the Perloff-Salop model, but not in ours. Interestingly, the opposite case can arise: a pure-price equilibrium exists in our model, but not in the corresponding Perloff-Salop model. This is because a firm's deviation incentive depends on whether the other firms play  $F$  or  $G^*$ . Finally, we show that, despite these complications, sufficient regularity on  $F$  ensures the existence of pure-price equilibrium in our model.

**Related literature.** As explained above, the interaction between advertising and pricing has been extensively studied in the monopoly context but not in oligopoly settings. To the best of our knowledge, the only exception is [Ivanov \(2013\)](#). He also adopts the Perloff-Salop framework but considers the case in which firms are endowed with a restricted set of advertising strategies that are rotation-ordered in the sense of [Johnson and Myatt \(2006\)](#). Our full information result for  $n$  sufficiently large ([Corollary 1](#)) can be interpreted as showing that the same economic conclusion can be drawn without restriction on feasible advertising content. Our information design approach permits a more comprehensive characterization than his structural approach: [Ivanov \(2013\)](#) shows only that the maximal information equilibrium exists when  $n$  is sufficiently large. By contrast, we provide a necessary and sufficient condition for full information disclosure and also characterize the equilibrium structure when the full information equilibrium does not exist.

Our advertising game can be interpreted as an information disclosure game with multiple senders. There are two strands in that literature, one in which the senders have access to the same information or the full state of nature (e.g., [Gentzkow and Kamenica, 2017](#); [Li and Norman, 2021](#)) and another in which each sender controls only his own information. Our model belongs to the latter category because each firm only provides information about its own product.

Our equilibrium characterization of the advertising game closely resembles that of [Ostrovsky and Schwarz \(2010\)](#). They consider a competitive information disclosure problem in the context of school grading policies and also find that the equilibrium aggregate distribution has an alternating structure between coinciding with the primitive aggregate distribution and being locally linear. The main difference is that they adopt a Walrasian equilibrium approach, assuming each school chooses its transcript structure while taking the desirability mapping (the aggregate outcome) as given, and the desirability mapping should be consistent with schools' choices in equilibrium. Their approach enables them to consider a more general environment with heterogeneous schools and any reward structure. However, it is unsuitable for our oligopoly problem, in which an individual firm's limited market power is a key element. For example, their approach cannot be used for comparative static

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that each firm's best response depends on the entire distribution, so it may not be well behaved without some strong regularity on  $F$ . See [Caplin and Nalebuff \(1991\)](#) and [Quint \(2014\)](#) for some important contributions.

results regarding the number of firms.

Other papers in the same category, such as [Boleslavsky and Cotton \(2015, 2018\)](#) and [Au and Kawai \(2020, 2021\)](#), consider a fully strategic setting but assume discrete distributions. They also show that the equilibrium distributions are such that each player faces a locally linear value function.<sup>9</sup> Our analysis reveals that the equilibrium structure—in particular, the relationship between  $F$  and the equilibrium distribution  $G^*$ —is more transparent when  $F$  is a continuous distribution. It also allows us to directly link our work to the large discrete-choice literature, most of which considers continuous distributions.

We benefit from recent technical developments in the literature on information design. In particular, we utilize a tractable verification technique developed by [Dworczak and Martini \(2019\)](#) (DM hereafter).<sup>10</sup> Our analysis differs from DM’s in two ways. First, they consider an exogenously given programming problem, while we search for Nash equilibrium in a strategic environment, so each firm’s problem is endogenously determined in our model. Second, in our model, an individual firm chooses not only its advertising strategy but also its price, which necessitates combining DM’s result with the traditional pricing argument.

More broadly, this paper belongs to the growing literature that combines information design with classical industrial organization problems. Most papers in this literature consider the simpler monopoly setting,<sup>11</sup> but there is growing interest in oligopoly problems.<sup>12</sup> Particularly related to our paper is [Armstrong and Zhou \(2022\)](#), who also consider Bertrand competition under horizontal differentiation. They consider an independent information designer with full access to the consumer’s value profile and solve for the *optimal* information structures (maximizing producer surplus or consumer surplus); by contrast, we characterize the *equilibrium* information structure that arises when each firm controls the consumer’s information about its own product. Their paper and ours complement each other not only by studying conceptually different problems, but also by employing different frameworks and techniques: they consider Hotelling’s linear city model (effectively) and analyze the information design problem given firms’ equilibrium pricing incentives. By contrast, we build upon [Perloff and Salop](#)’s random-utility model and derive the equilibrium information structure based on firms’ informational incentives.

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<sup>9</sup>The same linear property of the equilibrium value function appears in the literature on risk-taking contests (see, e.g., [Myerson, 1993](#); [Ray and Robson, 2012](#)). This similarity is not surprising because mathematically, the game of competitive information disclosure differs from risk-taking contests only in that the former features the (more demanding) MPC constraint while the latter involves the mean constraint.

<sup>10</sup>Other relevant contributions include [Gentzkow and Kamenica \(2016\)](#), [Kolotilin \(2018\)](#), [Dworczak and Kolotilin \(2019\)](#), [Dizdar and Kováč \(2020\)](#), and [Ivanov \(2021\)](#).

<sup>11</sup>This literature is already too extensive to summarize here. See [Anderson and Renault \(2006\)](#), [Bergemann et al. \(2015\)](#), and [Roesler and Szentes \(2017\)](#) for some seminal contributions.

<sup>12</sup>For instance, [Armstrong and Vickers \(2019\)](#) and [Shi and Zhng \(2022\)](#) study the welfare implications of market segmentation structures with price-discriminating firms, and [Condorelli and Szentes \(2022\)](#) characterize the set of all possible equilibrium outcomes in Cournot competition given maximal willingness to pay and maximal total demand.

The remainder of this paper is organized as follows. [Section 2](#) introduces the formal model. [Section 3](#) analyzes the full information equilibrium. [Section 4](#) provides a general characterization for the advertising game. [Section 5](#) studies the price and welfare effects of strategic advertising and addresses the equilibrium existence problem of our full game. [Section 6](#) considers three variants of our baseline model and examines the robustness of our full information result. [Section 7](#) concludes. All the proofs not in the main text are relegated to [Appendix A](#) unless noted otherwise.

## 2 The Model

Our model builds upon the random-utility oligopoly framework of [Perloff and Salop \(1985\)](#). There are  $n(\geq 2)$  firms and a risk-neutral consumer.<sup>13</sup> Each firm, indexed by  $i$ , supplies a product with marginal cost normalized to zero, and the consumer demands one unit. The firms' products are horizontally differentiated.<sup>14</sup> Specifically, the consumer's value for each product is independently and identically drawn according to the distribution function  $F$ . We assume that  $F$  has compact and convex support  $[\underline{v}, \bar{v}]$  with mean  $\mu_F$ ,<sup>15</sup> and its density  $f$  is positive and continuously differentiable over  $[\underline{v}, \bar{v}]$ . We also assume that both  $f$  and  $f'$  are bounded, and  $f$  has a finite number of peaks.

Firm  $i$ 's strategy consists of its price  $p_i$  and advertising strategy. For the latter, we assume that each firm can choose any mean-preserving contraction (MPC, hereafter)  $G_i$  of the underlying distribution  $F$ ; that is, the set of feasible advertising strategies is given by the set of all MPCs of  $F$ , henceforth denoted by  $MPC(F)$ . By now, it is well-known that this MPC formulation captures, without loss, full flexibility in information choice when an agent's decision depends on her posterior mean. It is suitable for our model, because the consumer's purchase decision relies on her expected values for the products and their prices. Note that providing full information corresponds to choosing  $F$  itself, and more information increases  $G_i$  in convex order.

The timing of the game is as follows. First, the firms simultaneously choose their prices  $p_i \in \mathcal{R}_+$  and advertising strategies  $G_i \in MPC(F)$ .<sup>16</sup> Second, the consumer draws her interim value for each product  $i$  according to  $G_i$ . Her interim values are independent across the products. Finally,

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<sup>13</sup>The model can be interpreted as one with a continuum of ex ante homogeneous consumers. Our single-consumer setup allows us to avoid a measurability issue inherent in the continuum model (see, e.g., [Sun, 2006](#)).

<sup>14</sup>This is the standard interpretation of the Perloff-Salop model. As usual, the model can be reinterpreted as one of vertical differentiation, assuming that the quality of each product is realized after the firms set their prices.

<sup>15</sup>As usual,  $\mu_F$  can be interpreted as representing the common value (vertical) component of the products, while  $v_i - \mu_F$  as capturing the match-specific (horizontal) component. In our model,  $\mu_F$  is identical across the products and commonly known among all participants. One could consider a setting in which  $\mu_F$  is known only to the firms and they decide how much information to provide about  $\mu_F$ . Note that it is not immediate that  $\mu_F$  would be fully revealed.

<sup>16</sup>One may consider the sequential cases in which the firms choose prices and then advertising strategies, or vice versa. While the most appropriate timing assumption depends on the context, our simultaneity assumption reflects an agnostic view about the timing. Also, our model is relevant to the sequential cases if the firms' first actions remain private. Finally, our assumption simplifies the technical analysis, as it does not involve numerous subgames (following each price or advertising strategy profile) in the sequential case.

the consumer decides which product to purchase based on prices and her interm values. We assume that the consumer must purchase one of the products, so that she purchases product  $i$  if and only if  $v_i - p_i > v_j - p_j$  for all  $j \neq i$ .<sup>17</sup>

We study symmetric pure-price equilibria of this market game. Since the consumer's purchase problem is straightforward, we focus on Nash equilibria played by the firms. Let  $D(p_i, G_i, p, G)$  denote firm  $i$ 's demand—the probability that the consumer purchases product  $i$ —given its strategy  $(p_i, G_i)$  and the other firms' common strategy  $(p, G)$ . Given the consumer's optimal choice rule,<sup>18</sup>

$$D(p_i, G_i, p, G) = Pr\{v_i - p_i > v_j - p, \forall j \neq i\} = \int G(v_i - p_i + p)^{n-1} dG_i(v_i). \quad (1)$$

A tuple  $(p, G)$  is a *symmetric pure-price equilibrium* if  $(p, G)$  solves the following individual firm's profit maximization problem:

$$\max_{(p_i, G_i)} \pi(p_i, G_i, p, G) \equiv p_i D(p_i, G_i, p, G) \text{ s.t. } G_i \in MPC(F).$$

We refer to a symmetric pure-price equilibrium as an *equilibrium* unless it creates confusion.

It is useful to define a game in which the firms choose only their advertising strategy  $G_i$ , taking the symmetric price as given. We refer to this restricted game as the *advertising game*. By definition, if  $(p, G)$  is an equilibrium in our full game, then  $G$  must be an equilibrium in the advertising game. In other words,  $G$  being an equilibrium in the advertising game is a necessary condition for it to constitute an equilibrium in the full game. Note that in the advertising game, an individual firm's demand in (1) simplifies to

$$D^A(G_i, G) \equiv D(p, G_i, p, G) = \int G(v_i)^{n-1} dG_i(v_i), \quad (2)$$

and each firm can be interpreted as to maximize its demand  $D^A(G_i, G)$ .

### 3 Full information Equilibrium

We first study the case in which the firms *choose* to provide full product information. Specifically, we characterize a condition under which  $F$  itself is an equilibrium of the advertising game.

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<sup>17</sup>See Section 6.1 for a detailed discussion on the case where the consumer can choose not to purchase any product. As shown below, in any equilibrium of this paper, the probability that the consumer is indifferent among multiple products is negligible. Therefore, for notational and expositional simplicity, we ignore the event throughout the paper.

<sup>18</sup>Note that our derivation of  $D(p_i, G_i, p, G)$  does not account for the possibility of atoms in  $G$ . This significantly simplifies the notation but incurs no loss of generality because the equilibrium distribution has no atom.

In the advertising game, the full information equilibrium exists if and only if

$$D^A(F, F) = \int F(v)^{n-1} dF(v) \geq D^A(G_i, F) = \int F(v)^{n-1} dG_i(v) \text{ for any } G_i \in MPC(F).$$

One evident case in which this condition holds is when the integrand  $F^{n-1}$  is convex: in this case, the firm is “risk-loving” and so prefers  $F$  to any MPC of it (see Proposition 6.D.2 in [Mas-Colell et al., 1995](#)). In fact, it is the only relevant case, as formally stated in the following result.

**Proposition 1** *In the advertising game, the full information equilibrium, in which all firms choose  $F$ , exists if and only if  $F^{n-1}$  is convex over its support  $[\underline{v}, \bar{v}]$ .*

To see why convexity of  $F^{n-1}$  is necessary for the full information equilibrium, consider the polar opposite case in which  $F^{n-1}$  is strictly concave. In that case, provided that the other firms choose  $F$ , an individual firm is “risk-averse” and so prefers any  $G_i \in MPC(F)$  to  $F$  itself. In fact, it is easy to see that the degenerate distribution  $\delta_{\mu_F}$  is the firm’s optimal choice. The desired result follows from the fact that this logic applies whenever  $F^{n-1}$  has a non-convex region, as an MPC that concentrates probability over the non-convex region would outperform  $F$ .

For an economic intuition, notice that  $F^{n-1}$  represents the distribution of the consumer’s best alternative to product  $i$ . Convex  $F^{n-1}$  means that its density is increasing in  $v$ , so the consumer likely has an attractive outside option. In this case, the firm should aggressively take a risk and maximize the probability of offering the highest possible value to the consumer; despite its risk of sometimes giving a low value to the consumer, it is more profitable, on average, than taking no risk and just offering an intermediate value. When  $F^{n-1}$  is convex, this argument continues to apply from  $\bar{v}$  to  $\underline{v}$ , resulting in the optimality of full information provision.

If  $F^{n-1}$  is not convex and so has non-monotone density, then the above “unraveling” argument does not survive the entire support  $[\underline{v}, \bar{v}]$ , in which case  $F$  is no longer a best response to  $F^{n-1}$ . Note that when  $F$  is not convex, characterizing a firm’s best response to  $F^{n-1}$  is not sufficient for *equilibrium* characterization, because a firm’s not playing  $F$  changes the other firms’ problems and in equilibrium each firm should best respond to the other firms’ equilibrium strategies. This equilibrium characterization is the main subject of [Section 4](#).

Although simple to establish, [Proposition 1](#) has an important and robust implication on the effects of competition on strategic advertising, as formally reported in the following result.<sup>19</sup>

**Corollary 1** *If  $n$  is sufficiently large, then the full information equilibrium necessarily exists in the advertising game.*

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<sup>19</sup>As illustrated in the introduction, this result complements the seminar finding by [Ivanov \(2013\)](#).



**Proof.** We show that  $F^{n-1}$  is convex for  $n$  sufficiently large. Let  $\varepsilon \equiv \min_{v \in [\underline{v}, \bar{v}]} f(v)$  and  $M \equiv \max_{v \in [\underline{v}, \bar{v}]} |f'(v)|$ . Since  $f$  is positive and  $f'$  is bounded on  $[\underline{v}, \bar{v}]$ ,  $\varepsilon > 0$  and  $M < \infty$ . Then,

$$(F(v)^{n-1})'' = (n-1)F(v)^{n-3}((n-2)f(v)^2 + F(v)f'(v)) \geq (n-1)F(v)^{n-3}((n-2)\varepsilon^2 - M).$$

The result follows because the last expression is always positive for  $n$  sufficiently large.  $\blacksquare$

Technically, this result holds simply because the power function  $x^{n-1}$  becomes more convex as  $n$  rises, so if  $n$  is sufficiently large then  $F^{n-1}$  is convex, irrespective of the detailed shape of  $F$ . Economically, this is precisely due to the effects of competition on firms' advertising incentives. When there are many competitors, the probability of each firm's winning the consumer is low. In that case, an individual firm should be aggressive and attempt to give the highest value to the consumer; otherwise, the consumer would almost surely choose another product.

Two remarks are in order. First, [Corollary 1](#) is robust along multiple dimensions. In [Section 6](#) and [Appendix G](#), we consider various extensions of the model and show that the equilibrium distribution—although it may not coincide with  $F$  for any finite  $n$ —converges to  $F$  as  $n$  tends to infinity. Second, imposing certain regularity on  $F$ , a result stronger than [Corollary 1](#) holds, namely, that firms provide more information as  $n$  increases (see [Corollary 2](#)).

## 4 Competitive Advertising: General Characterization

This section provides a general characterization for competitive advertising. In particular, we determine the general structure of the equilibrium advertising strategy, which we denote by  $G^*$ .

Clearly,  $G^*$  should be an individual firm's optimal advertising strategy when all other firms choose  $G^*$ . This means that  $G^*$  is a fixed point that solves

$$\max_{G_i \in MPC(F)} D^A(G_i, G^*) = \int_{\underline{v}}^{\bar{v}} G^*(v)^{n-1} dG_i(v).$$

Exploiting this fixed-point nature of  $G^*$ , we uncover two necessary properties of  $G^*$ .

**Lemma 1**  $(G^*)^{n-1}$  is convex over its convex support.

If  $(G^*)^{n-1}$  is not convex anywhere, then it is optimal for a firm to concentrate all local probability mass on one point, just as the degenerate distribution is optimal when  $(G^*)^{n-1}$  is concave. The resulting optimal solution has a mass point, and thus cannot coincide with the given  $G^*$ . Intuitively, in equilibrium no firm should have an incentive to contract its distribution by providing less information. Convexity of  $(G^*)^{n-1}$  ensures that by rendering an individual firm “risk-loving.”

**Lemma 1** implies that locally,  $(G^*)^{n-1}$  is either strictly convex or linear. Consider any interval over which  $(G^*)^{n-1}$  is strictly convex. For the same reason as given in **Section 3**, a firm’s (local) best response is to disperse its distribution as much as possible by providing full information. This suggests that  $G^*$  must coincide with  $F$  over the interval. Now consider an interval over which  $(G^*)^{n-1}$  is linear. Over the interval, an individual firm is “risk-neutral” and so indifferent over all distributions with the same mean. Therefore, the given  $G^*$ —constructed so that  $(G^*)^{n-1}$  is linear over the interval—is an individual firm’s local best response provided that it is an MPC of  $F$ . This yields the following second necessary property of  $G^*$ .

**Lemma 2** *For almost all  $v \in \text{supp}(G^*)$ , there exists  $\varepsilon > 0$  such that either  $G^*$  coincides with  $F$  or  $(G^*)^{n-1}$  is linear over  $(v - \varepsilon, v + \varepsilon)$ . Over each interval on which  $(G^*)^{n-1}$  is linear,  $G^*$  must be an MPC of  $F$ .<sup>20</sup>*

Intuitively, in equilibrium a firm should have no incentive to disperse its distribution by providing more information, or should not be able to do so even if it wants. Given **Lemma 1**, the former holds only when  $(G^*)^{n-1}$  is linear, so an individual firm is “risk-neutral.” The latter corresponds to the case where  $G^* = F$ ; in this case, the MPC constraint binds, so a firm simply cannot disperse its distribution further.

The following theorem states that the necessary conditions in **Lemmas 1** and **2** are in fact sufficient for equilibrium advertising, and the advertising game always has a unique equilibrium.

**Theorem 1** *A distribution  $G^* \in \text{MPC}(F)$  is a symmetric equilibrium of the advertising game if and only if it satisfies **Lemmas 1** and **2**. For each  $F$ , such  $G^*$  uniquely exists.*

Given the previous discussions, it is natural that **Lemmas 1** and **2** are sufficient for the equilibrium advertising strategy: **Lemma 1** implies that no firm has an incentive to contract its distribution, while **Lemma 2** ensures that no firm benefits from dispersing its distribution.<sup>21</sup> The existence and uniqueness of  $G^*$  follow from our constructive proof in **Appendix C**; we show that starting from  $\underline{v}$  and proceeding forward, there is a unique way to construct an MPC of  $F$  that satisfies **Lemmas 1** and **2**. For the essence of our construction, see the proof of **Corollary 2** below.

**Theorem 1** immediately strengthens our results in **Section 3**: if  $F^{n-1}$  is convex, then the full information equilibrium is the *unique* symmetric equilibrium of the advertising game. In addition, firms *necessarily* provide full product information when  $n$  is sufficiently large.

<sup>20</sup>If  $G^*$  has a strictly smaller support than  $F$ , then the MPC condition over the last interval with linear  $(G^*)^{n-1}$  should hold over the interval including  $\bar{v}$ , not just over the linear interval. For example, in **Figure 1**,  $G^*$  is an MPC of  $F$  over  $[v^*, \bar{v}]$ , not over  $[v^*, \bar{v}^*]$ .

<sup>21</sup>In the proof of **Theorem 1** in **Appendix A**, we prove this sufficiency result by applying a verification theorem by **Dworczak and Martini (2019)**, which enables us to avoid the general fixed-point equilibrium existence argument.

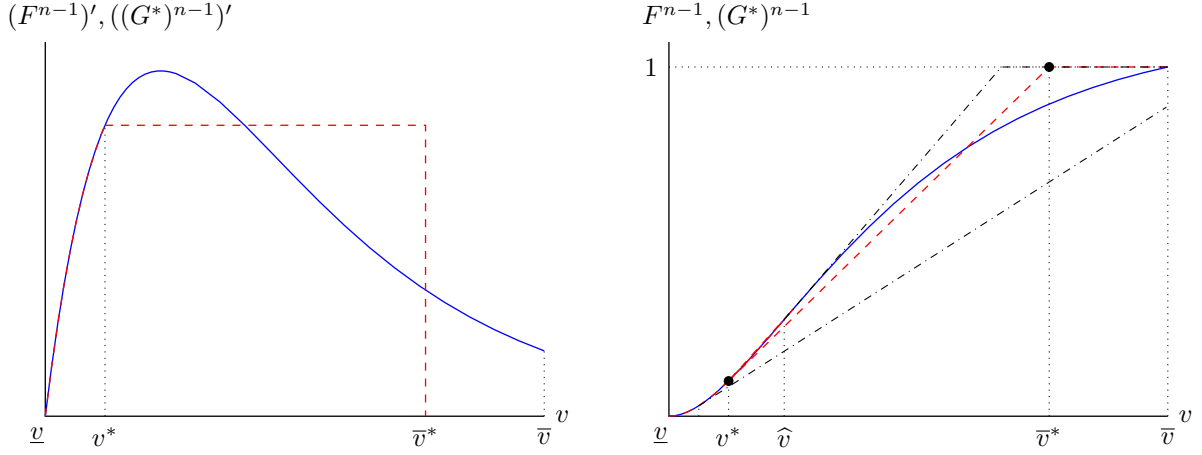


Figure 1: Equilibrium advertising strategy when  $F^{n-1}$  has quasiconcave density. In both panels, the blue solid curves represent  $F$ , while the red dashed curves depict  $G^*$ . In the right panel, the dashdotted curves show how the linear region of  $(G^*)^{n-1}$  depends on its starting point,  $v^*$ . In this figure,  $n = 3$  and  $F(v) = (1 - e^{-v})/(1 - e^{-\bar{v}})$  over  $[0, 3]$

The following result applies [Theorem 1](#) to the canonical case where  $F^{n-1}$  has strictly quasiconcave density and shows that in that case, the convergence of  $G^*$  to  $F$  is particularly transparent.

**Corollary 2** (1) *If  $F^{n-1}$  has strictly quasi-concave density, then there exists  $v^* \in [\underline{v}, \bar{v}]$  such that  $G^*(v) = F(v)$  for  $v \leq v^*$  and  $(G^*)^{n-1}$  is linear above  $v^*$ .*

(2) *Suppose  $F^{n-1}$  has strictly quasi-concave density for any  $n$ .<sup>22</sup> Then as  $n$  increases,  $v^*$  rises and  $G^*$  increases in convex order.*

**Proof.** We prove (1) here and relegate the proof of (2) to [Appendix A](#). For (1), it suffices to show that there exists  $G^* \in MPC(F)$  with the stated cutoff structure. We can set  $v^* = \underline{v}$  if  $G^*$  such that  $(G^*)^{n-1}$  is linear over its support belongs to  $MPC(F)$ .<sup>23</sup> Otherwise, let  $\hat{v}$  denote the unique peak of  $(F^{n-1})'$ , and continuously raise  $v^*$  from  $\underline{v}$  to  $\hat{v}$ . This lowers  $G^*$  in the sense of first-order stochastic dominance, because the slope of the linear part of  $(G^*)^{n-1}$  increases in  $v^*$  (see the dashdotted curves in the right panel of [Figure 1](#)). Combining this with the fact that such a distribution dominates  $F$  if  $v^* = \underline{v}$  and is dominated by  $F$  if  $v^* = \hat{v}$ , it follows that there exists a unique  $v^* \in (\underline{v}, \hat{v})$  that satisfies  $G^* \in MPC(F)$ . ■

One immediate implication of [Corollary 2](#).(2) is that consumer surplus strictly increases in  $n$ : as  $n$  rises, the consumer not only has more options to choose from, but also receives more accurate information about each product.

<sup>22</sup>This assumption is satisfied by many standard distributions. In particular, it holds whenever  $f$  is log-concave.

<sup>23</sup>This applies whenever  $F^{n-1}$  has decreasing density. It also applies if  $(F^{n-1})'$  increases only close to  $\underline{v}$ .

The final corollary considers the case where  $F^{n-1}$  is concave. This is a special case of [Corollary 2](#) and provides a simple sufficient condition under which  $(G^*)^{n-1}$  is linear over its support.

**Corollary 3** *If  $F^{n-1}$  is concave, then  $(G^*)^{n-1}$  is linear over its support, whose lower bound is  $\underline{v}$ .*

**Example 1 (Truncated Exponential Distributions)** *Suppose  $F(v) = \gamma(1 - e^{-\lambda v})$  over  $[0, \bar{v}]$  for some  $\lambda > 0$ , where  $\gamma(1 - e^{-\lambda \bar{v}}) = 1$ . If  $n = 2$ , then  $F^{n-1} = F$  is concave, in which case  $G^* = U[0, 2\mu_F]$  by [Corollary 3](#). If  $n \geq \bar{n} \equiv e^{\lambda \bar{v}} + 1 (> 2)$ , then  $F^{n-1}$  is convex, so  $G^* = F$  by [Proposition 1](#).*

*Suppose  $n \in (2, \bar{n})$ . In this case,  $F^{n-1}$  is initially convex and then concave. By [Corollary 2](#), there exists  $v^* \in (0, \bar{v})$  such that  $G^* = F$  below  $v^*$ , while  $(G^*)^{n-1}$  is linear above  $v^*$ . The cutoff  $v^*$  and the upper bound  $\bar{v}^*$  of  $\text{supp}(G^*)$  are implicitly defined by<sup>24</sup>*

$$\int_{v^*}^{\bar{v}} v dF(v) = \int_{v^*}^{\bar{v}^*} v dG^*(v) \text{ and } 1 = G^*(\bar{v}^*) = F(v^*)^{n-1} + (F(v^*)^{n-1})' (\bar{v}^* - v^*).$$

*Specifically, suppose  $\lambda = 1$  and  $\bar{v} = 3$ , as in [Figure 1](#). Then,  $\bar{n} \approx 21.09$ . Letting  $v^*(n)$  denote  $v^*$  with  $n$  firms,  $v^*(3) = 0.3589$ ,  $v^*(7) = 1.5024$ ,  $v^*(14) = 2.4348$ , and  $v^*(21) = 2.9943$ .*

## 5 Competitive Pricing

Notice that the equilibrium distribution  $G^*$  in [Theorem 1](#) is independent of the given symmetric price. In this section, we characterize the equilibrium price  $p^*$  that corresponds to  $G^*$  and use the result to study the effects of strategic advertising on market prices and consumer welfare. We also investigate the equilibrium existence problem by considering a firm's compound deviations.

Given  $G^*$ , our problem reduces to the canonical model of [Perloff and Salop \(1985\)](#). Assuming that all other firms charge  $p^*$ , an individual firm's optimal pricing problem is given by

$$\max_{p_i} \pi(p_i, G^*, p^*, G^*) = p_i D(p_i, G^*, p^*, G^*) = p_i \int G^*(v_i - p_i + p^*)^{n-1} dG^*(v_i).$$

Deriving the firm's first-order condition and imposing the symmetric equilibrium requirements, we arrive at the following familiar equilibrium pricing formula.

**Proposition 2** *In equilibrium, each firm charges*

$$p^* = \frac{1/n}{\int (G^*(v)^{n-1})' dG^*(v)}. \quad (3)$$

<sup>24</sup>The first condition is simply due to  $G^* \in MPC(F)$ . The second condition follows from  $G^*(v^*) = F(v^*)$  and  $(G^*(v^*)^{n-1})' = (F(v^*)^{n-1})'$ , which are necessary for  $G^*$  to satisfy the conditions in [Theorem 1](#).

As usual, this pricing formula shows that the equilibrium price is inversely proportional to the measure of *marginal consumers*,  $\int (G^*(v)^{n-1})' dG^*(v)$ , who are indifferent between a firm's product and the best alternative.<sup>25</sup> Intuitively, when there are more marginal consumers (i.e., it is more likely that the consumer is indifferent between purchasing and not purchasing its product), a firm's marginal benefit from price cutting is larger, leading to a lower  $p^*$ .

To assess the economic effects of strategic advertising, we compare the resulting market outcome to that of the full information benchmark. Let  $B^*$  denote the consumer's expected valuation of the purchased product—equivalent to social surplus—and  $CS^*$  denote her consumer surplus given  $G^*$ . Since the consumer purchases the product with the highest realization among  $n$  independent draws from  $G^*$ ,

$$B^* = \int v dG^*(v)^n \text{ and } CS^* = B^* - p^*. \quad (4)$$

Let  $p^F$ ,  $B^F$  and  $CS^F$  denote the corresponding quantities for the full information benchmark.

It is easy to see that  $B^* \leq B^F$ , because whenever  $G^* \neq F$ , the consumer selects a suboptimal product with a positive probability. In other words, information loss through strategic advertising causes deadweight loss.

In contrast, the effect of strategic advertising on the market price is unclear. The next example demonstrates that depending on the shape of  $F$ , strategic advertising could lower or raise the price. The example further shows that strategic advertising could benefit the consumer—even with the aforementioned deadweight loss—when it induces a significant price drop.<sup>26</sup>

**Example 2 (Power distributions)** Consider the duopoly case ( $n = 2$ ) with  $F(v) = v^\alpha$  over  $[0, 1]$  for some  $\alpha > 0$ . Clearly,  $F(v) = v^\alpha$  is convex if  $\alpha \geq 1$  and strictly concave otherwise. By **Proposition 1**,  $G^* = F$  is the unique equilibrium in the former (convex) case. In the latter (concave) case, **Corollary 3** implies that  $G^*(v) = \beta v$  over  $[0, 1/\beta]$  for some  $\beta > 0$ . Since  $G^* \in MPC(F)$  (in particular,  $\mathbb{E}_F[x] = \mathbb{E}_{G^*}[v]$ ),  $\beta = (\alpha + 1)/(2\alpha)$ . Applying this to (3) and (4) yields

$$p^F = \frac{2\alpha - 1}{2\alpha^2}, B^F = \frac{2\alpha}{2\alpha + 1}; p^* = \frac{\alpha}{\alpha + 1}, B^* = \frac{4\alpha}{3(\alpha + 1)}.$$

It follows that strategic advertising lowers the market price ( $p^* < p^F$ ) if and only if  $\alpha > 1/\sqrt{2} \approx 0.7071$ ; and it benefits consumers ( $CS^* > CS^F$ ) if and only if  $\alpha > 0.7928$ .

To understand when strategic advertising lowers/raises the market price, we report two results

<sup>25</sup>Notice that  $g^*(v)$  is the density of consumers who assign value  $v$  to the firm's product, while  $(G^*(v)^{n-1})'$  is the density of those whose best alternative has value  $v$ .

<sup>26</sup>In **Appendix F**, we consider truncated exponential distributions and show that strategic advertising always lowers the market price, while its effect on consumer surplus varies depending on  $\bar{v}$ .

regarding how the equilibrium price  $p^F$  depends on  $F$  in the Perloff-Salop model.<sup>27</sup> Whereas the first result is a corollary of a recent theoretical result by Zhou (2017) and Choi et al. (2018), the second result is—to our best knowledge—novel in the literature and also of independent interest.

**Proposition 3** (1) *The measure of marginal consumers,  $\int (F^{n-1})' dF$ , decreases if  $F$  is proportionally stretched over a larger support.*<sup>28</sup>

(2) *Among all distributions whose support belongs to  $[\underline{v}, \bar{v}]$ , the power distribution  $F(v) = \left(\frac{v-\underline{v}}{\bar{v}-\underline{v}}\right)^{2/n}$  minimizes  $\int (F^{n-1})' dF$ .*

As is well-known, preference diversity is the driving force for positive markups in the Perloff-Salop model. One clear way to increase preference diversity is to lengthen the support  $[\underline{v}, \bar{v}]$ . Proposition 3.(1) confirms that such a change indeed raises the equilibrium price.<sup>29</sup> Unfortunately, such a clear comparative statics result is not available over distributions with the same support. Proposition 3.(2) complements this deficiency by characterizing the price-maximizing distribution among all distributions over  $[\underline{v}, \bar{v}]$ . The crucial property of the power distribution in Proposition 3.(2) is that marginal consumers are *uniformly distributed* over the support (i.e., the conditional measure of marginal consumers is constant).<sup>30</sup> If marginal consumers are not uniformly distributed, then adjusting  $F$  to “flatten” the density of marginal consumers would lead to a smaller measure of marginal consumers.<sup>31</sup>

Proposition 3 enables us to identify two effects of strategic advertising on the market price. On one hand, since  $G^* \in MPC(F)$ , strategic advertising yields a value distribution with a narrower support compared to the full information benchmark. Proposition 3.(1) implies that this effect necessarily lowers the equilibrium price. On the other hand, strategic advertising induces a different distribution of marginal consumers, whose effect on the market price is ambiguous in general. Specifically, whenever  $G^*$  differs from  $F$  and so  $(G^*)^{n-1}$  has a locally constant slope of  $\beta$ , the density of marginal consumers  $((G^*)^{n-1})' g^* = \beta g^*$  is decreasing, while  $(F^{n-1})' f$  can exhibit any

<sup>27</sup>Perloff and Salop (1985) already point out that the equilibrium price may or may not increase when  $F$  changes in convex order. Their observation does not directly apply to our model, because  $G^*$  is a particular type of MPC.

<sup>28</sup>To be precise, we say that  $F_2[\underline{v}_2, \bar{v}_2]$  is a proportional stretch of  $F_1[\underline{v}_1, \bar{v}_1]$  if  $\bar{v}_2 - \underline{v}_2 > \bar{v}_1 - \underline{v}_1$  and

$$F_2(v) = F_1\left(\frac{\bar{v}_1 - \underline{v}_1}{\bar{v}_2 - \underline{v}_2}(v - \underline{v}_2) + \underline{v}_1\right) \text{ for all } v \in [\underline{v}_2, \bar{v}_2].$$

In other words,  $F_2$  is obtained when  $F_1$  is extended to a larger support while keeping its shape. Note that there are no restrictions on the locations of the bounds, that is, it is possible that  $\underline{v}_1 > \underline{v}_2$  or  $\bar{v}_1 > \bar{v}_2$ .

<sup>29</sup>More generally, the equilibrium price increases if  $F$  rises in *dispersive order*, that is,  $F^{-1}(a)$  increases at every quantile  $a \in (0, 1)$  (see Zhou, 2017; Choi et al., 2018). Proportional stretching is one (representative) way to do so.

<sup>30</sup>It is easy to show that  $(F(v)^{n-1})' f(v) = 4(n-1)/n^2(\bar{v}-\underline{v})^2$  for all  $v \in [\underline{v}, \bar{v}]$ .

<sup>31</sup>For example, consider the case with  $n = 2$ , so that the density of marginal consumers is  $(F^{n-1})' f = f^2$ . Then  $\int f^2$  (subject to  $\int f = 1$ ) is minimized when  $f$  is independent of  $v$ : if  $f(v) \neq f(v')$  then one can reduce  $f(v)^2 + f(v')^2$  by equally splitting combined density between the two. A similar logic applies to the case with  $n > 2$ .

given pattern. By [Proposition 3.\(2\)](#), this may or may not increase  $p^*$ , depending on the detailed shape of  $F$ . This second effect renders the comparison between  $p^*$  and  $p^F$  ambiguous.

**Equilibrium existence.** [Theorem 1](#) and [Proposition 2](#) identify the unique candidate for the symmetric pure-price equilibrium: if such an equilibrium exists then it must be as characterized in those two results. However, the existence of symmetric pure-price equilibrium is not guaranteed in our model. This non-existence problem is already present in the Perloff-Salop model (PS, hereafter). However, strategic advertising in our model introduces some novel features into the problem, so that conditions for equilibrium existence are not transferable between the two models.

Consider the case where  $F^{n-1}$  is convex, so that  $G^* = F$  and  $p^* = p^F$ . In this case, equilibrium existence in PS is necessary for that in our model, as it prevents profitable *price deviations*. It is not sufficient, however, because there may exist a profitable *compound deviation* in which a firm deviates both from  $G^*$  and from  $p^*$ . This implies that a pure-price equilibrium may exist only in PS, and our model can produce a different market outcome from PS even if  $F^{n-1}$  is convex.

Conversely, it is also possible that a symmetric pure-price equilibrium exists in our model, and not in PS. Recall that if  $F^{n-1}$  is not convex then  $G^*$  induces a different distribution of marginal consumers from  $F$ , generally leading to  $p^* \neq p^F$ . These changes directly affect a firm's deviation incentives, thereby sometimes sustaining a pure-price equilibrium even when such an equilibrium does not exist in PS. In [Appendix E](#), we offer a specific example for this phenomenon.

Despite these complications, sufficient (but commonly imposed) regularity on  $F$  ensures the existence of symmetric pure-price equilibrium in our model, as reported in the following result.

**Theorem 2** *If  $f$  is log-concave, then there exists a unique symmetric pure-price equilibrium  $(p^*, G^*)$  characterized by [Theorem 1](#) and [Proposition 2](#).*

**Proof.** See [Appendix C](#). ■

In the Perloff-Salop model, log-concavity of  $f$  is sufficient for equilibrium existence: It ensures that a firm's demand function is log-concave, so the first-order condition (3) is sufficient. In our model, it implies a simple structure of  $G^*$ : If  $f$  is log-concave then  $(G^*)^{n-1}$  has a convex-linear structure as characterized in [Corollary 2](#). This structure preserves log-concavity of a firm's profit function, even after accounting for compound deviations, making the first-order conditions in [Theorem 1](#) and [Proposition 2](#) sufficient for optimality.

## 6 Discussion

In this section, we examine the robustness of our central economic lessons. Specifically, we consider three economically relevant variations of our baseline model and study whether—and how—

our full information result under intense competition depends on each variation.

## 6.1 Consumer Outside Option

In our baseline model, the consumer does not have an option of not purchasing any product.<sup>32</sup> Normalizing the value of her outside option to zero, the assumption is inconsequential if  $\underline{v} - p^* \geq 0$ . If  $\underline{v} - p^* < 0$ , however, it binds and has non-trivial effects on competitive advertising.

Suppose  $F^{n-1}$  is convex over its support, so that  $G^* = F$  is the unique equilibrium in the baseline advertising game. Now, suppose the consumer has an outside option of value zero, and  $p^* \in (\underline{v}, \bar{v})$ . In this case, she opts out if the value of the best product is below  $p^*$ . This implies that, assuming that all other firms provide full information, firm  $i$ 's problem is given by

$$\max_{G_i \in MPC(F)} \int_{p^*}^{\bar{v}} F(v)^{n-1} dG_i(v) = \int 1_{\{v \geq p^*\}} F(v)^{n-1} dG_i(v). \quad (5)$$

The consumer purchases a product when her value for the product exceeds not only her values for the other products ( $F(v)^{n-1}$ ) but also the price ( $1_{\{v \geq p^*\}}$ ). As depicted in the left panel of [Figure 2](#), this introduces a discrete jump for the firm's value function, yielding the following result.

**Proposition 4** *If the consumer has a binding outside option (i.e.,  $p^* \in (\underline{v}, \bar{v})$ ) then there never exists a full information equilibrium in the advertising game.*

**Proof.** Because of the structure of  $1_{\{v \geq p^*\}} F(v)^{n-1}$ ,  $F$  is dominated by a distribution that pools values around  $p^*$  and puts an atom on  $p^*$ . Therefore,  $F$  cannot be a solution to (5).<sup>33</sup> ■

This result, however, does not exclude the possibility of *asymptotic full information*; that is, it is still possible that the equilibrium distribution  $G^*$  converges to  $F$  as  $n$  tends to infinity. Intuitively, the discrete jump at  $p^*$  has the size of  $F(p^*)^{n-1}$ , which vanishes exponentially fast as  $n$  increases; and the limiting value function—which stays equal to 0 and then jumps at  $\bar{v}$ —is convex. Therefore, if  $n$  is sufficiently large then  $G^*$  could be similar to  $F$ . This convergence indeed holds.

**Proposition 5** *In the advertising game with a binding outside option, the equilibrium distribution  $G^*$  converges (pointwise) to  $F$  as  $n$  tends to infinity.*

<sup>32</sup>This is a common simplifying assumption in the literature. In the standard Perloff-Salop model, its main advantage is to simplify the equilibrium pricing formula: absent the assumption, the right-hand side in (3) involves  $p^*$ , so  $p^*$  cannot be expressed in closed form. In our model, together with our focus on pure-price equilibria, it allows us to identify  $G^*$  independently of  $p^*$ . As we elaborate in [Appendix H](#), if the consumer's outside option is binding then  $G^*$  and  $p^*$  should be jointly determined, which significantly complicates the technical analysis.

<sup>33</sup>Applying Theorems 1 and 2 of [Dworczak and Martini \(2019\)](#), the unique solution  $G_i^*$  to (5) has the following structure: for some  $v_1 < p^* < v_2$ ,  $G_i^*(v) = F(v)$  if  $v \leq v_1$  or  $v \geq v_2$ , and  $G_i^*$  puts all remaining probability on  $p^*$ . See the dashed brown line in the left panel of [Figure 2](#) for the structure of the solution to (5).



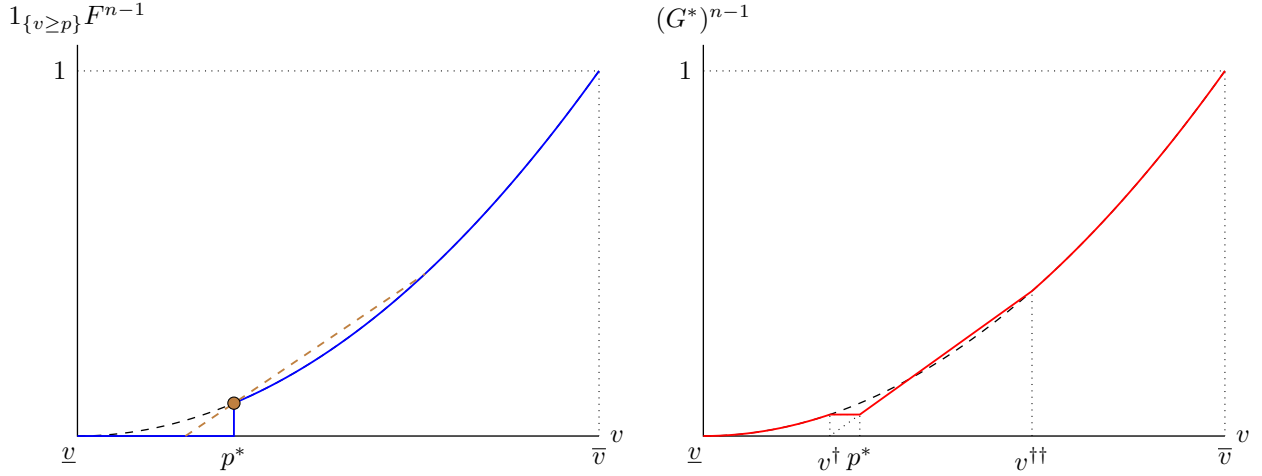


Figure 2: The left panel illustrates why the full information equilibrium does not exist when the consumer has a binding outside option, while the right panel depicts the resulting equilibrium structure. In each panel, the black dashed curve represents the underlying convex distribution.

Note that [Proposition 5](#) requires a full characterization of the equilibrium distribution  $G^*$ , which is different from characterizing the solution to (5). The right panel of [Figure 2](#) illustrates the main features of  $G^*$ : a firm has no incentive to offer values below  $p^*$ , so  $G^*$  is flat right below  $p^*$ . Meanwhile, by [Lemma 2](#) (which still applies to  $[p^*, \bar{v}]$ ),  $(G^*)^{n-1}$  must be locally linear above  $p^*$ . Combining these properties with the MPC constraint yields the unique structure of  $G^*$ . See [Appendix H](#) for a comprehensive equilibrium characterization.

[Proposition 5](#) considers the advertising game with fixed  $p^*$ , but  $n$  certainly affects the equilibrium price  $p^*$  as well. This consideration, however, does not inhibit our convergence result; in fact, it reinforces [Proposition 5](#). The equilibrium price  $p^*$  typically decreases in  $n$  and necessarily converges to 0 in the limit. This makes the consumer's outside option less binding, thereby facilitating the convergence of  $G^*$  to  $F$ . To be specific, suppose  $\underline{v} = 0$ . As  $n$  tends to infinity,  $p^*$  approaches 0, so the consumer's outside option is not binding in the limit. [Proposition 5](#) provides a stronger result: even if  $p^* > 0$  is independent of  $n$ ,  $G^*$  converges to  $F$ .

## 6.2 Multi-unit Demand

In our baseline model, the consumer has a unit demand, purchasing only one out of  $n$  products. This means that the firms effectively engage in a winner-take-all contest, whose stark reward structure induces the firms to be particularly aggressive. In this subsection, we study the extent to which our full information result depends on this winner-take-all structure.

We consider the advertising game in which the consumer purchases  $k$  (different) products out of  $n$ ; our baseline model is a special case with  $k = 1$ . Suppose all other firms play  $G \in MPC(F)$ .

If firm  $i$  chooses  $G_i$ , then its demand is given by

$$D^A(G_i, G) \equiv \int \Phi(v_i; G) dG_i(v_i),$$

where  $\Phi_k(v_i; G)$  denotes the probability that  $v_i$  exceeds at least  $n - k$  draws from  $G$ , that is,

$$\Phi(v_i; G) \equiv \sum_{\ell=0}^{k-1} \binom{n-1}{\ell} (1 - G(v_i))^\ell G(v_i)^{n-1-\ell}.$$

Clearly, (2) is a special case of  $\Phi(v_i; G)$  in which  $k = 1$ .

For the same reason as given in Section 3, the full information equilibrium exists if and only if  $\Phi(\cdot; F)$  is convex over  $[\underline{v}, \bar{v}]$ . As demonstrated before, this property often holds with single-unit demand: it suffices that  $F$  satisfies a certain convexity property or  $n$  is sufficiently large. This is no longer the case with multi-unit demand, as formally stated in the following result.

**Proposition 6** *In the advertising game with multi-unit demand (i.e.,  $k > 1$ ), there never exists a full information equilibrium.*

**Proof.** By direct calculus, one can show that

$$\Phi'(v; F) = \frac{(n-1) \cdots (n-k)}{(k-1)!} (1 - F(v))^{k-1} F(v)^{n-1-k} f(v). \quad (6)$$

Clearly,  $\Phi'(v; F) > 0$  for all  $v \in (\underline{v}, \bar{v})$  but whenever  $k > 1$ ,  $\Phi'(\bar{v}; F) = 0$ . This implies that if  $k > 1$  then  $\Phi(\cdot; F)$  cannot be convex around  $\bar{v}$ , regardless of  $F$  and  $n$ . ■

This result highlights the intuitive fact that multi-unit demand softens competition. Each firm now competes against the  $k$ -th best product among the other firms' products (i.e., the  $k$ -th order statistic among  $(n - 1)$  independent draws), which is less demanding than competing against the first best product. In particular, a firm has no incentive to offer the maximum possible value  $\bar{v}$  to the consumer: a slightly lower value would still make it into the consumer's purchase list, so it is more profitable for the firm to pool values around  $\bar{v}$ .

Proposition 6 suggests that regarding firms' advertising incentives, the intensity of competition cannot be measured by the relative firm size to demand. While firms may provide full information when there are  $n$  firms and the consumer has a unit demand, they never do the same when  $nk$  firms compete for  $k(> 1)$  demands by the consumer. The detail of the demand structure affects firms' advertising incentives.

Despite the above difference, our equilibrium characterization of the advertising game in Section 4 can be readily extended into the current multi-unit case. The only necessary change is to

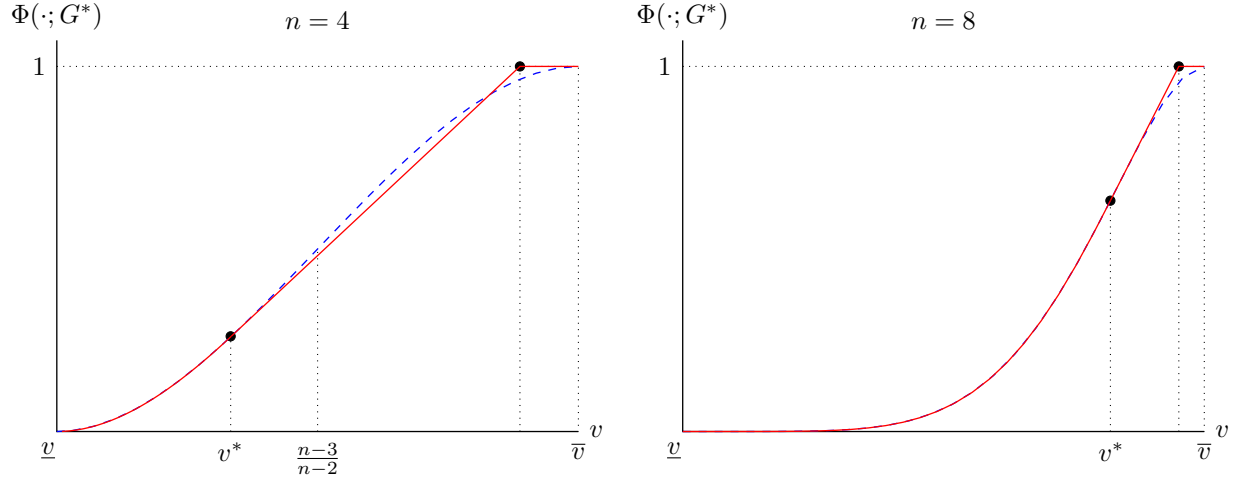


Figure 3: The equilibrium value function  $\Phi(\cdot; G^*)$  with multi-unit demand when  $F = U[0, 1]$  and  $k = 2$ . In both panels, the dashed curve represents  $\Phi(\cdot; F)$ .

impose convexity in [Lemma 1](#) and linearity in [Lemma 2](#) on  $\Phi(\cdot; G^*)$ . We skip a formal analysis—which is a tedious but conceptually straightforward extension of [Section 4](#)—and, instead, provide a simple but representative example below.

**Example 3** Suppose  $F = U[0, 1]$ . In the single-unit case,  $\Phi(v; F) = F(v)^{n-1} = v^{n-1}$  is convex over  $[0, 1]$  for any  $n$ . Therefore, by [Theorem 1](#),  $F$  is the unique equilibrium.

Suppose  $k = 2 < n$ . In this case,  $\Phi(v; F) = v^{n-1} + (n-1)(1-v)v^{n-2}$ , and thus

$$\Phi'(v; F) = (n-1)(n-2)(1-v)v^{n-3}.$$

This expression is single-peaked at  $\frac{n-3}{n-2}$ , so  $\Phi(\cdot; F)$  is convex until  $\frac{n-3}{n-2}$  but concave thereafter (see the dashed curves in [Figure 3](#)). This implies that, as in [Corollary 2](#) and depicted in [Figure 3](#), there exists  $v^* \in (0, \frac{n-3}{n-2})$  such that  $G^*(v) = F^*(v)$  if  $v \leq v^*$ , and  $\Phi(\cdot; G^*)$  is linear above  $v^*$ .

[Figure 3](#) suggests that, despite [Proposition 7](#), the equilibrium distribution  $G^*$  becomes closer to  $F$  as  $n$  rises; in particular,  $v^*$ —below which  $G^* = F$ —increases in  $n$ . The following result shows that this is indeed the case in general, just as for the case of binding outside option in [Section 6.1](#).

**Proposition 7** In the advertising game with multi-unit demand, the equilibrium distribution  $G^*$  converges (pointwise) to  $F$  as  $n$  tends to infinity.

Intuitively, if there are infinitely many firms then, even with multi-unit demand, a firm's product can be selected only when it yields the highest possible value  $\bar{v}$  to the consumer. As in the unit-demand case, this forces firms to provide precise information when  $n$  is large and almost all

information in the limit as  $n$  tends to infinity.<sup>34</sup>

### 6.3 Asymmetric Firms

Our clean characterization of competitive advertising owes much to our focus on symmetric environments/equilibria. It is technically far challenging to allow for firm asymmetries.<sup>35</sup> In this subsection, we consider a relatively simple asymmetric case and study how firm asymmetries affect our full information result.

Suppose there are two types of firms, type 1 and type 2.<sup>36</sup> Let  $n_q$  denote the number of type- $q$  firms. Each type- $q$  firm is endowed with the distribution  $F_q$  over  $[v_q, \bar{v}_q]$ ; that is, the consumer's value for a type- $q$  firm's product is drawn from  $F_q$ . To avoid triviality, assume that the two supports are not disjoint (i.e.,  $[v_1, \bar{v}_1] \cap [v_2, \bar{v}_2] \neq \emptyset$ ). In addition, without loss of generality, assume  $\bar{v}_1 \leq \bar{v}_2$ .

We focus on the advertising game, in which each firm maximizes its total demand.<sup>37</sup> Let  $H_q(v; G_1, G_2)$  denote the probability that the consumer purchases a firm's product with (net) value  $v$  for it. For each  $q = 1, 2$ ,  $H_q(v; G_1, G_2)$  is given as follows:

$$H_1(v; G_1, G_2) \equiv G_1(v)^{n_1-1} G_2(v)^{n_2} \text{ and } H_2(v; G_1, G_2) \equiv G_1(v)^{n_1} G_2(v)^{n_2-1}.$$

As before, the full information equilibrium exists if and only if both  $H_1(\cdot; F_1, F_2)$  and  $H_2(\cdot; F_1, F_2)$  are convex over their support. The following result shows that the property crucially depends on whether  $F_1$  and  $F_2$  have a common upper bound or not.

**Proposition 8** *Consider the advertising game with two types of firms. If the supports of  $F_1$  and  $F_2$  have the same upper bound (i.e.,  $\bar{v}_1 = \bar{v}_2$ ), then the full information equilibrium exists whenever*

<sup>34</sup>For [Proposition 7](#), it is crucial that  $k$  is fixed as  $n$  increases (or, more generally,  $k$  does not increase as fast as  $n$ ). If  $k$  increases at the same rate as  $n$  then  $\Phi(\cdot; F)$  is not convex—so full information cannot be an equilibrium—even in the limit as  $n \rightarrow \infty$ . As a specific example, consider  $k = n - 1$  and  $F = U[0, 1]$ . In this case, for any  $n$ ,  $\Phi(\cdot; F)$  is concave, so  $\Phi(v; G^*) = 1 - (1 - G^*(v))^{n-1}$  should be linear over its support. It can be shown that

$$G^*(v) = 1 - \left(1 - \frac{2(n-1)}{n}v\right)^{n-1} \Leftrightarrow \Phi(v; G^*) = \frac{2(n-1)}{n}v \text{ for } v \in \left[0, \frac{n}{2n(n-1)}\right].$$

In fact, this explains why the full information result does not always hold in the large market model of [Ostrovsky and Schwarz \(2010\)](#): their model can be interpreted as the limit in which both  $n$  and  $k$  tend to infinity.

<sup>35</sup>It is well known that even in the standard Perloff-Salop model (i.e., when firms' distributions are exogenously given), it is technically hard to analyze the asymmetric case and obtain substantive results. To our knowledge, [Quint \(2014\)](#) provides the most general and comprehensive set of (equilibrium existence and comparative statics) results for the Perloff-Salop framework. His analysis, which utilizes the supermodular-game structure of the model, relies on certain regularity assumptions on the distributions, which do not hold under our equilibrium distributions. In [Appendix I](#), we provide a partial characterization of equilibrium advertising in the duopoly case.

<sup>36</sup>All subsequent arguments can be easily generalized into the case with more than two types.

<sup>37</sup>Clearly, different firm types may choose different prices. One can assume that such a price difference is already reflected in the difference between  $F_1$  and  $F_2$ .

both  $F_1^{n_1-1}$  and  $F_2^{n_2-1}$  are convex. Otherwise (i.e., if  $\bar{v}_1 < \bar{v}_2$ ), there never exists a full information equilibrium, regardless of the values of  $n_1$  and  $n_2$ .

**Proof.** The first result follows from the following two facts: (i) if  $F_q^{n_q-1}$  is convex then  $F_q^{n_q}$  is also convex; and (ii) the product of two non-negative monotone convex functions is also convex. For the second result, observe that

$$\begin{aligned} \lim_{v \rightarrow \bar{v}_1^-} H_2'(v; F_1, F_2) &= \lim_{v \rightarrow \bar{v}_1^-} (F_1(v)^{n_1} F_2(v)^{n_2-1})' \\ &= n_1 F_2(\bar{v}_1)^{n_2-1} f_1(\bar{v}_1) + (n_2 - 1) F_2(\bar{v}_1)^{n_2-1} f_2(\bar{v}_1) \\ > \lim_{v \rightarrow \bar{v}_1^+} H_2'(v; F_1, F_2) &= \lim_{v \rightarrow \bar{v}_1^+} (F_2(v)^{n_2-1})' = (n_2 - 1) F_2(\bar{v}_1)^{n_2-1} f_2(\bar{v}_1). \end{aligned}$$

In other words,  $H_2(\cdot; F_1, F_2)$  has a downward kink at  $\bar{v}_1$  and so is not convex over  $[\underline{v}_2, \bar{v}_2]$ , which suffices for the desired result. ■

The economic force behind the full information equilibrium in our baseline symmetric case continues to apply to the current asymmetric case. If there are many competing firms, then a firm can attract the consumer only when she has a particularly high value for its product, so all firms would provide precise information. When  $\bar{v}_1 = \bar{v}_2$ , this “top-down unraveling” logic—always providing the highest value to the consumer—applies equally to all firms. If  $\bar{v}_1 < \bar{v}_2$ , however, the logic does not apply to type-2 firms around  $\bar{v}_1$ . Due to type-1 firms, they face more competition below  $\bar{v}_1$  than above  $\bar{v}_1$  and so have an incentive to pool values around  $\bar{v}_1$ , in order to increase their winning probability against type-1 firms. This incentive is independent of both  $n_1$  and  $n_2$ , so the full-informative equilibrium *never* exists in this case.

As for the previous two cases in [Sections 6.1](#) and [6.2](#), even if  $\bar{v}_1 < \bar{v}_2$  and so full information can never be an equilibrium, it seems likely that the equilibrium distributions,  $G_1^*$  and  $G_2^*$ , still converge to  $F_1$  and  $F_2$ , as  $n_1$  and  $n_2$  tend to infinity. This is certainly a reasonable conjecture, given that  $H_2(\cdot; F_1, F_2)$  is convex in the limit. A formal result, however, requires comprehensive equilibrium characterization, which is technically too challenging and so we are unable to provide.<sup>38</sup>

<sup>38</sup>Given the proof of [Proposition 8](#), one may think that (at least) type-1 firms could provide full information. This is usually not the case; the result holds only when the underlying distributions satisfy some very specific (not usual) properties. For a basic idea, suppose  $n_1$  is sufficiently large, while  $n_2 = 1$ . If all type-1 firms provide full information then the (unique) type-2 firm would pool values around  $\bar{v}_1$  and put positive mass on  $\bar{v}_1$ . Unless the type-2 firm can win probability 1, there exists  $\tilde{v} (< \bar{v}_1)$  such that  $G_2(v) = F_2(v)$  for  $v < v^*$  and  $G_2(v)$  is constant over  $[\tilde{v}, \bar{v}_1)$ . This makes  $H_1(\cdot; F_1, G_2)$  non-convex around  $\tilde{v}$ : just as in the proof of [Proposition 8](#),  $H_1'(\tilde{v}-; F_1, G_2) > H_1'(\tilde{v}+; F_1, G_2)$ .

## 7 Conclusion

This paper studies an oligopoly model in which the firms compete not only on price but also through informative advertising (i.e., by choosing how much product information to disclose). Unlike most existing studies on informative advertising, but as is common in the recent literature on information design, we impose no structural restriction on feasible advertising content; this enables us to focus on firms' strategic incentives, not subject to any technological constraints, and also obtain a relatively tractable characterization of the equilibrium advertising content. Our main economic insight is that intense competition induces firms to provide precise product information. As elaborated in [Section 1](#), this economic result was also obtained by [Ivanov \(2013\)](#), but our analysis strengthens it in multiple ways. In particular, we show that the result is robust along multiple dimensions. We also demonstrate that strategic advertising, which creates deadweight loss through information loss, has ambiguous implications for both prices and consumer surplus; note that this offers a cautionary tale for policies regarding firms' disclosure requirements.

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## A Omitted Proofs

**Proof of Proposition 1.** The argument before **Proposition 1** establishes the sufficiency of convex  $F^{n-1}$ . Its necessity follows from **Lemma 1**: if  $G^*$  is an equilibrium in the advertising game then  $(G^*)^{n-1}$  must be convex. Therefore, if  $F^{n-1}$  is not convex, then  $F$  cannot be an equilibrium. For an explicit construction of  $G_i \in MPC(F)$  such that  $D^A(G_i, F) > D^A(F, F)$  when  $F^{n-1}$  is not convex, see the proof of **Lemma 1**. ■

**Proof of Lemma 1.** Let  $\overline{supp}(G^*)$  denote the convex closure (hull) of  $supp(G^*)$ , and suppose that  $(G^*)^{n-1}$  is not convex over  $\overline{supp}(G^*)$ . Then, there exist  $v_1, v_2, v_3$ , and  $\varepsilon$  (sufficiently small) such that  $v_1 < v_2 < v_3$ ,  $[v_1, v_1 + \varepsilon] \cup (v_3 - \varepsilon, v_3] \subset supp(G^*)$ , and

$$G^*(v_2)^{n-1} > \frac{v_3 - v_2}{v_3 - v_1} G^*(v_1)^{n-1} + \frac{v_2 - v_1}{v_3 - v_1} G^*(v_3)^{n-1}. \quad (7)$$

Let  $\delta \in (0, \varepsilon)$  and  $\delta' \in (0, \varepsilon)$  be the values such that

$$v_2 = \mathbb{E}_{G^*}[v | v \in [v_1, v_1 + \delta] \cup (v_3 - \delta', v_3]] = \frac{\int_{v_1}^{v_1 + \delta} v dG^*(v) + \int_{v_3 - \delta'}^{v_3} v dG^*(v)}{\Delta + \Delta'},$$

where  $\Delta \equiv \int_{v_1}^{v_1 + \delta} dG^*(v)$  and  $\Delta' \equiv \int_{v_3 - \delta'}^{v_3} dG^*(v)$ . The pair exists because  $\mathbb{E}_{G^*}[v | v \in [v_1, v_1 + \delta] \cup (v_3 - \delta', v_3]]$  can take any value in  $(v_1, v_3)$ , depending on the relative weights of  $\delta$  and  $\delta'$ .

Consider the following alternative distribution function  $G_i$ :

$$G_i(v) = \begin{cases} G^*(v) & \text{if } v < v_1, \\ G^*(v_1) & \text{if } v \in [v_1, v_1 + \delta), \\ G^*(v) - \Delta & \text{if } v \in [v_1 + \delta, v_2), \\ G^*(v) + \Delta' & \text{if } v \in [v_2, v_3 - \delta'), \\ G^*(v_3) & \text{if } v \in [v_3 - \delta', v_3), \\ G^*(v) & \text{if } v \geq v_3. \end{cases}$$

In other words,  $G_i$  differs from  $G^*$  only in that it takes probability mass from  $[v_1, v_1 + \delta) \cup (v_3 - \delta', v_3]$  and assigns it to  $v_2$ . By construction,  $G_i \in MPC(G^*) \subset MPC(F)$ . In addition,

$$\int (G^*)^{n-1} dG_i - \int (G^*)^{n-1} dG^* = (\Delta + \Delta') G^*(v_2)^{n-1} - \left( \int_{v_1}^{v_1 + \delta} (G^*)^{n-1} dG^* + \int_{v_3 - \delta'}^{v_3} (G^*)^{n-1} dG^* \right),$$

which is strictly positive for  $\varepsilon$  sufficiently small, due to (7). This violates the equilibrium requirement that  $D^A(G^*, G^*) \geq D^A(G_i, G^*)$ , establishing that  $(G^*)^{n-1}$  being convex over  $\overline{supp}(G^*)$  is necessary for  $G^*$  to be an equilibrium.

Since  $(G^*)^{n-1}$  must have non-decreasing density over  $\overline{supp}(G^*)$ ,  $supp(G^*) = \overline{supp}(G^*)$ . ■

**Proof of Lemma 2.** Define a function  $W : [\underline{v}, \bar{v}] \rightarrow \mathcal{R}$  as follows:

$$W(v) = \int_{\underline{v}}^v (F(v) - G^*(v)) dv.$$

Since  $G^* \in MPC(F)$ ,  $W(v) \geq 0$  for any  $v$  and  $W(\bar{v}) = 0$  (see Section 6.D in Mas-Colell et al., 1995). We show that whenever  $W(v_1) = W(v_2) = 0$  and  $W(v) > 0$  for all  $v \in (v_1, v_2)$ ,  $(G^*)^{n-1}$  must be linear over  $[v_1, \min\{v_2, \bar{v}^*\}]$ , where  $\bar{v}^*$  denotes the upper bound of  $supp(G^*)$ . This suffices for Lemma 2, because  $W(v) = 0$  only when  $F(v) = G(v)$ , and  $W(v) = 0$  if and only if  $F$  and  $G^*$  have the same mean over  $[\underline{v}, v]$ .

Observe that the continuity of  $G^*$  (from Lemma 1) implies that  $G^*$  must coincide with  $F$  at  $v_1$  and  $v_2$ . We consider the following two cases.

(i) Suppose that  $v_2 \leq \bar{v}^*$ . Let  $G_i$  denote the distribution that coincides with  $F$  on  $[v_1, v_2]$  and follows  $G^*$  elsewhere; that is,  $G_i(v) = F(v)$  if  $v \in [v_1, v_2]$  and  $G_i(v) = G^*(v)$  otherwise. By construction,  $G_i$  is an MPC of  $F$  (because  $G^*$  is an MPC of  $F$  over  $[\underline{v}, v_1]$  and  $[v_2, \bar{v}]$ ), while it is a mean-preserving spread of  $G^*$  (because  $G^*$  is an MPC of  $F$  over  $[v_1, v_2]$ ).

Since  $G^*$  must be a best response to itself, we have

$$\int G^*(v)^{n-1} dG_i(v) \leq \int G^*(v)^{n-1} dG^*(v) \Leftrightarrow \int_{v_1}^{v_2} G^*(v)^{n-1} dG_i(v) \leq \int_{v_1}^{v_2} G^*(v)^{n-1} dG^*(v).$$

On the other hand, since  $(G^*)^{n-1}$  is convex over  $(v_1, v_2)$  and  $G^* \in MPC(G_i)$ , we also have

$$\int G^*(v)^{n-1} dG_i(v) \geq \int G^*(v)^{n-1} dG^*(v) \Leftrightarrow \int_{v_1}^{v_2} G^*(v)^{n-1} dG_i(v) \geq \int_{v_1}^{v_2} G^*(v)^{n-1} dG^*(v),$$

leading to  $\int_{v_1}^{v_2} G^*(v)^{n-1} dG_i(v) = \int_{v_1}^{v_2} G^*(v)^{n-1} dG^*(v)$ . Since  $G_i = F \neq G^*$  on  $(v_1, v_2)$  and  $(G^*)^{n-1}$  is convex, this equality holds only when  $(G^*)^{n-1}$  is linear over  $(v_1, v_2)$ .

(ii) Suppose that  $v_2 > \bar{v}^*$ . Since  $G^* \in MPC(F)$ , this case arises only when  $v_2 = \bar{v}$  (see **Figure 1**). The previous proof does not directly apply here, because  $(G^*)^{n-1}$  is convex only over  $[v_1, \bar{v}^*] \subsetneq [v_1, v_2]$ . To fix this problem, let  $\tilde{v}$  be the value such that

$$\mathbb{E}_F[v|v \geq \tilde{v}] \equiv \int_{\tilde{v}}^{\bar{v}} v \frac{dF(v)}{1 - F(\tilde{v})} = \bar{v}^*.$$

This value is well defined, because  $\mathbb{E}_F[v|v \geq \tilde{v}]$  is strictly increasing in  $\tilde{v}$ ,  $\mathbb{E}_F[v|v \geq v_1] = \mathbb{E}_{G^*}[v|v \geq v_1] < \bar{v}^*$ , and  $\mathbb{E}_F[v|v \geq \bar{v}^*] > \bar{v}^*$ .

Let  $G_i$  denote the distribution that coincides with  $G^*$  below  $v_1$ , follows  $F$  until  $\tilde{v}$ , and then places all remaining mass on  $\bar{v}^*$ . Formally,

$$G_i(v) = \begin{cases} G^*(v) & \text{if } v < v_1, \\ \min\{F(v), F(\tilde{v})\} & \text{if } v \in [v_1, \bar{v}^*), \\ 1 & \text{if } v \geq \bar{v}^*. \end{cases}$$

By construction,  $G_i \in MPC(F)$ , while  $G^* \in MPC(G_i)$ : this latter result holds because  $\mathbb{E}_{G_i}[v|v \geq v_1] = \mathbb{E}_F[v|v \geq v_1] = \mathbb{E}_{G^*}[v|v \geq v_1]$  and, by the definition of  $G_i$  and the fact that  $G^*$  is an MPC of  $F$  over  $[v_1, v_2]$ ,  $G_i$  crosses  $G^*$  only once from above. Given this, a similar proof to the previous one applies: by the optimality of  $G^*$  over  $G_i$  and the convexity of  $(G^*)^{n-1}$  over  $[v_1, \bar{v}^*]$ ,

$$\int_{v_1}^{\bar{v}^*} G^*(v)^{n-1} dG_i(v) = \int_{v_1}^{\bar{v}^*} G^*(v)^{n-1} dG^*(v).$$

Since  $G^* \neq G_i$  over  $[v_1, \bar{v}^*]$ , this equality holds only when  $(G^*)^{n-1}$  is linear over  $(v_1, \bar{v}^*)$ . ■

**Proof of Sufficiency in Theorem 1.** Suppose that  $G^*$  is an MPC of  $F$  that satisfies **Lemmas 1** and **2**.<sup>39</sup> It is an equilibrium in the advertising game if and only if it is a solution to

$$\max_{G_i \in MPC(F)} D^A(G_i, G^*) = \int G^*(v)^{n-1} dG_i(v). \quad (8)$$

<sup>39</sup>See **Appendix B** for our construction of  $G^*$  and the uniqueness proof.

We show that  $G^*$  solves this programming problem by applying the following theorem.

**Theorem 3** (*Dworczak and Martini, 2019*) Consider the following programming problem, where  $F$  is a distribution over  $[\underline{v}, \bar{v}]$ :

$$\max_{G \in MPC(F)} \int_{\underline{v}}^{\bar{v}} u(x) dG(x).$$

A distribution  $G^* \in MPC(F)$  solves the above problem if there exists a convex function  $\phi : [\underline{v}, \bar{v}] \rightarrow \mathcal{R}$  such that (i)  $\phi(x) \geq u(x)$  for all  $x \in [\underline{v}, \bar{v}]$ , (ii)  $\text{supp}(G^*) \subset \{x \in [\underline{v}, \bar{v}] : u(x) = \phi(x)\}$ , and (iii)  $\int_{\underline{v}}^{\bar{v}} \phi(x) dG^*(x) = \int_{\underline{v}}^{\bar{v}} \phi(x) dF(x)$ .

In what follows, we construct a function  $\phi$  that satisfies all three properties in the theorem. Let  $\underline{v}^*$  and  $\bar{v}^*$  denote the lower and upper bounds of  $\text{supp}(G^*)$ , respectively. By [Lemma 1](#),  $\text{supp}(G^*) = [\underline{v}^*, \bar{v}^*]$  and, since  $G^* \in MPC(F)$ ,  $[\underline{v}^*, \bar{v}^*] \subseteq [\underline{v}, \bar{v}]$ . For each  $v \in [\underline{v}, \bar{v}]$ , let

$$\phi(v) \equiv \begin{cases} G^*(v)^{n-1} & \text{if } v \in [\underline{v}, \bar{v}^*], \\ \alpha(v - \bar{v}^*) + 1 & \text{if } v \in (\bar{v}^*, \bar{v}], \end{cases}$$

where

$$\alpha \equiv \limsup_{v \rightarrow \bar{v}^* -} \frac{G^*(\bar{v}^*)^{n-1} - G^*(v)^{n-1}}{\bar{v}^* - v}. \quad (9)$$

If  $\bar{v}^* = \bar{v}$  then  $\phi(v) = G^*(v)^{n-1}$  over  $[\underline{v}, \bar{v}]$ . If  $\bar{v}^* < \bar{v}$ , then  $G^*(\bar{v}^*) = 1 > F(\bar{v}^*)$ , in which case, by [Lemma 2](#),  $(G^*)^{n-1}$  must be linear right below  $\bar{v}^*$  and  $\phi(v)$  linearly extends  $(G^*)^{n-1}$  above  $\bar{v}^*$ .

By construction,  $\phi(v) \geq G^*(v)^{n-1}$  for all  $v \in [\underline{v}, \bar{v}]$ , and  $\phi(v) = G^*(v)^{n-1}$  for all  $v \in [\underline{v}^*, \bar{v}^*]$ . Therefore, (i) and (ii) in [Theorem 3](#) hold. For (iii), notice that [Lemma 2](#) and our construction of  $\phi(v)$  imply that  $\phi$  can be described by a partition  $\{v_0 = \underline{v}, v_1, \dots, v_m = \bar{v}\}$  such that for each  $k = 1, \dots, m$ ,  $G^*(v) = F(v)$  over  $[v_{k-1}, v_k]$ , or  $\phi$  is linear over  $[v_{k-1}, v_k]$  and  $G^*$  is an MPC of  $F$  over  $[v_{k-1}, v_k]$ . Either case, we have

$$\int_{v_{k-1}}^{v_k} \phi(v) dG^*(v) = \int_{v_{k-1}}^{v_k} \phi(v) dF(v), \text{ leading to } \int_{\underline{v}}^{\bar{v}} \phi(v) dG^*(v) = \int_{\underline{v}}^{\bar{v}} \phi(v) dF(v).$$

■

**Proof of Corollary 2.(2).** For each  $n \geq 2$ , let  $G_n^*$  denote the corresponding equilibrium distribution and  $v_n^*$  denote the value of  $v^*$  over which  $(G_n^*)^{n-1}$  is linear. Fix  $k \geq 2$ . The problem is trivial if  $v_k^* = \underline{v}$  or  $v_{k+1}^* = \bar{v}$ , so assume that  $v_k^* > \underline{v}$  and  $v_{k+1}^* < \bar{v}$ . Given the proof for (1) in the main text, it suffices to show that if the cutoff  $v^*$  remains constant as  $n$  increases from  $k$  to  $k+1$ , then the resulting distribution, denoted by  $\tilde{G}$ , requires a higher mean than  $F$ . Here, we prove a stronger result, namely, that  $\tilde{G}$  first-order stochastically dominates  $G_k^* \in MPC(F)$ .

For notational simplicity, let  $a \equiv F(v_k^*)$  and  $b \equiv f(v_k^*)(v - v_k^*)/F(v_k^*)$ . Then,

$$G_k^*(v) = \begin{cases} F(v) & \text{if } v \leq v_k^* \\ \min \left\{ (a^{k-1} + (k-1)a^{k-1}b)^{1/(k-1)}, 1 \right\} & \text{if } v > v_k^*, \end{cases} \quad (10)$$

while

$$\tilde{G}(v) = \begin{cases} F(v) & \text{if } v \leq v_k^* \\ \min \left\{ (a^k + ka^kb)^{1/k}, 1 \right\} & \text{if } v > v_k^*. \end{cases}$$

We show that for any  $v > v_k^*$  such that  $G_k^*(v) \leq 1$ ,

$$G_k^*(v) = (a^{k-1} + (k-1)a^{k-1}b)^{\frac{1}{k-1}} > \tilde{G}(v) = (a^k + ka^kb)^{\frac{1}{k}},$$

which is equivalent to

$$(1 + (k-1)b)^k > (1 + kb)^{k-1} \Leftrightarrow (1 + kb) \left( \frac{1 + (k-1)b}{1 + kb} \right)^{k-1} > 1.$$

The desired result follows from the fact that the left-hand side in the final expression is equal to 1 if  $b = 0$  (i.e.,  $v = v_k^*$ ) and strictly increasing in  $b$  (i.e., in  $v$ ).

It remains to prove that  $G_k^* \in MPC(G_{k+1}^*)$ . The problem is trivial if  $v_k^* = \bar{v}$  (i.e.,  $G_k^* = F$ ). There are the following two cases to consider.

*Case 1:*  $v_k^* < v_{k+1}^*$ . By their definitions,  $G_k^*(v) = G_{k+1}^*(v) = F(v)$  for  $v \leq v_k^*$ , and  $G_{k+1}^*(v) = F(v)$  over  $[v_k^*, v_{k+1}^*]$ . Combining this with  $G_k^* \in MPC(F)$  implies that there exists  $\varepsilon > 0$  such that  $G_{k+1}^*(v) > G_k^*(v)$  whenever  $v \in (v_k^*, v_k^* + \varepsilon)$ . Since  $G_k^*, G_{k+1}^* \in MPC(F)$ , it suffices to show that  $G_{k+1}^*$  crosses  $G_k^*$  once from above.

Suppose to the contrary that  $G_{k+1}^*$  crosses  $G_k^*$  more than once. Let  $\hat{v}$  be the smallest value at which  $G_{k+1}^*$  crosses  $G_k^*$  from above. Recall that  $(G_k^*)^{k-1}$  is linear above  $v_k^*$ . Furthermore, since  $(G_{k+1}^*)^k$  is linear above  $v_{k+1}^*$ ,  $(G_{k+1}^*)^{k-1} = ((G_{k+1}^*)^k)^{(k-1)/k}$  is strictly concave above  $v_{k+1}^*$ . Therefore, it must be that  $\hat{v} \in (v_k^*, v_{k+1}^*)$ ; otherwise,  $G_{k+1}^*$  crosses  $G_k^*$  only once. Then,  $F^{k-1}$  must be concave at some  $\tilde{v} \leq \hat{v}$ . Since  $F^{k-1}$  has quasi-concave density,  $F^{k-1}$  is concave for all  $v \geq \tilde{v}$ . However, this implies that  $G_{k+1}^*$  is concave for all  $v \geq \tilde{v}$  as  $G_{k+1}^*$  is smooth at  $v_{k+1}^*$ , and thus  $G_{k+1}^*$  does not cross  $G_k^*$  for all  $v > \hat{v}$ , leading to a contradiction.

*Case 2:*  $v_k^* = v_{k+1}^*$ . The above argument implies that this case occurs only when  $v_k^* = v_{k+1}^* = \bar{v}$ . Then [Corollary 3](#) implies that  $(G_k^*)^{k-1}$  and  $(G_{k+1}^*)^k$  are linear over  $[\underline{v}, \bar{v}_k^*]$  and  $[\underline{v}, \bar{v}_{k+1}^*]$ , respectively, where  $\bar{v}_k^*$  and  $\bar{v}_{k+1}^*$  are determined by the condition that  $G_k^*, G_{k+1}^* \in MPC(F)$ . Similarly to Case 1,  $(G_k^*)^{k-1}$  is linear, while  $(G_{k+1}^*)^{k-1}$  is strictly concave (below 1). Combining this with the fact that  $G_k^*$  and  $G_{k+1}^*$  have the same mean, it follows that  $G_{k+1}^*$  crosses  $G_k^*$  exactly once from

above, which suffices for  $G_k^* \in MPC(G_{k+1}^*)$ . ■

**Proof of Proposition 3.** (1) Suppose  $F_2$  is a proportional stretch of  $F_1$ , as defined in [Footnote 28](#). By a recent result by Zhou (2017) and Choi, Dai, and Kim (2018), it suffices to show that  $F_2$  dominates  $F_1$  in dispersive order; that is, for any  $0 < a < b < 1$

$$F_2^{-1}(b) - F_2^{-1}(a) \geq F_1^{-1}(b) - F_1^{-1}(a).$$

Let  $x_{1,a} \equiv F_1^{-1}(a)$ ,  $x_{1,b} \equiv F_1^{-1}(b)$ ,  $x_{2,a} \equiv F_2^{-1}(a)$ , and  $x_{2,b} \equiv F_2^{-1}(b)$ . Since  $F_2$  is a proportional stretch of  $F_1$ , we have

$$F_1(x_{1,a}) = a = F_2(x_{2,a}) = F_1\left(\frac{\bar{v}_1 - \underline{v}_1}{\bar{v}_2 - \underline{v}_2}(x_{2,a} - \underline{v}_2) + \underline{v}_1\right) \Rightarrow x_{1,a} = \frac{\bar{v}_1 - \underline{v}_1}{\bar{v}_2 - \underline{v}_2}(x_{2,a} - \underline{v}_2) + \underline{v}_1.$$

Similarly, we have

$$x_{1,b} = \frac{\bar{v}_1 - \underline{v}_1}{\bar{v}_2 - \underline{v}_2}(x_{2,b} - \underline{v}_2) + \underline{v}_1.$$

It then follows that

$$F_1^{-1}(b) - F_1^{-1}(a) = x_{1,b} - x_{1,a} = \frac{\bar{v}_1 - \underline{v}_1}{\bar{v}_2 - \underline{v}_2}(x_{2,b} - x_{2,a}) = \frac{\bar{v}_1 - \underline{v}_1}{\bar{v}_2 - \underline{v}_2}(F_2^{-1}(b) - F_2^{-1}(a)) \leq F_2^{-1}(b) - F_2^{-1}(a),$$

where the last inequality holds because  $\bar{v}_1 - \underline{v}_1 \leq \bar{v}_2 - \underline{v}_2$ .

(2) Let  $\bar{\mathcal{F}}$  denote the set of all distributions defined over  $[\underline{v}, \bar{v}]$  and  $\mathcal{F}$  denote the subset of  $\bar{\mathcal{F}}$  that includes only those distributions with piecewise continuous density. Any distribution  $F \in \bar{\mathcal{F}}/\mathcal{F}$  can be approximated by those in  $\mathcal{F}$ , so we focus on  $\mathcal{F}$ . Then, the problem of minimizing  $\int (F^{n-1})' dF$  can be written as

$$\max_{f(\cdot)} - \int_{\underline{v}}^{\bar{v}} (n-1)F(v)^{n-2} f(v)^2 dv \text{ s.t. } \dot{F}(v) = f(v) \geq 0, F(\underline{v}) = 0, \text{ and } F(\bar{v}) = 1.$$

The corresponding Hamiltonian is given by  $H = -(n-1)F^{n-2}f^2 + \xi f$ .

First, consider the case when  $n = 2$ . Since the Hamiltonian is simply  $H = -f^2 + \xi f$ , the optimal solution  $f$  and the corresponding costate variable  $\xi$  satisfy

$$\frac{\partial H}{\partial f} = -2f + \xi \leq 0, = 0 \text{ if } f > 0, \text{ and } \dot{\xi} = 0.$$

It is easy to see that  $\xi(v) = \xi_0$  and  $f(v) = \xi_0/2$  for some  $\xi_0$ . The value of  $\xi_0$  is determined from

the boundary conditions:

$$1 = F(\bar{v}) = \int_{\underline{v}}^{\bar{v}} f(v)dv = \frac{(\bar{v} - \underline{v})\xi_0}{2} \Rightarrow \xi_0 = \frac{2}{\bar{v} - \underline{v}} > 0.$$

It follows that  $F(v) = \frac{v-\underline{v}}{\bar{v}-\underline{v}}$  uniquely satisfies Pontryagin's maximum principle.

Now consider the case when  $n > 2$ . In this case, the Hamiltonian is  $H = -(n-1)F^{n-2}f^2 + \xi f$ , so the optimal solution  $f$  and the corresponding costate variable  $\xi$  satisfy

$$\frac{\partial H}{\partial f} = -2(n-1)F^{n-2}f + \xi(v) \leq 0, = 0 \text{ if } f > 0 \text{ and}$$

$$\dot{\xi} = -\frac{\partial H}{\partial F} = (n-1)(n-2)F^{n-3}f^2.$$

The second condition can be rewritten as  $\xi(v) = (n-1) \int_{\underline{v}}^v (n-2)F(x)^{n-3}f(x)^2 dx + c$  for some constant  $c$ . Combining this with the first condition leads to

$$2(n-1)F(v)^{n-2}f(v) - (n-1) \int_{\underline{v}}^v (n-2)F(x)^{n-3}f(x)^2 dx = c.$$

This equation should hold for any  $v$ , so

$$\begin{aligned} 0 &= \frac{d}{dv} \left( 2(n-1)F(v)^{n-2}f(v) - (n-1) \int_{\underline{v}}^v (n-2)F(x)^{n-3}f(x)^2 dx \right) \\ &= (n-1)(n-2)F(v)^{n-3}f(v)^2 + 2F(v)^{n-2}f'(v) = (n-1) (F(v)^{n-2}f(v)^2)'. \end{aligned}$$

This means that at the optimal solution,  $F(v)^{n-2}f(v)^2$  should be independent of  $v$ . It can be directly shown that the given power function is the unique distribution over  $[\underline{v}, \bar{v}]$  that possesses the property.

Note that  $-F^{n-2}f^2$  is concave in  $(F, f)$  and, whether  $n = 2$  or  $n > 2$ ,  $\xi \geq 0$ . This ensures that Pontryagin's maximum principle is not only necessary but also sufficient (see, e.g., Theorem 4.6.2. in [Leonard and Van Long, 1992](#)) so the distribution characterized above is indeed an optimal solution. ■

**Proof of Proposition 5.** Recall that we consider the case where  $n$  is so large that  $F^{n-1}$  is convex over its support. By [Proposition 9](#) in [Appendix H](#), for any (large)  $n$ , there exist  $v_n^\dagger$  and  $v_n^{\dagger\dagger}$  such that

$v_n^\dagger < p^* < v_n^{\dagger\dagger}$  and  $G_n^*$  is given by

$$G_n^*(v)^{n-1} = \begin{cases} F(v)^{n-1}, & \text{if } v \leq v_n^\dagger, \\ F(v_n^\dagger)^{n-1}, & \text{if } v \in (v_n^\dagger, p^*], \\ F(v_n^\dagger)^{n-1} + \beta_n(v - p^*), & \text{if } v \in (p^*, v_n^{\dagger\dagger}), \\ F(v)^{n-1}, & \text{if } v > v_n^{\dagger\dagger}, \end{cases}$$

where

$$F(v_n^\dagger)^{n-1} + \beta_n(v_n^\dagger - p^*) = 0 \Leftrightarrow \beta_n \equiv \frac{F(v_n^\dagger)^{n-1}}{p^* - v_n^\dagger}.$$

Clearly,  $G_n^*$  should be an MPC of  $F$  over  $[v_n^\dagger, v_n^{\dagger\dagger}]$ .

Given the above characterization, it suffices to show that  $v_n^\dagger$  approaches  $p^*$  as  $n \rightarrow \infty$ . Suppose, to the contrary, there exist  $\varepsilon > 0$  and a subsequence  $\{v_{n_k}^\dagger\}$  such that  $p^* - v_{n_k}^\dagger > \varepsilon$  for all  $k$ . Then

$$\frac{(F(p^*)^{n_k-1})'}{\beta_{n_k}} = \frac{(n_k - 1)f(p^*)F(p^*)^{n_k-2}}{F(v_{n_k}^\dagger)^{n_k-1}/(p^* - v_{n_k}^\dagger)} = \frac{f(p^*)(n_k - 1)(p^* - v_{n_k}^\dagger)}{F(p^*)} \left( \frac{F(p^*)}{F(v_{n_k}^\dagger)} \right)^{n_k-1} > 1,$$

for sufficiently large  $k$ . However, this implies that  $F(v_{n_k}^\dagger)^{n_k-1} + \beta_{n_k}(v - p^*)$  stays uniformly below  $F(v)^{n_k-1}$  over  $[v, \bar{v}]$ , which contradicts the requirement that  $G_{n_k}^*$  is an MPC of  $F$  over  $[v_{n_k}^\dagger, v_{n_k}^{\dagger\dagger}]$ . ■

**Proof of Proposition 7.** For each  $n$ , let  $v_n^*$  be the first point at which  $\Phi(\cdot; G^*)$  is locally linear:

$$v_n^* \equiv \min\{v \in [v, \bar{v}] : \text{there exists } \varepsilon > 0 \text{ s.t. } \Phi(\cdot; G^*) \text{ is linear over } [v, v + \varepsilon]\}.$$

For each  $n$ , let  $\bar{v}_n^*$  denote the maximal value such that  $\Phi(\cdot; G^*)$  is linear over  $[v_n^*, \bar{v}_n^*]$ . For the same reason as in our baseline model,  $G^*$  must be an MPC of  $F$  over  $[v_n^*, \bar{v}_n^*]$ . Note that this implies that  $\Phi(\cdot; G^*)$  must cross  $\Phi(\cdot; F)$  at least once from below over the interval, as the same property should hold between  $G^*$  and  $F$ .

Since  $G^*(v) = F(v)$  for all  $v \leq v_n^*$ , it suffices to show that  $v_n^*$  approaches  $\bar{v}$  as  $n$  tends to  $\infty$ . Toward a contradiction, suppose there is an infinite subsequence  $\{n_k\}$  such that  $v_{n_k}^* \leq v^\dagger$  for all  $n_k$  and some  $v^\dagger < \bar{v}$ . We show that if  $n_k$  is sufficiently large then  $\Phi(\cdot; G^*)$  stays uniformly below  $\Phi(\cdot; F)$ , and thus  $G^*$  cannot be an MPC of  $F$ , which is a contradiction.

We first show that for  $n$  sufficiently large,  $\Phi(\cdot; F)$  is convex over  $[v, v^\dagger]$ . From (6), we get

$$\begin{aligned} \Phi''(v; F) &= \frac{(n-1) \cdots (n-k)}{(k-1)!} (1 - F(v))^{k-2} F(v)^{n-2-k} f(v)^2 \\ &\quad \times \left( (n-1-k)(1 - F(v))f(v)^2 - (k-1)F(v)f(v)^2 + (1 - F(v))F(v)f'(v) \right). \end{aligned}$$



To show that  $\Phi''(v; F) > 0$  for all  $v \in [\underline{v}, v^\dagger]$ , let  $\varepsilon \equiv \min\{f(v) : v \in [\underline{v}, \bar{v}]\}$  and  $M \equiv \max\{|f'(v)| : v \in [\underline{v}, \bar{v}]\}$  (as in the proof of **Corollary 1**). Then, we have

$$\begin{aligned} & (n-1-k)(1-F(v))f(v)^2 - (k-1)F(v)f(v)^2 + (1-F(v))F(v)f'(v) \\ > & (n-1-k)(1-F(v^\dagger))\varepsilon^2 - (k-1)\varepsilon^2 - M \end{aligned}$$

which is positive for  $n$  sufficiently large; note that for this argument, it is crucial that  $v^\dagger < \bar{v}$ , so  $F(v^\dagger) < 1$ .

Fix any sufficiently large  $n_k$ . Then, by the equilibrium structure, and because  $\Phi(\cdot; F)$  is convex until  $v^\dagger$ , it must be that for all  $v \in [v_{n_k}^*, \bar{v}_{n_k}^*]$ ,

$$\Phi(v; G^*) = \Phi(v_{n_k}^*; F) + \Phi'(v_{n_k}^*; F)(v - v_{n_k}^*) \leq \begin{cases} \Phi(v; F) & \text{if } v \in [v_{n_k}^*, v^\dagger] \\ \Phi(v^\dagger; F) + \Phi'(v^\dagger; F)(v - v^\dagger) & \text{if } v \in (v^\dagger, \bar{v}_{n_k}^*]. \end{cases}$$

From (6),

$$\Phi'(v^\dagger; F) = \frac{(n-1) \cdots (n-k)}{(k-1)!} (1-F(v^\dagger))^{k-1} F(v^\dagger)^{n-1-k} f(v^\dagger)$$

which converges to 0 as  $n$  tends to  $\infty$ , because  $(n-1) \cdots (n-k)$  increases polynomially fast, while  $F(v^\dagger)^{n-1-k}$  decreases exponentially fast. Combining this with the above arguments, it follows that  $\Phi(v; G^*)$  stays uniformly below  $\Phi(v; F)$  over  $[v_{n_k}^*, \bar{v}]$ , which is a contradiction. ■