Chain Stability in Trading Networks*

John William Hatfield[†] Scott Duke Kominers[‡] Alexandru Nichifor[§]
Michael Ostrovsky[¶] Alexander Westkamp[‖]

April 2, 2015

Abstract

We show that in general trading networks with bilateral contracts, a suitably adapted *chain stability* concept (Ostrovsky, 2008) is equivalent to *stability* (Hatfield and Kominers, 2012; Hatfield et al., 2013) if all agents' preferences are fully substitutable and satisfy the Laws of Aggregate Supply and Demand. Furthermore, in the special case of trading networks with transferable utility, an outcome is consistent with competitive equilibrium if and only if it is not blocked by any chain of contracts.

*We are grateful to Dave Donaldson, Alvin Roth, Alex Teytelboym, and Rakesh Vohra for helpful comments and suggestions. Kominers thanks the National Science Foundation (grant CCF-1216095 and a graduate research fellowship), the Harvard Milton Fund, the Yahoo! Key Scientific Challenges Program, the John M. Olin Center (a Terence M. Considine Fellowship), the American Mathematical Society, and the Simons Foundation for support. Nichifor received funding from the People Programme (Marie Curie Outgoing International Fellowship) of the European Union's Seventh Framework Programme (FP7/2007-2013) under REA grant agreement 625718. Ostrovsky thanks the Alfred P. Sloan Foundation for financial support. Westkamp received funding from the German Science Foundation and from the People Programme (Marie Curie Intra-European Fellowship) of the European Union's Seventh Framework Programme (FP7/2007-2013) under REA grant agreement 628276.

[†]University of Texas at Austin; `john.hatfield@utexas.edu`.

[‡]Harvard University; `kominers@fas.harvard.edu`.

[§]University of St Andrews and Stanford University; `nichifor@stanford.edu`.

[¶]Stanford University; `ostrovsky@stanford.edu`.

[‖]Maastricht University; `acwestk@gmail.com`.

1 Introduction

Cooperative solution concepts in game theory often rely on coordinated deviations by large groups of agents, including, in some cases, all the agents in the economy. A natural question when considering such deviations is how (and whether) such coalitions can in fact form. Do all the agents in the economy need to consider all the possible deviations by all the possible coalitions? Or is it perhaps sufficient for them to only consider smaller or more structured types of deviations? Do the agents need to reason about the structure of the entire economy in order to discover a profitable deviation, or is it sufficient for each of them to only consider his or her "local" environment?

A different yet related issue arises when considering the concept of competitive equilibrium. Competitive equilibrium requires specifying prices for *all* possible goods and trades in the economy—even those that are not actually traded. Reliance on "hypothetical" prices for untraded goods may be a relatively minor issue in economies with homogeneous goods and centralized trading, where most (or all) goods are actually traded and thus have observed prices. However, this issue is more problematic in economies with heterogeneous goods and personalized pricing, where only a small fraction of all possible trades in fact take place. In such environments, the reliance of competitive equilibrium on "unobserved" prices raises a number of questions and concerns. Do the agents in the economy know what the "unobserved" prices are? Do they need to? Can agents tell whether a particular set of realized contracts is consistent with competitive equilibrium, given that to verify consistency, they would need data that does not actually exist?

A parallel set of issues arises in applied work: An econometrician may want to estimate parameters of interest and would like to use conditions arising from cooperative solution concepts or from equilibrium requirements. Similarly, an econometrician may want to test whether an outcome in a particular market satisfies those conditions. In such cases, when using cooperative solution concepts, does an econometrician need to consider deviations by all possible groups of agents? When studying competitive equilibrium, does he need to "fill out" all the "missing" prices for the trades that do not take place?

Shapley and Shubik (1971), Crawford and Knoer (1981), and Kelso and Crawford (1982) have shown that in two-sided matching environments with substitutable preferences, the answer to the preceding questions is "No": one needs neither rely on coordinated deviations by very large groups of agents nor on optimization under "missing" prices to perform competitive equilibrium analysis. These authors show that in two-sided one-to-one and many-to-one matching markets, competitive equilibrium is, in essence, equivalent to *pairwise stability*, i.e., the requirement that no pair of agents wants to mutually deviate from their assignments, in favor of trading with each other. The pairwise stability requirement does not require specifying any "missing" prices, and does not require coordinated deviations by complex coalitions. Beyond their conceptual interest, over the last decade these results have been extensively used in the empirical and econometric literature on matching markets, in such diverse settings as marriage and dating markets (Choo and Siow, 2006; Hitsch et al., 2010; Galichon and Salanié, 2010; Chiappori et al., 2015), medical residency matching (Agarwal, 2015), the matching of teachers to schools (Boyd et al., 2013), the matching of athletes to professional sports teams (Yang et al., 2009), the market for venture capital investment (Sørensen, 2007), mergers and acquisitions (Akkus et al., 2013), and markets for automotive parts (Fox, 2010a).[1]

In this paper, we establish analogous results for a very rich setting—trading networks with bilateral contracts. We allow agents to be buyers in some contracts and sellers in other contracts, and do not impose *any* restrictions on the network of possible trades. In particular, the market is neither required to have a two-sided structure, nor is the network of possible trades required to have a vertical structure. Our model subsumes settings with discrete and continuous prices, with quasilinear and non-quasilinear utility functions, and with and without indifferences in agents' preferences. We prove two equivalence results. Our main result shows that if all agents' preferences are fully substitutable (Definition 1 of Section 2.1) and satisfy *the Laws of Aggregate Supply and Demand* (Definition 2 of Section 2.1), the notion of *stability* (under which all possible deviations by groups of agents need to be considered) is equivalent to *chain stability*, under which only deviations by agents along a *chain* of contracts need to be considered.[2] Our second equivalence result is a corollary of the main result of the current paper and the results of Hatfield et al. (2013). We show that in trading environments with continuously transferable utility, if all agents' preferences are fully substitutable, then an outcome is consistent with competitive equilibrium[3] if and only if it is not blocked by any chain. Note that this is not a limit result for large economies in the spirit of Debreu and Scarf (1963)—our equivalence result holds for fixed, finite economies.

[1] Fox (2010a,b), Echenique et al. (2013), Agarwal and Diamond (2014), and Menzel (2015), among others, develop econometric techniques for the empirical analysis of two-sided matching markets based on pairwise stability.

[2] Chain stability was originally introduced by Ostrovsky (2008) for a more restrictive, vertical environment in which all trade flows in one direction, from the suppliers of basic inputs to the consumers of final outputs. In the Ostrovsky (2008) environment, any chain of contracts has a beginning and an end, and passes "through" every agent at most once. In the current, richer environment, we adapt the definition of a chain to allow a chain to end at the same node as where it began (thus becoming a "loop"), and to cross itself several times. However, as before, the essential feature of a chain is that it is a "linked" sequence of relationships, such that the agent who is the buyer in a particular relationship is the seller in the next relationship in the sequence. We discuss the concept of chain stability in more detail in Section 2.2 after introducing it formally.

[3] That is, one can "fill out" the missing prices to obtain a competitive equilibrium.

We present three examples that demonstrate the role that our assumptions play in the main equivalence result. The first example shows that if preferences of some agents do not satisfy the Laws of Aggregate Supply and Demand, then chain stable outcomes may not be stable. The second example shows that if preferences of some agents are not fully substitutable, then chain stable outcomes may likewise not be stable. The third example illustrates that if blocking sets are restricted to chains that do not "cross" themselves (i.e., chains that involve each agent in at most two contracts), then an outcome that is robust to such blocks may not be robust to richer blocks. This last example, combined with our equivalence results, illustrates that chain stability plays the same role in the trading network setting as pairwise stability does in two-sided settings: chains are the "essential" blocking sets that one needs to consider to evaluate the stability of an outcome or its consistency with competitive equilibrium.

The remainder of the paper is organized as follows. Section 1.1 provides an overview of related literature. Section 2 introduces our general model. Section 3 states and proves the main result on the equivalence of stability and chain stability. Section 4 states and proves the result on the equivalence of chain stability and consistency with competitive equilibrium for the special case of quasilinear preferences and fully transferable utility. Section 5 presents the examples that show the roles of our assumptions. Section 6 concludes.

1.1 Related Literature

The concept of blocking is fundamental in the analysis of matching markets. In the original papers of Gale and Shapley (1962) and Shapley and Shubik (1971) on stability in two-sided markets, attention is restricted to *pairwise blocks*, i.e., pairs of agents who mutually prefer each other to their assigned partners. The requirement that a two-sided matching be pairwise stable—i.e., be robust to pairwise blocks—seems much weaker than the requirement that a matching be robust to deviations by arbitrary sets of agents. Indeed, in general, in markets where some agents are allowed to match with multiple partners, a matching that is robust to deviations by pairs may not be robust to richer deviations.[4] However, as we discussed in the Introduction, a series of key results in the theory of two-sided, many-to-one matching shows that when agents' preferences are *substitutable* (Kelso and Crawford, 1982; Roth, 1984), pairwise relationships are in fact the essential blocking sets: any pairwise stable matching is also robust to larger deviations.[5]

[4]For example, if every firm in an economy is only interested in hiring an even number of workers, then an empty matching will always be pairwise stable, even in the cases where another, non-empty matching makes all agents in the economy strictly better off.

[5]Hatfield and Kominers (2014) prove this result in a general two-sided matching setting with contracts, and provide an overview of earlier literature on related results in other two-sided settings.

Ostrovsky (2008) introduced a generalization of two-sided matching to "supply chain" environments. In supply chain matching, goods flow downstream from initial producers to end consumers, potentially with numerous intermediaries in between. In the Ostrovsky (2008) framework, attention is restricted to blocking *chains*—sequences of agents who could benefit from re-contracting with each other along a vertical chain. Outcomes robust to chain deviations are said to be *chain stable*. Chain stability is a natural extension of pairwise stability to the setting in which an agent can be both a buyer and a seller; for example, an agent may be willing to sell a unit of output only if he can buy a unit of input required to produce that output. Ostrovsky (2008) showed that when the preferences of all agents in the economy are fully substitutable (see Definition 1 in Section 2.1), chain stable outcomes are guaranteed to exist. Again, chain stability appears to be a much weaker condition than the requirement that an outcome be robust to deviations by arbitrary sets of agents. However, as in the case of pairwise stability, under the assumption that agents' preferences are fully substitutable, chains are in fact the essential blocking sets: Hatfield and Kominers (2012) showed that in the supply chain setting, any chain stable outcome is in fact *stable*, in the sense that it is robust to blocks by arbitrary groups of agents.[6]

Hatfield et al. (2013) dispensed with the vertical structure of the supply chain environment, and instead considered arbitrary trading networks. They also assumed that prices can vary freely (instead of being restricted to a finite discrete set), and that agents' preferences are quasilinear.[7] In their analysis, Hatfield et al. (2013) considered a stability concept analogous to that of Hatfield and Kominers (2012), allowing for re-contracting by arbitrary groups of agents. They showed that when agents' preferences are fully substitutable, stable outcomes exist and are essentially equivalent to competitive equilibria with personalized prices. Our model includes the setting of Hatfield et al. (2013) as a special case, and for that special case, the corollary of our main result is that an outcome is consistent with competitive equilibrium if and only if it is not blocked by any chain of contracts.

Our paper contributes to the literature on the relationships between different solution concepts in matching environments (see, e.g., Sotomayor (1999), Echenique and Oviedo

[6]The setting of Hatfield and Kominers (2012) is a special case of our framework, and for that special case, the definition of stability coincides with ours (Definition 3 in Section 2.2). Note, however, that even in the case of vertical networks, our setting is substantially more general than that of Ostrovsky (2008) and Hatfield and Kominers (2012): we allow for arbitrary sets of contracts (as opposed to just finite ones) and explicitly incorporate the case in which an agent may be indifferent between two different sets of contracts (as opposed to having strict preferences). These generalizations are necessary to define the concept of competitive equilibrium and to establish the connections between chain stable outcomes and competitive equilibria.

[7]If one dispenses with the supply chain structure without assuming that prices can vary freely, then stable outcomes may not exist (Hatfield and Kominers, 2012). Teytelboym (2014) introduced a weaker concept, *path stability*, for settings without a supply chain structure and showed that path stable outcomes are guaranteed to exist.

(2006), Klaus and Walzl (2009), Westkamp (2010), and Hatfield and Kominers (2014)). It also has parallels in the operations research literature on flows in networks (see, e.g., a textbook treatment by Ahuja et al. (1993)); the "flow decomposition lemma" in that literature states that any "flow" in a network can be "decomposed" into a collection of simple "paths" and "cycles," resembling the decomposition of any blocking set into a collection of blocking chains in our Theorem 2 in Section 3. Note, however, that "paths" and "cycles" in the flow decomposition lemma cannot cross themselves, while in our environment, we need to allow for the possibility of chains crossing themselves (see Example 3 in Section 5). This is due to the fact that in the "network flows" environment, there is a single type of good "flowing" through the network, and the objective function is the maximization or minimization of the aggregate flow, whereas in our setting many different types of goods may be present, and the preferences of agents in the market may be more complex.

2 Model

Our model includes as special cases the discrete-price models of Ostrovsky (2008) and Hatfield and Kominers (2012), as well as the continuous-price model of Hatfield et al. (2013). Our notation is closely based on that of Hatfield et al. (2013).

There is an economy with a finite set I of *agents*. Pairs of agents can participate in bilateral *trades*. Each trade ω is associated with a *buyer* $b(\omega) \in I$ and a *seller* $s(\omega) \in I$, with $b(\omega) \neq s(\omega)$. The set of possible trades, denoted Ω, is finite. Note that we require that the buyer and the seller associated with a trade be distinct agents, but we allow Ω to contain multiple trades associated with the same agents, and allow the possibility of trades $\omega \in \Omega$ and $\psi \in \Omega$ such that $s(\omega) = b(\psi)$ and $s(\psi) = b(\omega)$.

A *contract* x is a pair $(\omega, p_\omega) \in \Omega \times \mathbb{R}$ that specifies a trade and an associated price. For a contract $x = (\omega, p_\omega)$, we denote by $b(x) \equiv b(\omega)$ and $s(x) \equiv s(\omega)$ the buyer and the seller associated with the trade ω of x. We denote by $X \subseteq \Omega \times \mathbb{R}$ the set of all contracts available to the agents; this set is fixed and exogenously given. It can be infinite (as, e.g., in the setting of Hatfield et al. (2013), where all prices are allowed for all trades, and thus $X = \Omega \times \mathbb{R}$), or finite (as, e.g., in the settings of Ostrovsky (2008) and Hatfield and Kominers (2012)).

For each agent $i \in I$ and set of contracts $Y \subseteq X$, we let $Y_{i\to} \equiv \{y \in Y : i = s(y)\}$ denote the set of contracts in Y in which i is the seller and $Y_{\to i} \equiv \{y \in Y : i = b(y)\}$ denote the set of contracts in Y in which i is the buyer. We let $Y_i \equiv Y_{i\to} \cup Y_{\to i}$. We let $a(Y) \equiv \bigcup_{y\in Y}\{b(y), s(y)\}$ denote the set of agents involved in Y as either buyers or sellers. Slightly abusing notation, for a contract $x \in X$, we write $a(x) \equiv a(\{x\})$. We use analogous notation for various properties of trades $\omega \in \Omega$ and sets of trades $\Psi \subseteq \Omega$: e.g.,

$a(\omega) = \{b(\omega), s(\omega)\}$ and $\Psi_i \equiv \{\omega \in \Psi : i \in a(\omega)\}$. Finally, we denote by $\tau(Y)$ the set of trades involved in contracts in Y: $\tau(Y) \equiv \{\omega \in \Omega : (\omega, p_\omega) \in Y \text{ for some } p_\omega \in \mathbb{R}\}$.

A set of contracts $Y \subseteq X$ is *feasible* if it does not contain two or more contracts associated with the same trade: formally, $Y \subseteq X$ is feasible if $(\omega, p_\omega), (\omega, \hat{p}_\omega) \in Y$ implies that $p_\omega = \hat{p}_\omega$; equivalently, $Y \subseteq X$ is feasible if $|Y| = |\tau(Y)|$. An *outcome* is a feasible set of contracts.

2.1 Preferences

Each agent i has a utility function U_i over feasible sets $Y \subseteq X_i$ of contracts that involve i as the buyer or the seller. For a feasible set $Y \subseteq X_i$, $U_i(Y) \in \mathbb{R} \cup \{-\infty\}$, with the value of $-\infty$ used to denote sets of contracts that are technologically impossible for the agent to undertake (e.g., selling the same object to two different buyers). We assume that $U_i(\varnothing) \in \mathbb{R}$, i.e., any agent's utility from the "outside option" of not forming any contracts is finite.

The *choice correspondence* of agent i from a set of contracts $Y \subseteq X_i$ is defined as the collection of sets of contracts maximizing the utility of agent i:

$$C_i(Y) \equiv \{Z \subseteq Y : Z \text{ is feasible; } \forall \text{ feasible } Z' \subseteq Y,\ U_i(Z) \geq U_i(Z')\}.^8$$

For notational convenience, we also extend the choice correspondence to sets of contracts that do not necessarily involve agent i: for a set of contracts $Y \subseteq X$, $C_i(Y) \equiv C_i(Y_i)$.

We now introduce our first key condition on preferences, *full substitutability*.[9]

Definition 1. The preferences of agent i are *fully substitutable* if:

1. for all finite sets of contracts $Y, Z \subseteq X_i$ such that $Y_{i\rightarrow} = Z_{i\rightarrow}$ and $Y_{\rightarrow i} \subseteq Z_{\rightarrow i}$, for every $Y^* \in C_i(Y)$ there exists $Z^* \in C_i(Z)$ such that $(Y_{\rightarrow i} \smallsetminus Y^*_{\rightarrow i}) \subseteq (Z_{\rightarrow i} \smallsetminus Z^*_{\rightarrow i})$ and $Y^*_{i\rightarrow} \subseteq Z^*_{i\rightarrow}$;
2. for all finite sets of contracts $Y, Z \subseteq X_i$ such that $Y_{\rightarrow i} = Z_{\rightarrow i}$ and $Y_{i\rightarrow} \subseteq Z_{i\rightarrow}$, for every $Y^* \in C_i(Y)$ there exists $Z^* \in C_i(Z)$ such that $(Y_{i\rightarrow} \smallsetminus Y^*_{i\rightarrow}) \subseteq (Z_{i\rightarrow} \smallsetminus Z^*_{i\rightarrow})$ and $Y^*_{\rightarrow i} \subseteq Z^*_{\rightarrow i}$.

Informally, the choice correspondence C_i is fully substitutable if, when the set of options available to i on one side expands, i both rejects a (weakly) larger set of contracts on that side

[8]Note that $C_i(Y)$ may be empty if Y is infinite.

[9]For the case of quasilinear utility functions, the full substitutability definition we use here corresponds to the (CEFS) condition of Hatfield et al. (2015). Thus, the results of Hatfield et al. (2015) imply that (again, for the case of quasilinear utility functions) our definition is equivalent to a number of other substitutability concepts that have originated in several distinct literatures. Ostrovsky (2008) and Hatfield et al. (2013) provide detailed discussions of the implications of full substitutability in various environments.

and selects a (weakly) larger set of contracts on the other side, where "larger" is understood in a set-inclusion sense.

The second property important for our results is that the preferences of all agents satisfy the Laws of Aggregate Supply and Demand.

Definition 2. The preferences of agent i satisfy the *Law of Aggregate Demand* if for all finite sets of contracts $Y, Z \subseteq X_i$ such that $Y_{i\to} = Z_{i\to}$ and $Y_{\to i} \subseteq Z_{\to i}$, for every $Y^* \in C_i(Y)$, there exists $Z^* \in C_i(Z)$ such that $|Z^*_{\to i}| - |Y^*_{\to i}| \geq |Z^*_{i\to}| - |Y^*_{i\to}|$.

The preferences of agent i satisfy the *Law of Aggregate Supply* if for all finite sets of contracts Y and Z such that $Y_{i\to} \subseteq Z_{i\to}$ and $Y_{\to i} = Z_{\to i}$, for every $Y^* \in C_i(Y)$, there exists $Z^* \in C_i(Z)$ such that $|Z^*_{i\to}| - |Y^*_{i\to}| \geq |Z^*_{\to i}| - |Y^*_{\to i}|$.

Informally, the choice correspondence C_i satisfies the Laws of Aggregate Supply and Demand if, when the set of options available to i on one side expands, the net change in the number of contracts chosen by the agent on that side of the market is at least as large as the net change in the number of contacts chosen by the agent on the other side of the market. These conditions extend the canonical Law of Aggregate Demand (Hatfield and Milgrom (2005); see also Alkan and Gale (2003)) to the current setting, in which each agent can be both a buyer in some trades and a seller in others. Hatfield et al. (2015) showed that in quasilinear settings, full substitutability implies the Laws of Aggregate Supply and Demand; however, full substitutability does not imply the Laws of Aggregate Supply and Demand in general (see, e.g., Example 1 of Section 5 below).

2.2 Stability and Chain Stability

Our main result connects two solution concepts for trading network settings: *stability*, based on the concepts introduced by Hatfield and Kominers (2012) and Hatfield et al. (2013), and *chain stability*, based on the concept introduced by Ostrovsky (2008).

We begin with the definition of stability.

Definition 3. An outcome A is *stable* if it is

1. *Individually rational:* $A_i \in C_i(A)$ for all i;
2. *Unblocked:* There is no feasible nonempty *blocking set* $Z \subseteq X$ such that
   (a) $Z \cap A = \varnothing$, and
   (b) for all $i \in a(Z)$, for all $Y \in C_i(Z \cup A)$, we have $Z_i \subseteq Y$.

Individual rationality is a voluntary participation condition based on the idea that an agent can always unilaterally drop contracts if doing so increases his welfare. The unblockedness condition states that when presented with a stable outcome A, one cannot propose a new set of contracts such that all the agents involved in these new contracts would strictly prefer to execute all of them (and possibly drop some of their existing contracts in A) instead of executing only some of them (or none).

To introduce our second solution concept, chain stability, we first need to formalize the notion of a *chain*.

Definition 4. A non-empty set of contracts Z is a *chain* if its elements can be arranged in some order $y^1, \ldots, y^{|Z|}$ such that $s(y^{\ell+1}) = b(y^{\ell})$ for all $\ell \in \{1, 2, \ldots, |Z| - 1\}$.

Note that because there is no vertical ordering of agents in our framework, Definition 4 adapts the "chain" concept of Ostrovsky (2008) to our current framework by allowing chains to cross themselves: the buyer in contract $y^{|Z|}$ is allowed to be the seller in contract y^1 (in which case the chain becomes a cycle), and also a given agent can be involved in the chain multiple times. Example 3 of Section 5 illustrates the role of "self-crossing" chains in our results.

We now define chain stability.

Definition 5. An outcome A is *chain stable* if it is

1. *Individually rational:* $A_i \in C_i(A)$ for all i;
2. *Not blocked by a chain*: There does not exist a feasible *blocking chain* $Z \subseteq X$ such that
   (a) $Z \cap A = \varnothing$, and
   (b) for all $i \in a(Z)$, for all $Y \in C_i(Z \cup A)$, we have $Z_i \subseteq Y$.

The essential difference between the definitions of stability (Definition 3) and chain stability (Definition 5) consists of just one word: "set" in requirement 2 in the definition of stability versus "chain" in requirement 2 in the definition of chain stability. Substantively, however, these definitions are very different. Blocking sets considered in Definition 3 can be arbitrarily complex, involving any sets of contracts and agents. By contrast, blocking chains considered in Definition 5 have a well-defined linear structure. As Ostrovsky (2008) argued, blocking chains are much easier to identify and organize than arbitrary blocking sets. An agent can contact a potential supplier and propose a possible contract. The supplier would then contact one of his suppliers, and so on, and the process would proceed in a linear fashion until a blocking chain is identified. An important difference in our setting is that in the case of loops, the agent initiating the communication may need to make his initial offer tentative: instead of proposing a contract outright, he would have to say something along the lines of: "I may be interested in buying contract x from you, if I get a request from a customer for one of the contracts I am offering." An initial agent may also try initiating the deviation in both directions at the same time, making tentative offers to a supplier and a customer.[10] While this type of communication is more complicated than identifying pairwise blocks in two-sided settings or chain blocks in the setting of Ostrovsky (2008), it is still relatively simple and natural, compared to trying to identify grand coalitions and large, complex, "non-linear" sets of blocking contracts.

3 Main Result: Equivalence of Stability Concepts

Stability appears substantively different and noticeably stronger than chain stability: the former requires robustness to all blocking sets, while the latter requires robustness only to specific blocking sets—chains of contracts. It is immediate that any stable outcome is chain stable, even if agents' preferences are not fully substitutable or do not satisfy the Laws of Aggregate Supply and Demand. Our main result shows that when agents' preferences *are* fully substitutable and *do* satisfy the Laws of Aggregate Supply and Demand, the two solution concepts are in fact equivalent.

Theorem 1. *If all agents' preferences are fully substitutable and satisfy the Laws of Aggregate Supply and Demand, then any chain stable outcome is stable.*

Theorem 1 is an immediate corollary of a stronger result: when agents' preferences are fully substitutable and satisfy the Laws of Aggregate Supply and Demand, any set blocking an outcome A can be "decomposed" into blocking chains.

Theorem 2. *Suppose that all agents' preferences are fully substitutable and satisfy the Laws of Aggregate Supply and Demand. Consider any outcome A that is blocked by some nonempty set Z. Then for some $M \geq 1$, we can partition the set Z into a collection of M chains W^m such that $Z = \cup_{m=1}^{M} W^m$, A is blocked by W^1, and for any $m \leq M-1$, the set of contracts $A \cup W^1 \cup \cdots \cup W^m$ is blocked by chain W^{m+1}.*

To prove Theorem 2, we prove the following Lemma, which implies the statement of the Theorem 2 by a natural inductive argument.

Lemma 1. *For any feasible outcome A blocked by a nonempty set Z, if Z is not itself a chain, then there exists a nonempty chain $W \subsetneq Z$ such that set A is blocked by $Z \smallsetminus W$ and set $A \cup (Z \smallsetminus W)$ is blocked by W.*

[10] In a sense, this is similar to how the proof of our main result works, although there are of course some important differences: e.g., our proof starts with a complex blocking set and iteratively simplifies it by "extracting" chains one by one, instead of immediately constructing a single blocking chain.

Before proceeding to the formal proof of Lemma 1, we explain the intuition behind the result and highlight the step in the proof where our assumptions on preferences are used. Suppose for simplicity that there are no indifferences in agents' preferences over any relevant sets of contracts, that is, that all agents' choice correspondences over subsets of $A \cup Z$ are single-valued. Our goal is to "peel off" a chain, W, from the set Z in such a way that the remaining set $Z \smallsetminus W$ still blocks A.

Take any contract $y \in Z$ and consider its buyer, $j = b(y)$, and seller, $k = s(y)$. From the definition of blocking, we know that all of the contracts in Z_j are in the choice of j from $A \cup Z$, and all of the contracts in Z_k are in the choice of k from $A \cup Z$.

What happens if we remove contract y from Z? If it so happens that all of the contracts in $Z_j \smallsetminus \{y\}$ are in the choice of j from $A \cup Z \smallsetminus \{y\}$, and likewise all of the contracts in $Z_k \smallsetminus \{y\}$ are in the choice of k from $A \cup Z \smallsetminus \{y\}$, then we are done—contract y is a (trivial) chain that can be "peeled off" from Z.

Suppose, to the contrary, that not all of the contracts in $Z_j \smallsetminus \{y\}$ are in the choice of j from $A \cup Z \smallsetminus \{y\}$. This is the critical step, where we use our assumptions on preferences: First, as the preferences of j are fully substitutable, we know that all of the contracts in $Z_{\to j} \smallsetminus \{y\}$ are in the choice of j from $A \cup Z \smallsetminus \{y\}$. Second, by the Law of Aggregate Demand, we know that there is exactly one contract in $Z_{j\to}$—say, y'—that is not in the choice of j from $A \cup Z \smallsetminus \{y\}$. Thus, all contracts in $Z_j \smallsetminus \{y, y'\}$ are in the choice of j from $A \cup Z \smallsetminus \{y, y'\}$—which is exactly the blocking condition (for agent j).

Of course, this condition may still not be satisfied for the seller of contract y or the buyer of contract y', in which case we need to continue expanding the "peeled off" chain in the analogous manner (possibly in both directions). At some point, the process has to stop (because set Z is finite), giving us the required chain W such that A is blocked by $Z \smallsetminus W$.

Proof of Lemma 1. The second part of the statement of Lemma 1, that $A \cup (Z \smallsetminus W)$ is blocked by W, is true for any $W \subseteq Z$: Set Z blocks A, and thus for every agent i involved in W (and hence in Z), every choice of i from $A \cup Z = (A \cup (Z \smallsetminus W)) \cup W$ contains every contract in Z_i, and thus contains every contract in $W_i \subseteq Z_i$. So we only need to prove the first part of the statement, i.e., that we can "remove" some chain W from Z so that the remaining set $Z \smallsetminus W$ still blocks A.

For the following, it is convenient to introduce additional notation. Let $\widehat{C}_i(\cdot)$ denote the *modified choice correspondence* defined on subsets of Z as follows: for $T \subseteq Z$,

$$\widehat{C}_i(T) = \{T' \subseteq T : \text{for some set } Y \in C_i(A \cup T),\ T' = Y \cap Z\}.$$

Note that from the definition of a blocking set, it follows that $\widehat{C}_i(Z) = \{Z_i\}$.

Take any contract $y \in Z$, and let $y^0 = y$. We will algorithmically "grow" a chain

$$W^{\ell_s,\ell_b} \equiv \{y^{\ell_s}, \dots, y^0, \dots, y^{\ell_b}\} \subseteq Z$$

by applying (generally in both directions from y^0) the iterative procedure below, starting with $\ell_s = \ell_b = 0$ and $W^{0,0} = \{y^0\}$. (During the procedure, ℓ_s will be decreasing, and thus will always be less than or equal to 0, while ℓ_b will be increasing, and thus will always be greater than or equal to 0.) We ensure that at every step of the procedure, the following four conditions hold for every agent i:

1. If $i \neq b(y^{\ell_b})$, $i \neq s(y^{\ell_s})$, then $\widehat{C}_i(Z \smallsetminus W^{\ell_s,\ell_b}) = \{(Z \smallsetminus W^{\ell_s,\ell_b})_i\}$.
2. If $i = b(y^{\ell_b}) \neq s(y^{\ell_s})$, then $\widehat{C}_i((Z \smallsetminus W^{\ell_s,\ell_b}) \cup \{y^{\ell_b}\}) = \{(Z \smallsetminus W^{\ell_s,\ell_b})_i \cup \{y^{\ell_b}\}\}$.
3. If $i = s(y^{\ell_s}) \neq b(y^{\ell_b})$, then $\widehat{C}_i((Z \smallsetminus W^{\ell_s,\ell_b}) \cup \{y^{\ell_s}\}) = \{(Z \smallsetminus W^{\ell_s,\ell_b})_i \cup \{y^{\ell_s}\}\}$.
4. If $i = s(y^{\ell_s}) = b(y^{\ell_b})$, then $\widehat{C}_i((Z \smallsetminus W^{\ell_s,\ell_b}) \cup \{y^{\ell_b}, y^{\ell_s}\}) = \{(Z \smallsetminus W^{\ell_s,\ell_b})_i \cup \{y^{\ell_b}, y^{\ell_s}\}\}$.

These conditions are satisfied in the beginning when $\ell_s = \ell_b = 0$ and $W^{0,0} = \{y^0\}$. Suppose now that we have a non-empty chain W^{ℓ_s,ℓ_b} (with $\ell_s \leq 0$ and $\ell_b \geq 0$) that satisfies conditions (1)–(4). Let $j = b(y^{\ell_b})$ and $k = s(y^{\ell_s})$.

Stopping Condition. If, in addition to satisfying conditions (1)–(4), W^{ℓ_s,ℓ_b} is such that $\widehat{C}_j(Z \smallsetminus W^{\ell_s,\ell_b}) = \{(Z \smallsetminus W^{\ell_s,\ell_b})_j\}$ and $\widehat{C}_k(Z \smallsetminus W^{\ell_s,\ell_b}) = \{(Z \smallsetminus W^{\ell_s,\ell_b})_k\}$, then set $Z \smallsetminus W^{\ell_s,\ell_b}$ blocks outcome A, and we are done.[11]

Incrementing Step. Suppose the Stopping Condition is not satisfied. To simplify notation, let $S = Z \smallsetminus W^{\ell_s,\ell_b}$. We have three cases to consider.

If $j \neq k$ and $\widehat{C}_j(S) \neq \{S_j\}$, take any $T \in \widehat{C}_j(S)$ such that $T \neq S_j$. By condition (2) of the inductive assumption on chain W^{ℓ_s,ℓ_b}, we have $\widehat{C}_j(S \cup \{y^{\ell_b}\}) = \{S_j \cup \{y^{\ell_b}\}\}$. Combined with the full substitutability of the original choice function C_j, this implies $S_{\to j} \subseteq T$. Also, by the Law of Aggregate Demand for choice function C_i, it implies that T excludes at most one contract in $S_{j\to}$—and since $T \neq S_j$, it must exclude exactly one such contract. Denote that excluded contract by y^{ℓ_b+1}, resulting in the extended chain $W^{\ell_s,\ell_b+1} = W^{\ell_s,\ell_b} \cup \{y^{\ell_b+1}\}$. Note that by construction, $\widehat{C}_j(Z \smallsetminus W^{\ell_s,\ell_b+1}) = \{(Z \smallsetminus W^{\ell_s,\ell_b+1})_j\}$,[12] and thus all four required

[11]Note that it may happen that $j = k$, in which case the two conditions $\widehat{C}_j(Z \smallsetminus W^{\ell_s,\ell_b}) = \{(Z \smallsetminus W^{\ell_s,\ell_b})_j\}$ and $\widehat{C}_k(Z \smallsetminus W^{\ell_s,\ell_b}) = \{(Z \smallsetminus W^{\ell_s,\ell_b})_k\}$ become identical.

[12]It is immediate that $(Z \smallsetminus W^{\ell_s,\ell_b+1})_j = T \in \widehat{C}_j(Z \smallsetminus W^{\ell_s,\ell_b+1})$. To see that there are no other elements in set $\widehat{C}_j(Z \smallsetminus W^{\ell_s,\ell_b+1})$, note that for any $T' \subsetneq T$, we must have $S_{\to j} \nsubseteq T'$ or $|S_{j\to} \smallsetminus T'| \geq 2$. By the previous arguments, this implies $T' \notin \widehat{C}_j(S)$. Since $T \in \widehat{C}_j(S)$, we have $U_j(T) > U_j(T')$, and thus $T' \notin \widehat{C}_j(Z \smallsetminus W^{\ell_s,\ell_b+1})$.

conditions are satisfied for W^{ℓ_s,ℓ_b+1}. This chain is strictly longer than the initial chain W^{ℓ_s,ℓ_b}, and we can now go the next iteration of our algorithm (checking whether W^{ℓ_s,ℓ_b+1} satisfies the Stopping Condition above and applying the Incrementing Step if it does not).

If $j \neq k$, $\widehat{C}_j(S) = \{S_j\}$, and $\widehat{C}_k(S) \neq \{S_k\}$, we can use arguments analogous to those used to identify y^{ℓ_b+1} so as to identify a contract $y^{\ell_s-1} \in S_{\to k}$ such that $W^{\ell_s-1,\ell_b} = \{y^{\ell_s-1}\} \cup W^{\ell_s,\ell_b}$ is a strictly larger chain satisfying conditions (1)–(4), and we can again go to the next iteration of our algorithm, applying it to W^{ℓ_s-1,ℓ_b}.

Finally, we must consider the case in which $j = k$ and $\widehat{C}_j(S) \neq \{S_j\}$. Note that as before, by the full substitutability of the original choice function C_j and the fact that C_j satisfies the Laws of Aggregate Supply and Demand, each $T \in \widehat{C}_j(S)$ excludes at most one upstream contract and at most one downstream contract from S_j.[13] Take any such excluded contract y (i.e., a contract y such that for some $T \in \widehat{C}_j(S)$, $y \in (S \smallsetminus T)_j$), and suppose it is a downstream contract for agent j, i.e., $y \in S_{j\to}$ (the argument for the case $y \in S_{\to j}$ is completely analogous). Let $y^{\ell_b+1} = y$. Each $T \in \widehat{C}_j(S \smallsetminus \{y^{\ell_b+1}\})$ is, by construction, also in $\widehat{C}_j(S)$. Each such T contains $(S \smallsetminus \{y^{\ell_b+1}\})_{j\to}$ as a subset and excludes at most one contract from $(S \smallsetminus \{y^{\ell_b+1}\})_{\to j}$. If it so happens that each T contains $(S \smallsetminus \{y^{\ell_b+1}\})_{\to j}$ as a subset (i.e., $\widehat{C}_j(S \smallsetminus \{y^{\ell_b+1}\}) = \{(S \smallsetminus \{y^{\ell_b+1}\})_j\}$), then the extended chain $W^{\ell_s,\ell_b+1} = W^{\ell_s,\ell_b} \cup \{y^{\ell_b+1}\}$ satisfies conditions (1)–(4), and we can again go to the next iteration of our algorithm, applying it to $W^{\ell_s,\ell_b} \cup \{y^{\ell_b+1}\}$. Otherwise, we can select any $T \in \widehat{C}_j(S \smallsetminus \{y^{\ell_b+1}\})$ that does not contain $(S \smallsetminus \{y^{\ell_b+1}\})_{\to j}$ as a subset. By the preceding arguments, there is a unique contract y' in $(S \smallsetminus \{y^{\ell_b+1}\})_{\to j} \smallsetminus T_{\to j}$; we set $y^{\ell_s-1} = y'$, thus extending the chain to $W^{\ell_s-1,\ell_b+1} = \{y^{\ell_s-1}\} \cup W^{\ell_s,\ell_b} \cup \{y^{\ell_b+1}\}$. This extended chain satisfies conditions (1)–(4), and we can go to the next iteration of our algorithm, applying it to W^{ℓ_s-1,ℓ_b+1}. This concludes the Incrementing Step.

Since set Z is finite, this algorithm must terminate, resulting in some chain W^{ℓ_s,ℓ_b}. At every iteration, we ensured that for each agent $i \notin \{b(y^{\ell_b}), s(y^{\ell_s})\}$, $\widehat{C}_i(Z \smallsetminus W^{\ell_s,\ell_b}) = \{(Z \smallsetminus W^{\ell_s,\ell_b})_i\}$. The algorithm's Stopping Condition ensures that the same equality also holds for $i \in \{b(y^{\ell_b}), s(y^{\ell_s})\}$. Thus, set A is blocked by $Z \smallsetminus W^{\ell_s,\ell_b}$. □

[13] By condition (4) of the inductive assumption on chain W^{ℓ_s,ℓ_b}, we have $\widehat{C}_j(S \cup \{y^{\ell_b}, y^{\ell_s}\}) = \{S_j \cup \{y^{\ell_b}, y^{\ell_s}\}\}$. By full substitutability and the Law of Aggregate Supply, any $T \in \widehat{C}_j(S \cup \{y^{\ell_b}\})$ has to be such that $S_{j\to} \subseteq T$ and $|(S_{\to j} \cup \{y^{\ell_b}\}) \smallsetminus T| \leq 1$. Now pick an arbitrary $T' \in \widehat{C}_j(S)$. Since $S_{j\to} \subseteq T$ for all $T \in \widehat{C}_j(S \cup \{y^{\ell_b}\})$, the Law of Aggregate Demand implies $|S_{j\to} \smallsetminus T'| \leq 1$. Since $|(S_{\to j} \cup \{y^{\ell_b}\}) \smallsetminus T| \leq 1$ for all $T \in \widehat{C}_j(S \cup \{y^{\ell_b}\})$, full substitutability implies $|S_{\to j} \smallsetminus T'| \leq 1$: If $|S_{\to j} \smallsetminus T'| \geq 2$, there would have to exist a $T \in \widehat{C}_j(S \cup \{y^{\ell_b}\})$ such that $S_{\to j} \smallsetminus T' \subseteq (S_{\to j} \cup \{y^{\ell_b}\}) \smallsetminus T$ and hence $|(S_{\to j} \cup \{y^{\ell_b}\}) \smallsetminus T| \geq 2$.

4 Chain Stability and Competitive Equilibrium

The results of Section 3 hold for general sets of contracts and general utility functions (assuming full substitutability and the Laws of Aggregate Supply and Demand). In this section, we present a corollary of those general results for a more restricted environment, in which prices are continuous and unrestricted (i.e., the set of contracts X is $X = \Omega \times \mathbb{R}$) and agents' preferences are fully substitutable and quasilinear in prices. For this environment, Hatfield et al. (2013) showed that an outcome is stable if and only if it is consistent with competitive equilibrium. Thus, a corollary of Theorem 1 is that in the trading network setting of Hatfield et al. (2013), an outcome is consistent with competitive equilibrium if and only if it is not blocked by a chain of contracts. Before stating this result formally, we need to give several definitions.

Definition 6. Utility function U_i is *quasilinear in prices* if there exists a valuation function u_i from the sets of trades involving agent i to $\mathbb{R} \cup \{-\infty\}$ such that for any feasible set $Y \subseteq X_i$,

$$U_i(Y) = u_i(\tau(Y)) + \sum_{(\omega, p_\omega) \in Y_{i\rightarrow}} p_\omega - \sum_{(\omega, p_\omega) \in Y_{\rightarrow i}} p_\omega.$$

Definition 7. Outcome Y is *consistent with competitive equilibrium* if there exists a vector of prices of all trades in the economy, $p \in \mathbb{R}^{|\Omega|}$, such that

- for every $\omega \in \tau(Y)$, $(\omega, p_\omega) \in Y$, and
- for every agent i, for every set of trades $\Phi \subseteq \Omega_i$,

$$U_i(Y_i) \geq u_i(\Phi) + \sum_{\omega \in \Phi_{i\rightarrow}} p_\omega - \sum_{\omega \in \Phi_{\rightarrow i}} p_\omega.$$

An outcome Y only specifies prices for the trades that are in fact executed under the outcome, while a competitive equilibrium specifies prices for all the trades in the economy. For an outcome to be consistent with competitive equilibrium it has to be the case that one can specify prices for the trades that are not executed in such a way that under the resulting vector of prices p, for each agent i, selecting the contracts specified by outcome Y is in fact consistent with profit maximization. Definition 7 formalizes this requirement.

We are now ready to state and prove the main result of this Section.

Corollary 1. *Suppose that the set of contracts is $X = \Omega \times \mathbb{R}$, and that all agents' preferences are fully substitutable and quasilinear in prices. Then, an outcome is consistent with competitive equilibrium if and only if it is chain stable.*

Proof. Under the assumed conditions on X and agents' preferences, Theorem 9 of Hatfield et al. (2015) implies that all agents' utility functions satisfy the Laws of Aggregate Supply and Demand. Thus, by our Theorem 1, an outcome is chain stable if and only if it is stable. Moreover, by Theorems 5 and 6 of Hatfield et al. (2013), an outcome is stable if and only if it is consistent with competitive equilibrium. Thus, an outcome is chain stable if and only if it is consistent with competitive equilibrium. □

Checking for consistency with competitive equilibrium becomes particularly simple in vertical environments, with a flow of goods and services from the suppliers of basic inputs to the consumers of final outputs. One only needs to look at "simple" chain blocks that start at an agent and go downstream to another agent, with no loops, and passing through every agent on the way at most once. Corollary 1 also implies that checking for consistency with competitive equilibrium becomes straightforward in two-sided environments with gross substitutes and complements (Sun and Yang, 2006, 2009). In such environments, one side of the market contains two groups of objects. Agents on the other side of the market view objects in the same group as substitutes for each other, but view objects in different groups as complements (a natural example of such preferences is a firm that has two types of complementary inputs). As Hatfield et al. (2013) showed, such environments are special cases of our trading network framework.[14] And chains in two-sided environments with gross substitutes and complements become particularly simple: they consist either of one agent and one object (or more formally, one contract between an agent and an object) or of one agent and one object from each of the two groups (again, more formally, two contracts, involving the same agent and two objects from different groups). Thus, checking for consistency with competitive equilibrium reduces to checking one- and two-contract blocking chains.

5 Examples

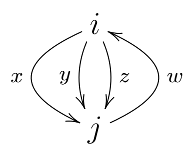
The proof of our main equivalence result (Theorem 1) depends both on the full substitutability of preferences and on the Laws of Aggregate Supply and Demand. In this section, we show that without our assumptions on preferences, the equivalence result may not hold. We also show that it is essential that the definition of chain stability allow chains to cross themselves, i.e., that we allow an agent to be involved in more than two contracts in a chain.[15]

[14] The embedding of Hatfield et al. (2013) allows for much more general environments than those considered by Sun and Yang (2006, 2009): e.g., "objects" may have preferences over whom they match with, and may be involved in multiple contracts.

[15] For convenience, we give our examples in terms of ordinal preference relations over sets of contracts. In these examples, it is straightforward to construct corresponding cardinal utility functions over sets of contracts that give rise to these ordinal preference relations, and we omit those constructions.

We start with an example of preferences that are fully substitutable, but for which the Laws of Aggregate Supply and Demand do not hold—and the equivalence result does not hold either.

Example 1. There are two agents, i and j. There are four contracts between these two agents: x, y, z, and w. Agent i is the seller of x, y, and z, and is the buyer of w, while agent j is thus the buyer of x, y, and z, and the seller of w. The economy is depicted in the diagram below, with each arrow denoting a contract from a seller to a buyer:



The preferences of the agents are as follows. Informally, agent i is happy to sign contract w in which he is the buyer, regardless of what his options are on the other side of the market, and if he is able to sign contract w, then he is also happy to sign any subset of the other three contracts (in which he is the seller)—the more, the better. Formally, the preferences of i over acceptable bundles of contracts are as follows:

$$\{w,x,y,z\} \succ \{w,x,y\} \succ \{w,x,z\} \succ \{w,y,z\} \succ \{w,x\} \succ \{w,y\} \succ \{w,z\} \succ \{w\} \succ \varnothing.$$

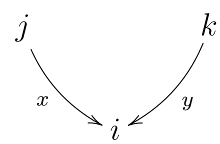
Agent j is happy to sign any subset of $\{x,y,z\}$ (in which he is the buyer)—the more, the better—no matter what his options are on the other side of the market. If he has access to all three of these contracts, then he is also happy to sign contract w (in which he is the seller). Formally, the preferences of j over acceptable bundles of contracts are as follows:

$$\{w,x,y,z\} \succ \{x,y,z\} \succ \{x,y\} \succ \{x,z\} \succ \{y,z\} \succ \{x\} \succ \{y\} \succ \{z\} \succ \varnothing.$$

Note first that the preferences of agents i and j are fully substitutable (but also note that the preferences of agent i do not satisfy the Law of Aggregate Demand). Consider now the empty set of contracts. It is not stable: it is blocked by the full set of contracts in the economy, $\{w,x,y,z\}$, which is the most preferred set of contracts for both agents. At the same time, the empty set of contracts is not blocked by any chain, and is therefore chain stable. To see this, note first that any blocking set would of course have to involve both agents. Second, every non-empty set acceptable to agent i must include contract w, so w would have to be a part of the blocking chain. Third, the only set of contracts involving contract w that is acceptable to agent j is the full set of contracts $\{w,x,y,z\}$. Thus, $\{w,x,y,z\}$ is the only blocking set in this example—and it cannot be represented as a chain.

Our second example shows that full substitutability likewise plays a critical role for the equivalence result: without it, chain stability is strictly weaker than stability, even when all agents' preferences satisfy the Laws of Aggregate Supply and Demand.

Example 2. There are three agents: i, j, and k. There are two contracts, x and y. Agent i is the buyer of both x and y, agent j is the seller of x, and agent k is the seller of y. The economy is depicted in the diagram below:

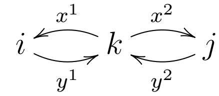


The preferences of agents j and k are straightforward, and satisfy full substitutability: each one prefers to sign the contract in which he is the seller to not signing it: $\{x\} \succ_j \varnothing$ and $\{y\} \succ_k \varnothing$. The preferences of agent i are not fully substitutable: i prefers signing both contracts to signing none, but prefers not signing any contracts to signing only one. That is, the preferences of agent i over acceptable bundles of contracts are given by $\{x, y\} \succ_i \varnothing$.

Consider now the empty set of contracts. It is not stable: it is blocked by the full set of contracts in the economy, $\{x, y\}$. At the same time, the empty set is chain stable: any chain in this market involves agent i and contains exactly one contract—which is not an acceptable deviation for agent i.

Our third and final example shows that even when full substitutability and the Laws of Aggregate Supply and Demand are satisfied, it may not be sufficient to restrict attention to blocking chains that do not "cross" themselves. Specifically, if attention is restricted to chains in which each agent appears in at most two consecutive contracts, then an outcome that is robust to deviations by such chains may be blocked by richer sets of contracts.

Example 3. There are three agents: i, j, and k. There are four contracts: x^1, x^2, y^1, and y^2. Agent i is the buyer in contract in x^1 and the seller in contract y^1. Agent j is the buyer in contract x^2 and the seller in contract y^2. Agent k is the seller in contracts x^1 and x^2 and the buyer in contracts y^1 and y^2. The economy is depicted in the diagram below:



The preferences of agents i and j are straightforward: each one prefers to sign both contracts that he is associated with and is not interested in any other non-empty set of contracts. The preferences of agent k are as follows:

$$\{x^1, x^2, y^1, y^2\} \succ \{x^1, y^2\} \succ \{x^2, y^1\} \succ \varnothing;$$

agent k finds other non-empty sets of contracts unacceptable.

In this example, all agents' preferences are fully substitutable and satisfy the Laws of Aggregate Supply and Demand. Also, the empty set of contracts is unstable: it is blocked by the chain (x^1, y^1, x^2, y^2). Note, however, that this chain is "self-crossing"—it involves agent k in all four contracts. If attention were restricted to chains that do not cross themselves (i.e., in this particular example, chains that involve agent k in at most two contracts), the empty set would be robust to deviations by such chains. For example, the chain (x^1, y^1) does not block the empty set, because $C_k(\{x^1, y^1\}) = \{\varnothing\}$, and so $\{x^1, y^1\}_k = \{x^1, y^1\} \notin C_k(\{x^1, y^1\})$.

6 Conclusion

In this paper, we show that when every agent has fully substitutable preferences satisfying the Laws of Aggregate Supply and Demand, every chain stable outcome is stable. As a corollary of this result, we also show that in quasilinear environments with transferable utility and fully substitutable preferences, an outcome is consistent with competitive equilibrium if and only if it is chain stable.

In practice, blocking chains may be relatively easy to form: they require much less coordination than general blocking sets, and do not rely on "hypothetical," "unobserved" prices for untraded goods. Our work shows that under standard assumptions on preferences, ruling out these particularly natural blocks in fact guarantees that there are no possible blocks by groups of agents. It also ensures that the outcome is in fact consistent with competitive equilibrium, and satisfies the various properties of such equilibria (such as, e.g., efficiency).

Our results may also be useful in facilitating empirical and econometric work on the analysis of trading environments beyond two-sided markets. Instead of considering all possible blocks to a particular outcome, or trying to reconstruct "missing" prices to check for consistency with equilibrium, an econometrician only needs to check for the existence of blocking chains of contracts.

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